Lotka-Volterra models with fractional diffusion

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Abstract

In this paper we study the Lotka-Volterra models with fractional Laplacian. For that, we study in detail the logistic problem and show that the sub-supersolution method works for the scalar problem and in case of systems as well. We apply this method to show existence and non-existence of positive solutions in terms of the system parameters.

Key Words. Fractional Laplacian, Lotka-Volterra models, sub-supersolution method.

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1 Introduction

In this paper we study the following systems

$$\begin{cases} (-\Delta)^{\alpha} u = u(\lambda - u - bv) & \text{in } \Omega, \\ (-\Delta)^{\beta} v = v(\mu - v - cu) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

uno

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded and regular domain, $\lambda, \mu, b, c \in \mathbb{R}$ and $\alpha, \beta \in (0,1)$. Here, u and v denote the densities of two species inhabiting in Ω, the habitat, which is surrounded by inhospitable areas, due to the homogeneous Dirichlet boundary conditions. In (1.1) we are assuming that the species diffuse following the fractional laplacian, see Section 2 where we have defined this non-local operator.

When $\alpha = \beta = 1$, (1.1) is the classical Lotka-Volterra system with random walk, widely studied in the last years in all the cases: competition ($b, c > 0$), predator-prey ($b > 0$ and $c < 0$) and symbiosis ($b, c < 0$), see [8] and references therein.

Fractional operators are used in different contexts: physics, finance and ecology; see [14] and [21] for the ecological meaning of the fractional diffusion. For many years, the non-oriented animal movement was modelled by the classical Brownian motion. However, it seems that when the species is searching for resources, the strategy based on Lévy flights (supported in long jumps) could be more appropriate in some situations. This kind of strategy is optimal for the location of targets which are randomly and sparsely distributed, but the Brownian motion is optimal where the resources are abundant. The Lévy diffusion processes are generated by fractional powers of the Laplacian $(-\Delta)^{\gamma}$ for $\gamma \in (0,1)$.

We are interested in the existence of non-negative solutions of (1.1). It is clear that (1.1) possesses the trivial solution $(u, v) = (0, 0)$ for all $\lambda, \mu \in \mathbb{R}$, since when $u \equiv 0$ (resp. $v \equiv 0$) then v (resp. u) verifies an equation of type

$$\begin{cases} (-\Delta)^{\gamma} w + c(x) w = w(\sigma - w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

log

where $\gamma \in (0,1)$, $\sigma \in \mathbb{R}$ and $c \in L^{\infty}(\Omega)$. This is the classical logistic equation, studied in [19] and [20] with homogeneous Dirichlet and Neumann boundary conditions, respectively, with $\gamma = 1/2$ in both papers. To study this equation, we previously analyze the eigenvalue problem

$$\begin{cases} (-\Delta)^\gamma w + c(x)w = \lambda w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

eigenintro

We study the existence of a principal eigenvalue, the unique eigenvalue of (1.3) having a positive eigenfunction, denoted by $\lambda_1[\gamma; c]$. This problem has been analyzed in [1] and [19] (for $\gamma = 1/2$) and in [20] for the Neumman case. We study some properties of this eigenvalue and of its eigenfunction associated.

Then, we prove that (1.2) possesses a positive solution if and only if $\sigma > \lambda_1[\gamma; c]$. Moreover, it is unique and we denote it by $\theta_{[\gamma,\sigma-c]}$.

Moreover, we try to give an ecological interpretation of the result, comparing our results with the obtained in local operator case, in which the fractional Laplacian is substituted by the classical Laplacian operator.

For the existence, we employ the sub-supersolution method. Let us point some remarks. The sub-supersolution method has been used previously in non-linear fractional diffusion problem, see for instance [3] and [9]. In both papers, the method is consequence of a maximum principle and a classical iterative argument. However, we present a different proof which is also valid, with minor technical changes, for systems.

Once studied in detail (1.2), we analyse the existence of solutions with both positive components of (1.1). For that, we apply the sub-supersolution method. We first show that this method works for systems, and then we apply it to (1.1). For that, we have to find appropriate sub-supersolutions of (1.1) using the results obtained for the logistic equations. We prove the following results:

a) If $b, c > 0$ or $b, c < 0$ and $bc < C(\alpha, \beta)$ for some positive constant (detailed in Section 6) and λ and μ verify

$$\lambda > \lambda_1[\alpha; b\theta_{[\beta,\mu]}] \quad \text{and} \quad \mu > \lambda_1[\beta; c\theta_{[\alpha,\lambda]}], \tag{1.4}$$

condigeneralintro

b) or $b > 0$ and $c < 0$ and λ and μ verify

$$\lambda > \lambda_1[\alpha; b\theta_{[\beta,\mu-c\theta_{[\alpha,\lambda]}]}] \quad \text{and} \quad \mu > \lambda_1[\beta; c\theta_{[\alpha,\lambda]}], \tag{1.5}$$

condigeneralppint

then there exists at least a positive solution of (1.1). We show that conditions (1.4) and (1.5) define regions on the $\lambda - \mu$ plane.

The paper is organized as follows. In Section 2 we present the functional setting necessary for the remainder of the work. Section 3 is devoted to the eigenvalue problem. We study the existence and main properties of the principal eigenvalue. In Section 4 we study equation (1.2). The sub-supersolution method for systems is shown in Section 5. Finally, in the last Section we study the existence of positive solution of (1.1).

2 Preliminaries

In this section we begin introducing the functional framework necessary to develop the theory, and recover some known results about the different forms to define the fractional power of the Laplacian with Dirichlet boundary condition.

2.1 Functional setting

Consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$. Since in bounded domains there are some non-equivalent definitions of the fractional laplacian operator, let us explain what we mean by the symbol $(-\Delta)^\alpha$. For $u \in C_0^\infty(\Omega)$ such that $u = \sum_{k=1}^\infty b_k \varphi_k$, where λ_k, φ_k are the eigenpairs of $(-\Delta, H_0^1(\Omega))$, (λ_k repeated as much as its multiplicity and $\{\varphi_k\}$ forming an ortonormal basis of $L^2(\Omega)$), we define

$$(-\Delta)^\alpha u := \sum_{k=1}^\infty \lambda_k^\alpha b_k \varphi_k.$$

Then the operator $(-\Delta)^\alpha$ is defined on $D((-\Delta)^\alpha) = \{u \in L^2(\Omega);\, \sum_{k=1}^\infty \lambda_k^\alpha b_k^2 < +\infty\}$ by density.

Now, let us consider the half cylinder with base Ω,

$$\mathcal{C} := \Omega \times (0, +\infty),$$

and denote its lateral boundary by

$$\partial_L \mathcal{C} := \partial\Omega \times [0, +\infty).$$

We denote $(x,y) \in \mathcal{C}$, $x \in \Omega$ and $y > 0$ and define

$$\mathcal{H}^\alpha(\mathcal{C}) := \left\{ v \in H^1(\mathcal{C});\ \|v\|_\alpha < +\infty \right\},$$

$$\mathcal{H}_0^\alpha(\mathcal{C}) := \{ v \in \mathcal{H}^\alpha(\mathcal{C});\ v = 0 \text{ on } \partial_L \mathcal{C} \},$$

where

$$\|v\|_\alpha := \left(k_\alpha^{-1} \int_{\mathcal{C}} y^{1-2\alpha} |\nabla v|^2 dxdy \right)^{\frac{1}{2}},$$

$k_\alpha = \dfrac{2^{1-2\alpha}\Gamma(1-\alpha)}{\Gamma(\alpha)}$, $\alpha \in (0,1)$ and Γ is the Gamma function. It is not difficult to see that $\mathcal{H}_0^\alpha(\mathcal{C})$ is a Hilbert space when endowed with the norm $\|\cdot\|_\alpha$, which comes from the following inner product

$$\langle v, w \rangle_\alpha = k_\alpha^{-1} \int_{\mathcal{C}} y^{1-2\alpha} \nabla v \cdot \nabla w dxdy.$$

Consider the following subspace of the fractional Sobolev space $H^\alpha(\Omega)$,

$$\mathcal{V}_0^\alpha(\Omega) := \{ tr_\Omega v;\ v \in \mathcal{H}_0^\alpha(\mathcal{C}) \}$$

which is a Banach space when endowed with the norm

$$\|u\|_{\mathcal{V}_0^\alpha(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} dxdy \right)^{\frac{1}{2}},$$

where tr_Ω is the trace operator defined by

$$tr_\Omega v = v(\cdot, 0) \quad \text{for } v \in \mathcal{H}_0^\alpha(\mathcal{C}).$$

Moreover, by Trace Theorem (see Proposition 2.1 in [9]) and embeddings for fractional Sobolev spaces (see Theorem 6.7 in [12]) it follows that

$$\|tr_\Omega v\|_{L^p(\Omega)} \le C \|v\|_\alpha, \quad \forall v \in \mathcal{H}_0^\alpha(\mathcal{C}), \quad \text{where } p \in (1, 2_\alpha) \tag{2.1}$$ tracetheorem

where $2_\alpha = \frac{2N}{N-2\alpha}$.

By Proposition 2.1 in [9] it holds that

$$\mathcal{V}_0^\alpha(\Omega) = \left\{ u \in L^2(\Omega);\ u = \sum_{k=1}^\infty b_k \varphi_k \ \textit{satisfying} \ \sum_{k=1}^\infty b_k^2 \lambda_k^\alpha < +\infty \right\}.$$

As far as the following scalar nonlocal problem is concerned,

$$\begin{cases} (-\Delta)^\alpha u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$ P2

the approach we are going to follow is by associating to (2.2) a one-more dimensional local problem in $\mathcal{C}$. This can be made by considering the procedure to get a local realization of $(-\Delta)^\alpha$ described beneath.

As proved in [9] [Section 2.1], for each $u \in \mathcal{V}_0^\alpha(\Omega)$, there exists a unique $v \in \mathcal{H}_0^\alpha(\mathcal{C})$, called its $\alpha-$*harmonic extension* such that

$$\begin{cases} -\mathrm{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } \mathcal{C}, \\ v = 0 & \text{on } \partial_L \mathcal{C}, \\ v(\cdot, 0) = u & \text{on } \Omega. \end{cases}$$

Moreover, if $u = \sum_{k=1}^{\infty} b_k \varphi_k$ is its spectral decomposition, then

$$v(x,y) = \sum_{k=1}^{\infty} b_k \varphi_k(x) \psi(\lambda_k^{\frac{1}{2}} y), \quad \forall (x,y) \in \mathcal{C}, \tag{2.3}$$

harmonicextension

where ψ solves the Bessel equation

$$\begin{cases} \psi'' + \dfrac{(1-2\alpha)}{s}\psi' = \psi & s > 0 \\ -\lim\limits_{s\to 0^+} s^{1-2\alpha}\psi'(s) = k_\alpha \\ \psi(0) = 1. \end{cases} \tag{2.4}$$

besselequation

Let $u \in \mathcal{V}_0^\alpha(\Omega)$ and $v \in \mathcal{H}_0^\alpha(\mathcal{C})$ its $\alpha-$harmonic extension. Define the functional $\left.\frac{1}{k_\alpha}\frac{\partial v}{\partial y^\alpha}\right|_{\Omega\times\{0\}} \in \mathcal{V}_0(\Omega)^*$ by

$$\left\langle \frac{1}{k_\alpha}\frac{\partial v}{\partial y^\alpha}(\cdot,0), g \right\rangle := \frac{1}{k_\alpha}\int_{\mathcal{C}} y^{1-2\alpha}\nabla v . \nabla \tilde{g}\, dx dy,$$

where $\tilde{g}$ is the $\alpha-$harmonic extension of $g \in \mathcal{V}_0^\alpha(\Omega)$ and

$$\frac{\partial v}{\partial y^\alpha}(x,0) = -\lim_{y\to 0^+} y^{1-2\alpha}\frac{\partial v}{\partial y}(x,y), \quad \forall x \in \Omega.$$

Then we can define an operator $A_\alpha : \mathcal{V}_0^\alpha(\Omega) \to \mathcal{V}_0^\alpha(\Omega)^*$ such that

$$A_\alpha u := \left.\frac{1}{k_\alpha}\frac{\partial v}{\partial y^\alpha}\right|_{\Omega\times\{0\}},$$

where v is the $\alpha-$harmonic extension of u to $\mathcal{C}$. Let us prove that the operators A_α and $(-\Delta)^\alpha$ are in fact the same, i.e., that for all $u \in \mathcal{V}_0^\alpha(\Omega)$,

$$A_\alpha u = \sum_{k=1}^{\infty} b_k \lambda_k^\alpha \varphi_k, \quad \text{where } u = \sum_{k=1}^{\infty} b_k \varphi_k.$$

By linearity, it is enough to prove that for all φ_k,

$$\left\langle \frac{1}{k_\alpha}\frac{\partial v}{\partial y^\alpha}(\cdot,0), \varphi_k \right\rangle = \langle (-\Delta)^\alpha u, \varphi_k \rangle_{L^2(\Omega)}, \quad \text{for all } k \in \mathbb{N},$$

where v is the $\alpha-$harmonic extension of u.

For $u \in \mathcal{V}_0^\alpha(\Omega)$ and $k \in \mathbb{N}$, let v and $\tilde{\varphi}_k$ be the $\alpha-$harmonic extensions of u and φ_k, respectively. By (2.3), $v(x,y) = \sum_{k=1}^\infty b_k \varphi_k(x)\psi(\lambda_k^{1/2} y)$ and $\tilde{\varphi}_k(x,y) = \varphi_k(x)\psi(\lambda_k^{1/2} y)$. Now, integration by parts and properties of φ_k imply that for each $y > 0$, it holds

$$\int_\Omega y^{1-2\alpha} \nabla_x v(x,y) \cdot \nabla_x \tilde{\varphi}_k(x,y) dx = y^{1-2\alpha} b_k \left(\lambda_k \psi(\lambda_k^{\frac{1}{2}} y)^2 + \psi'(\lambda_k^{\frac{1}{2}} y)^2 \right).$$

Then, by (2.4)

$$\begin{aligned}
\left\langle \frac{1}{k_\alpha}\frac{\partial v}{\partial y^\alpha}(\cdot,0), \varphi_k \right\rangle &= \frac{1}{k_\alpha}\int_{\mathcal{C}} y^{1-2\alpha} \nabla v \cdot \nabla \tilde{\varphi}_k dxdy \\
&= \frac{1}{k_\alpha}\int_0^{+\infty} y^{1-2\alpha} b_k \left(\lambda_k \psi(\lambda_k^{\frac{1}{2}} y)^2 + \psi'(\lambda_k^{\frac{1}{2}} y)^2 \right) dy \\
&= \frac{1}{k_\alpha} \lim_{\eta \to 0^+} y^{1-2\alpha} \lambda_k^{\frac{1}{2}} b_k \psi'(\lambda_k^{\frac{1}{2}} y) \psi(\lambda_k^{\frac{1}{2}} y) \Big|_{y=\eta} \\
&= b_k \lambda_k^\alpha \\
&= \langle (-\Delta)^\alpha u, \varphi_k \rangle_{L^2(\Omega)}.
\end{aligned}$$

Hence, in (2.2) we are going to understand $(-\Delta)^\alpha$ as A_α.

For simplicity, without loss of generality, we can assume throughout this paper that $k_\alpha = 1$. Then, we define

debil **Definition 2.1.** *$u \in \mathcal{V}_0(\Omega)$ is a weak solution of (2.2) if $u = tr_\Omega v$ where $v \in \mathcal{H}_0^\alpha(\mathcal{C})$ is a weak solution of*

$$\begin{cases} -div(y^{1-2\alpha}\nabla v) = 0 & \text{in } \mathcal{C}, \\ \dfrac{\partial v}{\partial y^\alpha}(x,0) = f(x, v(x,0)) & \text{on } \Omega. \end{cases}$$

In this case, v is such that

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla v \cdot \nabla \psi dxdy = \int_\Omega f(x, v(x,0))\psi(x,0)dx, \quad \forall \psi \in \mathcal{H}_0^\alpha(\mathcal{C}). \tag{2.5}$$ weakform

2.2 Maximum principle

Along the paper, the following maximum principle will be very useful, see Lemma 2.5 in [9] for a related result.

Proposition 2.2. *Let* $d \in L^{\infty}(\Omega)$ *and* $v \in \mathcal{H}^{\alpha}(\mathcal{C})$ *such that* $v \geq 0$ *in* $\partial_L \mathcal{C}$ *and*

$$\begin{cases} -div(y^{1-2\alpha}\nabla v) \geq 0 & in\ \mathcal{C}, \\ \dfrac{\partial v}{\partial y^{\alpha}}(x,0) + d(x)v(x,0) \geq 0 & on\ \Omega. \end{cases}$$

a) Assume that $d \geq 0$ *in* Ω*, then* $v \geq 0$ *in* $\mathcal{C}$*.*

b) Assume that $v \geq 0$ *in* $\mathcal{C}$*. Then, either* $v \equiv 0$ *or* $v > 0$ *in* $\mathcal{C}$*.*

ximumPrinciple

Proof. a) The proof follows just by using $-v^{-}$ as test function, where $v = v^{+} + v^{-}$.

b) In this paragraph we follow the proof of Lemma 4.9 in [6]. Define

$$w(x,y) := e^{Ay^{2\alpha}} v(x,y).$$

Then, w satisfies

$$\begin{cases} -\mathrm{div}(y^{1-2\alpha}\nabla(e^{-Ay^{2\alpha}}w)) \geq 0 & \text{in } \mathcal{C}, \\ \dfrac{\partial w}{\partial y^{\alpha}}(x,0) + (d(x) + 2A\alpha)w(x,0) \geq 0 & \text{on } \Omega. \end{cases}$$

We can choose A such that $d(x) + 2A\alpha \leq 0$ in Ω, and so

$$\frac{\partial w}{\partial y^{\alpha}}(x,0) \geq 0 \quad \text{in } \Omega.$$

Take $R > 0$, consider now the even extension of w in $\Omega \times (-R, R)$, defined by

$$\tilde{w}(x,y) = \begin{cases} w(x,y) & \text{if } y > 0, \\ w(x,-y) & \text{if } y \leq 0. \end{cases}$$

We can show that

$$-\mathrm{div}(|y|^{1-2\alpha}\nabla(e^{-A|y|^{2\alpha}}\tilde{w})) \geq 0 \quad \text{in } \Omega \times (-R, R).$$

Define now the problem

$$\begin{cases} -\mathrm{div}(|y|^{2\alpha}\nabla(e^{-A|y|^{2\alpha}}h) = 0 & \text{in } \Omega \times (-R,R), \\ h = \tilde{w} & \text{on } (\Omega \times \{-R\}) \cup (\Omega \times \{R\}). \end{cases}$$

The above problem possesses a solution by [13] (see also Theorem 3.2 in [6]) and by the maximum principle we get that

$$h \le \tilde{w} \quad \text{in } \Omega \times (-R,R).$$

On the other hand, by the strong maximum principle, see Lemma 2.3.5 in [13], we conclude that

$$h > 0.$$

This finishes the proof.

□

Remark 2.3. *Observe that Proposition 2.2 can be stated in an equivalent way: Assume $d \in L^\infty(\Omega)$ and $(-\Delta)^\alpha u + d(x)u \ge 0$ in Ω and $u \ge 0$ on $\partial\Omega$. Then,*

a) If $d \ge 0$ in Ω, then $u \ge 0$ in Ω.

b) Assume that $u \ge 0$ in Ω. Then, either $u \equiv 0$ or $u > 0$ in Ω.

2.3 Regularity results

The following result follows by Lemma 3.3 in [10], see also Proposition 5.1 in [2].

cotas1 **Lemma 2.4.** *Assume that $f \in C(\overline{\Omega} \times \mathbb{R})$ and that there exists a constant C and $p \in (2, 2N/(N-2\alpha))$ such that*

$$|f(x,t)| \le C(1 + |t|^{p-1}), \quad x \in \Omega,\ t \in \mathbb{R}.$$

If $v \in \mathcal{H}_0^\alpha(\mathcal{C})$ is a solution of (2.5) and $u = tr_\Omega v$, then $v \in L^\infty(\mathcal{C}) \cap C^\sigma(\overline{\mathcal{C}})$ and $u \in C^\sigma(\overline{\Omega})$ for some $\sigma \in (0,1)$.

Consider now the linear problem

$$\begin{cases} (-\Delta)^\alpha u = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$ P2line

The following result is also taken from [10] (Lemma 3.2), see also [7].

cotas2 **Lemma 2.5.** *Assume that $g \in H^{-\alpha}(\Omega)$ and $v \in \mathcal{H}_0^\alpha(\mathcal{C})$ is a solution of (2.6) and $u = tr_\Omega v$. Then,*

a) If $g \in L^r(\Omega)$ for $r > N/(2\alpha)$, then $v \in L^\infty(\mathcal{C})$ and $u \in L^\infty(\Omega)$.

b) If $g \in L^\infty(\Omega)$, then $v \in C^\sigma(\overline{\mathcal{C}})$ and $u \in C^\sigma(\overline{\Omega})$ for some $\sigma \in (0,1)$.

c) If $g \in C^\sigma(\overline{\Omega})$ and $g\rfloor_{\partial\Omega} \equiv 0$, then $v \in C^{1,\sigma}(\overline{\mathcal{C}})$ and $u \in C^{1,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$.

d) If $g \in C^{1,\sigma}(\overline{\Omega})$ and $g\rfloor_{\partial\Omega} \equiv 0$, then $v \in C^{2,\sigma}(\overline{\mathcal{C}})$ and $u \in C^{2,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$.

3 The eigenvalue problem

Given $c \in L^\infty(\Omega)$, we study the eigenvalue problem

$$\begin{cases} (-\Delta)^\alpha u + c(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$ eigen

where $\alpha \in (0,1)$ and $\lambda \in \mathbb{R}$. Recall that $u \in \mathcal{V}_0^\alpha(\Omega)$ is an eigenfunction associated to an eigenvalue λ of (3.1) if and only if $u = tr_\Omega v$ where $v \in \mathcal{H}_0^\alpha(\mathcal{C})$ is a solution of

$$\begin{cases} -\mathrm{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } \mathcal{C}, \\ v = 0 & \text{on } \partial_L\mathcal{C}, \\ \dfrac{\partial v}{\partial y^\alpha}(x,0) + c(x)v(x,0) = \lambda v(x,0) & \text{on } \Omega. \end{cases} \tag{3.2}$$ eigenextended

In the following result, we show the existence of principal eigenvalue and positive eigenfunction of (3.1) and their main properties.

Theorem 3.1. *There exists the principal eigenvalue of (3.1), denoted by $\lambda_1[\alpha; c]$. This eigenvalue is simple and possesses a unique eigenfunction Φ_1 of (3.2), up to multiplicative constants, which can be taken positive. Moreover, the principal eigenfunction Φ_1 is strongly positive, and $\lambda_1[\alpha; c]$ is the only eigenvalue of (3.1) possessing a positive eigenfunction. If we denote $\varphi_1 := tr_\Omega \Phi_1$, we have that $\varphi_1 \in C^\sigma(\overline{\Omega})$ and $\Phi_1 \in L^\infty(\mathcal{C}) \cap C^\sigma(\overline{\mathcal{C}})$ for some* teoremaeigen *$\sigma \in (0,1)$ Furthermore, the map from $c \in L^\infty(\Omega) \mapsto \lambda_1[\alpha; c]$ is increasing.*

Proof. For each $v \in \mathcal{H}_0^\alpha(\mathcal{C})$ such that $tr_\Omega v \neq 0$ in $L^2(\Omega)$, let us consider

$$J(v) := \frac{\displaystyle\int_{\mathcal{C}} y^{1-2\alpha}|\nabla v|^2 dxdy + \int_\Omega c(x)v(x,0)^2 dx}{\displaystyle\int_\Omega v(x,0)^2 dx} \tag{3.3}$$ J

and note that J if bounded from below. In fact, Trace Theorem and the boundedness of c imply that

$$\begin{aligned} \int_{\mathcal{C}} y^{1-2\alpha}|\nabla v|^2 dxdy + \int_\Omega c(x)v(x,0)^2 dx \;&\geq\; C\int_\Omega v(x,0)^2 dx + \int_\Omega c(x)v(x,0)^2 dx \\ &\geq\; K\int_\Omega v(x,0)^2 dx, \end{aligned}$$

where $K \in \mathbb{R}$, for every such v.

Let us define

$$\lambda_1[\alpha; c] := \inf\{J(v);\, v \in \mathcal{H}_0^\alpha(\mathcal{C}) \,\text{and}\; tr_\Omega v \neq 0 \text{ in } L^2(\Omega)\}. \tag{3.4}$$ mineigenvalue

Let $(v_n)_{n\in\mathbb{N}} \subset \mathcal{H}_0^\alpha(\mathcal{C})$ be such that $\int_\Omega v_n(x,0)^2 dx = 1$ and $J(v_n) \to \lambda_1[\alpha; c]$. It is straightforward to see that $(v_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_0^\alpha(\mathcal{C})$ and hence there exists $w \in \mathcal{H}_0^\alpha(\mathcal{C})$ such that $w_n \rightharpoonup w$ in $\mathcal{H}_0^\alpha(\mathcal{C})$. Since $\mathcal{H}_0^\alpha(\mathcal{C}) \hookrightarrow \mathcal{V}_0^\alpha(\Omega)$ continuously and $\mathcal{V}_0^\alpha(\Omega) \hookrightarrow L^2(\Omega)$ compactly, then $\int_\Omega w(x,0)^2 dx = 1$. Just by imitating the arguments of Section 8.12 in [18], one can show that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and then it strongly converges to v in $\mathcal{H}_0^\alpha(\mathcal{C})$. Hence $J(v) = \lambda_1[\alpha; c]$.

If $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$, setting $\varphi(t) = J(v + t\psi)$, it follows that

$$0 = \varphi'(0) = \int_{\mathcal{C}} y^{1-2\alpha}\nabla v \cdot \nabla\psi dxdy + \int_\Omega c(x)v(x,0)\psi(x,0)dx - \lambda_1[\alpha; c]\int_\Omega v(x,0)\psi(x,0)dx.$$

Hence v is a solution of (3.2) with $\lambda = \lambda_1[\alpha; c]$ and it is therefore an eigenfunction associated to $\lambda_1[\alpha; c]$.

Of course the definition implies that $\lambda_1[\alpha; c]$ is the smallest eigenvalue of (3.2).

Now let us prove that the eigenfunctions has at least $C^\gamma(\overline{\Omega})$ regularity, where $\gamma = \min\{1, 2\alpha\}$. This follows easily from Lemmas 2.4 and 2.5 once we prove that $\|tr_\Omega\phi\|_{L^\infty(\Omega)} < +\infty$, for every eigenfunctions ϕ. On the other hand, this L^∞ estimate can be obtained by a standard application of Moser iteration technique, which we describe below.

Let $v \in \mathcal{H}_0^\alpha(\mathcal{C})$ satisfying (3.2) for some λ and let $M > 0$. Denoting $v_M = \min\{v, M\}$, note that it is an $\mathcal{H}_0^\alpha(\mathcal{C})$ function. Let $b > 0$ a constant to be chosen conveniently and let us take ${v_M}^b$ as a test function in (3.2). Denoting $e(x) := (\lambda - c(x))$ it follows that

$$b\int_{\mathcal{C}} y^{1-2\alpha} {v_M}^{b-1}|\nabla v_M|^2 dxdy = \int_\Omega e(x)v(x,0)v_M(x,0)^b dx$$

which implies that

$$\frac{4b}{(b+1)^2}\int_{\mathcal{C}} y^{1-2\alpha}\left|\nabla (v_M{}^{\frac{b+1}{2}})\right|^2 dxdy \leq \int_\Omega e(x)v(x,0)^{b+1}dx.$$

By Trace Theorem and embedding of fractional Sobolev spaces, we have that

$$\frac{4b}{(b+1)^2}\left\|tr_\Omega v_M{}^{\frac{b+1}{2}}\right\|^2_{L^{2_\alpha}(\Omega)} \leq C\|tr_\Omega v\|^{b+1}_{L^{b+1}(\Omega)}.$$

Considering $M \to +\infty$ and using Fatou Lemma, we have that

$$\frac{4b}{(b+1)^2}\left\|tr_\Omega v^{\frac{b+1}{2}}\right\|^2_{L^{2_\alpha}(\Omega)} \leq C\|tr_\Omega v\|^{b+1}_{L^{b+1}(\Omega)}.$$

Then it follows that

$$\|tr_\Omega v\|_{L^{\frac{2_\alpha}{2}(b+1)}(\Omega)} \leq \left(C\frac{(b+1)^2}{4b}\right)^{\frac{1}{b+1}}\|tr_\Omega v\|_{L^{b+1}(\Omega)}. \tag{3.5}$$ moser

Let us consider a sequence $(\eta_k)_k$ defined by $\eta_0 = 2$ and $\eta_k = \frac{2_\alpha}{2}\eta_{k-1}$ for $k \geq 1$. Taking b in (3.5) such that $b + 1 = \eta_{k-1}$, we have that

$$\|tr_\Omega v\|_{L^{\eta_k}(\Omega)} \leq \left(C\frac{\eta_{k-1}^2}{4(\eta_{k-1}-1)}\right)^{\frac{1}{\eta_{k-1}}}\|tr_\Omega v\|_{L^{\eta_{k-1}}(\Omega)}.$$

Iterating this expression in k we get that

$$\|tr_\Omega v\|_{L^{\eta_k}(\Omega)} \leq \prod_{j=0}^{k-1}\left(C\frac{\eta_j^2}{4(\eta_j-1)}\right)^{\frac{1}{\eta_j}}\|tr_\Omega v\|_{L^2(\Omega)}.$$

Note that there exists a constant $C > 0$ such that $\frac{z^2}{4(z-1)} \leq Cz$, for all $z \geq 1$. Taking into account the fact that $\eta_j = \frac{2_\alpha^j}{2^{j-1}}$, it follows that

$$\begin{aligned}\|tr_\Omega v\|_{L^{\eta_k}(\Omega)} &\leq \prod_{j=0}^{k-1}\left(C\frac{2_\alpha^j}{2^{j-1}}\right)^{\frac{2^{j-1}}{2_\alpha^j}}\|tr_\Omega v\|_{L^2(\Omega)} \\ &\leq (2_\alpha C)^{A_k}\prod_{j=0}^{k-1}\left(\delta^{1-j}\right)^{\frac{1}{2\alpha}\delta^{j-1}}\|tr_\Omega v\|_{L^2(\Omega)},\end{aligned}$$

where $\delta = \frac{2}{2_\alpha} \in (0,1)$ and $A_k = \dfrac{1}{2_a}\displaystyle\sum_{j=1}^{k-1}\delta^{j-1}$. Now, since $0 < \delta < 1$, the series in A_k converges and

$$\prod_{j=0}^{k-1}\left(\delta^{1-j}\right)^{\frac{1}{2\alpha}\delta^{j-1}} < +\infty.$$

Now, observing that $\eta_k \to +\infty$, it follows that $\|tr_\Omega v\|_{L^\infty(\Omega)} < +\infty$.

If v is a minimizer for J, then it is straightforward to see that $|v|$ also is. Taking a constant $M > 0$ such that $M + c(x) > 0$ in Ω, Proposition 2.2 implies that $|v| > 0$ in $\mathcal{C}$. Since v is regular, it follows that v cannot change sign. As a consequence, two of them cannot be ortogonal and $\lambda_1[\alpha; c]$ is simple.

The same procedure employed to $\lambda_1[\alpha; c]$ applies to prove that, denoting by V_j the eigenspace associated to the j-th eigenvalue, the higher eigenvalues can be characterized as

$$\lambda_j = \inf\{J(u);\, u \neq 0,\ \langle u, v\rangle_{L^2(\Omega)} = 0\ \forall v \in span[V_1, ..., V_{j-1}]\}.$$

This characterization with the positiveness of the first eigenfunction implies that the first eigenvalue is the only one which has a one-signed eigenfunction.

In order to end up the proof, note that the variational characterization of the eigenvalues still implies that if $c_1, c_2 \in L^\infty(\Omega)$ and $c_1 < c_2$ in Ω, then $\lambda_1[\alpha; c_1] < \lambda_1[\alpha; c_2]$. In fact, let $w \in \mathcal{H}_0^\alpha(\mathcal{C})$ such that $tr_\Omega w \neq 0$ in $L^2(\Omega)$ and $J(w) = \lambda_1[\alpha; c_2]$, note that

$$\frac{\displaystyle\int_{\mathcal{C}} y^{1-2\alpha}|\nabla w|^2 dxdy + \int_\Omega c_1(x) w(x,0)^2 dx}{\displaystyle\int_\Omega w(x,0)^2 dx} < \frac{\displaystyle\int_{\mathcal{C}} y^{1-2\alpha}|\nabla w|^2 dxdy + \int_\Omega c_2(x) w(x,0)^2 dx}{\displaystyle\int_\Omega w(x,0)^2 dx}$$

and this finishes the proof. □

Let us point out that the behavior of $\lambda_1[\alpha; c]$ with respect to the weights is a challenging problem, see for instance Section 3 in [20]. In any case, we would like to study $\lambda_1[\alpha; c]$ in some particular case. When $c \equiv 0$ we denote $\lambda_1[\alpha] := \lambda_1[\alpha; 0]$. Finally, for $\alpha = 1$ we denote $\lambda_1[1; c]$ the principal eigenvalue of the local operator $-\Delta + c(x)$ under homogeneous Dirichlet boundary conditions and by $\lambda_1 := \lambda_1[1; 0]$. Recall that $\lambda_1[\alpha] = \lambda_1^\alpha$.

remark3.2 **Remark 3.2.** *Given $c \in L^\infty(\Omega)$, we denote*

$$c_L := ess\ inf_\Omega c(x) \quad and \quad c_M := ess\ sup_\Omega c(x).$$

Note that by the definition of J and the fact that $\lambda_1[\alpha; c]$ minimizes J, it follows that

$$\lambda_1[\alpha] + c_L \leq \lambda_1[\alpha; c] \leq \lambda_1[\alpha] + c_M.$$

It is not hard to show that when $c \in \mathbb{R}$ we get

$$\lambda_1[\alpha; c] = \lambda_1[\alpha] + c = \lambda_1^\alpha + c.$$

In the following result we show the dependence of $\lambda_1[\alpha; c]$ in $N = 1$ with respect to the domain $\Omega = B_r = (-r, r)$. Denote by $\lambda_1[\alpha; c; B_r]$ the principal eigenvalue of (3.1) in B_r and by $\lambda_1[1; c; B_r]$ the principal eigenvalue of the $-\Delta + c$ in B_r, that is, the principal eigenvalue of

$$-\Delta v + c(x)v = \lambda_1[1; c; B_r]v \quad \text{in } B_r, \quad v = 0 \quad \text{on } \partial B_r. \tag{3.6}$$

otra1

With this notation, we can prove:

bola **Proposition 3.3.** *It holds:*

$$\lambda_1[\alpha; c; B_r]r^{2\alpha} = \lambda_1[\alpha; r^{2\alpha}c(r\cdot); B_1], \tag{3.7}$$

bolita

and

$$\lambda_1[1; c; B_r]r^2 = \lambda_1[1; r^2 c(r\cdot); B_1]. \tag{3.8}$$

bolita1

As consequence,

$$\lim_{r\to 0} \lambda_1[\alpha; c; B_r]r^{2\alpha} = \lambda_1[\alpha; 0; B_1] = (\lambda_1[1; 0; B_1])^\alpha. \tag{3.9}$$

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Proof. By the definition of $\lambda_1[\alpha; c; B_r]$, there exists v such that

$$\begin{cases} -\operatorname{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } B_r \times (0, \infty), \\ v = 0 & \text{on } \partial B_r \times (0, \infty), \\ \dfrac{\partial v}{\partial y^\alpha}(x, 0) + c(x)v(x, 0) = \lambda_1[\alpha; c; B_r]v(x, 0) & \text{on } B_r. \end{cases} \tag{3.10}$$

parti

The change of variables

$$z = \frac{x}{r}, \quad t = \frac{y}{r}, \quad \text{and} \quad w(z, t) = v(zr, tr),$$

transforms (3.10) into

$$\begin{cases} -\operatorname{div}(t^{1-2\alpha}\nabla w) = 0 & \text{in } B_1 \times (0, \infty), \\ w = 0 & \text{on } \partial B_1 \times (0, \infty), \\ \dfrac{\partial w}{\partial t^\alpha}(z, 0) + r^{2\alpha}c(rz)w(z, 0) = r^{2\alpha}\lambda_1[\alpha; c; B_r]w(z, 0) & \text{on } B_1. \end{cases} \tag{3.11}$$

parti2

This concludes the proof of (3.7).

In a similar way, under the change of variable

$$z = \frac{x}{r}, \quad w(z) = v(zr),$$

in (3.6), we get (3.8). (3.9) is trivial from (3.7). $\square$

Let us compare the eigenvalues of the laplacian and fractional laplacian for the case $N = 1$, $c \in \mathbb{R}$ and $\Omega = B_r$.

casoconstante **Lemma 3.4.** *Assume $c \in \mathbb{R}$. Then,*

$$\lambda_1[\alpha; c; B_r] > \lambda_1[1; c; B_r] \quad \textit{(resp. } <, =\textit{)} \Longleftrightarrow r > \sqrt{\lambda_1[1; B_1]} \quad \textit{(resp. } <, =\textit{)}.$$

On the other hand, $\alpha \mapsto \lambda_1[\alpha; c; B_r]$ is decreasing when $r > \sqrt{\lambda_1[1; 0; B_1]}$ and increasing when $r < \sqrt{\lambda_1[1; 0; B_1]}$.

Proof. Observe that

$$\lambda_1[1; c; B_r]r^2 = \lambda_1[1; r^2 c(r\cdot); B_1],$$

and so, if c is a constant,

$$\lambda_1[1; c; B_r] = \frac{\lambda_1[1; 0; B_1]}{r^2} + c,$$

and by Proposition 3.3 we get

$$\lambda_1[\alpha; c; B_r] = \frac{\lambda_1[\alpha; 0; B_1]}{r^{2\alpha}} + c = \left(\frac{\lambda_1[1; 0; B_1]}{r^2}\right)^{\alpha} + c.$$

This concludes the result. $\square$

Remark 3.5. *Recall that $\lambda_1[1; 0; B_1] = \pi^2/4$.*

4 The logistic equation

In this section, we want to study the logistic equation

$$\begin{cases} (-\Delta)^\alpha u + c(x)u = \lambda u - u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$ logis

where $\alpha \in (0,1)$ and $c \in L^\infty(\Omega)$ or equivalently the equation

$$\begin{cases} -\operatorname{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } \mathcal{C}, \\ v = 0 & \text{on } \partial_L \mathcal{C}, \\ \dfrac{\partial v}{\partial y^\alpha}(x,0) + c(x)v(x,0) = \lambda v(x,0) - v(x,0)^2 & \text{on } \Omega. \end{cases} \tag{4.2}$$

logis2

Theorem 4.1. *Equation (4.1) possesses a positive solution if and only if $\lambda > \lambda_1[\alpha; c]$. Moveover, if it exists, this is the unique positive solution and we denote it by $\theta_{[\alpha,\lambda-c]}$. Furthermore, $\theta_{[\alpha,\lambda-c]} \in C^{2,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$ and the following property holds: if we denote by φ_1 the principal eigenfunction associated to $\lambda_1[\alpha; c]$ such that $\|\varphi_1\|_\infty = 1$, then*

$$(\lambda - \lambda_1[\alpha; c])\varphi_1(x) \le \theta_{[\alpha,\lambda-c]}(x) \le \lambda - c_L, \quad \forall x \in \Omega. \tag{4.3}$$

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theorem4.1

Remark 4.2. *A similar result holds for (4.2). In this case, we denote by $\Theta_{[\alpha,\lambda-c]}$ the unique positive solution of (4.2), that is, $\theta_{[\alpha,\lambda-c]} = tr_\Omega \Theta_{[\alpha,\lambda-c]}$. Moreover, $\Theta_{[\alpha,\lambda-c]} \in C^{2,\sigma}(\overline{\mathcal{C}}) \cap L^\infty(\mathcal{C})$.*

In the proof of Theorem 4.1 we are going to apply the well known sub-supersolution method. Despite of the definitions and results about this subject in the fractional setting are a rather standard adaptation of the sub-supersolution method to second order operators, we present them here for the sake of completeness.

Let us consider the problem (2.2) which is associated to the extension problem

$$\begin{cases} \operatorname{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } \mathcal{C}, \\ v = 0 & \text{on } \partial_L \mathcal{C}, \\ \dfrac{\partial v}{\partial y^\alpha}(x,0) = f(x, v(x,0)) & \text{on } \Omega, \end{cases} \tag{4.4}$$

subsuper

where $f \in C(\overline{\Omega} \times \mathbb{R})$. Recall the definition of solution of (4.4), Definition 2.1.

Definition 4.3. *We say that $(\underline{v}, \overline{v})$ is a sub-supersolution of (4.4) if $\underline{v}, \overline{v} \in \mathcal{H}^\alpha(\mathcal{C})$, $\underline{u} := tr_\Omega \underline{v}, \overline{u} := tr_\Omega \overline{v} \in L^\infty(\Omega)$ and:*

a) $\underline{v} \le \overline{v}$ in $\mathcal{C}$ and $\underline{v} \le 0 \le \overline{v}$ on $\partial_L \mathcal{C}$.

b) For all $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$, $\psi \geq 0$, it holds

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla \underline{v} \cdot \nabla \psi dxdy \leq \int_\Omega f(x, \underline{v}(x,0))\psi(x,0)dx \tag{4.5}$$

subsolution

and

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla \overline{v} \cdot \nabla \psi dxdy \geq \int_\Omega f(x, \overline{v}(x,0))\psi(x,0)dx. \tag{4.6}$$

supersolution

Theorem 4.4. *Assume that $(\underline{v}, \overline{v})$ is a sub-supersolution of (4.4). Then, there exists a solution v of (4.4) such that*

$$\underline{v} \leq v \leq \overline{v} \quad \text{in } \mathcal{C}.$$

In consequence, there exists a solution $u \in \mathcal{V}_0^\alpha(\Omega)$ of (2.2) such that

$$\underline{u} = tr_\Omega \underline{v} \leq u \leq \overline{u} = tr_\Omega \overline{v} \quad \text{in } \Omega.$$

eoremasubsuper

Proof. Let $\underline{v}$, $\overline{v}$ be such that (4.5) and (4.6) hold, respectively. Let us define for $x \in \Omega$ and $t \in \mathbb{R}$

$$\tilde{f}(x,t) := \begin{cases} f(x, \underline{u}(x)) & \text{if } t \leq \underline{u}(x), \\ f(x,t) & \text{if } \underline{u}(x) \leq t \leq \overline{u}(x), \\ f(x, \overline{u}(x)) & \text{if } t \geq \overline{u}(x), \end{cases}$$

and consider the problem

$$\begin{cases} \operatorname{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } \mathcal{C}, \\ v = 0 & \text{on } \partial_L \mathcal{C}, \\ \dfrac{\partial v}{\partial y^\alpha}(x,0) = \tilde{f}(x, v(x,0)) & \text{on } \Omega. \end{cases} \tag{4.7}$$

T

Observe that by the definition of $\tilde{f}$ we have that

$$\left| \int_\Omega \tilde{f}(x, u(x,0))\psi(x,0)dx \right| \leq C\|\psi(x,0)\|_{L^2(\Omega)}, \tag{4.8}$$

clave

for some positive constant C, for all $u \in \mathcal{H}^\alpha(\mathcal{C})$ and $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$. Here, we have used that $\underline{u}, \overline{u} \in L^\infty(\Omega)$ and $f \in C(\overline{\Omega} \times \mathbb{R})$

First, we show that (4.7) possesses at least a solution. Define the operator

$$T : \mathcal{H}_0^\alpha(\mathcal{C}) \mapsto (\mathcal{H}_0^\alpha(\mathcal{C}))'$$

given by

$$(Tu,v)=\int_{\mathcal{C}} y^{1-2\alpha}\nabla u\cdot\nabla v dxdy-\int_{\Omega}\tilde{f}(x,u(x,0))v(x,0)dx,\quad \forall u,v\in\mathcal{H}_0^\alpha(\mathcal{C}).$$

We study some properties of the map T.

- T is a bounded map. It is clear, using (4.8), that if u belongs to a bounded set of $\mathcal{H}_0^\alpha(\mathcal{C})$, then $T(u)$ is also bounded in $(\mathcal{H}_0^\alpha(\mathcal{C}))'$.

- T is pseudomonotone: given a sequence $u_n \rightharpoonup u$ in $\mathcal{H}_0^\alpha(\mathcal{C})$ such that

$$\limsup (Tu_n, u_n-u)\le 0,$$

we have to show that

$$\liminf (Tu_n, u_n-v)\ge (Tu,u-v)\quad \forall v\in\mathcal{H}_0^\alpha(\mathcal{C}). \tag{4.9}$$

claim

Observe that from (4.8) we have that

$$\left|\int_\Omega \tilde{f}(x,u_n(x,0))(u_n(x,0)-u(x,0))dx\right|\le C\|u_n-u\|_{L^2(\Omega)}\to 0,$$

hence using that $u_n\rightharpoonup u$ in $\mathcal{H}_0^\alpha(\mathcal{C})$

$$0\ge \limsup (Tu_n,u_n-u)=\limsup \int_{\mathcal{C}} y^{1-2\alpha}\nabla u_n\cdot\nabla(u_n-u)=\limsup \|u_n\|_\alpha^2-\|u\|_\alpha^2.$$

We can conclude that

$$\|u\|_\alpha^2\ge \limsup \|u_n\|_\alpha^2\ge \liminf \|u_n\|_\alpha^2 \ge \|u\|_\alpha^2,$$

and then

$$\lim \|u_n\|_\alpha^2=\|u\|_\alpha^2.$$

Consequently, $u_n\to u$ in $\mathcal{H}_0^\alpha(\mathcal{C})$ and we get that

$$\liminf (Tu_n,u_n-v)=\liminf\{(Tu_n,u_n-u)+(Tu_n,u-v)\}\ge (Tu,u-v).$$

- T is coercive, that is,

$$\lim_{\|v\|_\alpha\to\infty}\frac{(T(v),v)}{\|v\|_\alpha}=\infty.$$

It is clear that

$$(T(v),v)\ge \|v\|_\alpha^2-C\|v\|_{L^2(\Omega)}^2,$$

whence it follows that T is coercive.

Then, we can conclude from Theorem 2.7 in Chapter 2 of [15] that there exists a solution of (4.7), that is, $T(v) = 0$. Now, we show that

$$v \in [\underline{v}, \overline{v}],$$

and hence v is solution of (4.4). Indeed, define $\tilde{v} := \underline{v} - v$. Note that, for all $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$, $\psi \geq 0$,

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla \tilde{v} \cdot \nabla \psi dx dy \leq \int_{\Omega} \Big(f(x, \underline{v}(x,0)) - \tilde{f}(x, v(x,0)) \Big) \psi(x,0) dx.$$

Taking $\psi = (\underline{v} - v)^+$, we have that

$$\int_{\mathcal{C}} y^{1-2\alpha} |\nabla \tilde{v}^+|^2 dx dy \leq 0.$$

Then $\underline{v} \leq v$ in $\mathcal{C}$ and in a similar way one can prove that $v \leq \overline{v}$.

□

Now let us present the proof of the Theorem 4.1.

Proof of Theorem 4.1. First consider a positive solution $u \in \mathcal{V}_0^\alpha(\Omega)$ of (4.1), and consider $v \in \mathcal{H}_0^\alpha$ solution of (4.2). If $\lambda - c_L \leq 0$, then by the maximum principle it follows that $v \leq 0$. So, assume that $\lambda - c_L > 0$. Taking in (4.2) $\psi = (v - (\lambda - c_L))^+$,we can show that

$$v \leq \lambda - c_L \quad \text{in } \mathcal{C}.$$

By Lemma 2.4, we have that $u \in L^\infty(\Omega)$; and then, using Lemma 2.5 we arrive that u and v are regular functions.

Now, suppose that there exists a positive solution $u \in \mathcal{V}_0^\alpha(\Omega)$ of (4.1) for some $\lambda \in \mathbb{R}$. Then note that u is a positive solution of (3.1) with $c(x)$ substituted by $(c(x) + u(x))$. Then by Theorem 3.1

$$\lambda = \lambda_1[\alpha; c + u] > \lambda_1[\alpha; c].$$

Now let us prove that $\lambda > \lambda_1[\alpha; c]$ is sufficient to the existence of a positive solution. Let $\Omega \subset\subset \Omega'$, Ω' an open bounded set, $\mathcal{C}' = \Omega' \times (0, +\infty)$ and $E \in \mathcal{H}_0^\alpha(\mathcal{C}')$ the unique positive solution of

$$\begin{cases} \operatorname{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } \mathcal{C}', \\ v = 0 & \text{on } \partial_L \mathcal{C}', \\ \dfrac{\partial v}{\partial y^\alpha}(x,0) = 1 & \text{in } \Omega'. \end{cases} \tag{4.10}$$ e

Denote by

$$e(x) := tr_{\Omega'} E.$$

Observe that from the regularity results, $e \in L^\infty(\Omega')$ and by Proposition 2.2 we get that $E > 0$.

Note in particular that for $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$, we can extend it in such a way that $\psi \in \mathcal{H}_0^\alpha(\mathcal{C}')$ and then, it holds

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla E \cdot \nabla \psi dx dy = \int_\Omega \psi(x,0) dx.$$

Let us take $\overline{v} = KE$ where K is a positive constant to be chosen. Note that $\overline{v}$ is a supersolution of (4.2) if and only if for all $\psi \in \mathcal{H}_0^\alpha(\mathcal{C})$, $\psi \geq 0$

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla E \cdot \nabla \psi dx dy + K \int_\Omega c(x) E(x,0)\psi(x,0) dx \geq \int_\Omega (\lambda E(x,0) - K E(x,0)^2)\psi(x,0)) dx,$$

this is equivalent to

$$\int_\Omega \psi(x,0) \left(K e(x)^2 + e(x)(c(x) - \lambda) + 1 \right) dx \geq 0 \quad \forall \psi \in \mathcal{H}_0^\alpha(\mathcal{C}), \psi \geq 0.$$

It suffices that $K e(x)^2 + e(x)(c_L - \lambda) + 1 \geq 0$ a.e. in Ω, which is possible by choosing K large enough.

For the subsolution, let us take $\underline{v} = \epsilon \Psi_1$ where $\epsilon > 0$ is a constant to be chosen and $\Psi_1 \in \mathcal{H}_0^\alpha(\mathcal{C})$ is a positive eigenfunction associated to $\lambda_1[\alpha; c]$. Then, for all $\psi \in \mathcal{H}_0^\alpha$, $\psi \geq 0$, writing $\lambda_1 = \lambda_1[\alpha; c]$ we have

$$\begin{aligned} \int_{\mathcal{C}} y^{1-2\alpha} \nabla \underline{v} \cdot \nabla \psi dx dy + \int_\Omega c(x) \underline{v}(x,0)\psi(x,0) dx &= \epsilon \int_\Omega \lambda_1 \varphi_1 \psi(x,0) dx \\ &\leq \int_\Omega \epsilon \varphi_1 \psi(x,0)(\lambda - \epsilon \varphi_1) dx \end{aligned}$$

if and only if

$$\epsilon \varphi_1 \leq (\lambda - \lambda_1) \quad \text{in } \Omega, \tag{4.11}$$ cond1

where we have denoted $\varphi_1 = tr_\Omega \Psi_1$. Since $\varphi_1 \in \mathcal{V}_0^\alpha(\Omega)$, $\varphi_1 \in L^\infty(\Omega)$ and $\varphi_1 > 0$ in Ω, (4.11) is possible and it follows that we have a sub-supersolution pair if $\epsilon > 0$ is small enough. Now Theorem 4.4 implies the existence of solution if $\lambda > \lambda_1[\alpha; c]$.

To prove the uniqueness of positive solution, all the arguments of [4] (see also [5]) can be adapted to the fractional setting, see Lemma 5.2 in [3] or Proposition 4.2 in [19].

Then, there exists a solution $\theta_{[\alpha,\lambda-c]} \in \mathcal{V}_0^\alpha(\Omega)$ of (4.1) if and only if $\lambda > \lambda_1[\alpha; c]$.

We prove now (4.3). The first inequality follows since $\epsilon\varphi_1$ is a subsolution for all $\epsilon \in (0, \lambda - \lambda_1[\alpha; c]]$. For the second, note that $\theta_{[\alpha,\lambda-c]} \leq \lambda - c_L$. □

To compare different solutions of the logistic equation we need the following result:

compa **Proposition 4.5.** *Assume that $\underline{v}$ is a bounded subsolution of (4.2), then*

$$tr_\Omega \underline{v} \leq \theta_{[\alpha,\lambda-c]}.$$

Proof. Since $\underline{v}$ is bounded, it is clear that we can choose $K > 0$ such that KE is supersolution of (4.2) and $\underline{v} \leq KE$. By uniqueness, we conclude that $\underline{v}(x, 0) \leq \theta_{[\alpha,\lambda-c]}$. □

As a direct consequence of Proposition 4.5, we deduce

Corollary 4.6. *If $\lambda_1 \leq \lambda_2$ and $c_2 \leq c_1$ in Ω, then $\theta_{[\alpha,\lambda_1-c_1]} \leq \theta_{[\alpha,\lambda_2-c_2]}$.*

Let us give an interesting biological interpretation of this result, comparing with the linear diffusion case. Recall that the classical logistic equation

$$\begin{cases} -\Delta u + c(x)u = \lambda u - u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$ logiscla

possesses a unique positive solution if and only if

$$\lambda > \lambda_1[1; -c].$$

Let us compare this result with the obtained for (4.1) in the particular case $N = 1$, $c \in \mathbb{R}$ and $\Omega = B_r$. In Figure 1 we have represented by continuous line $G_1(r) := \lambda_1[1; c; B_r]$ and by pointed line $G_\alpha(r) := \lambda_1[\alpha; c; B_r]$ with $c = 0$ (a similar representation can be made with $c \neq 0$). Take Λ large ($\Lambda > 1$). Then, there exist $r_\alpha < r_1$ such that

$$\Lambda = G_1(r_1) = G_\alpha(r_\alpha).$$

Then,

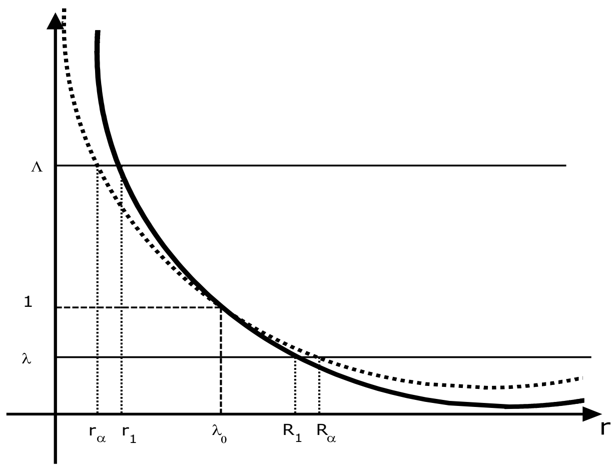

Figure 1: We have represented in continuous line the map $G_1(r) = \lambda_1[1; c; B_r]$ and by pointed line $G_\alpha(r) = \lambda_1[\alpha; c; B_r]$. We have denoted by $\lambda_0 = \sqrt{\lambda_1}$.

a) If $r < r_\alpha$, for (4.1) and (4.12) the species die.

b) If $r > r_1$, the species persists in both cases.

c) Assume that $r \in (r_\alpha, r_1)$. Then, the species disappears in the local diffusion and it persists in the fractional diffusion case.

Assume now λ small, ($\lambda < 1$). Then, there exist $R_1 < R_\alpha$ such that

$$\lambda = G_1(R_1) = G_\alpha(R_\alpha).$$

Moreover,

a) If $r < R_1$, for (4.1) and (4.12) the species die.

b) If $r > R_\alpha$, the species persists in both cases.

c) Assume that $r \in (R_1, R_\alpha)$. Then, the species disappears in the fractional diffusion and it persists in the local diffusion case.

Hence, in the case of favourable habitats (abundant resources) there exist domains such that the species with fractional diffusion persists, while the species with linear diffusion dies. In a contrary way, for unfavourable habitats, there exist domains when the opposite occurs.

5 The sub-supersolution method for systems

In this section we extend the sub-supersolution method employed in the last section to the system setting. Let us consider

$$\begin{cases} (-\Delta)^{\alpha} u = f(x,u,v) & \text{in } \Omega, \\ (-\Delta)^{\beta} v = g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$ subsupersystem

where $f, g \in C^0(\overline{\Omega} \times \mathbb{R}^2)$ and $\alpha, \beta \in (0,1)$.

definew **Definition 5.1.** *We say that $(u,v) \in \mathcal{V}_0^{\alpha}(\Omega) \times \mathcal{V}_0^{\beta}(\Omega)$ is a solution of (5.1) if there exists $(U,V) \in \mathcal{H}_0^{\alpha}(\mathcal{C}) \times \mathcal{H}_0^{\beta}(\mathcal{C})$ such that $tr_{\Omega} U := u$, $tr_{\Omega} V := v$ and (U,V) is solution of*

$$\begin{cases} div(y^{1-2\alpha}\nabla U) = div(y^{1-2\beta}\nabla V) = 0 & \text{in } \mathcal{C}, \\ U = V = 0 & \text{on } \partial_L \mathcal{C}, \\ \dfrac{\partial U}{\partial y^{\alpha}}(x,0) = f(x, U(x,0), V(x,0)) & \text{on } \Omega, \\ \dfrac{\partial V}{\partial y^{\beta}}(x,0) = g(x, U(x,0), V(x,0)) & \text{on } \Omega, \end{cases} \tag{5.2}$$ subsupersystem2

Definition 5.2. *We say that $\underline{U}, \overline{U} \in \mathcal{H}^{\alpha}(\mathcal{C})$, $\underline{V}, \overline{V} \in \mathcal{H}^{\beta}(\mathcal{C})$ is a pair of sub-supersolution of (5.1) if*

$$\underline{u} := tr_{\Omega}\underline{U},\ \overline{u} := tr_{\Omega}\overline{U},\ \underline{v} := tr_{\Omega}\underline{V},\ \overline{v} := tr_{\Omega}\overline{V} \in L^{\infty}(\Omega),$$

and

a) $\underline{U} \le \overline{U}$ and $\underline{V} \le \overline{V}$ in $\mathcal{C}$ and $\underline{U} \le 0 \le \overline{U}$ and $\underline{V} \le 0 \le \overline{V}$ on $\partial_L \mathcal{C}$.

b) For all $(\psi,\phi) \in \mathcal{H}_0^{\alpha}(\mathcal{C}) \times \mathcal{H}_0^{\beta}(\mathcal{C})$, $\psi, \phi \ge 0$ and $(u,v) \in [\underline{U}, \overline{U}] \times [\underline{V}, \overline{V}]$, it holds

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla \underline{U} \cdot \nabla \psi dxdy \le \int_{\Omega} f(x, \underline{U}(x,0), v(x,0)) \psi(x,0) dx,$$

$$\int_{\mathcal{C}} y^{1-2\alpha} \nabla \overline{U} \cdot \nabla \psi dxdy \ge \int_{\Omega} f(x, \overline{U}(x,0), v(x,0)) \psi(x,0) dx,$$

$$\int_{\mathcal{C}} y^{1-2\beta}\nabla \underline{V}\cdot\nabla\phi dxdy \le \int_\Omega f(x,u(x,0),\underline{V}(x,0))\phi(x,0)dx,$$
$$\int_{\mathcal{C}} y^{1-2\beta}\nabla \overline{V}\cdot\nabla\phi dxdy \ge \int_\Omega f(x,u(x,0),\overline{V}(x,0))\phi(x,0)dx,$$

where $[\underline{U},\overline{U}]=\{w\in\mathcal{H}^\alpha(\mathcal{C});\ \underline{U}\le w\le\overline{U}\ \text{in}\ \mathcal{C}\}$ *and analogous for* $[\underline{V},\overline{V}]$.

Theorem 5.3. *Assume that there exists a pair* $(\underline{U},\overline{U})$-$(\underline{V},\overline{V})$ *of sub-supersolution of (5.2). Then, there exists a solution* $(U,V)\in\mathcal{H}_0^\alpha(\mathcal{C})\times\mathcal{H}_0^\beta(\mathcal{C})$ *of (5.1) such that*

$$\underline{U}\le U\le\overline{U},\quad \underline{V}\le V\le\overline{V}\quad \text{in}\ \mathcal{C}.$$

Moreover, there exists a solution $(u,v)\in\mathcal{V}_0^\alpha(\Omega)\times\mathcal{V}_0^\beta(\Omega)$ *of (5.1) such that* $\underline{u}\le u\le\overline{u}$ *in* Ω *and* $\underline{v}\le v\le\overline{v}$ *in* Ω.

bsupersistemas

Proof. The proof is similar to Theorem 4.4. Define the operators T_1 and T_2 by

$$T_1(w)=\begin{cases}\underline{u} & \text{if } w\le\underline{u},\\ w & \text{if } \underline{u}\le w\le\overline{u},\\ \overline{u} & \text{if } w\ge\overline{u},\end{cases}\qquad T_2(z)=\begin{cases}\underline{v} & \text{if } z\le\underline{u},\\ z & \text{if } \underline{v}\le z\le\overline{v},\\ \overline{v} & \text{if } z\ge\overline{v},\end{cases}$$

and the functions

$$\tilde{f}(x,u,v)=f(x,T_1(u),T_2(v)),\qquad \tilde{g}(x,u,v)=g(x,T_1(u),T_2(v)).$$

Consider the problem

$$\begin{cases}\operatorname{div}(y^{1-2\alpha}\nabla U)=\operatorname{div}(y^{1-2\beta}\nabla V)=0 & \text{in } \mathcal{C},\\ U=V=0 & \text{on } \partial_L\mathcal{C},\\ \dfrac{\partial U}{\partial y^\alpha}(x,0)=\tilde{f}(x,U(x,0),V(x,0)) & \text{on } \Omega,\\ \dfrac{\partial V}{\partial y^\beta}(x,0)=\tilde{g}(x,U(x,0),V(x,0)) & \text{on } \Omega.\end{cases}\tag{5.3}$$

fractionalsystem

First, we prove that (5.3) has at least a solution. For that, consider the space

$$\mathcal{H}:=\mathcal{H}_0^\alpha(\mathcal{C})\times\mathcal{H}_0^\beta(\mathcal{C})$$

with the norm $\|(u,v)\|=\|u\|_\alpha+\|v\|_\beta$ and the map $T:\mathcal{H}\mapsto(\mathcal{H})'$ defined by

$$(T(u,v),(w,z))=\Big(\int_{\mathcal{C}} y^{1-2\alpha}\nabla u\cdot\nabla w dxdy-\int_\Omega\tilde{f}(x,u(x,0))w(x,0)dx,$$
$$\int_{\mathcal{C}} y^{1-2\beta}\nabla v\cdot\nabla z dxdy-\int_\Omega\tilde{g}(x,v(x,0))z(x,0)dx\Big).$$

Now, we can follow just the arguments of Theorem 4.4 and show that there exists (U,V) solution of (5.3), that is, $T(U,V)=(0,0)$. Again, we can prove that (U,V) is solution of (5.1), for that it suffices to show that $(U,V)\in[\underline{U},\overline{U}]\times[\underline{V},\overline{V}]$. Define $\tilde{U}=\underline{U}-U$, then taking $T_2(V)$ in the definition of sub-solution, we get that for all $\psi\in\mathcal{H}_0^\alpha$, $\psi\geq 0$,

$$\int_{\mathcal{C}} y^{1-2\alpha}\nabla\tilde{U}\cdot\nabla\psi dxdy\leq\int_\Omega\left[f(x,\underline{U},T_2(V))-\tilde{f}(x,U,V)\right]\psi(x,0)dx\leq 0.$$

Taking $\psi=(\underline{U}-U)^+$ we get that $\underline{U}\leq U$. The same argument can be used to the other inequalities. □

6 Application to the Lotka-Volterra systems

In this section we apply the above results to system (1.1), or equivalently, to the system

$$\begin{cases}\operatorname{div}(y^{1-2\alpha}\nabla U)=\operatorname{div}(y^{1-2\beta}\nabla V)=0 & \text{in } \mathcal{C},\\ U=V=0 & \text{on } \partial_L\mathcal{C},\\ \dfrac{\partial U}{\partial y^\alpha}(x,0)=U(x,0)(\lambda-U(x,0)-bV(x,0)) & \text{in } \Omega,\\ \dfrac{\partial V}{\partial y^\beta}(x,0)=V(x,0)(\mu-V(x,0)-cU(x,0)) & \text{in } \Omega,\end{cases}\tag{6.1}$$ uno2

First, we deduce some bounds of the solutions of (1.1).

cotassol **Proposition 6.1.** *a) Assume that $b,c>0$ and let (u,v) a positive solution of (1.1). Then,*

$$u\leq\theta_{[\alpha,\lambda]},\quad v\leq\theta_{[\beta,\mu]}.$$

b) Assume that $b>0$ and $c<0$ and let (u,v) a positive solution of (1.1). Then,

$$u\leq\theta_{[\alpha,\lambda-b\theta_{[\beta,\mu]}]}\leq\theta_{[\alpha,\lambda]},\quad\theta_{[\beta,\mu]}\leq v\leq\theta_{[\beta,\mu-c\theta_{[\alpha,\lambda]}]}.$$

c) Assume that $b,c<0$ and let (u,v) a positive solution of (1.1). Then,

$$\theta_{[\alpha,\lambda]}\leq u,\quad\theta_{[\beta,\mu]}\leq v.$$

Proof. a) Assume that $b,c>0$ and and let (u,v) a positive solution of (1.1), that is, $(u,v)=(tr_\Omega U,tr_\Omega V)$, being (U,V) solution of (6.1). With a similar reasoning to the used in Theorem 4.1 we can show that $U, V \in L^\infty(\mathcal{C})$. Moreover, $u \in L^\infty(\Omega)$. Now, it is clear that U is a bounded subsolution of (4.1) with $c \equiv 0$. Then, $U \leq \Theta_{[\alpha,\lambda]}$ and so

$$u \leq \theta_{[\alpha,\lambda]} \quad \text{in } \Omega.$$

In a similar way, we can show that $v \leq \theta_{[\beta,\mu]}$.

b) It is easy to show that $u \leq \theta_{[\alpha,\lambda]}$ and $\theta_{[\beta,\mu]} \leq v$, this last inequality showing that $\Theta_{[\beta,\mu]}$ is subsolution of $(-\Delta)^\beta v = v(\mu - v - cu)$. Moreover, using that $V \geq \Theta_{[\beta,\mu]}$, we can show that U is sub-solution of (4.2) with $c(x) = -b\theta_{[\beta,\mu]}$, and so $u \leq \theta_{[\alpha,\lambda - b\theta_{[\beta,\mu]}]}$.

c) Similar to the above paragraphs.

□

Corollary 6.2. *a) Assume that $b, c > 0$. If there exists a positive solution of (1.1), then $\lambda > \lambda_1[\alpha]$ and $\mu > \lambda_1[\beta]$.*

b) Assume that $b > 0$ and $c < 0$. If there exists a positive solution of (1.1), then $\lambda > \lambda_1[\alpha; b\theta_{[\beta,\mu]}]$ and $\mu > \lambda_1[\beta; c\theta_{[\alpha,\lambda]}]$.

We introduce now some notation. Denote by E_α the unique positive solution of (4.10) in $\mathcal{C}$ and $e_\alpha = tr_\Omega E$. We call

$$C(\alpha, \beta) := \left(\frac{e_\alpha}{e_\beta}\right)_M \left(\frac{e_\beta}{e_\alpha}\right)_M.$$

The main result is:

principal **Theorem 6.3.** *a) Assume $b, c > 0$ (Competitive case). Assume that $\lambda > \lambda_1[\alpha]$ and $\mu > \lambda_1[\beta]$. If (λ, μ) verifies*

$$\lambda > \lambda_1[\alpha; b\theta_{[\beta,\mu]}] \quad \textit{and} \quad \mu > \lambda_1[\beta; c\theta_{[\alpha,\lambda]}], \tag{6.2}$$ condigeneral

then there exists at least a coexistence state of (1.1).

b) Assume that $b > 0$ and $c < 0$ (Prey-predator case). If (λ, μ) verifies

$$\lambda > \lambda_1[\alpha; b\theta_{[\beta,\mu - c\theta_{[\alpha,\lambda]}]}] \quad \textit{and} \quad \mu > \lambda_1[\beta; c\theta_{[\alpha,\lambda]}], \tag{6.3}$$ condigeneralpp

then there exists at least a coexistence state of (1.1).

c) Assume that $b<0$, $c<0$ and $bc<C(\alpha,\beta)$ (Symbiosis case). If (λ,μ) verifies (6.2), then there exists at least a coexistence state of (1.1).

Proof. a) Assume that $b,c>0$. We te take following sub-supersolution

$$(\underline{U},\overline{U})=(\Theta_{[\alpha,\lambda-b\theta_{[\beta,\mu]}]},\Theta_{[\alpha,\lambda]}),\quad (\underline{V},\overline{V})=(\Theta_{[\beta,\mu-c\theta_{[\alpha,\lambda]}]},\Theta_{[\beta,\mu]}).$$

Indeed, observe that for $\psi\in\mathcal{H}_0^\alpha(\mathcal{C})$, $\psi\geq 0$

$$\begin{aligned}\int_{\mathcal{C}} y^{1-2\alpha}\nabla\overline{U}\cdot\nabla\psi dxdy = & \int_\Omega \overline{U}(x,0)(\lambda-\overline{U}(x,0))\psi(x,0)dx\\ \geq & \int_\Omega \overline{U}(x,0)(\lambda-\overline{U}(x,0)-bV(x,0))\psi(x,0)dx,\end{aligned}$$

for all $V\in[\underline{V},\overline{V}]$.

On the other hand, observe that if $V\in[\underline{V},\overline{V}]$, then $V\leq\Theta_{[\beta,\mu]}$; and so,

$$V(x,0)\leq\theta_{[\beta,\mu]}.$$

Hence, for $\psi\in\mathcal{H}_0^\alpha(\mathcal{C})$, $\psi\geq 0$

$$\begin{aligned}\int_{\mathcal{C}} y^{1-2\alpha}\nabla\underline{U}\cdot\nabla\psi dxdy = & \int_\Omega \underline{U}(x,0)(\lambda-\underline{U}(x,0)-b\theta_{[\beta,\mu]})\psi(x,0)dx\\ \leq & \int_\Omega \overline{U}(x,0)(\lambda-\overline{U}(x,0)-bV(x,0))\psi(x,0)dx,\end{aligned}$$

for all $V\in[\underline{V},\overline{V}]$.

In a completely similar way, we can proceed with $\underline{V}$ and $\overline{V}$.

Finally, observe that thanks to (6.2) we have that $\underline{U}>0$ and $\underline{V}>0$. Moreover, since $b,c>0$ then $\underline{U}\leq\overline{U}$ and $\underline{V}\leq\overline{V}$ in $\mathcal{C}$.

b) Assume that $b>0$, $c<0$ and (6.3). Now, we take as pair of sub-supersolution

$$(\underline{U},\overline{U})=(\Theta_{[\alpha,\lambda-b\overline{V}(x,0)]},\Theta_{[\alpha,\lambda]}),\quad (\underline{V},\overline{V})=(\Theta_{[\beta,\mu]},\Theta_{[\beta,\mu-c\theta_{[\alpha,\lambda]}]}).$$

First, since $b>0$ and $c<0$ it is clear that $\underline{U}\leq\overline{U}$ and $\underline{V}\leq\overline{V}$, and thanks to (6.3) we get that $\underline{U}>0$ and $\underline{V}>0$.

It is not hard to show that $\underline{V}$ and $\overline{U}$ are sub-supersolution. Consider $\overline{V}$. We have that for $\phi\in\mathcal{H}_0^\alpha(\mathcal{C})$, $\phi\geq 0$

$$\begin{aligned}\int_{\mathcal{C}} y^{1-2\alpha}\nabla\overline{V}\cdot\nabla\phi dxdy = & \int_\Omega \overline{V}(x,0)(\mu-\overline{V}(x,0)-c\theta_{[\alpha,\lambda]})\phi(x,0)dx\\ \geq & \int_\Omega \overline{V}(x,0)(\mu-\overline{V}(x,0)-cU(x,0))\phi(x,0)dx,\end{aligned}$$

for all $U \in [\underline{U}, \overline{U}]$ because $c < 0$.

Finally, we consider $\underline{U}$. In this case, we have

$$\begin{aligned}\int_{\mathcal{C}} y^{1-2\alpha} \nabla \underline{U} \cdot \nabla \phi dx dy = & \int_{\Omega} \underline{U}(x,0)(\lambda - \underline{U}(x,0) - b\overline{V}(x,0))\phi(x,0)dx \\ \leq & \int_{\Omega} \overline{U}(x,0)(\lambda - \overline{U}(x,0) - bV(x,0))\phi(x,0)dx,\end{aligned}$$

for all $V \in [\underline{V}, \overline{V}]$.

c) Assume $b, c < 0$, $bc < C(\alpha, \beta)$ and (6.2). Take

$$(\underline{U}, \overline{U}) = (\Theta_{[\alpha,\lambda - b\theta_{[\beta,\mu]}]}, M_1 E_\alpha), \quad (\underline{V}, \overline{V}) = (\Theta_{[\beta,\mu - c\theta_{[\alpha,\lambda]}]}, M_2 E_\beta),$$

where M_1, M_2 are positive constants to be chosen and E_α is the unique solution of (4.10). It is easy to show that $\underline{U}$ and $\underline{V}$ are sub-solutions. On the other hand, $\overline{U}$ and $\overline{V}$ are super-solutions provided of

$$M_1 e_\alpha^2 \geq e_\alpha \lambda + b M_2 e_\alpha e_\beta - 1 \quad \text{and} \quad M_2 e_\beta^2 \geq e_\beta \mu + c M_1 e_\alpha e_\beta - 1 \quad \forall x \in \Omega.$$

Since $bc < C(\alpha, \beta)$, we can take M_1 and M_2 large.

□

Remark 6.4. *Conditions (6.2) and (6.3) define a region in the $\lambda - \mu$ plane, they could eventually be empty. There are detailed studies in the case $\alpha = \beta = 1$ of these regions, see for example [8], [16], [17], [11]. This study is out of the scope of this paper, but let us only point out that if $b > 0$ the map*

$$\mu \in [\lambda_1[\beta], \infty) \mapsto \lambda_1[\alpha; b\theta_{[\beta,\mu]}] \in \mathbb{R}$$

is a increasing map, because $\mu \mapsto \theta_{[\beta,\mu]}$ is increasing and $c \mapsto \lambda_1[\alpha; c]$ is also increasing.

Similarly, it is decreasing when $b < 0$.

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