

Continuous surveillance of points by rotating floodlights

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Abstract

Let P and F be sets of $n \geq 2$ and $m \geq 2$ points in the plane, respectively, so that $P \cup F$ is in general position. We study the problem of finding the minimum angle $\alpha \in [2\pi/m, 2\pi]$ such that one can install at each point of F a stationary rotating floodlight with illumination angle α , initially oriented in a suitable direction, in such a way that, at all times, every target point of P is illuminated by at least one light. All floodlights rotate at unit speed and clockwise. We give an upper bound for the 1-dimensional problem and present results for some instances of the general problem. Specifically, we solve the problem for the case in which we have two floodlights and many points, and give an upper bound for the case in which there are many floodlights and only two target points.

1 Introduction

Illumination problems are well known in Discrete and Computational Geometry [4]. Let P be a set of $n \geq 2$ targets and F a set of $m \geq 2$ floodlights, both defined in the plane. We assume that $P \cup F$ is in general position and that all floodlights rotate clockwise at unit speed. We say that F covers P with illumination angle $\alpha \geq 2\pi/m$ if there is a suitable initial orientation of each light so that, at all times, each target point of P is illuminated by at least one floodlight. We

consider the problem of finding the minimum angle $\alpha = \alpha(P, F) \in [2\pi/m, 2\pi]$ and initial orientation of each floodlight so that F covers P .

Our general problem can be considered a discretization of the one studied by Kranakis et al. [2]. Given the locations of m floodlights (i.e. antennae) and a region, their problem asks to schedule the lights so that the entire region is covered at all times. The scheduling of static floodlights for covering a given region was first considered by Bose et al. [1]. Research related to our problem can be found in a number of domains, including art gallery and related problems, multi-target tracking, and multi-robot surveillance tasks [3, 4]. A complete review of these fields can be found in [4].

We present results for some cases of our general problem: the elements of $P \cup F$ are located on a given line (Section 2), the two-dimensional version with two floodlights (Section 3), and the problem in the plane with two target points (Section 4).

Given points u, v in the plane, $\ell(u, v)$ denotes the line containing both u and v . We identify any floodlight by the point where it is installed. For any floodlight f , at any instance of time, the region illuminated by f is delimited by rays f^- and f^+ , starting at f^- and ending at f^+ in the clockwise direction. We say that we configure f with angle β if the angle between f^+ and the positive x -axis is equal to β . Given that all floodlights rotate at the same speed, it suffices to consider only the interval of time $[0, 2\pi)$.

2 Points and floodlights on a line

We first consider the case in which the points of P and the floodlights of F lie on a line \mathcal{L} , say the x -axis. Kranakis et al. [2] considered the case in which the floodlights are located on a line. They showed that the entire line can be illuminated by m rotating floodlights using illumination angle $3\pi/m$ and this bound is tight. This can be viewed as a special case of our problem where $n \geq m + 1$ and each of the $m + 1$ segments determined by F contains at least one point of

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P . We consider other cases and show that the illumination angle is smaller than $3\pi/m$ for some of them.

Partition P into $k - 1$ ($k \geq 2$) maximal intervals s_1, s_2, \dots, s_{k-1} , from left to right, each of which contains elements of P but no elements of F . Let F_1 denote the elements of F to the left of s_1 , F_i ($i = 2, \dots, k-1$), the elements of F between s_{i-1} and s_i , and F_k , the elements of F to the right of s_{k-1} . Let $m_i = |F_i|$ for $i = 1, \dots, k$. Observe that $m_1, m_k \geq 0$, $m_i \geq 1$ for $i = 2, \dots, k-1$, and $m_1 + m_2 + \dots + m_k = m$.

Lemma 1 *Two floodlights f_1 and f_2 with illumination angle α , belonging to the same set among $F_1 \cup F_k, F_2, F_3, \dots, F_{k-1}$, can be configured so to cover P during 2α time in total. Furthermore, if two floodlights of F cover P with angle $\alpha < 3\pi/2$, then they must belong to a same set among $F_1 \cup F_k, F_2, F_3, \dots, F_{k-1}$.*

Proof. Suppose that both f_1 and f_2 belong to a set F_i ($i = 1, \dots, k$), and assume w.l.o.g. that f_1 is to the left of f_2 . Configure f_1 with angle zero and f_2 with angle π (see top of Figure 1a). At time $t = \pi$ the configuration of f_1 and f_2 is as shown in the bottom of Figure 1a. Since there is no element of P in the segment connecting f_1 and f_2 then all elements of P are illuminated during intervals $[0, \alpha]$ and $[\pi, \pi + \alpha]$, 2α time in total.

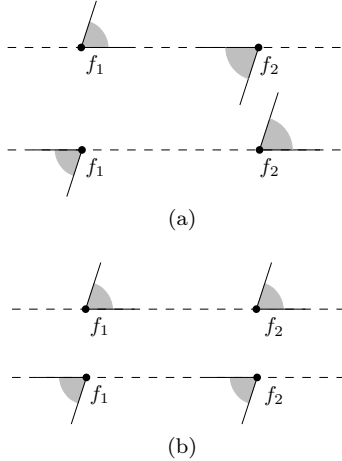


Figure 1: Proof of Lemma 1: (a) Case where f_1 and f_2 belong to a same set F_i . (b) Case where $f_1 \in F_1$ and $f_2 \in F_k$.

Suppose now that $f_1 \in F_1$ and $f_2 \in F_k$. Configure both f_1 and f_2 with angle zero (see top of Figure 1b). At time $t = \pi$ the configuration of f_1 and f_2 is as shown in the bottom of Figure 1b. Since all elements of P belong to the segment connecting f_1 and f_2 then all elements of P are illuminated during intervals $[0, \alpha]$ and $[\pi, \pi + \alpha]$, 2α time in total.

For the second part of the lemma, let f and g be the two lights that cover P . Clearly $\alpha \geq \pi$. Assume, w.l.o.g., that $g \notin F_1 \cup F_k$ and that f is to the left of g . Also, let $Q \subseteq P$ denote the set of targets to the

right of f . Light f alone can cover Q for an interval of length α , while g alone can cover Q for $\alpha - \pi$ only. Since $\alpha < 3\pi/2$, an interval of length $2\pi - \alpha > \pi/2$ is unaccounted for by f , but g can pick up at most $\alpha - \pi < \pi/2$ of this. Hence, f and g must belong to the same set. The claim does not hold if $\alpha \geq 3\pi/2$. \square

Lemma 2 *Three floodlights f_1, f_2 , and f_3 with illumination angle $\alpha < \pi$, can be configured so that, together, they cover the whole line \mathcal{L} (hence, P) during 2α time in total.*

Proof. The proof can be obtained from [2]. Assume, w.l.o.g., that f_1, f_2 , and f_3 appear in this order from left to right. Configure f_1, f_2 , and f_3 with angle zero, π , and zero, respectively (top of Figure 2). At time $t = \pi$ the configuration is as shown at the bottom of Figure 2. During interval $[0, \alpha]$ the line is illuminated by f_1 and f_2 , and during interval $[\pi, \pi + \alpha]$, by f_2 and f_3 . The result follows. \square

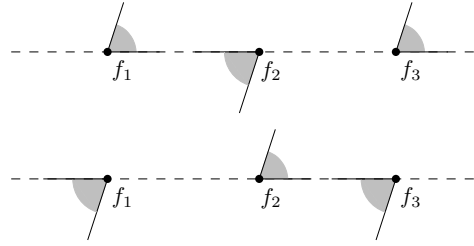


Figure 2: Proof of Lemma 2

Theorem 3 *If all floodlights belong to the same set among $F_1 \cup F_k, F_2, F_3, \dots, F_{k-1}$, then $\alpha(P, F) = 2\pi/m$, which is optimal. Otherwise, $\alpha(P, F)$ satisfies:*

$$\alpha(P, F) \leq \min \left\{ \frac{3\pi}{m}, \frac{2\pi}{m - Q + 2\lfloor \frac{Q}{3} \rfloor} \right\} \quad (1)$$

where Q denotes the number of odd numbers in the set $\{m_1 + m_k, m_2, \dots, m_{k-1}\}$.

Proof. Obviously $\alpha(P, F) \geq 2\pi/m$ in all cases. All lights belong to the same set iff $k = 2$, or $k = 3$ and $m_1 = m_3 = 0$. In both cases, illumination angle $\alpha = 2\pi/m$ is sufficient. Assume first that $k = 2$ and let $F_1 = \{f_1, \dots, f_{m_1}\}$ and $F_2 = \{f'_1, \dots, f'_{m_2}\}$. Floodlight f_i is configured with angle $(i-1)\alpha$ for $i = 1, \dots, m_1$, and floodlight f'_j , with angle $\pi - j\alpha$ for $j = 1, \dots, m_2$ (see Figure 3). Then, at any time $t \in [0, m_1\alpha]$ P is covered by a member of F_1 and, at any time $t \in [m_1\alpha, 2\pi]$, by a member of F_2 . Assume now that $k = 3$ and $m_1 = m_3 = 0$. Then, $F = F_2 = \{f_1, \dots, f_m\}$. By configuring f_i with angle $2i\pi/m$, P is covered by F . This proves the optimal result when all lights belong to the same set.

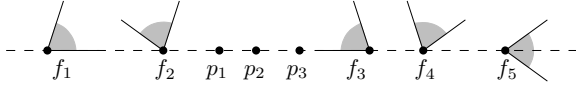


Figure 3: Two groups of floodlights $F_1 = \{f_1, f_2\}$ and $F_2 = \{f'_1, f'_2, f'_3\} = \{f_3, f_4, f_5\}$, where $m_1 = 2$ and $m_2 = 3$, and the configuration with angle $\alpha = 2\pi/5$.

If neither of the two cases above occurs, we configure the floodlights to satisfy inequality (1) as follows. We consider the case $m_1 = m_k = 0$ and $m_i = 1$ ($i = 2, \dots, k-1$) separately because the result follows immediately from [2], since our problem is equivalent to illuminating the whole x -axis. In this case $m = Q$ and $\alpha(P, F) = 3\pi/m = \min\{\frac{3\pi}{m}, \frac{2\pi}{m-Q+2\lfloor\frac{Q}{3}\rfloor}\}$ which is also optimal.

For the remaining cases, we can only obtain an upper bound on the optimal illumination angle. We proceed as follows. Pair the elements of F into $N = \lfloor\frac{m_1+m_k}{2}\rfloor + \lfloor\frac{m_2}{2}\rfloor + \dots + \lfloor\frac{m_{k-1}}{2}\rfloor = \frac{m-Q}{2}$ pairs $(f_{1,1}, f_{1,2}), (f_{2,1}, f_{2,2}), \dots, (f_{N,1}, f_{N,2})$ so that the elements of each pair belong to a same set among $F_1 \cup F_k, F_2, F_3, \dots, F_{k-1}$. Group the remaining Q floodlights into $M = \lfloor\frac{Q}{3}\rfloor$ triples $(f'_{1,1}, f'_{1,2}, f'_{1,3}), \dots, (f'_{M,1}, f'_{M,2}, f'_{M,3})$ (leaving at most two ungrouped). Let $\alpha = \frac{2\pi}{m-Q+2\lfloor\frac{Q}{3}\rfloor} = \frac{2\pi}{2N+2M}$. We now schedule the floodlights as follows. Configure $f_{i,1}$ and $f_{i,2}$ with angles $(i-1)\alpha$ and $\pi + (i-1)\alpha$, respectively, for $i = 1, \dots, N$; and configure $f'_{j,1}, f'_{j,2}$, and $f'_{j,3}$ with angles $(N+j-1)\alpha$, $(N+j-1)\alpha$, and $\pi + (N+j-1)\alpha$, respectively, for $j = 1, \dots, M$. Finally, arbitrarily configure the remaining floodlights (at most two). The correctness of this configuration follows from lemmas 1 and 2. \square

3 Many points and two lights

In this section we consider the case of two floodlights f_1 and f_2 , i.e., $m = 2$. Let p_1, \dots, p_n denote the elements of P . Assume w.l.o.g. that line $\ell(f_1, f_2)$ is horizontal and that f_1 is located to the left of f_2 . Given any target point p_i ($i = 1, \dots, n$), let $\theta_i \in [0, \pi)$ denote the angle at p_i in the triangle $\triangle p_i f_1 f_2$. If there are points from P on both sides of the line $\ell(f_1, f_2)$, then we define two angles β^+ and β^- as the maximum of θ_i over all points p_i above and below $\ell(f_1, f_2)$, respectively (see Figure 4a). Otherwise, all the points of P are on the same side of $\ell(f_1, f_2)$ and we define two angles, β_{max} and β_{min} , as the largest and smallest θ_i over all points p_i , respectively (see Figure 4b).

Theorem 4 (Two floodlights) For $m = 2$, $n \geq 2$:
 (1) If there are points of P on both sides of $\ell(f_1, f_2)$ then $\alpha(P, F) = \pi + \frac{\beta^+ + \beta^-}{2}$.
 (2) If all the points of P lie on one side of $\ell(f_1, f_2)$ then $\alpha(P, F) = \pi + \frac{\beta_{max} - \beta_{min}}{2}$.

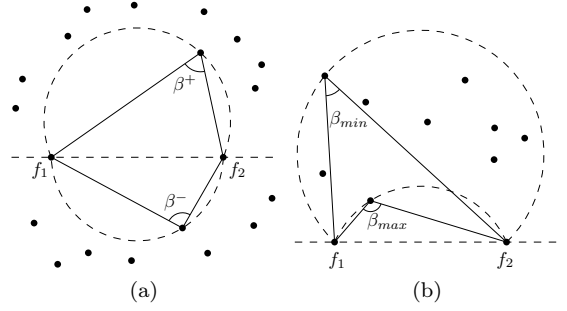


Figure 4: (a) β^+ and β^- . (b) β_{max} and β_{min} .

Proof. First, we prove part (1) of the theorem. We configure floodlights f_1 and f_2 initially as follows. Let Af_1Bf_2 be the quadrilateral such that angle $\angle f_1Af_2$ is equal to β^+ , angle $\angle f_2Bf_1$ is equal to β^- , and points f_1 and f_2 are symmetric with respect to the line $\ell(A, B)$ as shown in Figure 5a.

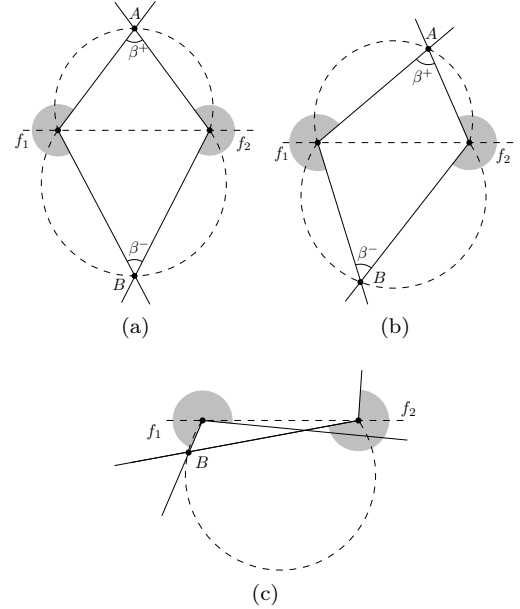


Figure 5: (a) Initial position. (b),(c) General position.

Since f_1 and f_2 rotate at unit speed, it always holds that $\angle f_1Af_2 = \beta^+$ and $\angle f_2Bf_1 = \beta^-$ (see Figure 5b). Furthermore, the region not illuminated by the lights is always a subset of the interior of the union of the triangles $\triangle f_1Af_2$ and $\triangle f_2Bf_1$ (see Figures 5b and 5c), and it never contains points of P by the definition of β^+ and β^- . It remains to show that any illumination angle smaller than $\pi + \frac{\beta^+ + \beta^-}{2}$ is not feasible.

Suppose that, initially, floodlight f_i , $i = 1, 2$ covers angles in the interval $[\alpha_i, \beta_i]$. First we show that these intervals cover all possible directions in $[0, 2\pi]$. If, to the contrary, a direction t is not covered by $[\alpha_1, \beta_1] \cup [\alpha_2, \beta_2]$, then there is a rotation such that p_i is not illuminated, where p_i is the point above the x -axis and $\angle f_2p_i f_1 = \beta^+$, a contradiction. Therefore, the

floodlight intervals overlap as shown in Figure 6a.

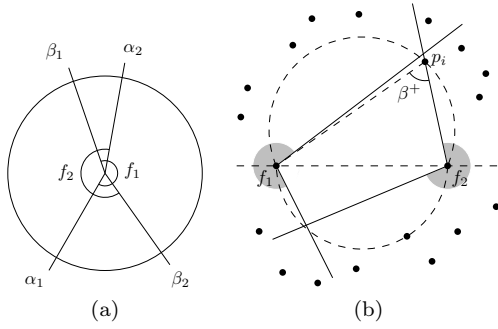


Figure 6: The floodlight intervals.

We show now that the overlapping interval $[\alpha_2, \beta_1]$ has length at least β^+ . Suppose to the contrary that it is smaller than β^+ . Consider the rotation by the angle γ such that the β_2 -ray of floodlight f_2 passes through the point p_i defining β^+ (see Figure 6b). Then the α_1 -ray of floodlight f_1 will not cover p_i since the angle between the two rays is less than β^+ . Therefore p_i is not covered if the rotation angle is slightly smaller than γ , a contradiction. Similarly, the overlapping interval $[\alpha_1, \beta_2]$ has length at least β^- . If the illumination angle is α then $2\alpha \geq 2\pi + \beta^+ + \beta_i$. The claim of part (1) follows.

To prove part (2) of the theorem we configure floodlights f_1 and f_2 at the beginning as follows. Let Af_1Bf_2 be a quadrilateral such that $\angle f_1Af_2 = \beta_{min}$ and $\angle f_2Bf_1 = \beta_{max}$ and points f_1 and f_2 are symmetric about line $\ell(A, B)$ as shown in Figure 7a. The argument is similar to the proof of part (1) since the points A and B move along arcs shown in Figure 7a. The uncovered part is either a region below Cf_1Bf_2D , shown in Figure 7a, or the wedge XYZ , shown in Figure 7c. In any case the area between the two arcs defined by β_{min} and β_{max} is always illuminated.

The optimality of angle $\pi + \frac{\beta_{max} - \beta_{min}}{2}$ can be shown similarly to the proof of part (1). \square

4 Many lights and two points

Let p_1 and p_2 denote the elements of P and f_1, \dots, f_m denote the elements of F . Let θ_i ($i = 1, \dots, m$) denote the angle by which line $\ell(f_i, p_1)$ has to be rotated clockwise with center f_i to become line $\ell(f_i, p_2)$. Assume w.l.o.g. that $\theta_1 \leq \theta_2 \leq \dots \leq \theta_m$.

Lemma 5 *If $n = 2$ then $\alpha(P, F) \leq \frac{2\pi}{m} + \frac{\theta_m - \theta_1}{m}$.*

Proof. It suffices to prove that for $\alpha = \frac{2\pi}{m} + \frac{\theta_m - \theta_1}{m}$ the m floodlights can be configured properly. Configure f_1 arbitrarily and, for $i = 1, \dots, m - 1$, configure f_{i+1} to start illuminating point p_2 at the time f_i stops illuminating it. Since $\alpha \geq 2\pi/m$ then p_2 is

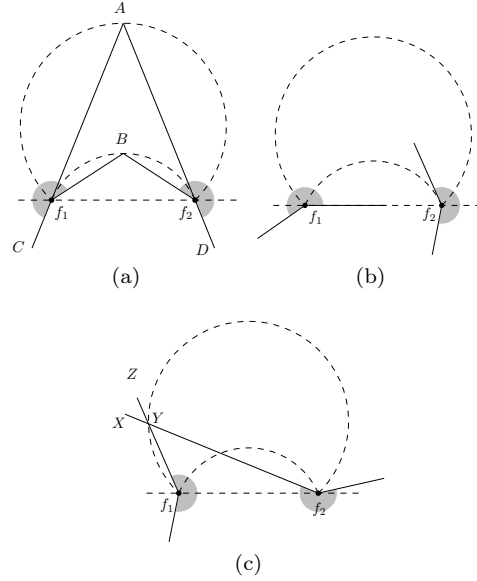


Figure 7: The floodlight intervals.

always illuminated. Observe that p_1 is illuminated at some time by both f_i and f_{i+1} since $\theta_i \leq \theta_{i+1}$. Then the time span in which p_1 is illuminated by f_i and not by f_{i+1} is equal to $\alpha + \theta_i - \theta_{i+1}$. Since $\sum_{i=1}^{m-1} (\alpha + \theta_i - \theta_{i+1}) + \alpha = m\alpha + \theta_1 - \theta_m = 2\pi$ then p_1 is always illuminated. \square

One can build examples, as the following one, in which $\theta_1 < \theta_m$ and α is the theoretical minimum (i.e. $\alpha = 2\pi/m$), showing that $\alpha = \frac{2\pi}{m} + \frac{\theta_m - \theta_1}{m}$ is not always optimal. Let $m = 8$ and $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8) = (\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4})$. Then f_1, \dots, f_8 can be configured with $\alpha = 2\pi/m$ so that p_1 is illuminated in the (circular) order $f_1, f_3, f_8, f_2, f_4, f_5, f_6, f_7$ and p_2 in the order $f_6, f_1, f_7, f_3, f_2, f_8, f_4, f_5$. Thus, it becomes interesting to decide whether $\alpha(P, F) = 2\pi/m$ when $\theta_1, \dots, \theta_m$ are all multiples of $2\pi/m$.

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