

## OPERATORS ON $L^p$ AND THE ROLE OF THE IMAGE OF THE UNIT BALL

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ABSTRACT. Let  $p \in [1, +\infty]$  such that its conjugate exponent  $q$  is not an even integer and let  $T$  be an operator defined on  $L^p(\lambda)$  with values in a Banach space. In this note we discuss how the image of the unit ball determines whether  $T$  belongs to some classes of operators such as operator ideals or the class of representable operators. We also study the monotonicity of these properties, proving that a Banach space is  $C$ -isomorphic to a subspace of an  $L^q$  space if and only if the representability of every operator on  $L^p$  is monotone with respect to the image of the unit ball.

### 1. OPERATORS ON $L^p(\lambda)$ AND THE IMAGE OF THE UNIT BALL

Let  $X$  be a Banach space. In this note we survey the following general question: Let  $T_1 : L^p(\lambda_1) \rightarrow X$  and  $T_2 : L^p(\lambda_2) \rightarrow X$  be two bounded linear operators. Suppose that  $\overline{T_1(B_{L^p(\lambda_1)})} = \overline{T_2(B_{L^p(\lambda_2)})}$  or  $\overline{T_1(B_{L^p(\lambda_1)})} \subseteq \overline{T_2(B_{L^p(\lambda_2)})}$ . Does then  $T_1$  belong to a given class of operators whenever  $T_2$  does? This question is closely related to earlier results by Grothendieck, Rodríguez Piazza and the author. In almost all the cases the results depend heavily on  $p$ . The essential tools here are previous results about equimeasurability and isometries in  $L^p$  spaces due to W. Rudin, W. Lusky, W. Linde and others.

The notation and terminology will follow that of [4]. We denote by  $L$  a Banach lattice. For a real, unless otherwise specified, Banach space  $X$ , we denote by  $X^*$  its dual space. The closed unit ball in  $X$  will be denoted by  $B_X$ .

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For an operator ideal  $\mathcal{A}$  we denote by  $\mathcal{A}(X, Y)$  the linear space of operators  $T: X \mapsto Y$  in  $\mathcal{A}$ . If in addition  $\mathcal{A}$  is a normed ideal, we denote its norm by  $\|\cdot\|_{\mathcal{A}}$ . Familiar examples of Banach operator ideals are the space of all (bounded linear) operators  $(\mathcal{L}, \|\cdot\|)$ , the space of compact operators  $(\mathcal{K}, \|\cdot\|)$  and the space of weakly continuous operators  $(\mathcal{W}, \|\cdot\|)$ , with the usual operator norm. Further, for  $1 \leq r, s < \infty$ , there are the Banach ideals  $(\Pi_{(r,s)}, \pi_{(r,s)})$  of all  $(r, s)$  summing operators and  $(\mathcal{I}_r, i_r)$  of all  $r$ -integral operators (see [4] for the definitions). The notion of an operator ideal is due to A. Pietsch [15]. It is a powerful notion, an elegant abstraction whose roots trace back to A. Grothendieck's Resumé [6]. The study of operator ideals is an important topic in Operator Theory which has generated problems of its own interest and has provided new insights into the theory of Banach spaces and their operators. The main difficulties in this area is recognizing and giving criteria for an operator to belong to a given ideal and the computation of its norm.

Some ideals are by definition determined by the image of the unit ball, for instance compact and weakly compact operators. However, one cannot determine whether an operator belongs to a given operator ideal by just looking to the image of the unit ball. For instance, if  $T_1$  is a quotient map from  $\ell_1$  onto  $\ell_2$  and  $T_2$  is the identity operator on  $\ell_2$ , then  $\overline{T_1(B_{\ell_1})} = B_{\ell_2} = \overline{T_2(B_{\ell_2})}$ ; by Grothendieck's Theorem,  $T_1$  is absolutely summing while  $T_2$  is not. Therefore, the fact that the images of the unit balls under two operators coincide does not always implies that the operators belong to the same ideal. However, if we restrict ourselves to some classes of spaces, some results can be obtained. The first result in this vein is due to Grothendieck (see [7] and [5] for the definition of equimeasurable set).

**Theorem 1.1.** *An operator  $T: X \rightarrow L^1(\lambda)$  is integral if and only if it is order bounded and in this case the integral norm satisfies  $i_1(T) = \|\sup_{x \in B_X} |Tx|\|$ . Also,  $T$  is nuclear if and only if  $T(B_X)$  is order bounded and equimeasurable.*

A general theory about properties of operators determined by the image of the unit ball has been developed by Rodríguez-Piazza and the author in the framework of vector measures. In [18] Rodríguez-Piazza, answering a question raised by Anantharaman and Diestel in [2], showed that the range of a vector measure determines its total variation; that is, if two measures with values in a Banach space have the same range, then they have the same total variation. Later, Rodríguez-Piazza in [19], proved that the range also determines the Bochner derivability. These results can be translated into the language of operators: if  $\lambda$  is a finite measure and  $T: L^\infty(\lambda) \rightarrow X$  is a weak\*-weakly continuous operator, then  $\overline{T(B_{L^\infty(\lambda)})}$  determines the 1-summing and the nuclear norms of  $T$ . In [20], similar results were obtained for  $(r, s)$ -summing norms and  $r$ -nuclear norms. The basic tool to obtain these results is the following result about determination of symmetric measures on the sphere.

**Theorem 1.2.** *If  $\mu$  and  $\nu$  are two finite, positive and symmetric measures on  $\mathbb{S}^{n-1}$  such that, for every  $\xi \in \mathbb{R}^n$ ,*

$$\int |\langle x, \xi \rangle| d\mu(x) = \int |\langle x, \xi \rangle| d\nu(x);$$

*then  $\mu = \nu$ .*

Theorem 1.2 goes back, at least, to Blaschke [3] who proved it for  $n = 3$  in 1916. The general theorem was first proved by Aleksandrov [1] in 1937, and was reproved by Petty [14] in 1961, Rickert in [16] and [17] in 1967 and Matheron [12] in 1971.

Let  $p \in [1, \infty]$ . Henceforth  $q$  will be the conjugate exponent of  $p$ , that is,  $1/p + 1/q = 1$ . The key to prove the most general theorem about properties of operators determined by the image of the unit ball is the following result due to W. Lusky, see [11].

**Theorem 1.3.** *Let  $q \in [1, +\infty]$  such that  $q \neq 4, 6, 8, \dots$ . Let  $\mu, \nu$  be two positive measures,  $E$  a subspace of  $L^q(\mu)$ , and  $S_0 : E \rightarrow L^q(\nu)$  an isometry. Then there exists an extension  $S : L^q(\mu) \rightarrow L^q(\nu)$  of  $S_0$  such that  $\|S\| = 1$ .*

For  $q = \infty$ , last result is a consequence of the injectivity of  $L^\infty(\nu)$ . For  $q = 2$ , it is a consequence of the fact that every subspace of a Hilbert space is complemented with a norm one projection. For the other values of  $q$ , Theorem 1.3 is due to W. Lusky [11] for complex  $L^q$  spaces; its proof makes use of a result of Rudin [23] for complex scalars. The real version of the last result was given by Linde in [9], that is, Theorem 1.3 holds for real  $L^q$  spaces. As a consequence, we have the main theorem in [21] stated as follows.

**Theorem 1.4.** *Let  $X$  be a Banach space and  $p \in (1, +\infty]$  such that its conjugate exponent  $q \neq 4, 6, 8, \dots$ . Let  $\mathcal{A}$  be an operator ideal. Let  $T_1 : L^p(\lambda_1) \rightarrow X$  and  $T_2 : L^p(\lambda_2) \rightarrow X$  be two operators such that  $\overline{T_1(B_{L^p(\lambda_1)})} = \overline{T_2(B_{L^p(\lambda_2)})}$ , and they are weak\*-weakly continuous if  $p = \infty$ . Then  $T_1 \in \mathcal{A}$  if and only if  $T_2 \in \mathcal{A}$ ; if in addition  $\mathcal{A}$  is a normed ideal, then  $\|T_1\|_{\mathcal{A}} = \|T_2\|_{\mathcal{A}}$ .*

*Proof.* Observe first that  $\overline{T_1(B_{L^p(\lambda_1)})} = \overline{T_2(B_{L^p(\lambda_2)})}$  if and only if  $\|T_1^*x^*\| = \|T_2^*x^*\|$ , for every  $x^* \in X^*$ . Therefore, there exists an isometry between  $T_1^*(X^*)$  and  $T_2^*(X^*)$  sending  $T_1^*x^*$  to  $T_2^*x^*$ . Using Theorem 1.3, this isometry can be extended to an operator  $S : L^q(\lambda_1) \rightarrow L^q(\lambda_2)$  such that  $\|S\| = 1$ , then  $T_2^* = S \circ T_1^*$  which implies  $T_2^{**} = T_1^{**} \circ S^*$ . From the fact that  $L^p$  is reflexive for  $p \in (1, +\infty)$  or the condition of being weak\*-weakly continuous for  $p = \infty$ , it follows that  $T_2 = T_1 \circ S^*$ . In particular, this implies that  $T_2 \in \mathcal{A}$  if  $T_1 \in \mathcal{A}$  and that  $\|T_2\|_{\mathcal{A}} \leq \|T_1\|_{\mathcal{A}}$  for a normed ideal. The same argument proves that  $T_1 \in \mathcal{A}$  if  $T_2 \in \mathcal{A}$  and that  $\|T_1\|_{\mathcal{A}} \leq \|T_2\|_{\mathcal{A}}$ .  $\square$

Theorem 1.4 does not hold for  $p = 1$  or  $p = \infty$  dropping the condition of being weak\*-weakly continuous, where we can only obtain that  $T_1^{**} \in \mathcal{A}$  if and only if  $T_2^{**} \in \mathcal{A}$  and  $\|T_1^{**}\|_{\mathcal{A}} = \|T_2^{**}\|_{\mathcal{A}}$  when  $\mathcal{A}$  is a normed ideal. The

same can be said for  $\mathcal{C}(K)$  spaces. Counterexamples for these values of  $p$  and  $q \neq 4, 6, 8, \dots$  are exhibited in [21].

The last result suggests many questions. At first glance, we can pose the following problems.

**Problem 1.** Once we know that the image of the unit ball determines the belonging to an operator ideal, it would be interesting to find geometrical conditions on the image of the unit ball to belong to a given ideal.

**Problem 2.** Theorem 1.4 applies to a large variety of properties of operators. However, it does not apply to some important classes of operators on Banach lattices. What are the analogues of this result for other classes of operators that are not ideals of operators?

In the remainder of this section, we will collect some results that solve affirmatively the second problem for concave and representable operators, which do not in general correspond to ideals of operators.

If  $1 \leq r, s < \infty$ , an operator  $T: L \rightarrow X$  is  $(r, s)$ -concave if and only if there exists a constant  $M < \infty$  such that the inequality

$$\left( \sum_{i=1}^n \|T\varphi_i\|^r \right)^{1/r} \leq M \left\| \left( \sum_{i=1}^n |\varphi_i|^s \right)^{1/s} \right\|$$

holds for every positive integer  $n$  and for all  $\varphi_1, \dots, \varphi_n \in L$ . The least of such  $M$  is denoted by  $\alpha_{(r,s)}(T)$ , the  $(r, s)$ -concave norm of  $T$  (or  $\alpha_r(T)$  if  $r = s$ ). Let  $\mathcal{C}_{(r,s)}(L, X)$  (or  $\mathcal{C}_r(L, X)$  if  $r = s$ ) be the space of  $(r, s)$ -concave operators from  $L$  to  $X$ . If we denote by  $\Pi_{(r,s)}(L, X)$  the space of  $(r, s)$ -absolutely summing operators then we have the following relationship between those classes of operators:

$$\Pi_{(r,s)}(L, X) \subseteq \mathcal{C}_{(r,s)}(L, X).$$

The following theorem, whose proof can be found in [4, Theorem 16.5], provides a characterization of  $(r, s)$ -concave operators in terms of  $(r, s)$ -summing operators.

**Theorem 1.5.** *Suppose that  $1 \leq r \leq s < \infty$  and  $C > 0$ . An operator  $T: L \rightarrow X$  is  $(r, s)$ -concave with  $\alpha_{(r,s)}(T) \leq C$  if and only if, for each compact Hausdorff space  $K$  and every positive operator  $P: \mathcal{C}(K) \rightarrow L$ , the composition  $T \circ P: \mathcal{C}(K) \rightarrow X$  is  $(r, s)$ -summing with  $\pi_{(r,s)}(T \circ P) \leq C\|P\|$ .*

Next theorem proves that the image of the unit ball of an operator defined on  $L^p(\lambda)$  determines whether the operator belongs to the space of  $(r, s)$ -concave operators when  $q \neq 2, 4, 6, \dots$ , see [22].

**Theorem 1.6.** *Suppose that  $X$  is a Banach space and  $p \in [1, +\infty]$  such that its conjugate exponent  $q$  is not an even integer. Let  $T_1: L^p(\lambda_1) \rightarrow X$  and  $T_2: L^p(\lambda_2) \rightarrow X$  be two operators such that  $\overline{T_1(B_{L^p(\lambda_1)})} = \overline{T_2(B_{L^p(\lambda_2)})}$ . Then  $\alpha_{(r,s)}(T_1) = \alpha_{(r,s)}(T_2)$ .*

The idea of the proof of Theorem 1.6 is the fact that it suffices to prove it for finite dimensional Banach spaces  $X$  and operators defined on  $L^p(\mu)$ , where  $\mu$  is a finite measure on the Euclidean unit sphere. Once this reduction is made, the theorem is proved using the following result about determination of symmetric measures on the sphere whose proof can be found in [8], [9] and [13].

**Theorem 1.7.** *Let  $q$  be a real number in  $[1, +\infty)$  which is not an even integer. If  $\mu$  and  $\nu$  are two finite, positive and symmetric measures on  $\mathbb{S}^{n-1}$  such that, for every  $\xi \in \mathbb{R}^n$ ,*

$$\int |\langle x, \xi \rangle|^q d\mu(x) = \int |\langle x, \xi \rangle|^q d\nu(x);$$

then  $\mu = \nu$ .

It is worth pointing out that Theorem 1.7 does not hold when  $q$  is an even integer, since in such a case the space generated by the functions  $|\langle \cdot, \xi \rangle|^q$ , with  $\xi \in \mathbb{R}^n$ , in the space of continuous and symmetric functions on the sphere is finite dimensional. This is the reason why Theorem 1.6 does not hold for these values of  $q$ . Counterexamples can be found in [22]. It is important to note that, in high contrast with Theorem 1.4, Theorem 1.7 fails to be true for  $p = 2$ .

For 1-concave operators  $T: L^p(\lambda) \rightarrow X$ , we can find a decomposition of the operator by means of a weak\* measure on  $X^{**}$ , [22]. We will denote by  $Ba(X^{**}, w^*)$  the Baire  $\sigma$ -algebra on  $X^{**}$  with respect to the weak\* topology in  $X^{**}$ , that is, the smallest  $\sigma$ -algebra making measurable every real-valued function which is continuous for the weak\* topology. In fact, this  $\sigma$ -algebra turns out to be the  $\sigma$ -algebra generated by the functionals in  $X^*$ , [24, 2-2-4]. It is important to note that, in general, the unit ball of  $X^{**}$  is not Baire measurable for  $Ba(X^{**}, w^*)$ . This is the reason why we have to consider the outer measure  $\mu_T^*(B_{X^{**}})$  in the following theorem. Observe that  $\alpha_1(T) < \infty$  means that the operator  $T^*: X^* \rightarrow L^q(\lambda)$  satisfies

$$\sup_{x^* \in B_{X^*}} |T^* x^*| = h \in L^q(\lambda),$$

see Proposition 1.d.4 in [10]. If  $\mu$  is a measure on  $Ba(X^{**}, w^*)$ ,  $\mu^s$  will denote its symmetrization defined as  $\mu^s(A) = \frac{1}{2}(\mu(A) + \mu(-A))$  for every  $A \in Ba(X^{**}, w^*)$ .

**Theorem 1.8.** *Let  $p \in (1, +\infty]$  such that  $q \neq 2, 4, 6 \dots$  and let  $T: L^p(\lambda) \rightarrow X$  be an operator with  $\alpha_1(T) < +\infty$  (weak\*-weak continuous if  $p = +\infty$ ). Then there exists a positive measure  $\mu_T$  on  $Ba(X^{**}, w^*)$  such that:*

- (a)  $(\alpha_1(T))^q = \mu_T(X^{**}) = \mu_{T^*}^*(B_{X^{**}})$ .
- (b)  $\|T^* x^*\|^q = \int |\langle x^*, x^{**} \rangle|^q d\mu_T(x^{**})$ .

Moreover, if  $\tilde{T}: L^p(\tilde{\lambda}) \rightarrow X$  is another operator and  $\mu_{\tilde{T}}$  satisfies (a) and (b) for  $\tilde{T}$ , then  $T(B_{L^p(\lambda)}) = \tilde{T}(B_{L^p(\tilde{\lambda})})$  if and only if  $\mu_T^s = \mu_{\tilde{T}}^s$ .

We denote by  $L^q(\lambda, X)$  the space of  $X$ -valued strongly measurable functions  $f$  such that

$$\|f\|_q = \left( \int \|f\|^q d\lambda \right)^{1/q} < \infty.$$

If  $T: L^p(\lambda) \rightarrow X$  is an operator such that

$$T\varphi = \int \varphi \cdot f d\lambda, \quad \text{for every } \varphi \in L^p(\lambda),$$

and  $f \in L^q(\lambda, X)$ , we say that  $T$  is represented by the function  $f$ . The representation of a linear operator on  $L^p(\lambda)$  by means of an strongly measurable function in  $L^q(\lambda, X)$  provides a very strong structural information about the operator under consideration. This is not the case for other weaker integral representation theories such as in Theorem 1.8 which fails to provide a good representation for well behaved operators.

Following the same argument as a result of Talagrand [24, 3-4-1], one can see that an operator  $T: L^p(\lambda) \rightarrow X$  is represented by a function  $f \in L^q(\lambda, X)$  if and only if there exists a separable subspace  $H$  of  $X$  such that  $\mu_T^*(H) = \mu_T(X^{**})$ . In this case,  $\mu_T$  is a Radon measure on  $X$ . Since  $H$  is a symmetric set, using the previous theorem we can deduce the following corollary, [22].

**Corollary 1.9.** *If we have two operators  $T_1: L^p(\lambda_1) \rightarrow X$  and  $T_2: L^p(\lambda_2) \rightarrow X$  such that  $\overline{T_1(B_{L^p(\lambda_1)})} = \overline{T_2(B_{L^p(\lambda_2)})}$  and  $p \in (1, \infty]$ ,  $q \neq 2, 4, 6 \dots$  ( $T_1, T_2$  weak\*-weak continuous if  $p = \infty$ ), then  $T_1$  is represented by a function in  $L^q(\lambda_1, X)$  if and only if  $T_2$  is represented by a function in  $L^q(\lambda_2, X)$ .*

This result can also be obtained as a consequence of Theorem 4 in [9]. For the other values of  $p$ , this theorem fails to be true using counterexamples by W. Linde in [9] and R. Sztencel (unpublished). Moreover, for  $p = 2$  this theorem is only valid when  $X$  is isomorphic to a Hilbert space.

## 2. MONOTONICITY WITH RESPECT TO THE IMAGE OF THE UNIT BALL

At this stage we know that, for many values of  $p$ , the image of the unit ball for an operator  $T: L^p(\lambda) \rightarrow X$  determines many of its properties. So one can wonder about the monotonicity of these properties with respect to the image of the unit ball. The precise questions follow for  $T_1: L^p(\lambda_1) \rightarrow X$  and  $T_2: L^p(\lambda_2) \rightarrow X$ :

**Question 1.** Does the condition  $\overline{T_1(B_{L^p(\lambda_1)})} \subseteq \overline{T_2(B_{L^p(\lambda_2)})}$  implies that  $T_1$  belongs to a given class of operators whenever  $T_2$  does?

**Question 2.** Does the condition

$$C_1 \overline{T_2(B_{L^p(\lambda_2)})} \subseteq \overline{T_1(B_{L^p(\lambda_1)})} \subseteq C_2 \overline{T_2(B_{L^p(\lambda_2)})},$$

where  $C_1$  and  $C_2$  are positive constants, implies that  $T_1$  belongs to a given class of operators if and only if  $T_2$  does?

The first authors to isolate these kinds of questions were Anantharaman and Diestel, see [2], in the context of vector measures, exhibiting two  $c_0$ -vector valued measures  $\mu$  and  $\nu$  such that  $\text{rg } \mu \subseteq \text{rg } \nu$ , being  $\nu$  of bounded variation while  $\mu$  is not. This provides a counterexample to Question 1 for the class of absolutely summing operators defined on  $L^\infty$ . Following this study, Rodríguez-Piazza showed in [18] that the monotonicity of the variation characterizes the subspaces of  $L^1$ . This means that the absolutely summing norm is monotone with respect to the image of the unit ball for operators defined on  $L^\infty$  if and only if the range is  $C$ -isomorphic to a subspace of  $L^1$ .

A result due to W. Linde in [9] leads us to the same conclusion for representable operators in  $L^p$ . Observe that no restriction on  $q$  is needed.

**Theorem 2.1.** *Let  $X$  be a Banach space and  $C \geq 1$ . Then the following statements are equivalent for  $1 < p \leq \infty$ :*

- (a)  *$X$  is  $C$ -isomorphic to a subspace of some  $L^q$  space.*
- (b) *For every pair of operators  $T_1 : L^p(\lambda_1) \rightarrow X$  and  $T_2 : L^p(\lambda_2) \rightarrow X$  (if  $p = \infty$ , then  $T_1, T_2$  must be weak\*-weak continuous), the condition  $\overline{T_1(B_{L^p(\lambda_1)})} \subseteq \overline{T_2(B_{L^p(\lambda_2)})}$  implies that  $T_1$  is represented by a function whenever  $T_2$  is. In this case if  $f_i$  is the function representing  $T_i$ , for  $i = 1, 2$ ,*

$$\|f_1\|_q \leq C \|f_2\|_q.$$

*Proof.* Suppose that condition (b) holds. Observe that two operators  $T_1$  and  $T_2$  satisfy  $\overline{T_1(B_{L^p(\lambda_1)})} \subseteq \overline{T_2(B_{L^p(\lambda_2)})}$  if and only if for every  $x^* \in X^*$  we have  $\|T_1^*x^*\| \leq \|T_2^*x^*\|$ . Indeed, since  $\overline{T_1(B_{L^p(\lambda_1)})}$  and  $\overline{T_2(B_{L^p(\lambda_2)})}$  are convex sets, we have  $\overline{T_1(B_{L^p(\lambda_1)})} \subseteq \overline{T_2(B_{L^p(\lambda_2)})}$  if and only if

$$\sup_{\varphi \in B_{L^p(\lambda_1)}} \langle x^*, T_1\varphi \rangle \leq \sup_{\psi \in B_{L^p(\lambda_2)}} \langle x^*, T_2\psi \rangle \quad \text{for every } x^* \in X^*.$$

The above display is equivalent to

$$\sup_{\varphi \in B_{L^p(\lambda_1)}} \langle T_1^*x^*, \varphi \rangle \leq \sup_{\psi \in B_{L^p(\lambda_2)}} \langle T_2^*x^*, \psi \rangle \quad \text{for every } x^* \in X^*$$

which, in turn, is equivalent to

$$\|T_1^*x^*\| \leq \|T_2^*x^*\| \quad \text{for every } x^* \in X^*.$$

If  $T_2$  is represented by a function  $f_2$  in  $L^q(\lambda_2, X)$ , then  $T_2^*x^* = x^*f_2$ , for any  $x^* \in X^*$ . In this case, since  $\mu_{T_2}$  can be extended to a Radon measure on  $X$ , we have

$$\|T_1^*x^*\|^q \leq \|T_2^*x^*\|^q = \int |\langle x^*, x \rangle|^q d\mu_{T_2}(x) \quad \text{and} \quad \int \|x\|^q d\mu_{T_2}(x) < \infty.$$

Using [9, Theorem 6], we easily deduce that condition (b) is equivalent to  $X$  being a subspace of some  $L^q$  space. □

**Remark 2.2.** *Another application of [9, Theorem 6] and an easy adaptation of the proof of Theorem 5 in [18], solves the same problem for other classes of*

operators. Indeed, one can prove that condition (b) in the last theorem can be replaced by the following: For every pair of operators  $T_1 : L^p(\lambda_1) \rightarrow X$  and  $T_2 : L^p(\lambda_2) \rightarrow X$  ( $T_1, T_2$  weak\*-weak continuous if  $p = \infty$ ), the condition  $\overline{T_1(B_{L^p(\lambda_1)})} \subseteq \overline{T_2(B_{L^p(\lambda_2)})}$  implies that  $T_1$  is  $p$ -summing whenever  $T_2$  is. In this case,  $\pi_p(T_1) \leq C\pi_p(T_2)$ , the same is true for  $p$ -integral operators. Indeed, this happens when the operators  $T_1$  and  $T_2$  are order bounded (see [4, Corollary 5.22]).

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