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Scale-Invariant Behavior in a Spatial Game of Prisoners' Dilemma

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A spatially extended version of the game of prisoner's dilemma, originally proposed by Nowak and May, is modified to include stochastic updating and found to exhibit scale-invariant behavior. Two critical regimes with different scaling behaviors are found; the corresponding exponents have been determined numerically. Spatially, the critical states are characterized by the existence of delicately balanced networks of defectors separating domains of cooperators; temporally, the evolution of the critical states following local perturbations is characterized by avalanches of various magnitudes, which cause restructuring of the networks of defectors on all scales.

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INTRODUCTION

The paradox of the Prisoners' dilemma has been much studied as a metaphor for the problems associated with the evolution of cooperative behavior in ecology, politics, and economics. The dilemma highlights the conflict between what is best from an individual point of view and that from a collective perspective, which cannot be resolved in the context of playing a single game. The pioneering work of Axelrod and Hamilton [1,2] in the late 1970s, in which the game is studied by playing it many times between players who remember past encounters, created an entirely different way to study the dilemma. The study of the evolutionarily stable strategies of the repeated game has provided insight into the understanding of sociobiological behaviors. When players play a single game, defection is the only evolutionarily stable strategy [3], in the sense that a population which always defects cannot be invaded by a mutant cooperator. For repeated games, evolutionarily stable strategies, such as tit-for-tat can lead to the emergence of cooperation among individuals; this provides a solution to the paradox and a mechanism for the emergence of cooperative behaviors.

Nowak and May and Nowak, Bonhoeffer, and May [4,5] noted that there is an another mechanism for the emergence of cooperative behavior. They considered a simple spatial game of the prisoners' dilemma, where the players only play with their neighbors. In a spatial game, the territorially structured interactions can promote cooperation (even if no follow-up encounter is expected). If the bonus for cheating is not too large, clusters of cooperators will grow. In addition, the spreading of defectors will be limited in the spatial game, because the returns diminish as they interact more with their likes. The actual evolution of the spatial game proposed by Nowak and May depends on the payoff values. In many cases, Nowak and May found the coexistence of defectors and cooperators in interesting evolving mosaics. They suggested that in simpler (biological) systems, cooperation persists by virtue of self-organized spatial structures generated by interactions with immediate neighbors, because the mechanism associated with repeated interaction among individuals who can recognize and remember one another are unlikely at work for simpler systems. The discovery of altruism and cheating in mixed clones of the amoeba, *dictyostelium discoideum* [6], renders the relevance of such spatially extended models to the study of sociobiology more plausible.

In addition to the emergence of cooperation, the evolution of biological, political, or economical systems is often characterized by ''punctuated equilibrium'' [7] with intermittent bursts of activity and volatility interrupting periods of relative tranquility (stasis). Using a simple model of biological evolution, Bak and Sneppen [8] suggested that ''punctuated equilibrium'' is an inherent property of self-organized critical systems [9,10], which naturally evolve into a highly interactive, critical state where minor perturbations lead to avalanches of all sizes. The model proposed by Nowak and May, however, does not exhibit self-organized criticality. Although under some special conditions the model can exhibit many fascinating Persian-carpetlike spatial patterns, the patterns are unstable against any perturbation. Typically, the model either evolves to a stable equilibrium state where a local perturbation only causes slight restructuring of the spatial pattern, or to a highly active state, which never settles down to a period of stasis. The main purpose of this paper is to show that when the effects of a random environment are incorporated appropriately into the model, the spatial game of the prisoners' dilemma can evolve to critical states, where small perturbations can lead to restructuring of the spatial patterns at all scales. Thus, the stochastic, spatial game of the prisoners' dilemma not only provides a mechanism for the emergence of cooperative behaviors, but also serves as an example of ''punctuated equilibrium'' in the evolution of cooperation.

SPATIAL GAME OF PRISONERS' DILEMMA

In an individual game of prisoner's dilemma with two players, each has two options: to cooperate (C or 1) or to defect (D or 0). If both cooperate, each receives one point; a player receives 0 points if the opponent defects and $b(>1)$ if the player defects and the opponent chooses to cooperate. The spatial game of prisoners' dilemma is defined on a two-

dimensional (square) lattice with evolution in discrete time. One assigns a player (who is in state C or D) to each site of the lattice who plays with the eight immediate neighbors and itself at each time step, i.e., the so-called Moore neighborhood is chosen for the interaction. Denoting the strategy of the player at site i at time t by $s_i(t)$ the payoff or the score earned by the player at site i is given by $P_i(t) = \Sigma_{j \in \nu_i}[s_i(t)s_j(t) + b[1 - s_i(t)]s_j(t)]$, where ν_i is the Moore neighborhood of i. In the original model studied by Nowak and May, the updating of the game is based on the average payoffs of individual players, $p_i(t) = P_i(t)/z_i$ where z_i is the number of neighbors (including itself) of site i. Clearly, for an interior player $z_i = 9$ and on the edges z_i depends on the boundary conditions. At the next time step, the player at each site adopts the strategy, $s_h(t)$, of the neighbor (excluding itself) at site h with the highest-average payoff if the highest payoff $p_h(t)$ is greater than the average payoff of the original player at the site, i.e., $s_i(t+1) = s_h(t)$ if $p_h(t) > p_i(t)$. The evolution is deterministic and there is no memory; the behavior is determined by the initial conditions and the parameter b which measures the benefit a defector obtains by exploiting a cooperator. We modify the model by introducing a *stochastic* element to simulate the influence of the environment by assigning an independent random variable δ_i at each site i. In our model, the player with the average payoff p_i will be replaced by the neighbor with the highest-average payoff p_h, if $p_h > p_i + \delta_i$. The variable δ_i is also changed to a new random number if the player is replaced, even if the strategy does not actually change. Thus, the dynamics in our model are no longer deterministic. In our simulation, we choose δ_i independently randomly for each player from the interval $(-\delta_0, \delta_0)$ with δ_0 chosen to be 2/9. One can choose periodic boundary conditions or choose open boundary conditions with players at the corners and on the edges playing only with four and six players, respectively. Most of our results are for the latter boundary condition. We have also studied the model with two boundary layers of players with a fixed strategy of cooperation.

Summary of different behaviors in the model. The goal of our paper is to characterize the asymptotic behavior of the spatial distribution of the strategies and the spatiotemporal response of the system to localized disturbances for various values of b (bonus for cheating) from one to two. There are roughly three regimes corresponding to different ranges of values for b. In the first regime, corresponding to $1 < b < 1.5$, the system evolves to a mostly static configuration (with occasional local periodic flips) dominated by cooperators with local perturbations leading only to a small restructuring of the system. In the second regime, neither cooperators nor defectors dominate. The system always settles down to a static or locally periodic state with a tightly connected network of defectors which extends over the entire system. A local perturbation can leads to restructuring of the network at all scales. The second regime is the most interesting: there are two ranges in which the system exhibits self-organized criticality and responds on all spatial scales when perturbed locally and concomitant power-law behavior, as discussed later. In the third regime, which occurs for $b > 1.8$, defectors clearly dominate and a lone defector can invade the entire population of cooperators. The state is stable against any local perturbation. Between the second and the third regime, there is actually a crossover regime $(1.75 < b < 1.8)$, where moving domains of defectors replace the mostly static network structure observed in the second regime and the system does not settle down to a steady state.

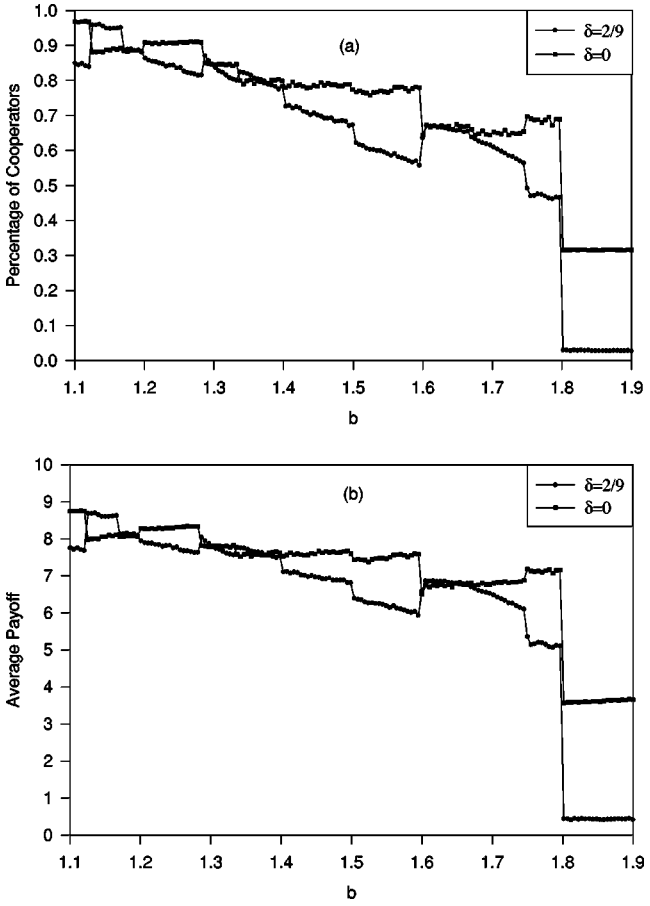

FIG. 1. (a) Concentration of cooperators and (b) average payoff as function of b. Results are obtained using 40×40 systems.

Macroscopic characterization of regimes. We first describe the general features focusing on the *macroscopic* quantities in the model. A quantitative characterization of the different regimes is obtained in terms of the average concentration of cooperators or defectors and the average payoff of the players as a function of b. The averages are obtained by performing both a spatial average at a given time and a temporal average over the states obtained from avalanches. The results are plotted in Figs. 1(a) and 1(b). The first regime is characterized by a relatively low concentration of defectors (less than 1/3) and high-average payoffs. For $b < 3/2$, the fraction of cooperators f_C decreases gradually from 1 to approximately 4/5; especially for b between 1.3 and 1.4, there is very little difference between the fraction of the cooperators in the deterministic and random update rules.

The transition to the second regime is sharp and is accompanied by a jump in the concentration of defectors and a sudden reduction in the average payoffs. At $b \approx 1.50$, there is a discontinuous jump in the fraction of cooperators to

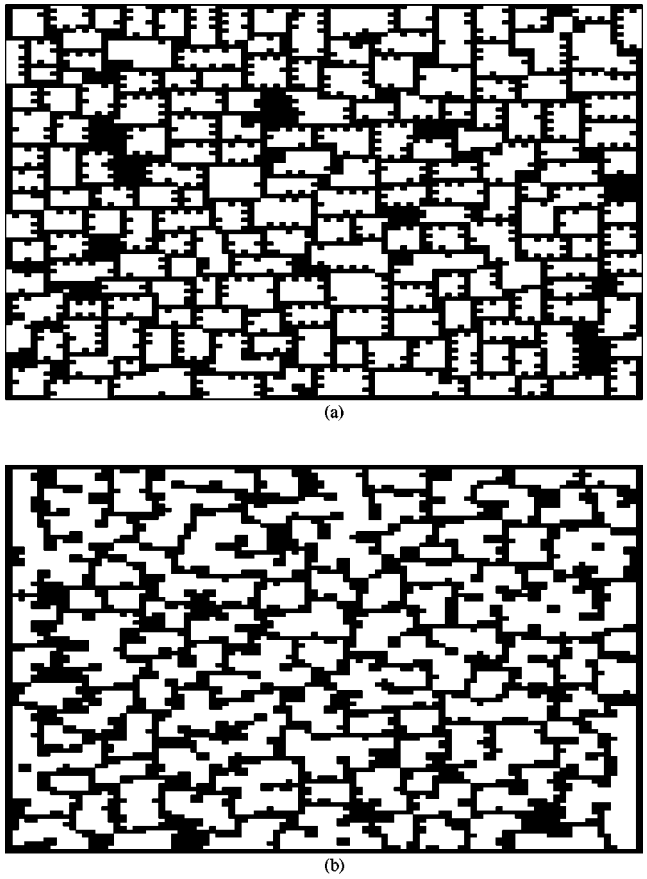

FIG. 2. Typical spatial patterns in the two distinct SOC regions, (a) $b=1.55$, (b) $b=1.70$. The defectors are in black.

roughly 2/3 followed by a steady decrease to approximately 0.60 as b is increased to $b=1.6$; this is in contrast to the behavior of the deterministic model where f_C fluctuates around 0.80. In this regime, the defectors form a network of thin domain walls surrounding more or less rectangular domains of co-operators. This is signaled by the fact that for $b>3/2$ in the deterministic case and for periodic boundary conditions, a row of defectors embedded in a sea of cooperators undergoes a two cycle between three adjacent rows of defectors and one row of defectors [11].

For $1.61<b<1.66$, f_C increases discontinuously to 0.70 and both the deterministic and random rules yield approximately the same density of cooperators. In this regime, the activity takes a very long time to stop in the random model in contrast to the pure case. At approximately $b=1.67$, f_C again increases with a jump to about 2/3 and in the regime $1.67<b<1.75$, f_C steadily decreases to 0.56 mirroring the behavior in the regime $1.50>b>1.60$. Again, in this second regime, we have found evidence for self-organized criticality as will be discussed below. The nature of the spatial patterns and structures is different from that in the regime $1.51<b<1.59$; the domain shapes of cooperators are no longer rectangular and the thickness of the domain walls varies. See Figs. 2(a) and 2(b). For larger b, a few domains of defectors (D) can be found.

The third regime obtains for $b>1.80$, wherein the system consists almost entirely of defectors, $f_C \approx 0.02$ and the average payoff is close to zero. In the crossover region between the second and third regimes, i.e., for $1.76<b<1.8$, both cooperators and defectors form large domains and f_C lie between 0.50 and 0.57. In addition, in this regime the background exhibits a great deal of periodic activity; nevertheless, the overall domain structures are quite stable.

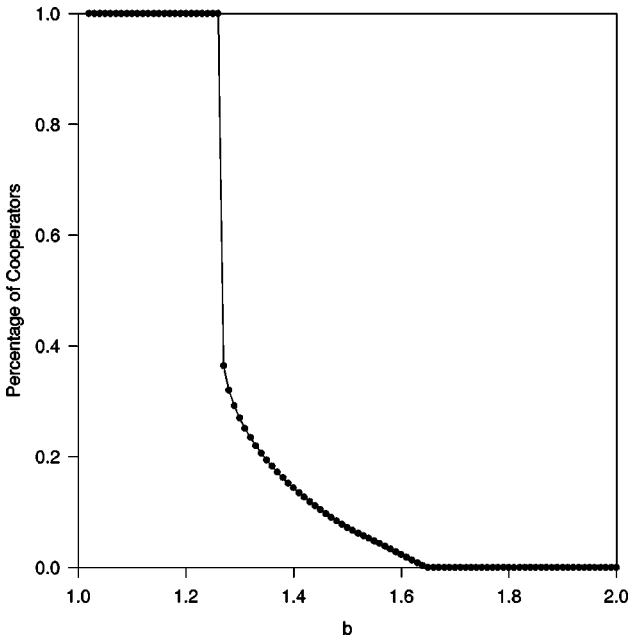

FIG. 3. Concentration of cooperators as a function of b in a simple mean-field theory

We point out in passing that in the deterministic case, the asymptotic concentration of cooperators can depend on initial conditions. For example, for $b>1.8$, the system can go to a state with density of cooperators around 0.3, a state described in Ref. [4], or to a state with zero cooperator density.

Mean-field results. We have also constructed a simple mean-field theory and studied it numerically. We take into account the fact that there are lines of defectors and study a one-dimensional system (i.e., fully correlated lines in two dimensions.) It is straightforward to write an evolution equation for the probability q that a site in one-dimension is a cooperator at the next time step by factorizing the probability distribution for the lattice (the mean-field assumption neglecting correlations.) The future of a site depends on the payoffs of the middle three sites of a five-site cluster: $P(i)=3s(i)+\{s(i)+b[1-s(i)]\}[s(i-1)+s(i+1)]$, where the factor of three accounts for the frozen neighbors in the orthogonal direction. By summing over all possible configurations of the five players appropriately weighted by the probabilities, we find the mean-field evolution of q. We use a value of $\delta=6/9$ and iterate the equation to find the fixed point or the steady-state value for different values of b. The results displayed in Fig. 3 reproduce the existence of three regimes, one in which $p=1$ for $b<1.30$ and one with $p=0$ for b beyond 1.7 and an intermediate regime where p varies continuously. The first transition is discontinuous while the second is continuous. The results for the values of b where the transitions occur are not quantitatively good re-

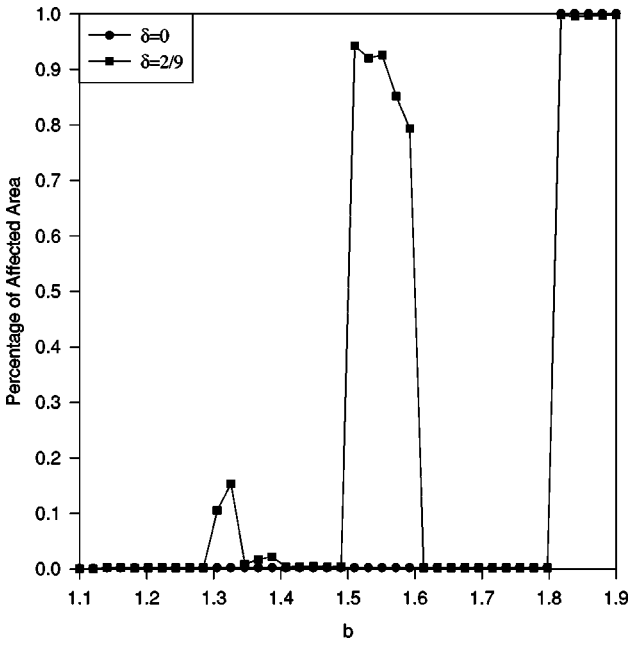

FIG. 4. Percentage of affected area due to a lone defector.

flecting the intricate geometry of the actual configurations which is not captured by the naive, local mean-field theory; however, incorporating the existence of the lines provides a simple description of the macroscopic features of the three regimes.

CRITICAL STATES OF THE MODEL

We now consider the critical regions of the model in more detail focusing on the spatio-temporal response of the system to localized external perturbations. As we mentioned in the previous section, there are two critical regions of the model, corresponding to $1.5<b<1.6$ and $1.67<b<1.75$, respectively. Not only are the spatial patterns in these two regions very different, but so is the dynamical behavior. One simple way of distinguishing the behavior of the system, especially in the two critical regimes, which we have found useful, is to study how an initial condition with a single defector in a sea of cooperators evolves in time. In the first critical region, a lone defector can invade almost the entire initial population consisting of all cooperators and generate a network of defectors covering the entire system; in contrast, in the second region, the lone defector generates only a small cluster of defectors. As we will see shortly, the critical behaviors are also very different. In Fig. 4, the percentage of the area affected by a single defector is plotted for the stochastic model as a function of b. For $b>1.8$, while a defector affects the entire population the resulting configuration is not spatially complex, the entire system becomes defectors.

We now present the results on the statistical critical properties of the model in the critical regions. The stochastic dynamical evolution is carried out on an $L\times L$ lattice until the transients have died down. Typically, the system settles into a state with a few periodic patterns. We make a local perturbation which changes a cooperator to a defector in the system or vice versa at one site. This initiates an avalanche consisting of a sequence of synchronous updates according to the evolution rule during which the effect of the perturbation spreads through the system. In contrast to models such as the Olami-Feder-Christensen [12] stick-slip model, the system does not reach a quiescent state in which the system is stationary. In this respect, this behavior is similar to that in the state which exhibits self-organized criticality in the game of Life [13]. Periodic local flips can continue to occur and therefore, the stochastic nature of the model poses technical difficulties in defining precisely when an avalanche ends. We have defined the end of the avalanche as occurring when the system reaches a periodic state. Once the avalanche stops, we perturb the system again, thus *driving the system slowly*. We have also performed a simulation on two copies of the system, one with and one without the perturbation and compared the two in order to remove spurious contributions to the avalanche due to local periodic patterns. The most commonly used diagnostic to study possible critical behavior is the probability distribution of various characteristics of the avalanches. We study the area affected by the avalanches A, defined as the number of sites where the values of the perturbed and unperturbed copies have differed at least once during the avalanche. We also monitor the the duration of the avalanche T, following the local perturbation and the magnitude (size) M, of the event defined as the total number of flips in the affected area (the periodic flipping outside the affected area is not included).

We have studied systems of sizes ranging from 20×20 up–160×160. For each size of the system, we generate up to 20 000 avalanches (and occasionally more) to obtain the statistics for M, A, and T. Power-law decay of the probability distribution of, say, M, i.e., $P(M)\sim M^{-\tau}$, up to a finite-size cutoff, $M^*(L)$ which grows algebraically with L, $M^*(L)\sim L^{\nu}$, is the typical signature of self-organized criticality. Computational limitations render it extremely difficult to distinguish such critical behavior from exponential decay, $M^{-\tau}\exp(-M/M_0)$ for very large avalanche sizes. Our results, as all numerical results, must therefore be treated with caution.

We found that the probability distribution of these three quantities can be fitted to a power law: $P(M)\propto M^{-\tau}$, $P(A)\propto A^{-\beta}$, and $P(T)\propto T^{-\alpha}$, respectively. Figures 5(a)–5(c) show the scaling plots for the probability distributions of these quantities for the case $b=1.55$ (in the first critical region). The exponents of the power laws are $\tau\approx 1.4$, $\beta\approx 1.65$, and $\alpha\approx 1.75$.

For the second critical region, we have a different set of exponents: $\tau\approx 1.1$, $\beta\approx 1.4$, and $\alpha\approx 1.05$. This clearly shows that these two critical regions are very different. In the second region, there is a significant chance that the system will not settle down to a periodic state, but to a pseudoperiodic state. This is due to the fact that the dynamics is not deterministic whenever a player is replaced. Numerically, we define the end of an avalanche when the system settles down to a pseudoperiodic state, in which the same configuration appears after certain number of time steps.

From a theoretical point of view, the understanding of scale-invariant behavior in systems which are driven slowly (i.e., where one perturbs, waits for the avalanche to end and

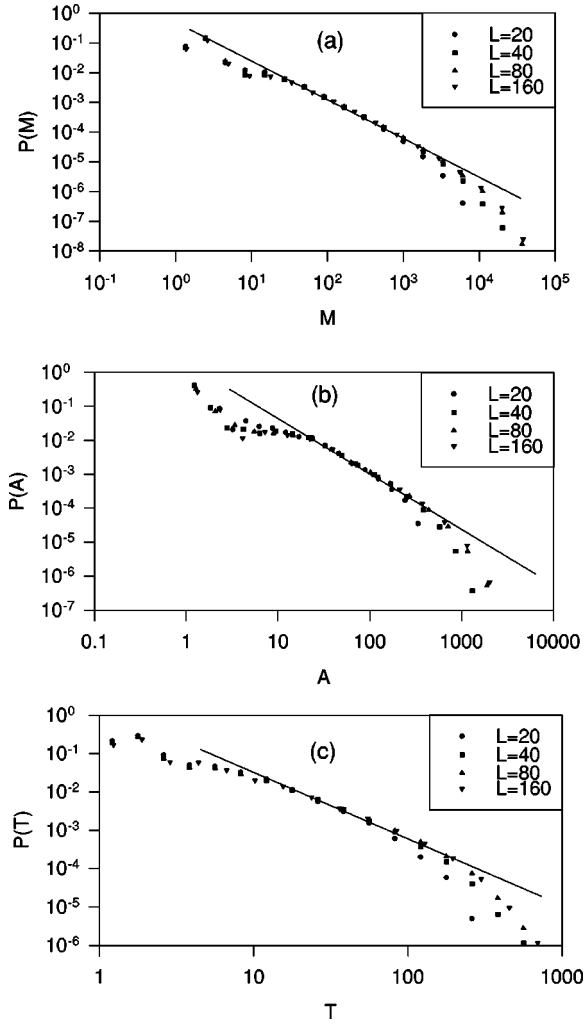

FIG. 5. Probability distribution of (a) M, (b) A, and (c) T for various sizes of the system in the first critical regime; $b=1.55$ is used.

then perturbs again) is meager. Numerically, the existence of power-law scaling in the stick-slip model [12], which do not have an obvious conservation law, as is the case with the model studied in this paper, has been established fairly convincingly numerically. In other models such as the game of life, there is also evidence for power laws. So the existence of power laws in our model is not unusual. However, the stick-slip model is deterministic with randomness occurring only through initial conditions. The original deterministic model does not exhibit scaling behavior for any b; the model studied in this paper has an intermittent underlying stochasticity, which nevertheless only leads to periodic patches in an otherwise quiescent background because of the discrete nature of the model. It is also worth noting that in contrast to the stick-slip model, there is not a great deal of sensitivity to boundary conditions although the scaling is better for some boundary conditions. We also recall that the finite-size scaling hypothesis postulates for any quantity x, (i.e., A, M, or T), $P(x,L)=L^{-p_x}F(x/L^{\nu_x})$. The other possibility is that the system obeys multifractal scaling as in the case of some systems exhibiting self-organized criticality [14]:

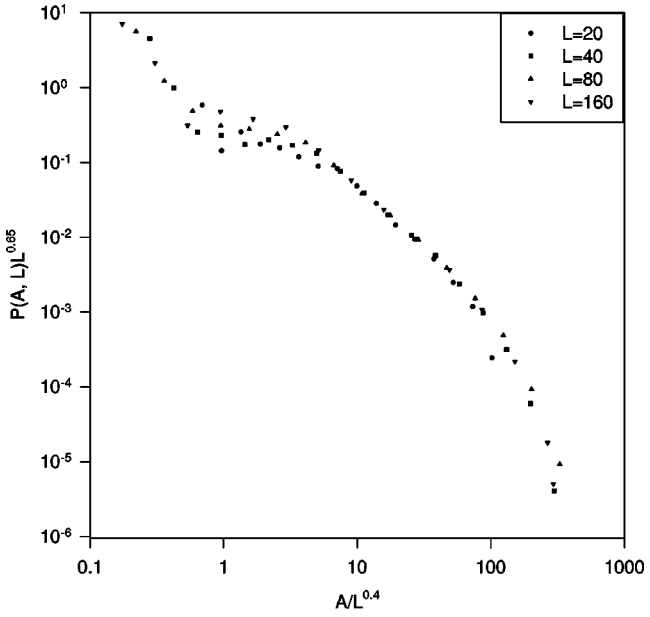

FIG. 6. Scaling plot of $P(A,L)L^{0.65}$ vs $A/L^{0.4}$.

$$\frac{\log_{10}P(x,\ L)}{\log_{10}L/L_0}=f(\alpha) \quad \text{where} \quad \alpha=\frac{\log_{10}(x/x_0)}{\log_{10}(L/L_0)}. \quad (1)$$

We display a plot of $L^{\beta}P(A,L)$ vs AL^{ν_A} with $\beta=0.65$ and $\nu_A=0.40$ and find reasonable data collapse for L ranging from 20–160. This is displayed in Fig. 6. The scaling fits for the other quantities are somewhat poorer or of comparable quality. We note that the value of ν is rather small in the first critical region and the plot implies that the scaling or metrical factor in A/L^{ν_A} is small. The fits to the multifractal form were not satisfactory. Given our limited theoretical understanding of scale-invariant behavior in discrete, driven dynamical systems and the limitations of numerical simulations (the number of avalanches needed to get statistics for large-system sizes is computationally prohibitive) the results must be treated with due caution. Nevertheless, the occurrence of scaling behavior over a wide range of scales is an interesting phenomenon in models used in the context of evolutionary biology.

CONCLUSION

In summary, we have studied a modified version of the spatial game of prisoners' dilemma originally proposed by Nowak and May by incorporating the effect of a random environment. We have shown that in the regime where neither cooperators nor defectors dominate, the spatial game naturally evolves to critical states in which a small perturbation can lead to restructuring of the defector network at all scales. Two different critical behaviors are found in the model, and the nature of these critical states remain to be understood. We have also tried to include other type of players, such as players using tit-for-tat and Pavlov strategies. Preliminary results show that similar self-organized critical states persist in these more complicated models. The crucial elements for self-organized criticality appear to be the play-

ers who always defect (all-*D* players). These players will form a tightly connected but delicately balanced network that separates the domains consisting of other type of players. It is the all-*D* players who force the system to a state of ''punctuated equilibrium.'' Without these players, we found that the evolution of the game to be quite random without the emergence of any particular spatial structures. While the occurrence of asymptotic criticality is an open question, very large correlation lengths and response on many scales clearly occur for a range of parameter values. These results may indicate that defectors such as parasites in real biological systems play a crucial role in evolution.

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