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HOW LARGE IS A RIEMANN SURFACE: THE TYPE PROBLEM

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Abstract. The uniformization theorem asserts that a simply-connected non-compact Riemann surface S is conformally equivalent to precisely one of the unit disk $\mathbb{D}$ or the finite complex plane $\mathbb{C}$. While this result (nearly a century old) closes one chapter in the theory of analytic functions of one complex variable, it opens another: given a surface S described in some explicit manner, determine from intrinsic considerations which of the conformal types S is. While this subject reached a zenith of activity in the 1930s, recent developments and the availability of new tools suggest a resurgence of interest.

§1.1 *Introduction.* Although general statements are to be avoided in surveys, the notions of convergence and divergence seem to be the fundamental issues of analysis. The history of the subject indicates that there cannot be a non-tautological general method to distinguish which will apply to a given process. However, much of the appeal of the subject arises from the many different contexts in which this question arises, and the rich techniques and connections developed to understand it.

A *Riemann surface* is a two-dimensional complex manifold. Culminating a long process that lasted during much of the nineteenth century, the uniformization theory [1] of Koebe established that a simply-connected Riemann surface S is conformally equivalent to one of three standard models (the sphere $\hat{\mathbb{C}}$ ['elliptic'], the finite complex plane $\mathbb{C}$ ['parabolic'], or the unit disk $\mathbb{D}$ ['hyperbolic']). These classes of surfaces are distinct, and the first has a purely

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topological characterization: S is elliptic precisely when S is compact. The distinction between the other two classes, however, is far more subtle, and in these lectures we shall study some methods developed to determine the '(conformal) type' of a given non-compact (open) S. Since every Riemann surface has a universal cover (simply-connected), this in principle gives information about any Riemann surface.

It is impossible to be exhaustive in the space of a few lectures, and we mention the classical study [15], the recent chapter [6] as well as the references therein in addition to those listed in these notes. Of course, there is in addition a large literature devoted to Riemann surfaces as a subject in itself.

Since much of this material is not frequently presented in courses or seminars in recent decades, much of the lectures were devoted to presenting examples; however, this will be greatly condensed for these notes.

The study of compact Riemann surfaces was one of the triumphs of nineteenth century mathematics: such a surface S is conformally equivalent to a sphere with n handles. Our discussion will center on non-compact simply-connected surfaces.

To determine the conformal type of non-compact (but always simply-connected!) surfaces is not likely to have a simple answer. Our most important example is the surface S_2 of $w = e^z$, which is the universal cover of $\hat{\mathbb{C}} \setminus \{0, \infty\}$. Since w is an entire function without branching ($(e^z)' \neq 0$!), we see at once that S_2 is (conformally equivalent to) the finite plane $\mathbb{C}$.

However, there are many functions analytic in $\mathbb{D}$ which cannot be continued beyond, for example $f(z) = \sum z^{n!}$. But then e^f omits $0, \infty$, its surface S_f is simply-connected, but now S_f is conformally the disk.

§1.2. *Speiser graphs.* The *type problem* asks for methods to determine the conformal type of a given Riemann surface S. This is of interest in the situation that S is given in some manner in which the uniformizing map $f : D \to S$ is not known precisely. That is, any Riemann surface is locally equivalent to an open set in the plane, but what is not known is the natural domain D of the uniformizing map $f : D \to S$.

Now S, the range of f, is a branched cover of the sphere, and so there is a natural projection $p : S \to \hat{\mathbb{C}}$ which is a local homeomorphism away from the branch points of S (these are the images of points at which $f'(z) = 0$, or at which f has multiple poles): if we write $S = \{(z, w) : z \in D, w = f(z)\}$, then

$$p(z, w) = w.$$

One useful class of surfaces to consider in this context is the surfaces $S \in \mathcal{S} = \cup_q F_q$; these may be represented by a *Speiser graph*. A simply-connected surface S is in F_q ($q \geq 2$) if there is a set $\mathcal{A} = \{a_1 \dots a_q\} \subset \hat{\mathbb{C}}$ such that if $a \notin \mathcal{A}$, then each component of the preimage of the disk $\{|f(z) - a| < \epsilon\}$ (for sufficiently small ϵ) is mapped by f conformally onto the disk $B(a, \epsilon) = \{|w - a| < \epsilon\}$ (for convenience, we may assume that $\mathcal{A} \subset \mathbb{C}$). For example, the surface of $w = e^z$ is in F_2, with $\mathcal{A} = \{0, \infty\}$.

Let L be a (positively oriented) Jordan curve passing through $\mathcal{A}$ with indexing chosen so that the points are encountered in the order $a_1, a_2, \ldots, a_q, a_1$. Then L separates $\mathbb{C}$ into components $\mathcal{I}$ ('inner') and $\mathcal{O}$ ('outer'). The preimages of L under $p \circ f$ decompose D into a network of cells, each of which is homeomorphic to $\mathcal{I}$ or $\mathcal{O}$. It is clear that S itself may be rebuilt by gluing these cells along the relevant arcs of L, which will include one or several segments $(a_\ell, a_{\ell+1})$ of L.

In turn this produces a graph, known as the Speiser graph. We choose base points $\circ \in \mathcal{I}$ and $\times \in \mathcal{O}$ and then a network of q Jordan arcs γ_ℓ on $\hat{\mathbb{C}}$ connecting $\times$ and $\circ$ having only endpoints in common, such that γ_ℓ meets L at a single point of the segment $(a_\ell, a_{\ell+1})$. The *Speiser graph* is the pull-back of these $\{\gamma_\ell, \times, \circ\}$ under p; note that it is possible that two such graphs may be graph-theoretically equivalent, but still may not be equivalent in the sense that as sets in the plane they might not be equivalent under isotopy; in this case the surfaces associated to these graphs could be of different conformal type.

Here are some examples of Speiser graphs.

(1) The exponential function $w = e^z$. Let L be the real axis, $\mathcal{A} = \{0, \infty\}$ and the points $\times, \circ$ be $\pm i$. The preimage of L in the plane is the familiar parallel network of horizontal lines separated by π, and the network of arcs connecting $\times$ and $\circ$ (which could be considered the unit circle) corresponds to the imaginary axis: thus the Speiser graph is a vertical line with the points $\ldots \times, \circ, \times, \ldots$ marked.

(2) The *Modular Function* is an important example in complex analysis. While the exponential function shows the plane is the conformal universal cover of the sphere punctured at $\{0, \infty\}$, the modular function exhibits the unit disk $\mathbb{D}$ as the universal cover of the sphere with $q \geq 3$ points deleted: its Riemann surface is hyperbolic. The classical situation is that $q = 3$, and in this setting the existence of the modular function provided a key ingredient for Picard's celebrated theorem that an entire function with two omitted values is constant (one of the earliest theorems of conformal type).

To construct this function, take $\mathcal{A} = \{0, 1, \infty\}$ and L the real axis. Then in the disk $\mathbb{D}$ let T_0 be a non-Euclidean triangle containing the origin and bounded by three arcs which meet $\partial\mathbb{D}$ orthogonally at the three roots of unity. If we map the upper half-plane to T_0 with $\mathcal{A}$ sent to the roots of unity and then perform successive reflection across the respective boundaries, we obtain the conformal mapping of $\mathbb{D}$ to this universal covering. In this case, again we may choose $\times, \circ$ as $\pm i$.

The corresponding Speiser graph is a tree such that three rays leave each vertex (a $\times$ or $\circ$), so that the graph has degree three at each point. The graph has no closed curves (loops).

(3) The Weierstrass $\wp$ function, which exhibits the plane as the universal cover of a torus. In complex analysis courses, we learn that $\wp$ is doubly

periodic, with its 'fundamental region' a parallelogram P. For example, P might first be conformally mapped to the upper half-plane. Then $\wp$ is extended to the full complex plane through successive reflection across the sides of these rectangles. The set $\mathcal{A}$ corresponds to the multiply-values of $\wp$, and so they are the images of the corners of P, so that $\mathcal{A} \subset \mathbb{R}$. Again $\times, \circ$ can be $\pm i$, and the graph consists of a network of squares, with each vertex alternating $\times$ or $\circ$, and four edges emerging from each vertex.

(4) The surface corresponding to $w = \sin z$ has some features of the previous examples. Its Speiser graph can be presented as a band Σ of rectangles stretching horizontally from left to right, each having two vertices $\times$ and $\circ$ alternating. In this case, $\mathcal{A} = \{\infty, \pm 1\}$ so that $S \in F_3$, and each rectangle corresponds to a circuit about one of the algebraic branch points ± 1.

The complement of the graph of the $\wp$ function is a network of bounded regions. In general, each such region will have an even number $(2m)$ of sides, which in turn will correspond to m circuits about some point of $\mathcal{A}$, and this point will be a branch point of order m. In all but the third example there are in addition (generalized) polygons having infinitely many sides, and in this situation (for Speiser graphs) we may take $m = \infty$ and view the corresponding point a of $\mathcal{A}$ as a *logarithmic branch point*. This is discussed in [11].

§2.1. *Some methods.* Here are some of the methods used to study the type problem (not exhaustive, however!):

- potential theory
- modulus, length-area
- comparisons
- differential-geometric methods
- the Ahlfors test
- graph theory
- circle-packing

§2.2. *Potential Theory.* This is an extremely classic notion, going back to the nineteenth century: whether the surface S can hold a charge. In modern terminology, this reduces to whether the non-compact surface S has a nonconstant positive superharmonic function: the disk obviously has $(u(z) = \log(1/|z|))$ and so is hyperbolic; the plane cannot and is parabolic. This is often used as the defining principle for the hyperbolic-parabolic dichotomy, since it makes sense even for surfaces that are not simply-connected as well as in other contexts, in particular for graphs (cf. §2.7 below).

However, given a concretely-presented surface, it is not evident how to relate it to a positive harmonic function, or show that such cannot exist.

§2.3. *Modulus.* This also goes back to the nineteenth century, but was systematically used in an *ad-hoc* manner by H. Grötzsch in his earliest work on quasiconformal mapping (cf. [7] and sources cited in [6]). It suffers from a nonintuitive definition, but is clearly a conformal invariant. Although it is not germane to our lectures, we mention that it is one of the few general methods that extend naturally to higher dimensions (quasiconformal and quasiregular mappings; cf. [14]).

Let Ω be a domain (open connected set) in the plane, and Γ a family of (locally rectifiable) curves in Ω. An admissible metric ρ is a measurable nonnegative function with sufficient regularity so the $\int_\gamma \rho\,|ds|$ can be defined for curves $\gamma \in \Gamma$ with

$$\int_\gamma \rho\,|ds| \geq 1 \qquad (\gamma \in \Gamma).$$

The modulus $M(\Gamma)$ is then defined as

$$M(\Gamma) = \inf_{\rho \in A(\Gamma)} \int_\Omega \rho^2\, dA,$$

where A is area measure and $A(\Gamma)$ are the admissible metrics.

If $f : \Omega' \to \Omega$ is a conformal mapping, so that $\Gamma' \subset \Omega'$ corresponds to $\Gamma \subset \Omega$, then the correspondence $\rho'(z') = \rho \circ f(z')$ establishes a $1-1$ correspondence between admissible metrics in Ω and Ω', from which we deduce that $M(\Gamma') = M(\Gamma)$: the modulus is a *conformal invariant*.

We then record a concrete test for type, whose effectiveness depends on sensitive choices of families $\{\Gamma\}$:

Theorem 1. *Let F be a noncompact simply-connected Riemann surface and D a simply-connected neighborhood of some $z_0 \in F$. Let Γ consist of all locally rectifiable curves $\gamma(t),, 0 \leq t < 1$, in S which "begin" on ∂D and leave any compact set of S as $t \to 1$. Then F is parabolic if and only if*

$$M(\Gamma) = 0.$$

Dually, let Γ' consist of all locally-rectifiable closed curves separating D from ∞ in F. Then F is hyperbolic if and only if

$$M(\Gamma') < \infty.$$

The proof is immediate, since if we delete a small neighborhood D^* of z_0, $F \setminus D^*$ is conformally equivalent to an annulus $A(1,R) \equiv \{1 < |z| < R\}$ for some $0 < R \leq \infty$. The modulus of curves joining the boundary components of $A(1,R)$ is $2\pi/\log R$, and the first part follows at once from the definition of modulus. For the second part, we have $M(\Gamma') = 1/M(\Gamma)$.

Once again, although this is an attractive defining characterization, choosing an appropriate ρ can be a sophisticated art.

Less than a decade after Grötzsch initiated what we call the study of quasiconformal mappings as a subject of its own, Teichmüller realized that these modulus estimates adapt readily to the situation that $f : \Omega' \to \Omega$ is only quasiconformal. To record what Teichmüller showed, we first recall the basic definitions. Let $f \in W^{1,2}_{loc}(\Omega)$ (Sobolev space). Then the dilatation of f at $z \in \Omega'$ is defined (a. e.) as

$$\mu(z) = \frac{f_{\bar{z}}(z)}{f_z(z)},$$

and f is K-quasiconformal if $K(z) \equiv (1 + |\mu(z)|)/(1 - |\mu(z)|) < K$ a. e. It follows that if Γ' and Γ correspond under a quasiconformal homeomorphism, the ratio $|\log(M(\Gamma')/M(\Gamma)|$ is bounded by a constant $C(K)$ which depends only on K: our Theorem 1 holds when f is K-qc. One simple formulation does not even require that f be K-qc: Teichmüller showed that if $f : \mathbb{C} \to \mathbb{D}$ is in $W^{1,2}_{loc}(\Omega')$ and $K(z)$ is defined as above, then we must have

$$\int_{\mathbb{C}} K(z)\, dxdy = \infty.$$

In recent years (perhaps [4] was the first) the issue has been raised as to how much of the theory can be recovered if we drop the assumption that $K \in L_\infty$, but in fact the importance of this insight had been appreciated years before.

§2.4. *Comparisons.* If we know that type of S we may be able to deduce from it the type of S': for example, metrics or superharmonic functions from S may suggest good choices for S'. A similar possibility arises in modulus computations, as we have already seen in our proof of Theorem 1.

§2.5. *Differential-geometric methods.* Recall that the model domains $\mathbb{C}$, $\hat{\mathbb{C}}$ and $\mathbb{D}$ carry conformal metrics of constant curvature (respectively $ds = |dz|$, $ds = 2|dz|/(1 + |z|^2)$, $ds = 2|dz|/(1 - |z|^2)$). This means that any parabolic surface admits a conformal metric of zero curvature, an elliptic surface one of positive curvature while a hyperbolic surface has one of constant negative curvature.

This has been used in recent times rather effectively, since the data of Speiser graphs may be interpreted locally using these notions (cf. [9]): let B be a Borel subset of the Riemann surface S, and recall from §1.2 that we may view S as a (branched) covering of the Riemann sphere with p the projection. Then the *integral curvature* of B is the signed measure ω with

$$\omega(B) = \int_B \left[\frac{|p'(w)|}{1 + |p(w)|^2}\right]^2 |dw|^2 - 2\pi \sum_{q \in B} (k-1)\,, \tag{1}$$

where the sum is over all branch points $q \in B$, and k the local degree of p at q). We list two such results; an exposition of the second was the main theme of the final lecture here and will be mentioned in §3.1.

First, we discuss the theorem of Bonk-Eremenko (cf. [3]). Once again, view S as ‘spread over the Riemann sphere,’ and assume this time that it arises as the image of some $f \in \mathcal{M}$, where $\mathcal{M}$ is the class of non-constant meromorphic functions in the plane (thus S is parabolic). Given $q \in S$, let $\beta(q)$ be the radius of the largest (spherical) disc about q such that the covering of $\{[\zeta, q] < \beta\}$

($[\cdot,\cdot]$ is the spherical distance) is unramified. We then set $\beta(f) = \sup_q \beta(q)$ and finally

$$\mathfrak{B} = \inf_{f \in \mathcal{M}} \beta(f).$$

This is the theorem of Bonk-Eremenko:

Theorem 2. *A nonconstant meromorphic function f defined in the plane $\mathbb{C}$ covers spherical disks Δ of radius arbitrarily close to*

$$\mathfrak{B} = \tan^{-1}\sqrt{8}.$$

Doubly-periodic elliptic functions show to the estimate for $\mathfrak{B}$ sharp. In fact, that it is sharp may be less of a surprise than it is true!

We show that $\mathfrak{B}$ cannot be replaced by any smaller number. To do this, consider two coverings by triangles. In the plane, we take an equilateral triangle T, and of course by repeatedly reflecting across the sides of T obtain a tiling of $\mathbb{C}$. To do the same on the sphere is a bit more delicate, but elementary spherical geometry shows that $\hat{\mathbb{C}}$ may be tiled by four equilateral spherical triangles $\mathcal{D}$, with each internal angle $4\pi/3$, so that each area of each triangle is π. Now map some T conformally to some $\mathcal{D}$ so that the vertices correspond. Then repeated reflections across the sides give a (doubly-periodic) elliptic function on the plane, and this function has all its branching at the vertices of the four spherical triangles. For this function f we have $\beta(f) = \tan^{-1}\sqrt{8}$.

While details – to say nothing of even an outline – of this work are beyond the scope of the lectures, I think one insight should be mentioned. The proof is by contradiction: assume that there is a surface S which is the image of some $f : \mathbb{C} \to \hat{\mathbb{C}}$, and suppose that for some $\varepsilon > 0$ every spherical disc $\mathcal{D}$ of radius $\tan^{-1}\sqrt{8}$ anchored on S meets a point at which f^{-1} cannot be defined. It follows from (1) (and a few reductions) that each such disk $\mathcal{D}$ must then contain point-masses of negative curvature: points which decrease the sum in (1). On the remainder of S we have the positive curvature inherited from S being a covering of the sphere, and the strategy of [3] is to modify S keeping track of the (algebraic) signs of curvature so that in the end we arrive at a surface with strictly negative curvature, which then must be hyperbolic. Thus the domain of the inverse map to S cannot be $\mathbb{C}$.

§2.6. *Ahlfors test.* Ahlfors later wrote that what in retrospect seems important about this is "not . . . the result, but because it may have been the first use of arbitrary conformal metrics in relation to Riemann surfaces," although he also notes that antecedents can be traced in the work of others. An exposition is in [11], Ch. XI., and we only list the key feature.

It depends on constructing a function $U(w)$ on S so that $U \to \infty$ as $w \to \partial S$ which is C^1 except that ∇U may fail to exist on certain curves and U may be infinite at isolated points. If for $\rho > 0$, we let $\Gamma(\rho)$ be the level-set $\{U = \rho\}$ and set

$$L(\rho) = \int_{\Gamma(\rho)} |\nabla U||dw|,$$

then S will be parabolic if

$$\int^{\infty} \frac{d\rho}{L(\rho)} = \infty.$$

2.7. *Graph theory.* Let Γ be an infinite connected graph in the plane whose vertices v are connected by a network of edges. It is natural to extend several of the notions discussed here, such as extremal length or subharmonic-harmonic functions to graphs (cf. [5], [9]). For example, here is the way to define superharmonic functions. We take a fixed (base) vertex v_0 and say that $d(v, v_0) \leq n$ if v and v_0 are linked by a chain of at most n edges, and then define $d(v, v_0)$ to be the least possible candidate. Let u be a real function defined on the vertices of Γ, and $B(v, 1)$ the unit ball about v in this graph-theoretic sense. Inspired by the mean-value criterion, we define the (graph-theoretic) Laplacian of u at each $v \in \Gamma$ as

$$\Delta u(v) = \frac{1}{\deg_v \Gamma}(u(w) - u(v)).$$

Then u is superharmonic if $\Delta u \leq 0$ on Γ, and harmonic if $\Delta u \equiv 0$. The hyperbolic-parabolic dichotomy makes sense in this context, although for a Speiser graph it is not precisely the same (this is discussed in [5], and given a function-theoretic proof with some extensions in [9]).

Benjamini, Merenkov and Schramm [2] have used insights from this perspective to produce an interesting example on the type problem: a surface can be parabolic even though on average it has negative curvature (thus, the insight of [3] has limitations). A discussion of this will be in §3.1.

§2.8. *Circle-packings.* Circle-packings came into complex analysis in the 1980s, and continue to be relevant, including a wide variety of applications [12]. For the type-problem we refer also to [8].

There is a simple relation between circle-packings and triangulations which has been central to the subject from the beginning. If $\mathcal{T}$ is a triangulation, then there is a packing of disjoint circles labelled by the vertices of $\mathcal{T}$ such that two circles are mutually tangent if and only if the associated vertices of $\mathcal{T}$ are connected by the edge of a triangle. Thus we may go back and forth from triangulations and circle packings.

Let S be a surface in F_q, and $\mathcal{T}_0$ a triangulation of the sphere such that the q points of $\mathcal{A}$ are contained in the vertex set of $\mathcal{T}_0$. Let $\mathcal{T}$ be the pull-back to S of $\mathcal{T}_0$ (and assume that $\mathcal{T}$ is a disk triangulation). If $\mathcal{P}$ is the packing corresponding to $\mathcal{T}$, then the carrier of $\mathcal{P}$ will be either $\mathbb{D}$ or $\mathbb{C}$, leading to two notions of the hyperbolic-parabolic dichotomy for S: in terms of complex analysis, and in the sense of circle-packing. Very recently, S. Merenkov and B-G Oh have shown that the circle-packing type of S need not always agree with the conformal type, although a complete answer is not clear at present.

§3.1. *A sample problem.*

We present one final topic which illustrates many of the techniques introduced in these lectures.

R. Nevanlinna realized from the beginning that his theory of meromorphic functions (the main theme of [11]) could be viewed as a transcendental form of the Riemann-Hurwitz theorem, which in turn can be seen as a purely combinatorial analogue of (1). Thus, let S be a surface in F_q with $\mathcal{G}$ its Speiser graph, and V its vertex set. Then $\mathcal{G}$ is a set in the plane, and each vertex v will bound one or several *faces* F, components of $\mathbb{C} \setminus \mathcal{G}$. Let v be a vertex and $F(v)$ the collection of faces having v on its boundary. From §1.2, we know that each F has $2m$ edges, and so we define the *excess* at each vertex v as

$$E(v) = 2 \sum_{F(v)} \left(1 - \frac{1}{m}\right).$$

Our examples of Speiser graphs show that $E(v) = 0$ when v is a vertex of the exponential or $\wp$ function, and $E(v) < 0$ for the modular function (or any universal cover of $\mathbb{C}$ with $q > 2$ points deleted). From this and other examples, Nevanlinna proposed that if the graph of $S \in F_q$ is sufficiently regular that it is possible to assign $E = E(F)$ as a suitable 'limiting average' of the $\{E(v)\}$ as $\mathcal{G}$ is exhausted by an increasing union of finite graphs, then the conformal type of S might be recovered from E: S is parabolic if $E = 0$ and hyperbolic if $E < 0$.

It has turned out that in neither direction is this implication valid. In his famous thesis [13] (which even today presents a wide perspective on the role of quasiconformal mappings) Teichmüller observed that there are many hyperbolic surface of excess zero, using ideas built from what we introduced in §2.6 (which in turn had received a great deal of development in the 1930s, by authors other than Ahlfors himself).

Here I want to indicate some of the key steps in the construction of Benjamini et al [2] which shows the failure of the other possibility in Nevanlinna's problem. They produce a parabolic surface with $E(S) < 0$. Rather than construct S directly, they produce a suitable Speiser graph, being careful that for this graph, the graph- and function-theoretic notions of type coincide.

First, let $\mathcal{G}_0$ be the graph of the (parabolic) modular function ($q = 3$), and let $\ell = \{v_0, v_1, \dots\}$ be a fixed infinite path in $\mathcal{G}_0$. Then take $\mathcal{G}'$ as the subtree of $\mathcal{G}$ consisting of vertices v such that $d(v, v_n) \leq n$. Of course, $\mathcal{G}'$ is not yet a Speiser graph, but since there is at most one infinite path from any vertex $v \in \mathcal{G}'$, an extremal-length argument will show that the final graph $\mathcal{G}$, built on $\mathcal{G}'$, will remain parabolic.

The problem now is to augment $\mathcal{G}'$ so that in addition to being a Speiser graph, we have $E(S) < 0$ for the associated S. The vertices of $\mathcal{G}'$ are of two types: those at which three edges meet and so have degree 3 (ordinary vertices), and endpoints of paths (leaves).

The authors now build the graph $\mathcal{G}$ on $\mathcal{G}'$. To this end, they will associate to each vertex a closed disk, each of which will be divided into suitable cells to give $\mathcal{G}$. This is illustrated in the original paper (and was carefully discussed in the lecture), but the key fact is that each leaf is mated with a standard cell which is divided into six polygons (of two and four sides) so that the total excess associated to each leaf will be $+2$ (this will come from the two two-sided polygons).

To force $E(S)$ to be negative, the authors divide each disk associated to the remaining (ordinary) vertices into $s(v)$ cells, and each of these $s(v)$ stages will contribute $-1/3$ to the excess associated to v. Since the number of stages at each vertex is at their disposal, by taking $s(v)$ large, it is then possible to arrange that the contribution to the total curvature from these ordinary vertices dominate that from the leaves: $E(S) < 0$.

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