Guarding the vertices of thin orthogonal polygons is NP-hard

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Abstract

An orthogonal polygon of P is called "thin" if the dual graph of the partition obtained by extending all edges of P towards its interior until they hit the boundary is a tree. We show that the problem of computing a minimum guard set for the vertices of a thin orthogonal polygon is NP-hard either for guards lying on the boundary, or on vertices or anywhere in the polygon.

Introduction

The classical art gallery problem for a polygon P asks for a minimum set of points $\mathcal{G}$ in P such that every point in P is seen by at least one point in $\mathcal{G}$ (the guard set). Many variations of art gallery problems have been studied over the years to deal with various types of constraints on guards and different notions of visibility. In the general visibility model, two points p and q in a polygon P *see* each other if the line segment $\overline{pq}$ contains no points of the exterior of P. The set $V(v)$ of all points of P visible to a point $v \in P$ is the *visibility region* of v. A guard set $\mathcal{G}$ for a set $S \subseteq P$ is a set of points of P such that $S \subseteq \cup_{v \in \mathcal{G}} V(v)$. Two points v_i and v_j are equivalent for the visibility relation if $V(v_i) \cap S = V(v_j) \cap S$. If $V(v_j) \cap S \subset V(v_i) \cap S$ then v_i strictly dominates v_j, and v_i can replace v_j in an optimal guard set of S. Guards that may lie anywhere inside P are called *point guards* whereas *vertex guards* are restricted to lie on vertices and *boundary guards* on the boundary. Combinatorial upper and lower bounds on the number of necessary guards are known for specific settings (for surveys, refer to e.g. [8, 10]). The fact that some art gallery problems are NP-hard [5, 9] motivates the design of heuristic and metaheuristic methods for finding approximate solutions and also the study of more specific classes of polygons where some guarding problems may be tractable [1, 2, 3, 6]. In this paper, we address the problem of guarding *the vertices* of orthogonal polygons, which is known to be NP-hard for generic orthogonal polygons [4]. We show that the problem is NP-hard also for the family of *thin orthogonal polygons*, which consists of the orthogonal polygons such that the dual graph of the corresponding *grid partition* $\Pi_{HV}(P)$ is a tree. $\Pi_{HV}(P)$ is obtained by adding all horizontal and vertical cuts incident to the reflex vertices of P (see Fig. 1). Our proof is inspired in [4]

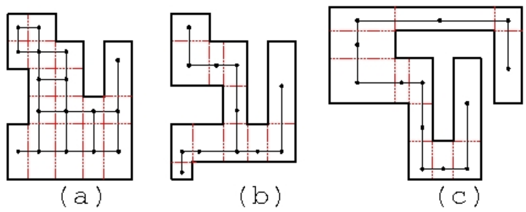

Figure 1: Orthogonal polygons, grid partitions and dual graphs: (a) $\Pi_{HV}(P)$ and its dual graph in general; (b) a thin orthogonal polygon; (c) a thin orthogonal polygon that is a path orthogonal polygon.

although the need to obtain thin orthogonal polygons led to novel aspects in the construction. The class of thin orthogonal polygons contains the class of thin polyomino trees introduced in [1], for which the authors conjecture that the guarding problem under the general visibility model has a polynomial-time (exact) algorithm. To the best of our knowledge, this problem is open. In [12], we give a linear-time algorithm for computing an optimal vertex guard set for any given path orthogonal polygon (for which the dual graph of $\Pi_{HV}(P)$ is a path graph), and prove tight lower and upper bounds of $\lceil n/6 \rceil$ and $\lfloor n/4 \rfloor$ for the optimal solution for the subclass where all horizontal and vertical cuts intersect the boundary at Steiner points. Since the *thin grid orthogonal polygons* belong to this class, our work extends results previously known for the spiral thin grid orthogonal polygons and the MinArea grid orthogonal polygons [6] (for which the minimum vertex guard sets have exactly $\lfloor n/4 \rfloor$ and $\lceil n/6 \rceil$ guards) and somehow explains why the MinArea grid orthogonal polygons were considered representative of extremal behaviour [11]. The result that supports our proof allows us to conclude that a minimum guard set for the vertices of a path orthogonal polygon can be found in linear-time.

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In rest of the paper, we show that computing a minimum guard set for *the vertices* of a thin orthogonal polygon (GVTP) is NP-hard, either for boundary guards, vertex guards or point guards.

1 Hardness for boundary guards

Theorem 1 GVTP *for thin orthogonal polygons is NP-hard for boundary guards.*

For the proof, we define a reduction from the vertex-cover problem in graphs (VC) to GVTP with boundary-guards. VC, known to be NP-complete, is the problem of deciding whether a graph $G = (V, E)$ has a vertex-cover S of size $|S| \leq k$, for k integer. A vertex-cover of G is a subset $S \subseteq V$ such that for each edge $(u, v) \in E$, either $u \in S$, or $v \in S$, or both.

The thin orthogonal polygon we construct for a given graph $G = (V, E)$ is a large square with $|E|$ tiny d-gadgets attached to its bottom. In Fig. 2 we sketch this construction and in Fig. 3 we present the *double gadget* (d-gadget) defined for the proof.

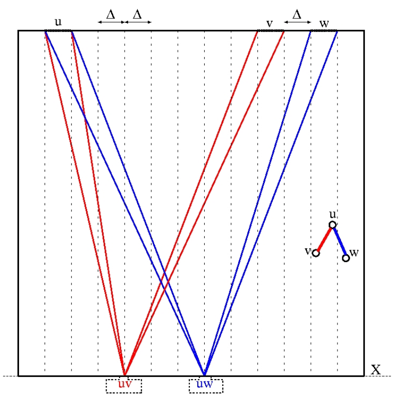

Figure 2: From VC to GVTP with boundary guards: the representation of $G = (\{u, v, w\}, \{(u, v), (u, w)\})$; the edges of G are mapped to the points uv and uw (that will be replaced by tiny d-gadgets) and the vertices are mapped to the segments u, v and w.

We define the side-length of this square to be $L\Delta$, with $L = 1 + 2|V| + 3|E|$ and $\Delta = 10L$. Considering $V = \{v_1, v_2, \dots, v_n\}$ sorted, we denote by E_i^+ the subset of all edges $(v_i, v_j) \in E$ such that $i < j$, also sorted by increasing value of j. In the construction we follow these orderings: for each i, we represent v_i by a segment of length Δ on the top edge of the square and the edges in E_i^+ as middle points of $|E_i^+|$ consecutive segments of length 2Δ on the bottom edge, placed between the projections of v_i and v_{i+1}, and with separation gaps of length Δ between each other. The square is implicitly divided into L slabs of length Δ, and we leave the first slab empty and an empty slab between consecutive items.

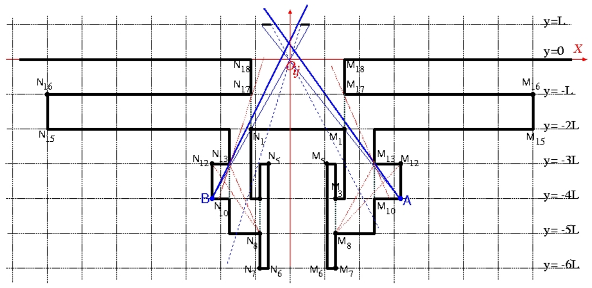

Figure 3: A sketch of the d-gadget Ξ_{ij} defined for the edge (v_i, v_j). The vertices on the left side are labelled from N_1 to N_{18} in CW order and on the right side are labelled from M_1 to M_{18} in CCW order. $B \equiv N_{11}$ and $A \equiv M_{11}$ are the two distinguished vertices.

The d-gadget associated to the edge $(v_i, v_j) \in E_i^+$, denoted by Ξ_{ij}, is defined as follows. Let O_{ij} be the point that represents the edge (v_i, v_j) and $\overline{A_iB_i}$ and $\overline{A_jB_j}$ the segments associated to v_i and v_j. Together with O_{ij}, these segments define two visibility cones with apex O_{ij}. By a slight perturbation, we can decouple the two cones, and move the new apexes to the distinguished vertices (B and A) of a tiny d-gadget Ξ_{ij}. The structure of this gadget will fix segment $\overline{A_iB_i}$ (resp. $\overline{A_jB_j}$) as the portion of the boundary of the polygon that A (resp. B) sees above line X (i.e, above the gadget). We use VC instead of the minimum 2-interval piercing problem used in [4] in order to be able to control the aperture of visibility cones and also the structure of the thin orthogonal polygon obtained in the reduction. Some of the vertices of a d-gadget can only be guarded by a *local guard* (i.e., a guard below line X), for instance, the vertices M_{16}, M_{12}, M_8, M_7 and M_5 on its right part and N_{16}, N_{12}, N_8, N_7 and N_5 on the left part. For every d-gadget, at least three local boundary-guards will be needed to guard these vertices and no three such guards can see both A and B if they see all these vertices. Moreover, one can always locate three local boundary-guards that see all the gadget vertices other than A (namely, at N_8, N_1 and M_8) or other than B (namely, at N_8, M_1 and M_8). Another guard is required to guard the unguarded vertex but it does not need to be local. As we will see, this guard can be located on the portion of the top edge of the polygon seen from the unguarded vertex.

We define the coordinates of the vertices of Ξ_{ij} w.r.t. a cartesian system $\mathcal{R}_{O_{ij}}$ with origin at O_{ij}. By construction, the x-coordinates of the points A_i, B_i and O_{ij} w.r.t. a cartesian system fixed at the bottom left corner of the large square are given by

$$\begin{aligned} x'_{A_i} &= (2i - 1 + 3\textstyle\sum_{k<i} |E_k^+|)\Delta \\ x'_{B_i} &= x'_{A_i} + \Delta \\ x'_{O_{ij}} &= x'_{B_i} + 2\Delta + 3\Delta\,|E_i^+ \cap \{(v_i, v_{j'}) : j' < j\}| \end{aligned}$$

and, consequently, if we define x_i and x_j as $x_i = (x'_{O_{ij}} - x'_{B_i})/\Delta$ and $x_j = (x'_{A_j} - x'_{O_{ij}})/\Delta$, then, w.r.t. the cartesian system $\mathcal{R}_{O_{ij}}$, we have

$$\begin{array}{ll} A_i = (-(x_i+1)\Delta,\ L\Delta) & A_j = (x_j\Delta,\ L\Delta) \\ B_i = (-x_i\Delta,\ L\Delta) & B_j = ((x_j+1)\Delta,\ L\Delta) \end{array}$$

for integers $x_i \geq 2$ and $x_j \geq 2$. Then, we define A and B as the intersection points of the supporting lines of $\overrightarrow{O_{ij}A_i}$ and $\overrightarrow{O_{ij}B_j}$ with the line $y = -4L$, that is, as $A = (4x_i + 4, -4L)$ and $B = (-4x_j - 4, -4L)$. So, the rays $\overrightarrow{AA_i}$ and $\overrightarrow{BB_j}$ share the supporting lines of the initial rays $\overrightarrow{O_{ij}A_i}$ and $\overrightarrow{O_{ij}B_j}$. The aperture of the visibility cone $\mathcal{C}_A = cone(A, \overline{A_iB_i})$ is determined by the vertices M_1 and M_{13}. We selected M_1 as the intersection of $\overrightarrow{AA_i}$ with the line $y = -2L$ and M_{13} as the intersection of $\overrightarrow{AB_i}$ with the line $y = -3L$. Therefore, $M_1 = (2x_i+2, -2L)$ and $M_{13} = (\tau_i, -3L)$, with $\tau_i = 3x_i + 3 + \frac{\Delta}{\Delta+4}$, because the straight lines AA_i and AB_i are given by the following equations.

$$\begin{aligned} AA_i : y &= \frac{-L}{x_i+1}x \\ AB_i : y &= \frac{-L(\Delta+4)}{x_i(\Delta+4)+4}x + \frac{4L\Delta}{x_i(\Delta+4)+4} \end{aligned}$$

Similarly, the vertices N_1 and N_{13} determine the aperture of the visibility cone $\mathcal{C}_B = cone(B, \overline{A_jB_j})$, being $N_1 = (-2x_j - 2, -2L)$ the intersection of $\overrightarrow{BB_j}$ with $y = -2L$ and $N_{13} = (\tilde{\tau}_j, -3L)$ the intersection of $\overrightarrow{BA_j}$ with $y = -3L$, with $\tilde{\tau}_j = -3x_j - 3 - \frac{\Delta}{\Delta+4}$. The coordinates of the vertices of Ξ_{ij} are

$$\begin{array}{ll} M_1 = (2x_i+2, -2L) & M_2 = (2x_i+2, -4L) \\ M_3 = (2x_i+1, -4L) & M_4 = (2x_i+1, -3L) \\ M_5 = (2x_i, -3L) & M_6 = (2x_i, -6L) \\ M_7 = (2x_i+1, -6L) & M_8 = (2x_i+1, -5L) \\ M_9 = (\tau_i, -5L) & M_{10} = (\tau_i, -4L) \\ A = (4x_i+4, -4L) & M_{12} = (4x_i+4, -3L) \\ M_{13} = (\tau_i, -3L) & M_{14} = (\tau_i, -2L) \\ M_{15} = (7L, -2L) & M_{16} = (7L, -L) \\ M_{17} = (2x_i+2, -L) & M_{18} = (2x_i+2, 0) \end{array}$$

with the $N_k = (-\alpha x_j - \beta, \gamma)$ iff $M_k = (\alpha x_i + \beta, \gamma)$, for $1 \leq k \leq 18$. Therefore, the coordinates of the vertices can be defined by rational numbers represented by pairs of integers bounded by a quadratic polynomial function on the size of the graph.

We can prove that: the dual graph of the grid partition of the resulting polygon is a tree; M_{16}, M_{12}, M_8, M_7, M_5, and N_{16}, N_{12}, N_8, N_7 and N_5 require local guards; the boundary of Ξ_{ij} imposes no restriction on the propagation of the corresponding visibility cones $\mathcal{C}_A$ and $\mathcal{C}_B$; the unique point on the boundary of Ξ_{ij} that sees both A and N_{16} is M_1 (similar for B, M_{16} and N_1); the three local guards N_8, N_1 and M_8 jointly see all the gadget vertices other than A (similar for N_8, M_1 and M_8 and B). Lemma 2 states the final result we need to conclude the proof and can be shown as Lemma 2.2. of [4].

Lemma 2 *The thin orthogonal polygon P that is obtained can be guarded by $3|E| + k$ boundary guards if and only if the there is a vertex-cover of size k for the instance graph $G = (V, E)$.*

2 Hardness for vertex guards

Theorem 3 GVTP *is NP-hard for thin orthogonal polygons with vertex guards.*

For the proof, we can adapt the previous construction, following the idea of [4], as sketched in Fig. 4.

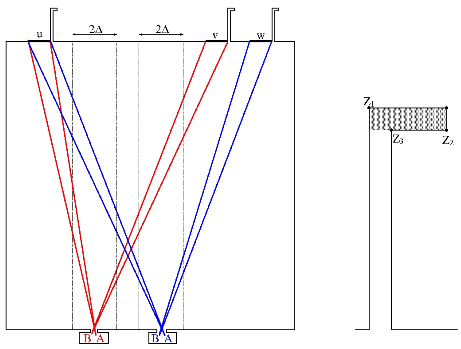

Figure 4: The reduction from VC to GVTP with vertex guards for $G = (\{u, v, w\}, \{(u, v), (u, w)\})$. Ear gadgets are attached to the right endpoints of the segments. Each ear gadget requires a local guard on a vertex of the shaded region (to guard Z_2).

We consider the polygon obtained previously and attach a tiny *ear gadget* to the right endpoint of each line segment $\overline{A_iB_i}$, for each $v_i \in V$. The local vertices of the ear gadget attached to B_j, w.r.t. the cartesian system fixed at the bottom left corner of the large square, can be defined as

$$\begin{aligned} Z_1 &= ((x'_j+1)\Delta,\ L(\Delta+1)+1) \\ Z_3 &= ((x'_j+1)\Delta+1,\ L(\Delta+1)) \\ Z_2 &= ((x'_j+1)\Delta+L,\ L(\Delta+1)) \end{aligned}$$

and $((x'_j+1)\Delta + L,\ L(\Delta+1)+1)$. The separation slabs guarantee that the dual graph of $\Pi_{HV}(P)$ for the new polygon P is still a tree, as required. The ear gadgets are defined in such a way that the vertex A of Ξ_{ij} cannot see any vertex of an ear-gadget except for B_i. Otherwise, A would see points on the boundary of P arbitrarily closed to B_i but to the right of B_i, which is impossible by the definition of the visibility cone $\mathcal{C}_A$. The height of the ear gadget prevents B from seeing any local vertex of the ear-gadget attached to A_jB_j. For each $j \geq 2$, it is sufficient to guarantee that, for all Ξ_{ij}, the intersection point of the ray $\overrightarrow{BB_j}$ with the vertical edge incident to the vertex labeled Z_3 is below Z_3. This holds for all i if it holds for B in the rightmost d-gadget $\Xi_{i'j}$, since the rays $\overrightarrow{BB_j}$ are sorted by slope around B_j.

Each ear-gadget needs a local guard that must be located in one of the vertices of the shaded region and none of these vertices sees a local vertex of a d-gadget. This means that these guards cannot replace any guard located in a segment. Since any guard located on a segment can move to the segment right endpoint to become a vertex-guard, without loss of visibility, we can adapt the proof of Lemma 2 to show Lemma 4.

Lemma 4 *The thin orthogonal polygon P that is obtained can be guarded by $|V|+3|E|+k$ boundary guards if and only if the there is a vertex-cover of size k for the instance graph $G = (V, E)$.*

3 Hardness for point guards

Theorem 5 GVTP *is NP-hard for thin orthogonal polygons with point guards.*

For the proof, we construct a reduction from the minimum line cover problem (MLCP), as in [4]. MLCP is NP-hard [7]. Given a set $\mathcal{L} = \{l_1, \ldots, l_n\}$ of lines in the plane, MLCP is the problem of finding a set of points of minimum cardinality such that each line $l \in \mathcal{L}$ contains at least one point in that set.

Without loss of generality, we consider that $\mathcal{L}$ contains neither vertical nor horizontal lines. The polygon constructed for the reduction is obtained by attaching *single-gadgets* (called *s-gadgets*) to a bounding box $\mathcal{B}(\mathcal{L})$ that contains all intersection points of pairs of lines in $\mathcal{L}$ in its interior. The idea of this construction is sketched in Fig. 5.

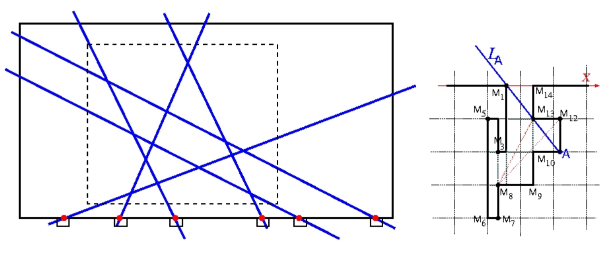

Figure 5: The reduction from MLCP to GVTP with guards anywhere. Each tiny box on the bottom represents an s-gadget (note that not all lines intersected the bottom edge of the dashed bounding box). On the right, an s-gadget in detail (the vertices are labelled from M_1 to M_{14}, in CCW order, $A \equiv M_{11}$).

In order to guarantee that a thin orthogonal polygon is obtained, we define a new type of s-gadget, sketched on Fig. 5, where M_1 and M_{13} reduce the visibility cone $\mathcal{C}_A$ to the line L_A. Moreover, we had to restrict the locations of s-gadgets to the bottom edge of $\mathcal{B}(\mathcal{L})$, in contrast to [4]. This can be done because, for a sufficiently large bounding box, all lines will intersect the bottom edge of $\mathcal{B}(\mathcal{L})$, as there are no horizontal lines in $\mathcal{L}$. At least a local guard is needed for each s-gadget. As for the d-gadgets, taking into account the relative positions of intersections of the lines with the bottom line (i.e., of vertices M_1), and their slopes, we can define the vertices of the tiny s-gadget in such a way that M_8 sees M_{12} and M_7, and all local vertices except for A. We can conclude that the vertices of P can be guarded by $n + k$ guards if and only if there is a cover for $\mathcal{L}$ of size k.

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