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ANALYTIC CONTRACTIONS AND BOUNDARY BEHAVIOUR-AN OVERVIEW

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1. Introduction

We start our discussion with the definition of two of the best known Hilbert spaces of convergent power series in the unit disc $\mathbb{D}$; the Hardy space and the Bergman space.

The Hardy space H^2 consists of all power series with square summable coefficients and becomes a Hilbert space with the norm

$$\|f\|^2 = \sum_{n\geq 0} |a_n|^2 = \lim_{r\to 1^-} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi},$$

where $f(z) = \sum_{n\geq 0} a_n z^n$.

The Bergman space L^2_a consists of all power series $f(z) = \sum_{n\geq 0} a_n z^n$ with the property that

$$\|f\|^2 = \sum_{n\geq 0} \frac{|a_n|^2}{n+1} = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty,$$

where dA denotes the normalized area measure on the unit disc.

In both cases, the shift operator, that is, the operator M_ζ of multiplication by the identity function ζ on $\mathbb{D}$, $\zeta(z) = z,\ z \in \mathbb{D}$ is deeply involved in the structure of these spaces. There are some classical results in complex analysis that reveal a great difference between the properties of functions in the Hardy space and the Bergman space. More precisely, it is well known that functions

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in H^2 have the nice property that the radial limits

$$f(e^{it}) = \lim_{r \to 1^-} f(re^{it})$$

exist a.e. on $\partial\mathbb{D}$ (w.r.t arclength or Lebesgue measure), while the Bergman space contains plenty of functions that do not share this property. Another fact that is less classical, but still well known is the following. If $\mathcal{M}$ is a shift invariant subspace ($\zeta\mathcal{M} \subset \mathcal{M}$) of H^2, then a zero $\lambda \in \mathbb{D}$ of a function $f \in \mathcal{M}$ is either a common zero of the whole subspace, or it can be divided out without leaving the subspace. Again for invariant subspaces of the Bergman space this property fails to hold. If one is looking for an operator-theoretic explanation to these phenomena, one makes the following simple observation that underlines the difference between the two shift operators: On H^2 the shift is an isometry (i.e. $\|\zeta f\| = \|f\|$), while on the Bergman space we have for all $f \in L^2_a$

$$\lim_{n\to\infty} \|\zeta^n f\|^2 = \lim_{n\to\infty} \int_{\mathbb{D}} |z^n f(z)|^2 dA(z) = 0,$$

by the dominated convergence theorem.

Thus one may be tempted to wonder whether for all Hilbert spaces of power series where M_ζ acts as a contraction there is a connection between the existence of nontangential limits, the fact that noncommon zeros can be divided out in shift invariant subspaces, and the behavior of $\|\zeta^n f\|$ as $n \to \infty$ for $f \in \mathcal{H}$.

The purpose of these notes is to show that under certain regularity assumptions on $\mathcal{H}$ the above three concepts can be used to state three conditions that are equivalent to one another. The paper is an overview of some recent results obtained in [ARS2], [ARS3], [ARS4] and presents some basic ideas as well as some of the main theorems in these papers in a more accessible form, that is, with stronger assumptions but less technical details. To set the stage let us now give the definition of a general space of analytic functions in $\mathbb{D}$ where M_ζ acts as a contraction. Throughout the paper we shall assume that $\mathcal{H}$ is a Hilbert space consisting of analytic functions on $\mathbb{D}$ which satisfies the following two axioms:

(1.1) for each $f \in \mathcal{H}$ we have $\zeta f \in \mathcal{H}$ and $||\zeta f|| \le ||f||$,

(1.2) for each $\lambda \in \mathbb{D}$ we have that $(\zeta-\lambda)\mathcal{H}$ is closed in $\mathcal{H}$ with $\dim \mathcal{H} \ominus (\zeta-\lambda)\mathcal{H} = \dim \mathcal{H} \cap ((\zeta-\lambda)\mathcal{H})^\perp = 1$.

The fact that $\mathcal{H}$ consists of analytic functions can also be expressed by the condition $\cap_{\lambda\in\mathbb{D}}(\zeta-\lambda)\mathcal{H} = \{0\}$. Moreover, these properties immediately imply that

(1.3) for each $\lambda \in \mathbb{D}$ the evaluation functional $f \to f(\lambda)$ is continuous on $\mathcal{H}$,

(1.4) for every $\lambda \in \mathbb{D}$ there is a $c_\lambda > 0$ such that $\left\| \frac{\zeta-\lambda}{1-\bar{\lambda}\zeta} f \right\| \ge c_\lambda \|f\|$ for all $f \in \mathcal{H}$.

The paper is organized as follows. The next two sections contain a detailed discussion of the two phenomena mentioned above, boundary behavior and index of shift invariant subspaces. In section 4 we state three theorems that offer some partial answers to the questions raised in the first two sections. In the rest of the paper we discuss the proofs of these theorems and mention some more technical results that strengthen their conclusions. There is a large class of such Hilbert spaces where we were able to obtain definitive results (see [ARS4]) without any additional conditions. These are obtained by taking the closure of polynomials in some $L^2(\mu)$-space and are denoted by $P^2(\mu)$. The last section contains a more or less heuristic description of the approach in [ARS4].

2. Basic questions about nontangential limits

Let us recall first the definitions of nontangential limits and some related notions. Given a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in $\mathbb{D}$ we say that it converges nontangentially to a point $z \in \partial\mathbb{D}$ if $\lim_{n\to\infty} \lambda_n = z$ and there is a fixed Stolz angle $\Gamma_r(z) =$ the interior of the convex hull of $\{z\} \cup r\mathbb{D}$, $0 < r < 1$ such that $\lambda_n \in \Gamma_r(z)$ for all n. Let k be a function defined on the unit disc with values on the extended positive axis. The nontangential limit superior $K(z)$ of k at $z \in \mathbb{T}$ is defined to be the supremum of $A \in (0, \infty]$ such that there is a sequence $\{\lambda_n\}$ of points in the open unit disc that converges to z nontangentially and such that $\{k(\lambda_n)\}_{n\in\mathbb{N}} \to A$. We write $K(z) = \mathrm{nt} - \limsup_{\lambda\to z} k(\lambda)$. A standard argument shows that if k is continuous, then K is measurable and for each $0 < r < 1$, $K(z)$ equals for a.e. $z \in \partial\mathbb{D}$ the limit superior of $k(\lambda)$, where z is approached from within a Stolz angle $\Gamma_r(z)$. The nontangential limit inferior and the nontangential limit of a function k as above are defined accordingly and share the above property if the function k is continuous.

Nontangential limits play a central role in the study of analytic functions on the unit disc because of the famous theorem of Privalov which says that meromorphic functions in $\mathbb{D}$ with nontangential limits zero on a set of positive Lebesgue measure on the unit circle must vanish identically. As is well known the conclusion fails if we consider only radial limits even of analytic functions in the disc. Much more was shown by Kahane and Katznelson [KK]. There exist analytic functions f in $\mathbb{D}$ such that

$$M_\infty(r, f) = \max_{|\lambda|=r} |f(\lambda)|$$

grows arbitrarily slow to infinity, but $\lim_{r\to 1^-} f(re^{it}) = 0$ a.e. on $[0, 2\pi]$. Of course, such analytic functions cannot have nontangential limits on any set of positive measure on the unit circle.

The oldest sufficient condition for the existence of the nontangential limit of an analytic function f in $\mathbb{D}$ at a point $z \in \partial\mathbb{D}$ is that the power series of f converges at z. This is a classical theorem of Abel which can be found in almost every text book in complex analysis. However, when combined with Carleson's famous theorem on pointwise convergence a.e. of the Fourier series of an L^2-function [C], this yields half of the following important result. The other half is part of the so-called Khinchin-Kolmogorov theorem (see [D]).

Theorem 2.1. *Let $f(z) = \sum_{n\geq 0} a_n z^n$ be analytic in the unit disc.*
(i) *If $\sum_{n\geq 0} |a_n|^2 < \infty$ then f has finite nontangential limits a.e. on $\partial\mathbb{D}$.*
(ii) *If $\sum_{n\geq 0} |a_n|^2 = \infty$ then there is a sequence $\{\varepsilon_n\}$ with $\varepsilon_n = \pm 1$ such that the power series $\sum_{n\geq 0} \varepsilon_n a_n z^n$ has no radial limit a.e. on $\partial\mathbb{D}$.*

To see (i) note that by Carleson's theorem such power series converge a.e. on the unit circle and then apply Abel's theorem. Thus, every function in H^2 has nontangential limits a.e. on $\partial\mathbb{D}$. On the other hand, if we consider slightly larger Hilbert spaces of analytic functions we encounter a completely different situation. For example, if $w = \{w_n\}$ is a decreasing sequence of positive numbers that converges to zero, one can easily verify that the space $\mathcal{H}_w$ of all power series $f(z) = \sum_{n\geq 0} a_n z^n$ with

$$\|f\|^2 = \sum w_n |a_n|^2 < \infty$$

satisfies all the axioms given in the introduction and contains functions whose Taylor coefficients are not square summable. Then Theorem 2.1 (ii) applies and it follows that $\mathcal{H}_w$ contains also functions that have no radial limits a.e. on $\partial\mathbb{D}$. A similar effect occurs when the "norms" under consideration are given by integration against positive (regular Borel) measures carried by $\mathbb{D}$. In fact, if μ is such a measure there always exists an analytic function f in $\mathbb{D}$ such that $f \in L^2(\mu)$, but f has no radial limits a.e. on $\partial\mathbb{D}$. This is a consequence of the following simple observation.

Proposition 2.2. *Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space of analytic functions in $\mathbb{D}$ which contains the constant functions and satisfies $\zeta\mathcal{B} \subset \mathcal{B}$. Suppose that for all $f \in \mathcal{B}$ we have*

$$\liminf_{n\to\infty} \|\zeta^n f\| = 0.$$

Then $\mathcal{B}$ contains functions that have no radial limits a.e. on $\partial\mathbb{D}$.

Proof. By assumption we can find a strictly increasing sequence of integers $\{n_k\}$ such that

$$\sum_k \|\zeta^{n_k}\| < \infty.$$

This immediately implies that $\sum_k \varepsilon_k z^{n_k} \in \mathcal{B}$ for any choice of the sequence $\{\varepsilon_k\}$ with $\varepsilon_k = \pm 1$ and the result follows by Theorem 2.1 (ii). □

The last example is related to some very interesting questions in polynomial approximation. Suppose that μ is now a regular Borel measure on $\overline{\mathbb{D}}$ and consider the space $P^2(\mu)$ defined as the closure of analytic polynomials in $L^2(\mu)$. Of course, $P^2(\mu)$ can be all of $L^2(\mu)$, for example if μ is the Lebesgue measure on $[-1, 1]$, but it can also be a Hilbert space of analytic functions in $\mathbb{D}$ (this is so when μ is the Lebesgue measure on $\partial\mathbb{D}$) or in a smaller domain. The remarkable structure theorem of Thomson [T] says that these two types of examples completely describe these spaces. More precisely, $P^2(\mu)$ admits a decomposition,

$$P^2(\mu) = L^2(\mu_0) \oplus \left(\bigoplus_{i=1}^{\infty} P^2(\mu_i) \right),$$

where μ_i are measures with mutually disjoint carriers whose sum is μ and for $i > 0$ the space $P^2(\mu_i)$ can be identified with a space of analytic functions satisfying the axiom (1.2) for a certain simply connected region Ω_i. Now let us assume for simplicity that the measure μ is such that $P^2(\mu)$ is such a space of analytic functions. Then by Proposition 2.2 we see that if μ is carried by the open unit disc then $P^2(\mu)$ contains functions with no radial limits a.e. on $\partial\mathbb{D}$. The interesting question arises when $\mu(\partial\mathbb{D}) > 0$. Note that in this case $\mu|\partial\mathbb{D}$ must be absolutely continuous w.r.t. Lebesgue measure on $\partial\mathbb{D}$ (see for ex. [G]). A function in $P^2(\mu)$ has an analytic restriction to $\mathbb{D}$ and a restriction on $\partial\mathbb{D}$ that belongs to $L^2(\mu|\partial\mathbb{D})$. How are these two functions related? For example, it follows immediately from (1.2) that the analytic restriction cannot vanish identically unless the original function does. Intuitively speaking, this seems to suggest that there should be a way to recapture the boundary restriction of a function in $P^2(\mu)$ from its values in the disc. The first thought that comes to mind is that this happens via nontangential limits, which leads us to the following question:

Question 1. If $\mu(\partial\mathbb{D}) > 0$ and $P^2(\mu)$ satisfies (1.1), (1.2), is it true that all functions $f \in P^2(\mu)$ have nontangential limits which equal $f|\partial\mathbb{D}$ a.e. on the set where $\frac{d\mu|\partial\mathbb{D}}{|dz|} > 0$?

If this question has an affirmative answer, we would get a dichotomy for such spaces of analytic functions: Either $\|\zeta^n f\| \to 0$ for all functions f in the space, or there is a set of positive measure on the circle such that all functions in the space have nontangential limits on that set. It is natural to ask whether this dichotomy actually holds for all spaces of analytic functions considered here. Also, if this is the case, can we find a formula for the values of these limits as we had in Question 1?

A good conjecture for such a formula can be deduced by analogy to the $P^2(\mu)$-case. To do this, observe first that for each $f \in P^2(\mu)$ we have

$$\|f\|_*^2 = \lim_{n\to\infty} \|\zeta^n f\|^2 = \lim_{n\to\infty} \int_{\overline{\mathbb{D}}} |\zeta|^{2n} |f|^2 d\mu = \int_{\partial\mathbb{D}} |f|^2 d\mu,$$

by the dominated convergence theorem. Thus, if Question 1 has an affirmative answer it follows by Privalov's theorem that $\|\cdot\|_*$ is a norm. Moreover, on the completion of $P^2(\mu)$ w.r.t. this norm the operator M_ζ becomes isometric and thus it has a minimal unitary extension U with spectral measure E. Clearly, in this case, U is just multiplication by ζ on $L^2(\mu|\partial\mathbb{D})$, while $E(\Delta)$ is the operator of multiplication by the characteristic function of the Lebesgue measurable set

Δ. We then obtain for every function $f \in P^2(\mu)$

$$f|\partial\mathbb{D} = \frac{d\langle E(\cdot)f, 1\rangle/|dz|}{d\langle E(\cdot)1, 1\rangle/|dz|}$$

a.e. on the set where the denominator is positive. Now the objects $\|\cdot\|_*, U, E$ can be defined in a natural way for any contraction T on a separable Hilbert space $\mathcal{H}$ exactly in the same way as above with the only difference that, in general, $\|f\|_* = \lim_{n\to\infty}\|\zeta^n f\|$ may not define a norm on $\mathcal{H}$. For this reason, we consider the closed subspace $\mathcal{M}_0$ of vectors $f \in \mathcal{H}$ for which $\|f\|_* = 0$ and note that the compression $P_{\mathcal{M}_0^\perp} T|\mathcal{M}_0^\perp$ is isometric with respect to the new norm $\|\cdot\|_*$. We then let U be the minimal unitary extension of this isometry and let E be the spectral measure of U. However, it turns out that in this general context the analogue of Question 1 has a negative answer. The example constructed below yields a Hilbert space $\mathcal{H}$ of analytic functions in $\mathbb{D}$ that satisfies (1.1) and (1.2) such that $\lim_{n\to\infty}\|\zeta^n f\| > 0$ for all $f \in \mathcal{H}\setminus\{0\}$, but at the same time contains functions that have no nontangential limits a.e. on $\partial\mathbb{D}$.

Example 2.3. Let $\Lambda = \{\lambda_n\}_{n\geq 0}$ be a sequence of distinct points in $\mathbb{D}$ such that Λ has no accumulation point in $\mathbb{D}$, let F be an analytic function in $\mathbb{D}$ with simple zeros at λ_n and no other zeros, and let $c, w_n > 0$ be such that $\sum_{n=0}^\infty \frac{w_n}{1-|\lambda_n|^2} \leq c$.

Note that for all $a_n \in \mathbb{C}$

$$\frac{1}{c}\left(\sum_{n=0}^\infty |a_n|\right)^2 \leq \sum_{n=0}^\infty (1-|\lambda_n|^2)\frac{|a_n|^2}{w_n} \leq \sum_{n=0}^\infty \frac{|a_n|^2}{w_n}. \tag{1}$$

Thus, whenever $\sum_{n=0}^\infty \frac{|a_n|^2}{w_n} < \infty$, then $\sum_{n=0}^\infty \frac{a_n}{\zeta-\lambda_n}$ defines a meromorphic function in $\mathbb{D}$ with a simple pole at each λ_n. We define $\mathcal{H}$ to be the set of analytic functions g of the form $g = F(u + \sum_{n=0}^\infty \frac{a_n}{\zeta-\lambda_n})$, where $u \in H^2$ and $\sum_{n=0}^\infty \frac{|a_n|^2}{w_n} < \infty$. Note that if $g \in \mathcal{H}$, then the function u and the coefficients $\{a_n\}_{n=0}^\infty$ are uniquely determined. Thus, we may define a Hilbert space norm on $\mathcal{H}$ by setting

$$||g||^2 = \left\|F\left(u + \sum_{n=0}^\infty \frac{a_n}{\zeta-\lambda_n}\right)\right\|^2 = \frac{1}{c}||u||_{H^2}^2 + \sum_{n=0}^\infty \frac{|a_n|^2}{w_n}.$$

We now verify that $\mathcal{H}$ is a space of analytic functions that satisfies conditions (1.1) and (1.2). Note first that if $g = F(u + \sum_{n=0}^\infty \frac{a_n}{\zeta-\lambda_n}) \in \mathcal{H}$, then $\zeta g = F\big(\zeta u + \sum_{n=0}^\infty a_n + \sum_{n=0}^\infty \frac{\lambda_n a_n}{\zeta-\lambda_n}\big)$, and by inequality (1) we have

$$||\zeta g||^2 = \frac{||u||_{H^2}^2 + |\sum_{n=0}^\infty a_n|^2}{c} + \sum_{n=0}^\infty |\lambda_n|^2 \frac{|a_n|^2}{w_n} \leq ||g||^2,$$

which implies (1.1). In order to verify (1.2) it suffices to prove that whenever $\lambda \in \mathbb{D}$ and $g \in \mathcal{H}$ with $g(\lambda) = 0$, then $g/(\zeta - \lambda) \in \mathcal{H}$ (see Lemma 2.1 of [R]). Let $g = F(u + \sum_{n=0}^{\infty} \frac{a_n}{\zeta - \lambda_n})$ with $g(\lambda) = 0$. We first assume $\lambda \neq \lambda_n$ for all n. Then $u(\lambda) + \sum_{n=0}^{\infty} \frac{a_n}{\lambda - \lambda_n} = 0$, so

$$\frac{g(z)}{z - \lambda} = F(z) \left(\frac{u(z) - u(\lambda)}{z - \lambda} + \sum_{n=0}^{\infty} \frac{a_n}{z - \lambda_n} \frac{1}{\lambda_n - \lambda} \right) \in \mathcal{H}.$$

If $\lambda = \lambda_k$ for some k, then $a_k = 0$. In this case

$$\frac{g(z)}{z - \lambda_k} = F(z) \left(\frac{u(z) - u(\lambda_k)}{z - \lambda_k} + \frac{u(\lambda_k) + \sum_{n=0}^{\infty} \frac{a_n}{\lambda_k - \lambda_n}}{z - \lambda_k} + \sum_{n \neq k}^{\infty} \frac{a_n}{z - \lambda_n} \frac{1}{\lambda_n - \lambda_k} \right)$$

and this function is in $\mathcal{H}$ also.

We will now specify choices of Λ, $\{w_n\}_{n \geq 0}$ and c as above such that $||\zeta^n g|| \nrightarrow 0$ for all $g \in \mathcal{H}$, $g \neq 0$. For this we choose Λ to be interpolating for the Bergman space L_a^2, that is, every l^2-sequence can be written as $((1 - |\lambda_n|^2) f(\lambda_n))$ with $f \in L_a^2$. Next we choose $\{w_n\}_{n \geq 0}$ and $c > 0$ such that $\sum_{n=0}^{\infty} \frac{w_n}{1 - |\lambda_n|^2} \leq c$ and $w_n \leq (1 - |\lambda_n|^2)^2$ for all n.

If $g = f(u + \sum_{n=0}^{\infty} \frac{a_n}{\zeta - \lambda_n}) \in \mathcal{H}$ such that $||\zeta^k g|| \to 0$ as $k \to \infty$, then $u = 0$ and $\sum_{n=0}^{\infty} a_n \lambda_n^k = 0$ for all $k \geq 0$. Thus $\sum_{n=0}^{\infty} a_n p(\lambda_n) = 0$ for every polynomial p. Furthermore, it is a standard fact that for any $f \in L_a^2$ we have

$$\sum_{n=0}^{\infty} (1 - |\lambda_n|^2)^2 |f(\lambda_n)|^2 \leq M ||f||_{L_a^2}^2$$

which gives

$$\begin{aligned} \left(\sum_{n=0}^{\infty} |a_n f(\lambda_n)| \right)^2 &\leq \sum_{n=0}^{\infty} \frac{|a_n|^2}{w_n} \sum_{n=0}^{\infty} w_n |f(\lambda_n)|^2 \\ &\leq ||g||^2 \sum_{n=0}^{\infty} (1 - |\lambda_n|^2)^2 |f(\lambda_n)|^2 \\ &\leq M ||g||^2 \; ||f||_{L_a^2}^2. \end{aligned}$$

Thus, $\sum_{n=0}^{\infty} a_n f(\lambda_n) = 0$ for every $f \in L_a^2$. Since $\Lambda = \{\lambda_n\}_{n \geq 0}$ is interpolating for L_a^2 for each n, we may choose $f_n \in L_a^2$ such that $f_n(\lambda_n) \neq 0$, but $f_n(\lambda_j) = 0$ for all $j \neq n$. Hence $a_n = 0$ for all n, i.e. $g = 0$.

In order to produce functions in $\mathcal{H}$ with no nontangential limits a.e. we choose an interpolating set Λ for L_a^2 that accumulates nontangentially at every boundary point. It is well known that such sequences exist (see [HKZ]). Then by Privalov's theorem F is a function in $\mathcal{H}$ that cannot have nontangential limits on any subset of $\partial\mathbb{D}$ with positive measure.

With this example in mind we now formulate the general version of Question 1 as follows.

Question 1a. Let $\mathcal{H}$ be a Hilbert space of analytic functions in $\mathbb{D}$ that satisfies (1.1), (1.2) and suppose there exists $f_0 \in \mathcal{H}$ such that $\lim_{n\to\infty} \|\zeta^n f_0\| \neq 0$. Under what additional assumptions is it true that for all functions $f, g \in \mathcal{H}$, $g \neq 0$ the meromorphic function f/g has the nontangential limit $\frac{d\langle E(\cdot)f,g\rangle/|dz|}{d\langle E(\cdot)g,g\rangle/|dz|}$ a.e. on some measurable set $\Sigma(\mathcal{H})$ with $E(\partial\mathbb{D} \setminus \Sigma(\mathcal{H})) = 0$?

The fact that we consider quotients of functions in $\mathcal{H}$ is just a normalization since we do not know whether $\mathcal{H}$ contains the constants or not.

3. The index of an invariant subspace

Consider a Hilbert space of analytic functions on $\mathbb{D}$ that satisfies the axioms (1.1) and (1.2) and let $\mathcal{M}$ be a nonzero invariant subspace for $M_\zeta|\mathcal{H}$. Does the restriction of M_ζ to $\mathcal{M}$ satisfy (1.1) and (1.2) as well? A simple inspection of these conditions reveals that the only part that might cause some problems is the condition on the codimension of the spaces $(\zeta-\lambda)\mathcal{M}$, $\lambda \in \mathbb{D}$. Elementary Fredholm theory implies that $\dim \mathcal{M} \ominus (\zeta-\lambda)\mathcal{M}$ is constant in $\mathbb{D}$, but it turns out that this constant may be any positive integer, or even ∞. Let us introduce the number

$$\mathrm{ind}\mathcal{M} = \dim \mathcal{M} \ominus (\zeta-\lambda)\mathcal{M}$$

which will be called the index of the invariant subspace $\mathcal{M}$. The fact that there exist invariant subspaces with arbitrary index follows from the deep work of Apostol, Bercovici, Foiaş and Pearcy [ABFP]. For the Hilbert spaces of analytic functions considered here, their result is as follows.

Theorem 3.1. *Let $\mathcal{H}$ be a Hilbert space of analytic functions satisfying (1.1) and (1.2) such that for every $f \in \mathcal{H}$ we have $\lim_{n\to\infty} \|\zeta^n f\| = 0$. Then given any $N \in \mathbb{N} \cup \{\infty\}$ there exists an invariant subspace $\mathcal{M}$ of $\mathcal{H}$ with $\mathrm{ind}\mathcal{M} = N$.*

This is merely an existence theorem, since the construction of these subspaces is based on the Hahn-Banach theorem, hence implicitly on the axiom of choice. Concrete examples of invariant subspaces with arbitrary index appeared much later. Hedenmalm [H] gave the first construction of invariant subspaces with arbitrary finite index in the Bergman space, and the argument was extended to cover the case of an infinite index in [HRS]. Several different methods to construct such subspaces have been developed (see, for example, [AB], [BHV], [ARS1]). As it will be shown in the next two sections, there is a class of such subspaces that arise from interpolating sequences that accumulate nontangentially at almost every boundary point.

It is also interesting to note that this very complex structure of the lattice of invariant subspaces is at its turn related to the condition $\lim_{n\to\infty} \|\zeta^n f\| = 0$, $f \in \mathcal{H}$ encountered in the previous section. In the Hardy space H^2 where

M_ζ is an isometry, every nonzero invariant subspace has index one. Moreover, if one looks at the examples of invariant subspaces with index higher than one, it appears immediately that the nonzero functions in such subspaces have a very irregular boundary behaviour, in particular, they have no nontangential limits a.e. on $\partial\mathbb{D}$. This raises the interesting question whether the existence of invariant subspaces with index ≥ 2 is related to the boundary behavior. A strong link between the two phenomena was first established for the Bergman space in [ARS1] via the so-called majorization function. Given an invariant subspace $\mathcal{M}$ of a space $\mathcal{H}$ of analytic functions the majorization function of $\mathcal{M}$ is defined by

$$k_{\mathcal{M}}(\lambda) = \frac{\sup\{|f(\lambda)| : \ f \in \mathcal{M}, \ \|f\| \leq 1\}}{\sup\{|f(\lambda)| : \ f \in \mathcal{H}, \ \|f\| \leq 1\}}, \quad \lambda \in \mathbb{D}.$$

Note that in H^2 we have by Beurling's theorem that any nonzero invariant subspace $\mathcal{M}$ has the form IH^2 for some inner function I, and this easily implies that $k_{\mathcal{M}}(\lambda) = |I(\lambda)|$. Consequently, $k_{\mathcal{M}}$ has nontangential limits equal to 1 a.e. on $\partial\mathbb{D}$. If $\mathcal{H} = L^2_a$ the functions $k_{\mathcal{M}}$ can be considerably smaller, for example, we can find nonzero invariant subspaces $\mathcal{M}$ such that

$$\text{nt-}\liminf_{\lambda \to z} k_{\mathcal{M}}(\lambda) = 0$$

for almost every $z \in \partial\mathbb{D}$. A concrete example of this type is obtained if we consider a zero sequence Λ for L^2_a that is at the same time dominating, that is, for each $h \in H^\infty$ we have

$$\sup_{\lambda \in \Lambda} |h(\lambda)| = \sup_{\lambda \in \mathbb{D}} |h(\lambda)|.$$

The following result was proved in [ARS1].

Theorem 3.2. *For a nonzero invariant subspace $\mathcal{M}$ of L^2_a the following are equivalent.*
(i) *Every invariant subspace that contains $\mathcal{M}$ has index one.*
(ii) *The majorization function $k_{\mathcal{M}}$ has a positive nontangential limit inferior on a set of positive measure in the unit circle.*

This theorem is a quite powerful tool in the study of the index of invariant subspaces. Several strong results are derived in [ARS1]. More precisely, it is shown that the two statements above are further equivalent to the fact that the extremal function of $\mathcal{M}$ (see [HKZ]) has nontangential limits on a set of positive measure in $\partial\mathbb{D}$. Moreover, the result was used to disprove the following natural conjecture suggested by earlier results from [AR] and [WY]: Every invariant subspace of the Bergman space that contains a nonzero function which has nontangential limits on a set of positive measure in the unit circle has index one.

Finally, let us now turn to a dichotomy similar to the one described in the previous section. In view of Theorem 3.1 one could ask whether the existence of an element $f_0 \in \mathcal{H}$ with $\|\zeta^n f_0\| \nrightarrow 0$ implies that every nonzero invariant subspace has index one. Recall that for $P^2(\mu)$-spaces the existence of such an $f_0 \in \mathcal{H}$ is equivalent to the fact that $\mu(\partial\mathbb{D}) > 0$. Conway and Yang [CY] conjectured that for these spaces the question has an affirmative answer and the problem was investigated in detail in [A1] and [A2]. In full generality the answer is again negative and counterexamples are provided by the spaces constructed in Example 2.1. Indeed, if the set Λ is at the same time interpolating for L^2_a and dominating for H^∞, one can show that the closed invariant subspace $\mathcal{M} = FH^2$ is contained in invariant subspaces of arbitrary index. The way of proving this last assertion is outlined in the next section. Another class of examples related to this question has been recently found by Esterle [E]. He constructed Hilbert spaces $\mathcal{H}$ of analytic functions where M_ζ is an expansive weighted shift which contain invariant subspaces with arbitrary index. This means that the norm of a power series $\sum_{n\geq 0} a_n z^n$ in $\mathcal{H}$ has the form

$$\|f\|^2 = \sum_{n\geq 0} w_n |a_n|^2,$$

where $\{w_n\}$ is a given increasing sequence of positive numbers (i.e. $\|\zeta f\| \geq \|f\|,\ f \in \mathcal{H}$). In particular, $\mathcal{H}$ is contained in H^2!

With these considerations we can formulate the following question that should be compared to Question 1a:

Question 2. Let $\mathcal{H}$ be a Hilbert space of analytic functions in $\mathbb{D}$ that satisfies (1.1), (1.2) and suppose there exists $f_0 \in \mathcal{H}$ such that $\lim_{n\to\infty} \|\zeta^n f_0\| \neq 0$. Under what additional assumption does every nonzero invariant subspace of $\mathcal{H}$ have index one?

4. Main results

In this section we will present three theorems that provide partial positive answers to the questions discussed in the previous sections. In the rest of the paper we shall outline the proofs and also point out several results that strengthen their conclusions. We shall first consider the case of a general Hilbert space of analytic functions and then turn to the spaces $P^2(\mu)$, where our results are conclusive, but, at the same time, more difficult to prove.

The discussion at the end of Section 3 not only reveals the difficulty of Question 2 in the general case, but it also suggests that the answer might depend on certain regularity conditions for the norm on the space in question. The condition we shall use in the sequel is nothing else than the uniform version of the property (1.4) from the introduction, that is:

(4.1) *There exists a constant $c > 0$ such that* $\left\| \frac{\zeta-\lambda}{1-\bar{\lambda}\zeta} f \right\| \geq c\|f\|$ *for all $f \in \mathcal{H}$ and all $\lambda \in \mathcal{H}$.*

Both questions above have affirmative answers for Hilbert spaces of analytic functions that satisfy this additional condition. Of course, one can consider local versions of this condition (see [ARS3] for details). The more restrictive global condition (4.1) has been prefered in order to avoid technicalities.

Theorem 4.1. *Let $\mathcal{H}$ be a Hilbert space of analytic functions satisfying* (1.1), (1.2) *and* (4.1). *If there exists $f_0 \in \mathcal{H}$ such that $\lim_{n\to\infty} \|\zeta^n f_0\| \neq 0$ then:*
(i) *For every $f, g \in \mathcal{H}$, $g \neq 0$ the meromorphic function f/g has the nontangential limit $\frac{d\langle E(\cdot)f,g\rangle/|dz|}{d\langle E(\cdot)g,g\rangle/|dz|}$ a.e. on some measurable set $\Sigma(\mathcal{H})$ with $E(\partial\mathbb{D}\setminus\Sigma(\mathcal{H})) = 0$.*
(ii) *Every nonzero invariant subspace of $M_\zeta|\mathcal{H}$ has index one.*

Our next theorem concerns the opposite situation, when $\lim_{n\to\infty} \|\zeta^n f\| = 0$ for all $f \in \mathcal{H}$. Note that the results below are independent of the condition (4.1). Let us recall first the notion of an interpolation sequence. Let $\mathcal{H}$ satisfy (1.1) and (1.2) and denote by k_λ, $\lambda \in \mathbb{D}$ the reproducing kernel at λ, that is k_λ is the function in $\mathcal{H}$ uniquely determined by the equality

$$f(\lambda) = \langle f, k_\lambda\rangle.$$

Given a sequence $\Lambda = \{\lambda_n\}$ in $\mathbb{D}$ consider the (interpolation) operator T_Λ which maps a function $f \in \mathcal{H}$ to the sequence $\{f(\lambda_n)/\|k_{\lambda_n}\|\}$. The sequence Λ is called interpolating for $\mathcal{H}$ if the operator T_Λ maps $\mathcal{H}$ onto l^2. The simple fact that T_Λ can never be invertible is left as an exercise to the reader.

Theorem 4.2. *Let $\mathcal{H}$ be a Hilbert space of analytic functions satisfying* (1.1), (1.2) *such that $\lim_{n\to\infty} \|\zeta^n f\| = 0$ for all $f \in \mathcal{H}$. Assume, in addition, that H^∞ is contained and dense in $\mathcal{H}$. Then there exists a sequence Λ in $\mathbb{D}$ which is at the same time dominating and interpolating for $\mathcal{H}$. In particular, if*

$$I(\Lambda) = \{f \in \mathcal{H} : \ f|\Lambda = 0\},$$

then each nonzero function in $I(\Lambda)$ has no nontangential limits a.e. on $\partial\mathbb{D}$, and $I(\Lambda)$ is contained in an invariant subspace with index greater than one.

Our third result concerns the special case when $\mathcal{H} = P^2(\mu)$ and provides affirmative answers to the questions above without any assumption about the underlying measure μ.

Theorem 4.3. *Let μ be a finite measure on $\overline{\mathbb{D}}$ such that $P^2(\mu)$ is a space of analytic functions on $\mathbb{D}$ that satisfies* (1.1) *and* (1.2). *If $\mu(\partial\mathbb{D}) > 0$ then:*
(i) *Every function $f \in P^2(\mu)$ has nontangential limits which equal $f|\partial\mathbb{D}$ a.e. on the set where $\frac{d\mu|\partial\mathbb{D}}{|dz|} > 0$.*
(ii) *Every nonzero invariant subspace for M_ζ has index one.*

Let us now begin the discussion of the proof of Theorem 4.1. It is probably clear from the statement that our first task is to get a better understanding of the spectral measure E defined in Section 2. Again, the spaces $P^2(\mu)$ will supply the necessary intuition. Suppose μ is a measure on $\overline{\mathbb{D}}$ such that its restriction on $\partial\mathbb{D}$ is absolutely continuous. Then its density can be recovered by means of the nice formula

$$\frac{d\mu|\partial\mathbb{D}}{|dz|}(z) = \text{nt-}\lim_{\lambda\to z}\int_{\overline{\mathbb{D}}}\frac{1-|\lambda|^2}{|1-\overline{\lambda}\zeta|^2}d\mu,$$

which holds for almost every $z \in \partial\mathbb{D}$ (see [ARS1] for a proof). This slightly improves the classical Fatou theorem about nontangential limits of Poisson integrals and, in view of this result, we see that it suffices to show that for any finite measure μ on $\mathbb{D}$ we have

$$\text{nt-}\lim_{\lambda\to z}\int_{\mathbb{D}}\frac{1-|\lambda|^2}{|1-\overline{\lambda}\zeta|^2}d\mu = 0$$

for a.e. $z \in T$. This last equality goes back to the work of Littlewood and his subordination principle [D]. An operator-theoretic counterpart of the above result is immediately seen. If T denotes the contraction $M_\zeta|P^2(\mu)$ and x denotes the constant function 1 then, with the notations from Section 2 we can rewrite (4.1) as

$$\text{nt-}\lim_{\lambda\to z}(1-|\lambda|^2)\|(1-\overline{\lambda}T)^{-1}x\|^2 = \frac{d\langle E(\cdot)x,x\rangle}{|dz|}(z) \quad \text{a.e.} \tag{2}$$

Lemma 4.4. *The equality* (2) *holds for every contraction T on a separable Hilbert space $\mathcal{H}$.*

Proof. For $x \in \mathcal{H}$ and $\lambda \in \mathbb{D}$ let

$$u_x(\lambda) = (1-|\lambda|^2)\|(1-\overline{\lambda}T)^{-1}x\|^2, \quad v_x(\lambda) = \text{Re}\langle(1+\overline{\lambda}T)(1-\overline{\lambda}T)^{-1}x, x\rangle.$$

By a direct computation we can verify that $u_x(\lambda) \le v_x(\lambda)$, and that v_x is harmonic in $\mathbb{D}$. Since v_x is nonnegative, it can be represented as the Poisson integral of a nonnegative finite measure on the unit circle (see [D]) whose total variation is $v_x(0) = \|x\|^2$. Then v_x has nontangential limits a.e. on $\partial\mathbb{D}$ which satisfy the weak-type inequality

$$|\{z \in \partial\mathbb{D} : \ v_x(z) > t\}| \le M\frac{\|x\|^2}{t},$$

with an absolute constant $M > 0$. Thus, we can conclude that the nontangential limit superior of u_x is finite a.e. and we have the estimate

$$|\{z \in \partial\mathbb{D} : \ \text{nt-}\limsup_{\lambda\to z} u_x(\lambda) > t\}| \le M\frac{\|x\|^2}{t}. \tag{3}$$

Next, let us check that

$$\text{nt-}\lim_{\lambda\to z}(u_x(\lambda) - u_{Tx}(\lambda)) = 0$$

a.e. Indeed, we have

$$\begin{aligned} 0 \le \|(1-\overline{\lambda}T)^{-1}x\|^2 - \|\overline{\lambda}T(1-\overline{\lambda}T)^{-1}x\|^2 &= -\|x\|^2 - 2\text{Re}\langle(1-\overline{\lambda}T)^{-1}x, x\rangle \\ &\le 2\|x\|\|(1-\overline{\lambda}T)^{-1}x\|, \end{aligned}$$

and the claim follows from above. With this information at hand we can conclude from (3) that for every positive integer n we have

$$|\{z \in \partial\mathbb{D} : \text{ nt-}\limsup_{\lambda\to z} u_x(\lambda) > t\}| \leq M\frac{\|T^n x\|^2}{t},$$

and by letting $n \to \infty$ we get

$$|\{z \in \partial\mathbb{D} : \text{ nt-}\limsup_{\lambda\to z} u_x(\lambda) > t\}| \leq M\frac{\|x\|_*^2}{t}.$$

The next step is to repeat the above reasoning with a new norm. Let $0 < c < 1$ and set $\|x\|_c^2 = \|x\|^2 - c\|x\|_*^2$. It is trivial to verify that this defines an equivalent norm on $\mathcal{H}$ and that T is a contraction with respect to this norm as well. If we denote by

$$u_x^*(\lambda) = (1 - |\lambda|^2)\|(1 - \overline{\lambda}T)^{-1}x\|_*^2 \leq u_x(\lambda),$$

then an application of the above argument to T on $(\mathcal{H}, \|\cdot\|_c)$ leads to the inequality

$$(4) \qquad |\{z \in \partial\mathbb{D} : \text{ nt-}\limsup_{\lambda\to z}(u_x(\lambda) - cu_x^*(\lambda)) > t\}| \leq M\frac{(1-c)\|x\|_*^2}{t}.$$

Finally, note that

$$u_x^*(\lambda) = \int_{\partial\mathbb{D}} \text{Re}\frac{1 + \overline{\lambda}z}{1 - \overline{\lambda}z} d\langle Ex, x\rangle$$

and apply Fatou's theorem together with (4) to obtain

$$\begin{aligned} \text{nt-}\lim_{\lambda\to z} u_x^*(\lambda) &\leq \text{nt-}\liminf_{\lambda\to z} u_x(\lambda) \\ &\leq \text{nt-}\limsup_{\lambda\to z} u_x(\lambda) \\ &\leq \text{nt-}\lim_{\lambda\to z} c\, u_x^*(\lambda) + t \end{aligned}$$

on a subset of $\partial\mathbb{D}$ whose complement has measure $< M\frac{(1-c)\|x\|_*^2}{t}$, and the result follows by choosing the parameters c and t in a suitable way. □

One additional result about the spectral measure E is needed. Its proof is more technical and will be omitted (see [ARS3]). Part of it asserts that the measures $\langle E(\cdot)x, x\rangle$ are absolutely continuous w.r.t. Lebesgue measure on $\partial\mathbb{D}$. If this is the case, we can easily construct the set $\Sigma(\mathcal{H})$. We consider measurable sets Σ such that $\chi_\Sigma(z)|dz|$ is a scalar-valued spectral measure for U. Such sets are unique only up to sets of measure zero, but if we let $\Sigma(\mathcal{H})$ denote the set of Lebesgue points of Σ, then this set is uniquely associated with the Hilbert space $\mathcal{H}$.

Lemma 4.5. *Let $\mathcal{H}$ be a Hilbert space of analytic functions that satisfies* (1.1) *and* (1.2). *If $T = M_\zeta|\mathcal{H}$ then the measures $\langle E(\cdot)f, f\rangle$ are absolutely continuous* *w.r.t. Lebesgue measure on $\partial\mathbb{D}$. If $\mathcal{D}$ is a dense subset of $\mathcal{H}$, then for every $\varepsilon > 0$, there is an $h \in \mathcal{D}$ such that*

$$|\Sigma(\mathcal{H}) \setminus \{z \in \partial\mathbb{D} : \frac{d\langle E(\cdot)h, h\rangle}{|dz|}(z) > 0\}| < \varepsilon.$$

Proof of Theorem 4.1-an outline. To prove (i) it will suffice to show that given $\varepsilon > 0$ there is a $g \in \mathcal{H}$ such that

$$|\Sigma(\mathcal{H}) \setminus \{z \in \partial\mathbb{D} : \frac{d\langle E(\cdot)g, g\rangle}{|dz|}(z) > 0\}| < \varepsilon \tag{5}$$

and (i) holds with this particular g a.e. on the set $\{z \in \partial\mathbb{D} : \frac{d\langle E(\cdot)g,g\rangle}{|dz|}(z) > 0\}$. This follows immediately by Privalov's theorem which says that nontangential limits of nonzero meromorphic functions cannot vanish on a set of positive measure. More precisely, if we assume the above claim, we can apply it to $f/g, h/g$ for arbitrary $f, h \in \mathcal{H},\ h \neq 0$ and conclude that $f/h = (f/g)/(h/g)$ will satisfy (i) on the set $\Sigma_g = \{z \in \partial\mathbb{D} : \frac{d\langle E(\cdot)g,g\rangle}{|dz|}(z) > 0\}$. To see the claim, fix $\varepsilon > 0$ and apply Lemma 4.5 to find a $g \in \mathcal{H}$ that satisfies (5). First we want to estimate the quantity

$$R(\lambda) = (1 - |\lambda|^2)\left\langle \frac{f - f/g(\lambda)g}{(\zeta - \lambda)(1 - \overline{\lambda}\zeta)}, g\right\rangle_* = (1 - |\lambda|^2)\left\langle \frac{f - f/g(\lambda)g}{(\zeta - \lambda)(1 - \overline{\lambda}\zeta)}, Ag\right\rangle,$$

where A is the bounded linear operator on $\mathcal{H}$ defined by the equality $\langle u, v\rangle_* = \langle u, Av\rangle$. If we apply the Cauchy-Schwarz inequality to the right hand side, we obtain

$$|R(\lambda)| \leq (1 - |\lambda|^2)\left\| \frac{(1 - \overline{\lambda}\zeta)(f - f/g(\lambda)g)}{(\zeta - \lambda)(1 - \overline{\lambda}\zeta)}\right\| \left\|(1 - \lambda M_\zeta^*)^{-1}Ag\right\|.$$

Note that $M_\zeta^*|\mathcal{H}$ is a contraction which satisfies

$$\lim_{n\to\infty} \|M_\zeta^{*n} f\| = 0, \quad f \in \mathcal{H}.$$

This can easily be seen if f is a reproducing kernel for $\mathcal{H}$ and then it follows from the fact that the reproducing kernels span $\mathcal{H}$. Then by Lemma 4.4 we have that

$$\text{nt-}\lim_{\lambda\to z}(1 - |\lambda|^2)^{1/2}\left\|(1 - \lambda M_\zeta^*)^{-1}Ag\right\| = 0$$

a.e. on $\partial\mathbb{D}$. Moreover, by (4.1) we have

$$\begin{aligned}\left\| \frac{(1 - \overline{\lambda}\zeta)(f - f/g(\lambda)g)}{(\zeta - \lambda)(1 - \overline{\lambda}\zeta)}\right\| &\leq \frac{1}{c}\left\|\frac{f - f/g(\lambda)}{1 - \overline{\lambda}\zeta}\right\| \\ &\leq \frac{1}{c}\left\|\frac{f}{1 - \overline{\lambda}\zeta}\right\| + \frac{|f/g(\lambda)|}{c}\left\|\frac{g}{1 - \overline{\lambda}\zeta}\right\|.\end{aligned}$$

Thus, we can apply again Lemma 4.4 to conclude that there is a function ρ with nontangential limits zero a.e. on Σ_g such that

$$|R(\lambda)| \leq \rho(\lambda)\left(1 + \left|\frac{f}{g}(\lambda)\right|\right). \tag{6}$$

On the other hand, if $P_\lambda(\zeta) = \frac{1-|\lambda|^2}{|1-\overline{\lambda}\zeta|^2}$ denotes the Poisson kernel, we can use the same elementary operator-theoretic arguments as in the proof of Lemma 4.4 to see that R can be written as

$$R(\lambda) = \int_{\partial\mathbb{D}} P_\lambda(z) d\langle E(\cdot)f, g\rangle - \frac{f}{g}(\lambda)\int_{\partial\mathbb{D}} P_\lambda(z) d\langle E(\cdot)g, g\rangle.$$

This means that R can be written also as

$$R(\lambda) = u_1(\lambda) + \frac{f}{g}(\lambda)u_2(\lambda) \tag{7}$$

where u_1, u_2 are harmonic functions that have nontangential limits a.e. on $\partial\mathbb{D}$ and, in addition, the limits of u_2 are positive a.e. on Σ_g. If we now compare (6) and (7) we conclude that

$$\text{nt-}\limsup_{\lambda\to z}\left|\frac{f}{g}(\lambda)\right| < \infty$$

a.e. on Σ_g. But then, by (6) R has nontangential limits 0 a.e. on Σ_g and the claim follows by Fatou's theorem.

Let us now turn to (ii). There is a standard way (see [R] for example) to prove that the index of a nonzero invariant subspace $\mathcal{M}$ of $\mathcal{H}$ is one. One needs to show that given $f, g \in \mathcal{M},\ g \neq 0$ we have that

$$\frac{f - f/g(\lambda)g}{\zeta - \lambda} \in \mathcal{M}.$$

Equivalently, if $h \in \mathcal{M}^\perp$ we want to prove that the meromorphic function

$$H(\lambda) = \left\langle \frac{f - f/g(\lambda)g}{\zeta - \lambda}, h\right\rangle$$

vanishes identically in $\mathbb{D}$. To this end, we shall show that this function has nontangential limits zero on a set of positive measure in $\partial\mathbb{D}$ and conclude the proof by Privalov's theorem. Let us start with the simple identity

$$\frac{1}{z-\lambda} + \frac{\overline{\lambda}}{1-\overline{\lambda}z} = \frac{1-|\lambda|^2}{(z-\lambda)(1-\overline{\lambda}z)}$$

which implies that our meromorphic function H can be written as

$$H(\lambda) = \left\langle \frac{f - f/g(\lambda)g}{\zeta - \lambda}, h\right\rangle = (1-|\lambda|^2)\left\langle \frac{f - f/g(\lambda)g}{(\zeta-\lambda)(1-\overline{\lambda}\zeta)}, h\right\rangle.$$

As we did in the proof of (i) we apply the Cauchy-Schwarz inequality to obtain

$$|H(\lambda)| \leq (1-|\lambda|^2)\left\|\frac{(1-\overline{\lambda}\zeta)(f - f/g(\lambda)g)}{(\zeta-\lambda)(1-\overline{\lambda}\zeta)}\right\| \left\|(1-\lambda M_\zeta^*)^{-1}h\right\|.$$

As pointed out above, from the fact that $\lim_{n\to\infty}\|M_\zeta^{*n}f\| = 0$, $f\in\mathcal{H}$, we deduce that

$$\text{nt-}\lim_{\lambda\to z}(1-|\lambda|^2)^{1/2}\left\|(1-\lambda M_\zeta^*)^{-1}h\right\| = 0$$

a.e. on $\partial\mathbb{D}$. Moreover, by (4.1) we have

$$\left\|\frac{(1-\overline{\lambda}\zeta)(f-f/g(\lambda)g)}{(\zeta-\lambda)(1-\overline{\lambda}\zeta)}\right\| \le \frac{1}{c}\left\|\frac{f-f/g(\lambda)}{1-\overline{\lambda}\zeta}\right\| \le \frac{1}{c}\left\|\frac{f}{1-\overline{\lambda}\zeta}\right\| + \frac{|f/g(\lambda)|}{c}\left\|\frac{g}{1-\overline{\lambda}\zeta}\right\|.$$

Since by (i) f/g has finite nontangential limits a.e. on $\Sigma(\mathcal{H})$ we can use Lemma 4.4. to conclude that the nontangential limits of H exist and are equal to zero on this set. The result now follows. □

5. The use of reproducing kernels

Recall that the reproducing kernel k_λ, $\lambda\in\mathbb{D}$ of a space $\mathcal{H}$ that satisfies (1.1) and (1.2) is defined by the relation

$$\langle f, k_\lambda\rangle = f(\lambda), \quad f\in\mathcal{H}.$$

Let us start with the simple observation that for every $g\in\mathcal{H}$ and $\lambda\in\mathbb{D}$ we have the inequality

$$|g(\lambda)| = (1-|\lambda|^2)\left|\left\langle \frac{g}{1-\overline{\lambda}\zeta}, k_\lambda\right\rangle\right| \le (1-|\lambda|^2)\left\|\frac{g}{1-\overline{\lambda}\zeta}\right\|\|k_\lambda\|. \tag{8}$$

This will be our main tool to describe the construction of the interpolating sequence required in the statement of Theorem 4.2.

Note that if

$$\lim_{n\to\infty}\|\zeta^n g\| = 0, \quad g\in\mathcal{H},$$

then by (8) and Lemma 4.4 we can conclude that

$$\text{nt-}\limsup_{\lambda\to z}\sqrt{1-|\lambda|^2}\,\frac{\|k_\lambda\|}{|g(\lambda)|} = \infty, \tag{9}$$

for a.e. $z\in\partial\mathbb{D}$. If $\mathcal{H}$ contains the constants we can choose $g = 1$ and obtain from (9) that

$$\text{nt-}\limsup_{\lambda\to z}\sqrt{1-|\lambda|^2}\|k_\lambda\| = \infty,$$

for a.e. $z\in\partial\mathbb{D}$. To simplify the exposition, we shall work with the stronger assumption that

$$\lim_{|\lambda|\to 1}\sqrt{1-|\lambda|^2}\|k_\lambda\| = \infty \tag{10}$$

and construct a dominating interpolating sequence for the space $\mathcal{H}$. One standard way of proving that a sequence Λ is interpolating for $\mathcal{H}$ is to show that the adjoint T_Λ^* of the interpolation operator T_Λ defined in the previous section is bounded above and below. This is equivalent to the inequality

$$K\sum_{\lambda\in\Lambda}|a_\lambda|^2 \ge \left\|\sum_{\lambda\in\Lambda}a_\lambda\frac{k_\lambda}{\|k_\lambda\|}\right\|^2 \ge \frac{1}{K}\sum_{\lambda\in\Lambda}|a_\lambda|^2 \tag{11}$$

for some positive constant K and for all l^2-sequences $\{a_\lambda\}_{\lambda\in\Lambda}$. We can now proceed to the construction of our sequence Λ in the following simple way: We let $\{r_n\}$ be a sequence of positive numbers that increases to 1, denote by p_n the integer part of $(1-r_n)^{-1}$ and set

$$\Lambda = \{r_n e^{\frac{2k\pi i}{p_n}} \,:\, 0 \le k < p_n, n \ge 1\}.$$

It is a simple exercise to show that this sequence accumulates nontangentially at every boundary point, hence it is dominating (see [BSZ]). We are going to show that the sequence $\{r_n\}$ can be chosen such that (11) holds. The following lemma is a direct application of the Carleson interpolation theorem (see [G]).

Lemma 5.1. *Let $\Lambda_n = \Lambda \cap \{|z| = r_n\}$. There exists an absolute constant $M > 0$ such that given $n \ge 1$ and complex numbers $w_\lambda,\ \lambda \in \Lambda_n$ there exists $f \in H^\infty$ with $f(\lambda) = w_\lambda,\ \lambda \in \Lambda_n$ and*

$$\|f\|_\infty \le M \max_{\lambda\in\Lambda_n} |w_\lambda|.$$

The next lemma is a part of the well-known Koethe-Toeplitz Theorem. An elementary proof can be based on a lemma by W. Orlicz (see [N], p. 159).

Lemma 5.2. *Let $K > 0$ and let $u_1, u_2, ..., u_n$ be unit vectors in $\mathcal{H}$ such that whenever $a_1, ..., a_n,\ b_1, ..., b_n \in \mathbb{C}$ with $|b_j| \le |a_j|, j = 1, ..., n$, we have*

$$\left\| \sum_{j=1}^n b_j u_j \right\| \le K \left\| \sum_{j=1}^n a_j u_j \right\|.$$

Then the following inequalities hold for all $a_1, ..., a_n \in \mathbb{C}$

$$\frac{1}{K}\left(\sum_{j=1}^n |a_j|^2\right)^{1/2} \le \left\| \sum_{j=1}^n a_j u_j \right\| \le K \left(\sum_{j=1}^n |a_j|^2\right)^{1/2}.$$

With these results at hand we can prove the estimate (11) for the sets Λ_n introduced in Lemma 5.1.

Corollary 5.3. *If M is the absolute constant from Lemma 5.1, then for every $n \in \mathbb{N}$ and every finite sequence $\{a_\lambda\}_{\lambda\in\Lambda_n}$ we have*

$$M \sum_{\lambda\in\Lambda} |a_\lambda|^2 \ge \left\| \sum_{\lambda\in\Lambda_n} a_\lambda \frac{k_\lambda}{\|k_\lambda\|} \right\|^2 \ge \frac{1}{M} \sum_{\lambda\in\Lambda_n} |a_\lambda|^2.$$

Proof. It suffices to verify the hypothesis of Lemma 5.2 with the constant M and $u_\lambda = \frac{k_\lambda}{||k_\lambda||}$, $\lambda \in \Lambda_n$. Let $a_\lambda, b_\lambda \in \mathbb{C},\ \lambda \in \Lambda_n$ be given with $|b_\lambda| \le |a_\lambda|,\ \lambda \in \Lambda_n$, and choose c_λ so that $c_\lambda a_\lambda = b_\lambda, \lambda \in \Lambda_n$. By Lemma 5.1 there exists a $\varphi \in H^\infty$ such that $\overline{\varphi(\lambda)} = c_\lambda$, $\lambda \in \Lambda_n$ and $||\varphi||_\infty \le M$, hence,

$$\left\| \sum_{\lambda\in\Lambda_n} b_\lambda \frac{k_\lambda}{\|k_\lambda\|} \right\| = \left\| M_\varphi^* \sum_{\lambda\in\Lambda_n} a_\lambda \frac{k_\lambda}{\|k_\lambda\|} \right\| \le \|M_\varphi^*\| \left\| \sum_{\lambda\in\Lambda_n} a_\lambda \frac{k_\lambda}{\|k_\lambda\|} \right\|$$

and by von Neumann's inequality we have $\|M_\varphi^*\| \leq \|\varphi\|_\infty \leq M$ which finishes the proof. $\square$

The next result gives us the final estimate needed for the proof of (11).

Lemma 5.4. *Suppose H^∞ is contained and dense in $\mathcal{H}$. Given a sequence of positive numbers $\{\varepsilon_n\}$ we can choose the sequence $\{r_n\}$ such that whenever $m, n \in \mathbb{N}$, with $m \neq n$ and*

$$u = \sum_{\lambda\in\Lambda_m} a_\lambda \frac{k_\lambda}{\|k_\lambda\|}, \quad v = \sum_{\mu\in\Lambda_n} b_\mu \frac{k_\mu}{\|k_m u\|}$$

we have $|\langle u, v\rangle| < \varepsilon_m\varepsilon_n\|u\|\|v\|$.

Proof. We proceed by induction. Suppose that $r_1, \ldots, r_{n-1}$ have been constructed such that the inequalities in the statement hold for $m < n-1$. We look for a number $1 > r_n > r_{n-1}$ such that if $m \leq n-1$,

$$u = \sum_{\lambda\in\Lambda_m} a_\lambda \frac{k_\lambda}{\|k_\lambda\|}, \quad v = \sum_{\mu\in\Lambda_n} b_\mu \frac{k_\mu}{\|k_\mu\|}$$

then $|\langle u, v\rangle| < \varepsilon_m\varepsilon_n\|u\|\|v\|$. Let $f_\lambda \in H^\infty,\ \lambda \in \Lambda_m$ and estimate first

$$\begin{aligned}
\left|\left\langle \sum_{\lambda\in\Lambda_m} a_\lambda f_\lambda, v\right\rangle\right| &\leq \sum_{\lambda\in\Lambda_m,\mu\in\Lambda_n} |a_\lambda||b_\mu|\frac{|f_\lambda(\mu)|}{\|k_\mu\|} \\
&\leq \frac{\sup_{\lambda\in\Lambda_m}\|f_\lambda\|_\infty}{\inf_{\mu\in\Lambda_n}\|k_\mu\|}\sum_{\lambda\in\Lambda_m,\mu\in\Lambda_n}|a_\lambda||b_\mu| \\
&\leq \frac{\sup_{\lambda\in\Lambda_m}\|f_\lambda\|_\infty}{\inf_{\mu\in\Lambda_n}\|k_\mu\|}\sqrt{p_m p_n}\left(\sum_{\lambda\in\Lambda_m}|a_\lambda|^2\right)^{1/2}\left(\sum_{\mu\in\Lambda_n}|b_\mu|^2\right)^{1/2} \\
&\leq 2M\frac{\sup_{\lambda\in\Lambda_m}\|f_\lambda\|_\infty}{(1-r_n^2)\inf_{\mu\in\Lambda_n}\|k_\mu\|}\|u\|\|v\|,
\end{aligned}$$

where we have used Corollary 5.3 and the fact that $p_j \leq (1-r_j)^{-1}$. Secondly, we have the inequality

$$\left|\langle u - \sum_{\lambda\in\Lambda_m} a_\lambda f_\lambda, v\rangle\right| \leq \sum_{\lambda\in\Lambda_m}|a_\lambda|\left\|\frac{k_\lambda}{\|k_\lambda\|} - f_\lambda\right\|\|v\|$$

and by Corollary 5.3 together with the Cauchy-Schwarz inequality we obtain

$$\left|\left\langle u - \sum_{\lambda\in\Lambda_m} a_\lambda f_\lambda, v\right\rangle\right| \leq \sqrt{M}\|u\|\|v\|\left(\sum_{\lambda\in\Lambda_m}\left\|\frac{k_\lambda}{\|k_\lambda\|} - f_\lambda\right\|^2\right)^{1/2}.$$

Now if H^∞ is dense in $\mathcal{H}$ we can choose f_λ such that for $m \leq n-1$ we have

$$\sqrt{M}\left(\sum_{\lambda\in\Lambda_m}\left\|\frac{k_\lambda}{\|k_\lambda\|}-f_\lambda\right\|^2\right)^{1/2} < \frac{\varepsilon_m\varepsilon_n}{2}$$

and then by (10) we can choose r_n close enough to 1 to ensure that

$$2M\frac{\sup_{\lambda\in\Lambda_m}\|f_\lambda\|_\infty}{(1-r_n^2)\inf_{\mu\in\Lambda_n}\|k_\mu\|} < \frac{\varepsilon_m\varepsilon_n}{2}.$$

Thus,

$$|\langle u,v\rangle| \leq \left|\left\langle u-\sum_{\lambda\in\Lambda_m} a_\lambda f_\lambda, v\right\rangle\right| + \left|\left\langle \sum_{\lambda\in\Lambda_m} a_\lambda f_\lambda, v\right\rangle\right| < \varepsilon_m\varepsilon_n\|u\|\|v\|$$

and the result follows. □

With these results we can immediately prove (11). Indeed, let $\{\varepsilon_n\}$ satisfy $\sum_n \varepsilon_n^2 \leq 1/2$. Then use Lemma 5.4 to obtain a sequence $\{r_n\}$ such that the inequalities in the statement are satisfied. Given an l^2-sequence $\{a_\lambda\}_{\lambda\in\Lambda}$ we set

$$u_i = \sum_{\lambda\in\Lambda_i} a_\lambda \frac{k_\lambda}{\|k_\lambda\|}$$

and apply Lemma 5.4 to obtain

$$\begin{aligned}
\left|\left\|\sum_{i=1}^n u_i\right\|^2 - \sum_{i=1}^n \|u_i\|^2\right| &\leq \sum_{i\neq j}^n |\langle u_i,u_j\rangle| \\
&\leq \sum_{i\neq j}^n \varepsilon_i\varepsilon_j\|u_i\|\|u_j\| \\
&\leq \left(\sum_{i=1}^n \varepsilon_i\|u_i\|\right)^2 \\
&\leq \sum_{i=1}^n \varepsilon_i^2 \sum_{i=1}^n \|u_i\|^2 \\
&\leq \frac{1}{2}\sum_{i=1}^n \|u_i\|^2.
\end{aligned}$$

This implies that

$$\frac{1}{2}\sum_{i=1}^n \|u_i\|^2 \leq \left\|\sum_{i=1}^n u_i\right\|^2 \leq \frac{3}{2}\sum_{i=1}^n \|u_i\|^2.$$

We now combine this with Corollary 5.3 to obtain

$$\frac{1}{2M}\sum_{\lambda\in\Lambda}|a_\lambda|^2 \le \left\|\sum_{\lambda\in\Lambda} a_\lambda \frac{k_\lambda}{\|k_\lambda\|}\right\|^2 \le \frac{3M}{2}\sum_{\lambda\in\Lambda}|a_\lambda|^2$$

and we are done. Once such a sequence Λ is constructed, the second part of Theorem 4.2 follows. By Privalov's theorem we see that the nonzero functions in $I(\Lambda)$ cannot have nontangential limits on any set of positive measure on the unit circle. Moreover, the restriction of M_ζ^* to $I(\Lambda)^\perp$ is similar to the diagonal operator with simple eigenvalues $\lambda \in \Lambda$ and by the work of Wermer [W] and the results in [BSZ] it follows that $M_\zeta^*|I(\Lambda)^\perp$ has an invariant subspace $\mathcal{N}$ such that the spectrum of its restriction to this subspace is the closed unit disc. Then another well known result from [R] implies that $\mathcal{N}^\perp$ has index greater than one. See [ARS1] for more details on this construction.

Let us now turn back to (9) and consider the case when the nontangential limits considered there are finite on sets of positive measure on the unit circle. More precisely, for a nonzero function $g \in \mathcal{H}$ consider the set

$$\Delta_g(\mathcal{H}) = \{z \in \partial\mathbb{D} : \mathrm{nt} - \limsup_{\lambda\to z}(1-|\lambda|^2)\frac{\|k_\lambda\|^2}{|g(\lambda)|^2} < \infty\}.$$

Roughly spoken, this is the boundary set where the normalized kernel $\frac{k_\lambda(w)}{\overline{g(\lambda)}g(w)}$ is nontangentially bounded by the Hardy kernel $(1-\overline{\lambda}w)^{-1}$. The remarks at the beginning of section 2 imply that $\Delta_g(\mathcal{H})$ is always measurable, and it can be shown that up to a set of measure 0 the set $\Delta_g(\mathcal{H})$ is independent of the choice of the nonzero function g. We shall thus drop the subscript g and write $\Delta(\mathcal{H})$ for the set of all Lebesgue points of $\Delta_g(\mathcal{H})$. One can also show (see [ARS3] for all these statements) that the complement of the set $\Sigma(\mathcal{H})$ in $\Delta(\mathcal{H})$ has measure zero. The following result proved in [ARS3] shows that if $\Delta(\mathcal{H})$ has positive measure then quotients of functions in $\mathcal{H}$ have a good nontangential boundary behavior on this set and more than that, locally near $\Delta(\mathcal{H})$ they behave like H^2-functions. Let us introduce first the following notation. For a closed set $F \subseteq \partial\mathbb{D}$ we let $\Omega_{F,r} = \bigcup_{z\in F}\Gamma_r(z)$. It is well known that $\Omega_{F,r}$ is a simply connected domain bounded by a rectifiable Jordan curve called the Stolz domain about F.

Theorem 5.5. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ satisfying conditions* (1.1) *and* (1.2) *such that $\Delta(\mathcal{H})$ has positive measure. If $0 < r < 1$ is fixed, then for every $\varepsilon > 0$ there exist a closed set $F \subset \Delta(\mathcal{H})$ with $|\Delta(\mathcal{H})\setminus F| < \varepsilon$, a function $h \in \mathcal{H}$ with $h \neq 0$ and a finite Blaschke product B such that $\mathcal{H} \subset \frac{h}{B}H^2(\Omega_{F,r})$.*

A direct consequence of this result is that the conclusion of Theorem 4.1 holds true if we replace $\Sigma(\mathcal{H})$ by $\Delta(\mathcal{H})$. In [ARS3] it is proved that although in general these two sets may differ by more than a set of measure 0, they do agree a.e. whenever $\mathcal{H}$ satisfies (4.1).

Finally, we should point out that the reproducing kernels are closely related to the majorization function introduced in Section 3. In [ARS3] we show that Theorem 3.2 continues to hold in any Hilbert space of analytic functions that satisfies (1.1), (1.2) and (4.1).

6. Some heuristics about the proof of Theorem 4.3

In this section we give a rough presentation of the strategy of proof of Theorem 4.3, more precisely, of its first part, the existence of nontangential limits of functions in $P^2(\mu)$. According to the comments at the end of the previous section it will suffice to show that

$$\text{nt-}\limsup_{\lambda\to z}\sqrt{1-|\lambda|^2}\|k_\lambda\|<\infty$$

a.e. on the set $\Sigma(P^2(\mu))=\{z:\ \frac{d\mu}{|dz|}>0\}$. Since $\|k_\lambda\|=\sup\{|f(\lambda)|:\ f\in P^2(\mu),\|f\|\le 1\}$, this means that we need to estimate the norms of the functionals of evaluation $f\mapsto f(\lambda)$, $f\in P^2(\mu)$ when λ approaches nontangentially the set considered above. We begin with the following simple scaling argument. Given $\lambda\in\mathbb{D}$ let $\varphi_\lambda(z)=\frac{z+\lambda}{1+\overline{\lambda}z}$ and consider the measure μ_λ defined by

$$d\mu_\lambda=\frac{1}{|\varphi_\lambda'|}d\mu\circ\varphi_\lambda.$$

Equivalently,

$$\int u\,d\mu_\lambda=\int u\left(\frac{z-\lambda}{1-\overline{\lambda}z}\right)\frac{1-|\lambda|^2}{|1-\overline{\lambda}z|^2}d\mu,$$

for all continuous functions u on $\overline{\mathbb{D}}$. It is now not difficult to show that the inequality

$$|f(\lambda)|^2\le\frac{C^2}{1-|\lambda|^2}\int|f|^2d\mu\quad f\text{ polynomial},$$

is equivalent to

$$|g(0)|^2\le C^2\int|g|^2d\mu_\lambda\quad g\text{ polynomial}.$$

Thus, we have to estimate the norm of the evaluation at the origin in the space $P^2(\mu_\lambda)$. The next step is to show that, for these last estimates, we can focus on measures that are absolutely continuous with respect to the area measure. To outline this we need to recall the definition of the Cauchy transform of a finite compactly supported measure on $\mathbb{C}$. If ν is such a measure then its Cauchy transform is defined a.e. w.r.t. area measure by

$$\hat{\nu}(z)=\int\frac{d\nu(w)}{w-z}.$$

Also recall that if $\nu=u\,dA$ with $u\in L^\infty(\mathbb{D})$, then $\hat{\nu}\in L^\infty(\mathbb{D})$ and there is a constant $C_1>0$ such that

$$\|\hat{\nu}\|_\infty\le C_1\|u\|_\infty. \tag{12}$$

Proposition 6.1. *Let ν be a positive measure on $\overline{\mathbb{D}}$ such that $P^2(\nu) \neq L^2(\nu)$, let $G \in P^2(\nu)^\perp$ and let ψ be the Cauchy transform of the measure $Gd\nu$. Assume that there is a constant $C > 0$ such that*

$$|f(0)| \leq C \int_{\mathbb{D}} |f||\psi| dA$$

for all polynomials f. If C_1 is the constant from (12)*, then for every polynomial f we have*

$$|f(0)| \leq CC_1 \left(\int_{\overline{\mathbb{D}}} |f|^2 d\nu \right)^{1/2} \left(\int_{\overline{\mathbb{D}}} |G|^2 d\nu \right)^{1/2}.$$

Proof. By the Hahn-Banach Theorem we can find a function $H \in L^\infty$ with $\|H\|_\infty \leq C$ such that for every polynomial f we have

$$f(0) = \int_{\mathbb{D}} fH\psi dA. \tag{13}$$

Since $Gd\nu$ annihilates the polynomials and ψ is its Cauchy transform we can use the equality

$$\int \frac{f(z) - f(w)}{z - w} G(w) d\nu(w) = 0$$

which holds whenever f is a polynomial and $z \in \mathbb{D}$ to conclude that $f(z)\psi(z)$ equals the value of the Cauchy transform of $fGd\nu$ at z. We replace this in (13) and obtain

$$f(0) = \int_{\mathbb{D}} H(z) \int_{\mathbb{D}} \frac{fG(w)}{w - z} d\nu(w) dA(z) = \int_D fG(w) \int_{\mathbb{D}} \frac{H(z)}{w - z} dA(z) d\nu(w),$$

and by (12) we have

$$|f(0)| \leq C_1 C \int_{\mathbb{D}} |fG(w)| d\nu(w)$$

and the result follows. □

Since for every polynomial f we have that

$$f(0) = \frac{1}{r^2} \int_{|\zeta|<r} f dA$$

we can guess that a function ψ will satisfy the hypothesis of Proposition 6.1 if the level sets of the form $\{z : \ |z| < r, \ |\psi(z)| < \delta\}$ are not too large. Thus, intuitively spoken, what we need to do is to look for functions $G_\lambda \in P^2(\mu_\lambda)^\perp$ such that the level sets of the their Cauchy transforms are small. It turns out that the appropriate way of measuring sets from this point of view is the analytic capacity. This is defined first for compact subsets F of the plane as

$$\gamma(F) = \sup\{|f'(\infty)| : \ f \in H^\infty(\mathbb{C} \setminus F), \|f\|_\infty \leq 1\}$$

and for arbitrary sets A as $\gamma(A) = \sup\{\gamma(F) : \ F \subset A \text{ is compact}\}$. Remarkable progress in the study of the analytic capacity has been recently made by Tolsa [XT]. He solved a long standing open problem by showing that γ is semiadditive.

Moreover, one of his major results in this direction asserts that if we replace in the definition of $\gamma(F)$, F compact, the set $H^\infty(\mathbb{C}\setminus F)$ by the set of Cauchy transforms of finite measures supported on F whose modulus is bounded by one in $\mathbb{C}\setminus F$, we obtain a comparable quantity. With this second powerful tool at hand we show in [ARS4] that there are functions $G_\lambda \in P^2(\mu_\lambda)^\perp$ such that the level sets of their Cauchy transforms have arbitrarily small analytic capacity. The rest of the proof can then be obtained from the following (main) lemma.

Lemma 6.2. *There are absolute constants C_0, ε_0 with the following property. Let ψ be measurable and set*

$$F = \{z : \ |\psi(z)| < 1\}.$$

If $\gamma(F) < \varepsilon_0$ then for every polynomial f we have

$$|f(0)| \le C_0 \int_{\mathbb{D}} |f||\psi| dA.$$

The proof of this last result is long, rather technical and uses again Tolsa's characterization of the analytic capacity in terms of Cauchy transforms of finite measures. Finally, we should also point out that Theorem 4.3 holds true for all spaces $P^t(\mu)$, $1 \le t < \infty$ which are defined as the closure of analytic polynomials in $L^t(\mu)$. Moreover, the result continues to hold with the appropriate modifications if μ is supported on the closure of a bounded simply connected domain.

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