# Equipartitioning triangles

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## Abstract

An intriguing conjecture of Nandakumar and Ramana Rao is that for every convex body $K \subseteq R^2$, and for any positive integer $n$, $K$ can be expressed as the union of $n$ convex sets with disjoint interiors and each having the same area and perimeter. The first difficult case - $n = 3$ - was settled by Bárány, Blagojević, and Szucs using powerful tools from algebra and equivariant topology. Here we give an elementary proof of this result in case $K$ is a triangle, and show how to extend the approach to prove that the conjecture is true for triangles.

## Introduction

Let $K$ be a convex body in the plane. Nandakumar and Ramana Rao [7] noticed that if a ham-sandwich cut for $K$ were rotated through $\pi$ radians - always maintaining a bisection of $K$ - then at some point in this process, $K$ is partitioned into two convex parts with disjoint interiors, and each having the same area and perimeter. A slightly more careful argument using this fact, along with induction, was given to show that for $n = 2^k$, $K$ can always be partitioned into $n$ convex subsets, each with the same area and perimeter. They made the intriguing

**Conjecture 1** *For every $n \in N$ and all convex bodies $K \subseteq R^2$, $K$ is the disjoint union of $n$ convex pieces, each with the same area and perimeter.*

The conjecture describes an $n-$*equipartition* of $K$ (because of the $n$ equal areas) which is in addition *fair*, by virtue of the equal perimeters. Observe that, in the related problems of *cake partitioning* [1] only the perimeter of $\partial K$ is taken into account.

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Bárány et. al. [3], using heavy-duty tools from algebra and equivariant topology settled the case $n = 3$: A $3-$*fan* is a point $P$ in the plane with three rays emanating from it. It is convex if all angles are at most $\pi$. It equipartitions $K$ if the three rays divide $K$ into three regions of equal area, and it is fair, if these regions also have equal perimeter. Bárány et. al. showed that there is a convex $3-$fan that makes a *fair equi-partition* of $K$. Subsequently, Aronov and Hubard [2] and then Karasev [6], showed that the conjecture was true for $n = p^k$, a prime power, and also in dimension $d \geq 2$, with "area" replaced by "volume" and "perimeter", by "surface area". Blagojević and Ziegler found some problems with the proofs in these two papers, so they established the results - and more - using different tools. In the present paper, in an attempt to understand some of the geometric features of this problem and why - or why not - it may be difficult, we use (only) elementary methods to study the conjecture for $R^2$, and when $K$ is a triangle. We call a $3-$fan *interior for K* if the apex $P$ is interior to $K$; otherwise it is *exterior*. In the first case, all three rays play a role in the partition. In the second case, the partition is simply via two chords (which might meet on the boundary of $K$, but not in its interior). Because our results concern only exterior 3-fans, we will state them using chords. A main result of this paper is

**Theorem 1** *Every triangle has a fair equi-partition defined by two disjoint chords.*

More importantly, the ideas used to prove this result can be extended to show that every triangle has a fair $n$-partition defined by $n-1$ disjoint chords.

**Corollary 2** *For every $n \geq 1$, every triangle has a fair, $n$-equipartition using $n-1$ disjoint chords.*

In [7] it was observed that when $K$ is a triangle, it can be covered by $n = k^2$ disjoint triangles similar to $K$, and all with the same area (and perimeter). But Theorem 1 is a new step toward resolving Conjecture 1, if only for triangles.

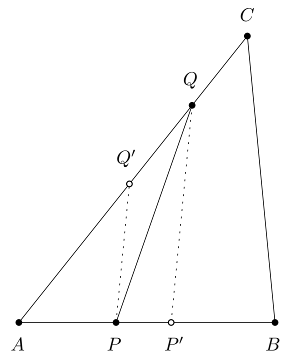


Figure 1: Illustration for Lemma 2.

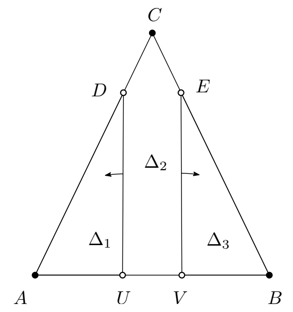


Figure 2: Rotation for an isosceles triangle.

# 1 Some details

In this section we describe some ideas behind the proofs for our results. The full proofs will appear in the actual paper.

We are able to understand this problem in the triangle case partly due to a simple tool that describes how perimeters change when a chord in a partition is moved slightly while still preserving the areas of all regions.

We use a Cartesian coordinate system in the Euclidean plane, and denote by $AB$ and $|AB|$, respectively, the segment defined by points $A$ and $B$ and its length. Vector $\overrightarrow{OP}$ and point $P$ will be identified if the context is clear enough. The list of points $AB \cdots D$ will be used to denote the corresponding polygon and, finally, $\pi(\cdot)$ will be the perimeter of the polygon.

Let $A$, $B$, $C$ be non collinear points, and fix $A$ as the origin of the coordinate system.

**Lemma 3** *Consider points $P \in AB$ and $Q \in AC$ such that $|AP| \leq |AQ|$. Let $P' = tP$ and let $Q' = \frac{1}{t}Q$ (then the area of $APQ$ equals the area of $AP'Q'$).*

1. *$\pi(AP'Q')$ and $\pi(BCQ'P')$ are convex functions of $t$, achieving their minima when $|AP'| = |AQ'|$.*
2. *If we write $\Delta\pi_1 = \pi(AP'Q') - \pi(APQ)$ and $\Delta\pi_2 = \pi(BCQ'P') - \pi(BCQ'P')$ then $|\Delta\pi_1| \geq |\Delta\pi_2|$.*

Consider a unit area triangle $\Delta = ABC$. Without loss of generality, we can assume that the smallest side is $AB$ and that the coordinates of its vertices are $A = (0, 0), B = (b, 0)$, and $C = (c, 2/b)$, with $b > 0$ and $c \geq b/2$, as in the figure above. If we take points $U = (b/3, 0)$ and $V = (2b/3, 0)$, $\Delta$ is partitioned into $\Delta_1 = CAU, \Delta_2 = CUV$, and $\Delta_3 = CVB$, *all with the same area.* The goal is to rotate the chords $CU$ and $CV$ maintaining equality of all three areas, but in such a way as to force all three perimeters to coincide. To describe this process unambiguously, we place points $D$ and $E$ on the boundary of $\Delta$. Initially both points are placed on $C$. We then manipulate the chords $DU$ and $EV$ by moving the endpoints along the boundary of $\Delta$, thus altering the three sets in the partition. We use the notation $\pi_i$ to denote the perimeter of region $i$.

The case where $x = b/2$, and $\Delta$ is isosceles, is easiest. Here, we move chord $DU$ counter-clockwise (maintaining the area of $\Delta_1 = DAU$), and chord $EV$ clockwise (maintaining the area of $\Delta_3 = EVB$) until $U$ and $V$ coincide at $F = (b/2, 0)$ (see Figure 2). During this process the middle region is a pentagon $\Delta_2 = CDUVE$, ending at $\Delta_2 = CDFE$. If we also keep $U + V = (b, 0)$, there will be a position where the partition is fair, by the intermediate value theorem, since $\Delta_1$ and $\Delta_3$ initially have equal perimeters, but larger than that of $\Delta_2$, and at the end, $\pi_1 = \pi_3 \leq \pi_2$, by virtue of $AB$ being the smallest side of $\Delta$.

When $\Delta$ is not isosceles, we rotate $DU$, again maintaining the area, till the perimeters of two regions are equal. Observe that, during the rotation, both $\pi_1$ and $\pi_2$ decrease. If $b/2 < c < 2b/3$, because we start with perimeters $\pi_1 > \pi_3 > \pi_2$, we must reach case 1 in Figure 3, where $\pi_1 = \pi_3 > \pi_2$. If $c \geq 2b/3$, we start with perimeters $\pi_1 > \pi_2 \geq \pi_3$. When rotating $DU$ we can get $\pi_2 = \pi_3 < \pi_1$ (case 2) or $\pi_1 = \pi_2 > \pi_3$ (case 3).

Using Lemma 3 we know how to proceed in each case. For case 1, we rotate the chord $EV$ clockwise, and the chord $DU$ counterclockwise, preserving equality for the areas of regions 1 and 3. For case 2, we rotate the chords in the same way, maintaining now equal areas for regions 2 and 3. Finally, for case 3 both chords are rotated counterclockwise, keeping areas of regions 1 and 2 equal. Theorem 1 will be proven if we show that during this rotation, equality of the three perimeters is achieved.

For cases 1 and 2, the key is to observe that the foot of the left chord reaches the midpoint of $AB$ first. If $U$ is the midpoint of $AB$, we consider the triangle $UBE'$, congruent with $AUD$ (see Figure 4,left). From this, it is not hard to see that $\Delta_3$ has to be $VBE$, with $V \in UB$ and that $\pi_1 < \pi_2$.

For case 3, consider a partition into three pieces of

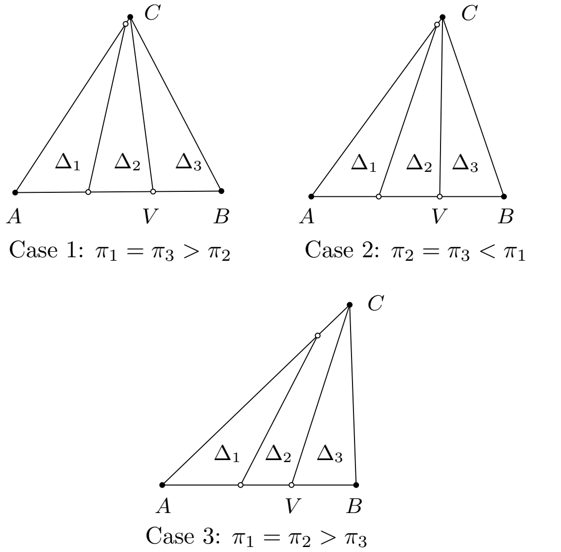


Figure 3: After the first rotation, two perimeters are equal.

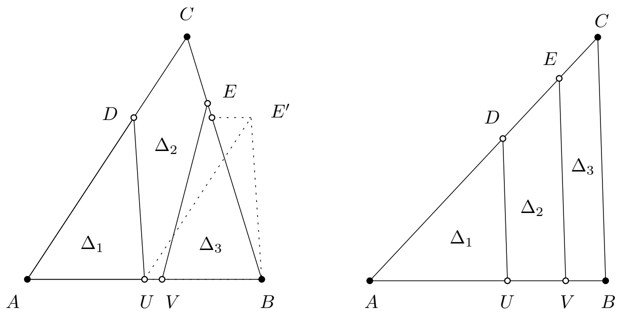


Figure 4: Situation at the end of the rotation.

equal area (but different perimeters) using two chords parallel to $BC$ (see Figure 4,right). In this setting, it can be shown that $\pi_1 + \pi_2 < 2\pi_3$. Therefore, if we consider now the chord that divides $AVE$ into two pieces of equal area and perimeter, from Lemma 3 it follows that perimeters of regions 1 and 2 are smaller and in that situation we must have $\pi_1 = \pi_2 < \pi_3$.

The approach of the arguments above can be applied to prove the existence of a fair $n$-equipartition of every triangle. Take points $V_i = (ib/n, 0)$ and points $U_i, i = 0, \ldots, n$; initially all $U_i = C$. The $n-1$ chords $U_iV_i, i = 1, \ldots, n-1$ partition $\Delta$ into $n$ triangles ($\Delta_i = U_{i-1}V_{i-1}V_i, i = 1, \ldots, n$), of equal area. We order the chords left to right, and so the chord with foot at $V_i$ is the $i$-th chord. The chords for which $V_i < c$ are called *left chords*, and the rest are the *right chords*.

We start by rotating the first chord counterclockwise, until the perimeter of the first region is equal, either to the perimeter of the second region, or to the perimeter of the last one. In the first case, we continue by rotating the first and the second chord (both counterclockwise), maintaining equality of the perimeters of the first two regions. In the second case, we continue by rotating the first chord counterclockwise, and the last chord clockwise, again maintaining equal perimeters for the involved regions. This process can be iterated, and when we reach the "central region" (the one in between the last left chord and the first right chord), we will have cases analogous to the ones in Figure 3. We proceed in the same way here, and it can be shown than equality of all perimeters is obtained before reaching a critical situation. The proof of this fact is more involved and will appear in the full version of this paper.

# 2 Discussion

It is clear that an equilateral triangle has two distinct fair 3-equipartitions that are interior (as well as three exterior ones - for each vertex, the process outlined in the previous section produces an equi-partition with an exterior 3-fan). It is easy to see that skinny triangles do not have fair interior 3-equipartitions. It would be interesting to characterize which triangles have equipartitions both exterior and interior, and which have only exterior ones.

It is easy to see that fair equipartitions of disks have to be radial, with the vertex at the center. Therefore, we cannot expect to extend the approach of this paper to the general case. Nevertheless, some convex bodies may have fair $n$-equipartitions produced by $n-1$ disjoint chords. It would be interesting to try to characterize this family.

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