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***K*-theory of derivators revisited**

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We define a K -theory for pointed right derivators and show that it agrees with Waldhausen K -theory in the case where the derivator arises from a good Waldhausen category. This K -theory is not invariant under general equivalences of derivators, but only under a stronger notion of equivalence that is defined by considering a simplicial enrichment of the category of derivators. We show that derivator K -theory, as originally defined, is the best approximation to Waldhausen K -theory by a functor that is invariant under equivalences of derivators.

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1. Introduction

Recent developments in the theory of derivators have shown both that the theory is sufficiently rich to contend for an independent approach to homotopical algebra and that its language is very useful in formulating precisely universal properties in homotopy theory. Since models for homotopical algebra typically give rise to derivators, the theory reflects a minimalist approach employing basically only purely (2-)categorical arguments, albeit technically quite complex at times, to address problems of abstract homotopy theory.

Derivators codify structure lying somewhere between the model and its associated homotopy category, but fairly closer to the model than the homotopy category,

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and restructure the presentation of the homotopy theory defined by the model in a surprisingly efficient way. This intermediate structure involves the collection of all homotopy categories of diagrams of various shapes in the model together with the network of restriction functors between them and their adjoint homotopy Kan extensions. The theory of derivators is based on an abstract axiomatization of collections of such (homotopy) Kan extensions (and their adjoints), which does not involve any underlying model. On the one hand, it is often the case that questions about the model are really questions about the associated derivator and thus they can instructively be handled more abstractly at this level of generality. On the other hand, for the theory to be successful, one is normally required to supply a large amount of data in order to compensate for the lack of an underlying homotopy theory and, consequently, working with these objects can be cumbersome.

The main problem is to understand how close this passage from the model to its associated derivator actually is to being faithful. This paper is a contribution to this problem in connection with Waldhausen K -theory regarded as an invariant of homotopy theories. The proven failure to reconstruct K -theory from the triangulated structure of the homotopy category in such a way that it satisfies certain desirable properties [Schlichting 2002] suggested turning to the more highly structured world of derivators for such a reconstruction. Indeed, it turned out that the structure of a derivator is rich enough to allow for a natural definition of K -theory. This was introduced by Maltsiniotis [2007], who also conjectured that it satisfies an *additivity* property, *agreement* with Quillen K -theory and a *localization* property. In previous work [Muro and Raptis 2011], we showed that *agreement* fails for Waldhausen K -theory and moreover, that derivator K -theory cannot satisfy both *agreement* with Quillen K -theory and the *localization* property. On the other hand, Cisinski and Neeman [2008] showed that *additivity* for derivator K -theory holds for triangulated derivators.

The purpose of this paper is twofold. First, we define a new K -theory of derivators, which we also call Waldhausen K -theory, and show in Theorem 4.3.1 that it agrees with the usual Waldhausen K -theory for all well-behaved Waldhausen categories. The proof rests crucially on the homotopically flexible versions of the \mathcal{S}_\bullet -construction due to Blumberg and Mandell [2011] and Cisinski [2010b].

The price to be paid for such a strong version of agreement is that this new definition is *provably* not invariant under equivalences of derivators. Here it is worthwhile to recall that Toën and Vezzosi [2004] showed that Waldhausen K -theory cannot factor through the 2-category of derivators. However, we consider here a simplicial enrichment of the category of derivators which enhances the 2-categorical structure. This leads to a stronger and more refined notion of equivalence of derivators which basically encodes higher coherence and is closer to an equivalence of homotopy theories. We show that Waldhausen K -theory of derivators is invariant under

this stronger notion of equivalence. We think that the simplicial enrichment of the category of derivators and the accompanying stronger notion of equivalence have independent interest and may prove useful also in other applications of the theory.

There is a natural comparison transformation from Waldhausen K -theory to derivator K -theory. The second main result of the paper (Theorem 5.2.2) says that this comparison transformation is homotopically initial among all natural transformations from Waldhausen K -theory to a functor which is invariant under equivalences of derivators.

The paper is organized as follows. In Section 2, we review some background material from the (2-categorical) theory of derivators and fix some notational conventions. In Section 3, we discuss the simplicial enrichment of the category of derivators, the associated notion of strong equivalence, and the comparison with the 2-categorical viewpoint.

Section 4 is concerned with the definition of Waldhausen K -theory for derivators and some of its general properties. We present two canonically homotopy equivalent models for Waldhausen K -theory, both of which we use in the paper. Then we show that the Waldhausen K -theory of derivators is invariant under strong equivalences of derivators and agrees with the Waldhausen K -theory of derivable strongly saturated Waldhausen categories. In Section 5, we recall the definition of derivator K -theory and discuss its dependence on the 2-categorical theory of derivators. Then we recall the definition of the comparison map from Waldhausen K -theory to derivator K -theory and show that derivator K -theory is the best approximation to Waldhausen K -theory by a functor that is invariant under equivalences of derivators.

There are several remaining open questions, regarding either the notion of strong equivalence or the K -theory of derivators, some of which are briefly mentioned in Section 6. The paper ends with two appendices on topics of related interest but which are, strictly speaking, independent of the rest of the paper. In Appendix A, we recall the results from the comparison between combinatorial model categories and the 2-category of derivators due to Renaudin [2009] and discuss some slight improvements with an eye towards understanding the comparison with the simplicial category of derivators. Appendix B is concerned with the approximation theorem in K -theory, which in a version due to Cisinski [2010b] shows that K -theory is invariant under derived equivalences, and a partial converse which shows that derived equivalences are detected by the homotopy type of the S_\bullet -construction.

2. Preliminaries on (pre)derivators

2.1. Prederivators. Let Cat denote the 2-category of small categories. We fix a 1- and 2-full sub-2-category of diagrams $Dia \subset Cat$ which is closed under all

required constructions appearing below (e.g., taking opposite categories or finite (co)products, passing to comma categories, etc.); see [Maltsiniotis 2007] for the precise list of axioms. We think of the collection of categories in Dia as possible shapes for indexing diagrams in other categories.

The smallest option for Dia is Pos_f , spanned by the finite posets, and the largest option is, of course, Cat itself. An intermediate option, which appears prominently in connection with K -theory, is the 2-category of diagrams Dir_f spanned by the finite direct categories. Recall that a *finite direct category* is a small category whose nerve has finitely many nondegenerate simplices. This is equivalent to saying that the underlying graph spanned by the nonidentity arrows of the category has no cycles. Every finite poset is a finite direct category.

A *prederivator* (with domain Dia) is¹ a strict 2-functor $\mathbb{D} : Dia^{op} \rightarrow Cat$. More explicitly, for every category X in Dia there is a small category $\mathbb{D}(X)$, for every functor $f : X \rightarrow Y$ in Dia there is an *inverse image* functor

$$f^* = \mathbb{D}(f) : \mathbb{D}(Y) \rightarrow \mathbb{D}(X),$$

for every natural transformation $\alpha : f \Rightarrow g$ in Dia there is a natural transformation $\alpha^* = \mathbb{D}(\alpha) : f^* \Rightarrow g^*$,

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} & Y, \\ \mathbb{D}(X) & \begin{array}{c} \xleftarrow{f^*} \\ \Downarrow \alpha^* \\ \xleftarrow{g^*} \end{array} & \mathbb{D}(Y), \end{array}$$

and all these are required to satisfy the obvious strict 2-functoriality properties.

A (1-)morphism of prederivators $\phi : \mathbb{D} \rightarrow \mathbb{D}'$ is a pseudonatural transformation of contravariant 2-functors, i.e., for every X in Dia there is a functor

$$\phi(X) : \mathbb{D}(X) \rightarrow \mathbb{D}'(X),$$

and for every $f : X \rightarrow Y$ in Dia there is a natural isomorphism $\phi(f)$,

$$\begin{array}{ccc} \mathbb{D}(Y) & \xrightarrow{\phi(Y)} & \mathbb{D}'(Y) \\ f^* \downarrow & \swarrow \phi(f) & \downarrow f^* \\ \mathbb{D}(X) & \xrightarrow{\phi(X)} & \mathbb{D}'(X) \end{array} \tag{2.1.1}$$

such that certain coherence laws are satisfied. The morphism ϕ is called *strict* if $\phi(X)f^* = f^*\phi(Y)$ and $\phi(f)$ is the identity natural transformation for every f .

¹The reader should be warned about the slight variations of this definition that appear in the literature. These pertain to the choice of domain and the different ways of forming the opposite of a 2-category.

A 2-morphism $\tau : \phi \Rightarrow \phi'$ between 1-morphisms of prederivators is a *modification* of pseudonatural transformations. This is defined by a collection of suitably compatible natural transformations in Cat ,

$$\begin{array}{ccc}
 & \phi(X) & \\
 \mathbb{D}(X) & \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \tau(X) \\ \xrightarrow{\quad} \end{array} & \mathbb{D}'(X) \\
 & \phi'(X) &
 \end{array}$$

for every X in Dia (see, e.g., [Borceux 1994, Section 7.5] for the precise definitions).

Let $PDer$ (resp. $PDer^{str}$) denote the resulting 2-category of prederivators, morphisms (resp. strict morphisms) and 2-morphisms. This is an example of a 2-category formed by 2-functors, pseudonatural (or 2-natural) transformations and modifications (see [Borceux 1994, Propositions 7.3.3 and 7.5.4]). The notion of *equivalence* of prederivators is defined in the usual way in terms of the 2-categorical structure of $PDer$. Equivalently, a morphism $\phi : \mathbb{D} \rightarrow \mathbb{D}'$ is an equivalence if and only if $\phi(X) : \mathbb{D}(X) \rightarrow \mathbb{D}'(X)$ is an equivalence of categories for every X in Dia . We also consider the 1-full sub-2-categories $PDer_{eq}$ and $PDer_{eq}^{str}$ of $PDer$ and $PDer^{str}$ respectively, which have the same objects and 1-morphisms but whose 2-morphisms are the invertible modifications. These are categories enriched in groupoids.

Remark 2.1.2. A basic example of a prederivator is the representable prederivator defined by a small category X :

$$Cat(-, X) : Dia^{op} \rightarrow Cat.$$

This construction yields a 2-categorical Yoneda functor

$$Cat \rightarrow PDer^{str},$$

which is 1- and 2-fully faithful when restricted to Dia . If we restrict to the 1-full sub-2-category whose 2-morphisms are the invertible natural transformations, we obtain a 2-functor to $PDer_{eq}^{str}$.

Let e denote the final category with one object e and one morphism id_e . Given a small category X , there is a canonical isomorphism of categories $i_{X,-} : X \cong Cat(e, X)$ defined as follows. An object $x \in Ob X$ defines a functor $i_{X,x} : e \rightarrow X$ with $i_{X,x}(e) = x$, and a morphism $g : x \rightarrow x'$ in X induces a natural transformation $i_{X,g} : i_{X,x} \Rightarrow i_{X,x'}$ with $i_{X,g}(e) = g$.

Let X be a category in Dia . For every prederivator \mathbb{D} there is a functor

$$dia_{X,e} : \mathbb{D}(X) \rightarrow Cat(X, \mathbb{D}(e)) \tag{2.1.3}$$

which sends an object F in $\mathbb{D}(X)$ to the functor $\text{dia}_{X,e}(F) : X \rightarrow \mathbb{D}(e)$ defined by

$$\text{dia}_{X,e}(F)(x) = i_{X,x}^* F, \quad \text{dia}_{X,e}(F)(g : x \rightarrow x') = i_{X,g}^* F,$$

and a morphism $\varphi : F \rightarrow G$ in $\mathbb{D}(X)$ to $\text{dia}_{X,e}(\varphi) : \text{dia}_{X,e}(F) \Rightarrow \text{dia}_{X,e}(G)$, the natural transformation given by

$$\text{dia}_{X,e}(\varphi)(x) = i_{X,x}^* \varphi.$$

This suggests a useful analogy, namely, to regard $\mathbb{D}(e)$ as the underlying category of \mathbb{D} and the elements of $\mathbb{D}(X)$ as X -indexed diagrams in \mathbb{D} . We will often write F_x for $i_{X,x}^*(F)$.

Remark 2.1.4. The functors (2.1.3) assemble to a morphism of prederivators

$$\text{dia}_{-,e} : \mathbb{D} \rightarrow \text{Cat}(-, \mathbb{D}(e))$$

which is the unit of the 2-adjoint pair

$$\text{PDer}^{\text{str}} \begin{array}{c} \xrightarrow{\text{evaluation at } e} \\ \xleftarrow{2\text{-Yoneda}} \end{array} \text{Cat}.$$

The counit is the natural isomorphism $\text{Cat}(e, X) \cong X$ described above.

The product of 2-categories is 2-functorial, hence for any Y in Dia and any prederivator \mathbb{D} , we obtain a new prederivator

$$\mathbb{D}_Y := \mathbb{D}(- \times Y) : \text{Dia}^{\text{op}} \rightarrow \text{Cat}.$$

The morphism $\text{dia}_{X,e}$ for this new prederivator will be denoted by

$$\text{dia}_{X,Y} : \mathbb{D}(X \times Y) \rightarrow \text{Cat}(X, \mathbb{D}(Y)). \tag{2.1.5}$$

Here we use the obvious isomorphism $e \times Y \cong Y$ as an identification. This functor sends an object F in $\mathbb{D}(X \times Y)$ to the functor $\text{dia}_{X,Y} : X \rightarrow \mathbb{D}(Y)$ defined by

$$\text{dia}_{X,Y}(F)(x) = (i_{X,x} \times Y)^* F, \quad \text{dia}_{X,Y}(F)(g : x \rightarrow x') = (i_{X,g} \times Y)^* F,$$

and a morphism $\varphi : F \rightarrow G$ in $\mathbb{D}(X \times Y)$ to $\text{dia}_{X,Y}(\varphi) : \text{dia}_{X,Y}(F) \Rightarrow \text{dia}_{X,Y}(G)$, the natural transformation given by

$$\text{dia}_{X,Y}(\varphi)(x) = (i_{X,x} \times Y)^* \varphi.$$

The functor $\text{dia}_{X,Y}$ may be viewed as taking an $(X \times Y)$ -indexed diagram to the underlying X -diagram of Y -indexed diagrams in \mathbb{D} .

2.2. Derivators. A (right or left, pointed, stable/triangulated) derivator is a pre-derivator that satisfies certain additional properties. We only briefly review the definitions here. With the exception of Appendix A, we are mainly concerned with the case of pointed right derivators.

A right derivator is a prederivator \mathbb{D} satisfying the following properties:

(Der1) For every pair of small categories X and Y in Dia , the functor induced by the inclusions of the factors to the coproduct $X \sqcup Y$,

$$\mathbb{D}(X \sqcup Y) \rightarrow \mathbb{D}(X) \times \mathbb{D}(Y),$$

is an equivalence of categories. Moreover, $\mathbb{D}(\emptyset)$ is the final category e .

(Der2) For every small category X in Dia , the functor

$$(i_{X,x}^*)_{x \in \text{Ob } X} : \mathbb{D}(X) \rightarrow \prod_{x \in \text{Ob } X} \mathbb{D}(e)$$

reflects isomorphisms.

(Der3) For every morphism $f : X \rightarrow Y$ in Dia , the inverse image $f^* : \mathbb{D}(Y) \rightarrow \mathbb{D}(X)$ admits a left adjoint $f_! : \mathbb{D}(X) \rightarrow \mathbb{D}(Y)$.

(Der4) Given $f : X \rightarrow Y$ in Dia and y an object of Y , consider the following diagram in Dia :

$$\begin{array}{ccc} f \downarrow y & \xrightarrow{j_{f,y}} & X \\ p_{f \downarrow y} \downarrow & \swarrow \alpha_{f,y} & \downarrow f \\ e & \xrightarrow{i_{Y,y}} & Y \end{array}$$

Here $f \downarrow y$ is the comma category whose objects $(x, f(x) \rightarrow y)$ are pairs given by an object x in X and a map $f(x) \rightarrow y$ in Y , $j_{f,y}$ is the functor $j_{f,y}(x, f(x) \rightarrow y) = x$, and $\alpha_{f,y}(x, f(x) \rightarrow y) = (f(x) \rightarrow y)$. Then the diagram obtained by applying \mathbb{D} satisfies the Beck–Chevalley condition, i.e., the mate natural transformation

$$c_{f,y} : (p_{f \downarrow y})_! j_{f,y}^* \Longrightarrow i_{Y,y}^* f_!,$$

which is the adjoint of

$$j_{f,y}^* \xrightarrow{\text{unit of } f_! \dashv f^*} j_{f,y}^* f^* f_! = (f j_{f,y})^* f_! \xrightarrow{\alpha_{f,y}^* f_!} (i_{Y,y} p_{f \downarrow y})^* f_! = p_{f \downarrow y}^* i_{Y,y}^* f_!,$$

is a natural isomorphism.

A left derivator \mathbb{D} is a prederivator whose opposite prederivator \mathbb{D}^{op} , defined by $\mathbb{D}^{\text{op}}(X) = \mathbb{D}(X^{\text{op}})^{\text{op}}$, is a right derivator. A prederivator which is both a left

and a right derivator is simply called a *derivator*. There is yet another axiom that a prederivator may satisfy:

(Der5) For every pair of small categories X and \mathcal{I} in Dia where \mathcal{I} is a free finite category, the canonical functor

$$\text{dia}_{\mathcal{I}, X} : \mathbb{D}(\mathcal{I} \times X) \rightarrow \text{Cat}(\mathcal{I}, \mathbb{D}(X))$$

is full and essentially surjective.

The inclusion of this axiom in the definition of derivator is somewhat controversial in the literature. Heller [1988] includes (Der5) as part of the definition. Other authors prefer either to omit it and reserve it for an additional “strongness” property of a derivator, or to replace it with the seemingly weaker version in which $\mathcal{I} = [1]$; see, e.g., [Maltsiniotis 2001; 2007; Franke 1996; Groth 2013]. The inclusion of (Der5) matters very little for our purposes here, but we choose to exclude it from the basic definition.

We recall that a small category is called *pointed* if it has a zero object. A functor between pointed categories is called pointed if it preserves zero objects. A prederivator \mathbb{D} is called *pointed* if $\mathbb{D}(X)$ is a pointed category and $f^* : \mathbb{D}(Y) \rightarrow \mathbb{D}(X)$ is a pointed functor for all X and $f : X \rightarrow Y$ in Dia . This definition follows Groth [2013], who showed that it is equivalent for derivators to the original definition; see, e.g., [Maltsiniotis 2007].

We recall the definition of cocartesian squares for right derivators. Consider the “commutative square” category $\square = [1] \times [1]$. A *commutative square* in a prederivator \mathbb{D} is an object F of $\mathbb{D}(\square)$. There is a subcategory $\ulcorner \subseteq \square$ which can be depicted as follows:

$$\begin{array}{ccc} (0, 0) & \longrightarrow & (1, 0) \\ \downarrow & \square & \downarrow \\ (0, 1) & \longrightarrow & (1, 1) \end{array} \quad \begin{array}{ccc} (0, 0) & \longrightarrow & (1, 0) \\ \downarrow & \ulcorner & \downarrow \\ (0, 1) & & (1, 1) \end{array}$$

Denote the inclusion functor by

$$i_{\ulcorner} : \ulcorner \rightarrow \square.$$

If \mathbb{D} is a right derivator, a commutative square F in \mathbb{D} is called *cocartesian* if the counit

$$(i_{\ulcorner})_! i_{\ulcorner}^* F \rightarrow F$$

is an isomorphism. If \mathbb{D} is a left derivator, a commutative square F in \mathbb{D} is called *cartesian* if it is cocartesian in \mathbb{D}^{op} . A pointed derivator which satisfies (Der5) is called *stable* (or *triangulated*) if cocartesian and cartesian squares coincide.

A crucial point to note regarding all of the above definitions is that only the notion of a prederivator constitutes *structure*, while the additional axioms assert *properties*. We also emphasize that the property of being a right (or left, pointed, stable/triangulated) derivator is invariant under equivalences of prederivators. Moreover, if \mathbb{D} is a pointed right derivator, then so is \mathbb{D}_Y for every Y in Dia .

A morphism of right derivators $\phi : \mathbb{D} \rightarrow \mathbb{D}'$ is called *cocontinuous* if for every $f : X \rightarrow Y$ in Dia , the canonical natural transformation

$$f_! \phi(X) \implies \phi(Y) f_!,$$

adjoint to

$$\phi(X) \xrightarrow{\text{unit of } f_!^{-1} f^*} \phi(X) f^* f_! \xrightarrow{\phi(f)^{-1} f_!} f^* \phi(Y) f_!,$$

is an isomorphism.² An easy application of (Der1) shows that the components of a cocontinuous morphism between pointed right derivators are automatically pointed functors.

Let Der and Der^{str} denote the 2-full sub-2-categories of $PDer$ and $PDer^{str}$, respectively, given by the pointed right derivators, cocontinuous (strict) morphisms and 2-morphisms. Let Der_{eq} and Der_{eq}^{str} denote the 1-full sub-2-categories of Der and Der^{str} whose 2-morphisms are the invertible modifications.

2.3. Examples. The examples of prederivators that we are interested in arise from categories with weak equivalences as follows. Let $(\mathcal{C}, \mathcal{W})$ be a pair consisting of a small category \mathcal{C} together with a subcategory \mathcal{W} which contains the isomorphisms. The morphisms of \mathcal{W} are called *weak equivalences*. The *homotopy category* of $(\mathcal{C}, \mathcal{W})$ is the localization

$$Ho \mathcal{C} := \mathcal{C}[\mathcal{W}^{-1}].$$

For every object X in Dia , the diagram category \mathcal{C}^X together with the subcategory of objectwise weak equivalences of functors is again a category with weak equivalences $(\mathcal{C}^X, \mathcal{W}^X)$. The choice of objectwise weak equivalences is natural in X , so there is a prederivator $\mathbb{D}(\mathcal{C}, \mathcal{W}) : Dia^{op} \rightarrow Cat$ given by the homotopy categories of all relevant diagram categories, i.e., it is defined on objects by

$$\mathbb{D}(\mathcal{C}, \mathcal{W})(X) := Ho(\mathcal{C}^X)$$

and on 1- and 2-morphisms in the canonically induced way.

A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ that preserves the weak equivalences $F(\mathcal{W}) \subset \mathcal{W}'$ induces a (strict) morphism of prederivators $\mathbb{D}(F) : \mathbb{D}(\mathcal{C}, \mathcal{W}) \rightarrow \mathbb{D}(\mathcal{C}', \mathcal{W}')$. Such functors are called *homotopical*. A homotopical functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is a *derived equivalence* if

²This condition is comparable to *right exactness* of a functor. This justifies the term *right derivator*.

it induces an equivalence of homotopy categories $\mathrm{Ho} F : \mathrm{Ho} \mathcal{C} \xrightarrow{\sim} \mathrm{Ho} \mathcal{C}'$. A natural transformation $\alpha : F \Rightarrow F'$ of homotopical functors $F, F' : (\mathcal{C}, \mathcal{W}) \rightarrow (\mathcal{C}', \mathcal{W}')$ defines a 2-morphism $\mathbb{D}(\alpha) : \mathbb{D}(F) \Rightarrow \mathbb{D}(F')$ in $\mathrm{PDer}^{\mathrm{str}}$. If the components of the natural transformation α are given by weak equivalences, then $\mathbb{D}(\alpha)$ is in $\mathrm{PDer}_{\mathrm{eq}}^{\mathrm{str}}$.

We note that if \mathcal{W} is the subcategory of isomorphisms, $\mathbb{D}(\mathcal{C}, \mathcal{W}) = \mathrm{Cat}(-, \mathcal{C})$ is the representable prederivator of Remark 2.1.2. We will normally write $\mathbb{D}(\mathcal{C})$ when the choice of \mathcal{W} is clear from the context.

For well-behaved categories with weak equivalences $(\mathcal{C}, \mathcal{W})$, the associated prederivator $\mathbb{D}(\mathcal{C})$ is a (right or left, pointed, stable/triangulated) derivator. We refer the reader to [Cisinski 2010a] for a systematic treatment of the results in this direction. Here we will be particularly concerned with categories with weak equivalences that arise from Waldhausen categories [Waldhausen 1985]. Following [Cisinski 2010b], we say that a Waldhausen category $(\mathcal{C}, \mathrm{co}\mathcal{C}, \mathrm{w}\mathcal{C})$ is *derivable* if it satisfies the “2-out-of-3” axiom and every morphism in \mathcal{C} can be written as the composition of a cofibration followed by a weak equivalence. The following theorem is a special case of results proved in [Cisinski 2010a; 2010b].

Theorem 2.3.1 (Cisinski). (a) *Let $(\mathcal{C}, \mathrm{co}\mathcal{C}, \mathrm{w}\mathcal{C})$ be a derivable Waldhausen category. Then the associated prederivator $\mathbb{D}(\mathcal{C}) : \mathrm{Dir}_f^{\mathrm{op}} \rightarrow \mathrm{Cat}$ is a pointed right derivator which also satisfies (Der5).*

(b) *An exact functor of derivable Waldhausen categories*

$$F : (\mathcal{C}, \mathrm{co}\mathcal{C}, \mathrm{w}\mathcal{C}) \rightarrow (\mathcal{C}', \mathrm{co}\mathcal{C}', \mathrm{w}\mathcal{C}')$$

induces a cocontinuous morphism $\mathbb{D}(F) : \mathbb{D}(\mathcal{C}) \rightarrow \mathbb{D}(\mathcal{C}')$ in $\mathrm{Der}^{\mathrm{str}}$.

(c) *Moreover, the morphism $\mathbb{D}(F) : \mathbb{D}(\mathcal{C}) \rightarrow \mathbb{D}(\mathcal{C}')$ is an equivalence in $\mathrm{Der}^{\mathrm{str}}$ if and only if $\mathrm{Ho} F : \mathrm{Ho} \mathcal{C} \rightarrow \mathrm{Ho} \mathcal{C}'$ is an equivalence of categories.*

A derivable Waldhausen category $(\mathcal{C}, \mathrm{co}\mathcal{C}, \mathrm{w}\mathcal{C})$ is called *strongly saturated* if it satisfies the property that a morphism in \mathcal{C} is a weak equivalence if and only if it becomes an isomorphism in the homotopy category. A derivable Waldhausen category with functorial factorizations is strongly saturated if and only if the weak equivalences are closed under retracts; see [Blumberg and Mandell 2011, Theorems 5.5 and 6.4].

3. Simplicial enrichments of (pre)derivators

We recall that the *simplex category* Δ consists of the finite ordinals

$$[n] = \{0 < \dots < n\}$$

for $n \geq 0$, and the nondecreasing maps between them. Thus, it is contained in Dir_f , and in fact also in any other possible category of diagrams Dia . The naturality of

the construction $Y \mapsto \mathbb{D}_Y$ shows that there is a 2-functor

$$Dia^{op} \times PDer^{str} \rightarrow PDer^{str}$$

which may be regarded as a ‘‘cotensor 2-structure’’ of $PDer^{str}$ over Dia . Using this, we can associate to every prederivator \mathbb{D} a simplicial object \mathbb{D}_\bullet in $PDer^{str}$ with

$$\mathbb{D}_n = \mathbb{D}([n] \times -).$$

In particular, we have $\mathbb{D}_0 = \mathbb{D}$. Faces and degeneracies are morphisms of prederivators since both \mathbb{D} and the product of 2-categories are 2-functorial. This natural simplicial object will be used to define an enrichment of the underlying category of $PDer^{str}$ over simplicial sets.

3.1. Definition of \underline{PDer}^{str} . We define a simplicially enriched category \underline{PDer}^{str} with prederivators as objects and morphism simplicial sets

$$\underline{PDer}^{str}(\mathbb{D}, \mathbb{D}')_\bullet = \text{Ob } PDer^{str}(\mathbb{D}, \mathbb{D}')_\bullet.$$

The composition is defined by simplicial maps

$$\underline{PDer}^{str}(\mathbb{D}', \mathbb{D}'')_\bullet \times \underline{PDer}^{str}(\mathbb{D}, \mathbb{D}')_\bullet \rightarrow \underline{PDer}^{str}(\mathbb{D}, \mathbb{D}'')_\bullet$$

which send pairs of strict morphisms

$$\begin{aligned} \phi &: \mathbb{D} \rightarrow \mathbb{D}'([n] \times -), \\ \psi &: \mathbb{D}' \rightarrow \mathbb{D}''([n] \times -) \end{aligned}$$

to the composite

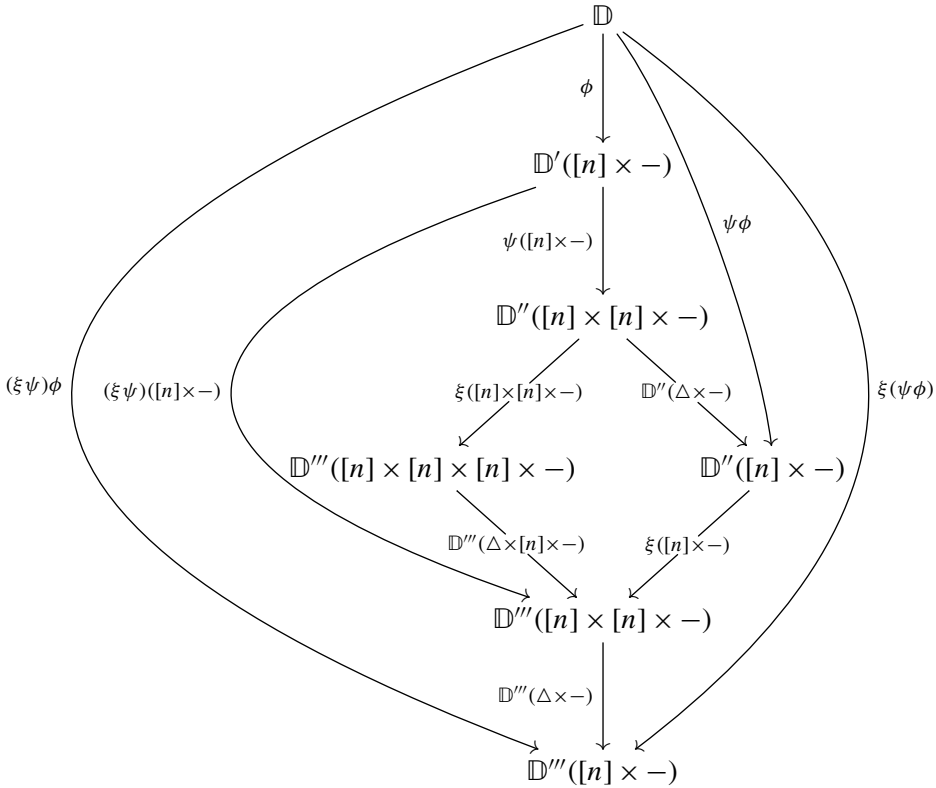
$$\mathbb{D} \xrightarrow{\phi} \mathbb{D}'([n] \times -) \xrightarrow{\psi([n] \times -)} \mathbb{D}''([n] \times [n] \times -) \xrightarrow{\mathbb{D}''(\Delta \times -)} \mathbb{D}''([n] \times -),$$

where $\Delta : [n] \rightarrow [n] \times [n]$ is the diagonal functor.

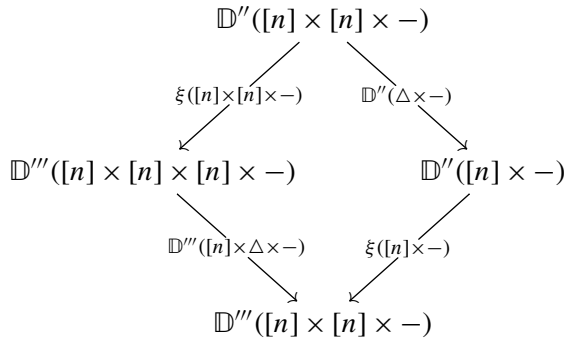
To see that the composition is associative, consider strict morphisms

$$\begin{aligned} \phi &: \mathbb{D} \rightarrow \mathbb{D}'([n] \times -), \\ \psi &: \mathbb{D}' \rightarrow \mathbb{D}''([n] \times -), \\ \xi &: \mathbb{D}'' \rightarrow \mathbb{D}'''([n] \times -). \end{aligned}$$

Then it suffices to show that the leftmost and rightmost morphisms in the following diagram coincide.



All cells in this diagram commute by definition, except for the inner square. If the inner square were commutative, the result would follow immediately. However, the post-composition of the square with $\mathbb{D}'''(\Delta \times -)$ yields a commutative square, and this suffices. Indeed, since the diagonal functor is coassociative, i.e., $(\Delta \times [n])\Delta = ([n] \times \Delta)\Delta$, it is enough to show that the slightly different square



commutes. Note that the only difference between this last square and the inner square in the previous diagram is in the lower left arrow. The latter square commutes because ξ is a strict morphism.

This simplicial enrichment can be used to introduce homotopy theoretic notions into the world of prederivators, but these will be too coarse for our purposes here. For a more appropriate notion of homotopy, we consider the subobject of \mathbb{D}_\bullet defined by “simplicially constant” diagrams.

3.2. Definition of $\underline{\text{PDer}}_{\text{eq}}^{\text{str}}$. Given a prederivator \mathbb{D} and Y in $\mathcal{D}ia$, there is a prederivator $\mathbb{D}(Y \times -)_{\text{eq}}$ equipped with a strict morphism $i_{\text{eq}} : \mathbb{D}(Y \times -)_{\text{eq}} \rightarrow \mathbb{D}(Y \times -)$ such that for all X in $\mathcal{D}ia$,

$$i_{\text{eq}}(Y) : \mathbb{D}(Y \times X)_{\text{eq}} \rightarrow \mathbb{D}(Y \times X)$$

is the inclusion of the full subcategory spanned by the objects F such that the underlying Y -diagram

$$\text{dia}_{Y,X}(F) : Y \longrightarrow \mathbb{D}(X)$$

sends each morphism of Y to an isomorphism in $\mathbb{D}(X)$.

To show that this is well-defined, it is enough to check that given $f : X \rightarrow Z$ in $\mathcal{D}ia$ and F in $\mathbb{D}(Y \times Z)_{\text{eq}}$, the object $(Y \times f)^*(F)$ in $\mathbb{D}(Y \times X)$ is actually in $\mathbb{D}(Y \times X)_{\text{eq}}$. Let $g : y \rightarrow y'$ be a morphism in Y . We have

$$\begin{aligned} (i_{Y,g} \times X)^*(Y \times f)^*(F) &= ((Y \times f)(i_{Y,g} \times X))^*(F) \\ &= (i_{Y,g} \times f)^*(F) \\ &= ((i_{Y,g} \times Z)(Y \times f))^*(F) \\ &= (e \times f)^*(i_{Y,g} \times Z)^*(F). \end{aligned}$$

Since F is in $\mathbb{D}(Y \times Z)_{\text{eq}}$, it follows that $(i_{Y,g} \times Z)^*(F)$ is an isomorphism. Hence, so is $(i_{Y,g} \times X)^*(Y \times f)^*(F)$ for any morphism g in Y , and therefore $(Y \times f)^*(F)$ is in $\mathbb{D}(Y \times X)_{\text{eq}}$. (See also Remark 2.1.4.)

Hence, for any prederivator \mathbb{D} , there is a simplicial prederivator $\mathbb{D}_{\text{eq},\bullet}$ with $\mathbb{D}_{\text{eq},n} = \mathbb{D}([n] \times -)_{\text{eq}}$ equipped with a morphism $i_{\text{eq}} : \mathbb{D}_{\text{eq},\bullet} \rightarrow \mathbb{D}_\bullet$ of simplicial prederivators. Note that $\mathbb{D}_{\text{eq},0} = \mathbb{D}$.

We define a simplicially enriched category $\underline{\text{PDer}}_{\text{eq}}^{\text{str}}$ with prederivators as objects and morphism simplicial sets

$$\underline{\text{PDer}}_{\text{eq}}^{\text{str}}(\mathbb{D}, \mathbb{D}')_\bullet = \text{Ob } \underline{\text{PDer}}^{\text{str}}(\mathbb{D}, \mathbb{D}'_{\text{eq},\bullet})$$

such that the morphisms of simplicial prederivators $i_{\text{eq}} : \mathbb{D}'_{\text{eq},\bullet} \rightarrow \mathbb{D}'_\bullet$ induce a simplicial functor $i_{\text{eq}} : \underline{\text{PDer}}_{\text{eq}}^{\text{str}} \rightarrow \underline{\text{PDer}}^{\text{str}}$.

To show that this is a well-defined simplicial subcategory, we check that the composition in $\underline{\text{PDer}}^{\text{str}}$ of two composable n -simplices in $\underline{\text{PDer}}_{\text{eq}}^{\text{str}}$ is again in $\underline{\text{PDer}}_{\text{eq}}^{\text{str}}$, i.e., given strict morphisms $\phi : \mathbb{D} \rightarrow \mathbb{D}'([n] \times -)_{\text{eq}}$ and $\psi : \mathbb{D}' \rightarrow \mathbb{D}''([n] \times -)_{\text{eq}}$,

we show that the composite

$$\mathbb{D} \xrightarrow{\phi} \mathbb{D}'([n] \times -) \xrightarrow{\psi([n] \times -)} \mathbb{D}''([n] \times [n] \times -) \xrightarrow{(\Delta \times -)^*} \mathbb{D}''([n] \times -)$$

takes values in $\mathbb{D}''([n] \times -)_{\text{eq}}$. Given an object F in $\mathbb{D}(X)$ and a morphism $g : x \rightarrow x'$ in $[n]$, consider the following diagram:

$$\begin{array}{ccccc}
 \mathbb{D}(X) & \xrightarrow{\phi(X)} & \mathbb{D}'([n] \times X) & \xrightarrow{\psi([n] \times X)} & \mathbb{D}''([n] \times [n] \times X) & \xrightarrow{(\Delta \times X)^*} & \mathbb{D}''([n] \times X) \\
 & & \downarrow (i_{[n],x} \times X)^* & \downarrow ([n] \times i_{[n],x} \times X)^* & \downarrow ([n] \times i_{[n],x'} \times X)^* & & \downarrow (i_{[n],x} \times X)^* \\
 & & \begin{array}{c} \curvearrowright \\ \xrightarrow{(i_{[n],g} \times X)^*} \\ \curvearrowleft \end{array} & & \begin{array}{c} \curvearrowright \\ \xrightarrow{([n] \times i_{[n],g} \times X)^*} \\ \curvearrowleft \end{array} & & \begin{array}{c} \curvearrowright \\ \xrightarrow{(i_{[n],g} \times X)^*} \\ \curvearrowleft \end{array} \\
 & & \mathbb{D}'(X) & \xrightarrow{\psi(X)} & \mathbb{D}''([n] \times X) & & \mathbb{D}''([n] \times X) \\
 & & & & \downarrow (i_{[n],x} \times X)^* & & \downarrow (i_{[n],x'} \times X)^* \\
 & & & & \begin{array}{c} \curvearrowright \\ \xrightarrow{(i_{[n],g} \times X)^*} \\ \curvearrowleft \end{array} & & \begin{array}{c} \curvearrowright \\ \xrightarrow{(i_{[n],g} \times X)^*} \\ \curvearrowleft \end{array} \\
 & & & & \mathbb{D}''(X) & \xlongequal{\quad\quad\quad} & \mathbb{D}''(X)
 \end{array}$$

This diagram satisfies several commutativity properties. The subdiagram of functors formed by the straight arrows and the arrows which are curved to the left is commutative, and likewise the subdiagram of straight arrows and arrows curved to the right. In the middle square, the natural transformations

$$\psi(X)(i_{[n],g} \times X)^* = ([n] \times i_{[n],g} \times X)^* \psi([n] \times X)$$

coincide, since ψ is a strict morphism. In the rightmost region, the two horizontally composable natural transformations compose to $(i_{[n],g} \times X)^*(\Delta \times X)^*$, since \mathbb{D}'' is a 2-functor.

Since ϕ takes values in $\mathbb{D}'([n] \times -)_{\text{eq}}$, we have that $(i_{[n],g} \times X)^*\phi(X)$ is a natural isomorphism. Moreover, ψ takes values in $\mathbb{D}''([n] \times -)_{\text{eq}}$, and therefore $(i_{[n],g} \times X)^*\psi(X)$ is also a natural isomorphism. This, together with the aforementioned commutativity properties, shows that

$$(i_{[n],g} \times X)^*(\Delta \times X)^* \psi([n] \times X) \phi(X)(F)$$

is an isomorphism.

The passage from PDer^{str} to $\text{PDer}_{\text{eq}}^{\text{str}}$ is reminiscent of the passage from the category of ∞ -categories, regarded as an $(\infty, 2)$ -category, to the associated ∞ -category defined by restriction to the maximal ∞ -groupoids of the morphism ∞ -categories. More on the viewpoint that regards well-behaved types of prederivators as models for homotopy theories will be discussed in Appendix A; see also [Renaudin 2009].

3.3. Strong equivalences. The prederivator $\mathbb{D}([1] \times -)_{\text{eq}}$ together with the factorization

$$\mathbb{D} \xrightarrow{s_0} \mathbb{D}([1] \times -)_{\text{eq}} \xrightarrow{(d_1, d_0)} \mathbb{D} \times \mathbb{D}$$

of the diagonal natural transformation will be regarded as a path object associated with \mathbb{D} . We can now introduce some basic homotopical notions in the context of prederivators.

Definition 3.3.1. Let $\phi_0, \phi_1 : \mathbb{D} \rightarrow \mathbb{D}'$ be two strict morphisms of prederivators. A *strong isomorphism from ϕ_0 to ϕ_1* is a 1-simplex of $\underline{\text{PDer}}_{\text{eq}}^{\text{str}}(\mathbb{D}, \mathbb{D}')$,

$$\Psi : \mathbb{D} \rightarrow \mathbb{D}'([1] \times -)_{\text{eq}},$$

such that $d_1(\Psi) = \phi_0$ and $d_0(\Psi) = \phi_1$. We say that ϕ_0 is *strongly isomorphic* to ϕ_1 , written $\phi_0 \simeq \phi_1$, if there is a zigzag of strong isomorphisms from ϕ_0 to ϕ_1 .

Obviously the relation \simeq is exactly the relation that two vertices of $\underline{\text{PDer}}_{\text{eq}}^{\text{str}}(\mathbb{D}, \mathbb{D}')$ lie on the same component.

Definition 3.3.2. Let \mathbb{D} and \mathbb{D}' be prederivators.

- (a) A strict morphism $\phi : \mathbb{D} \rightarrow \mathbb{D}'$ is called a *strong* (or *coherent*) *equivalence* if there is a strict morphism $\psi : \mathbb{D}' \rightarrow \mathbb{D}$ such that $\text{id}_{\mathbb{D}} \simeq \psi\phi$ and $\phi\psi \simeq \text{id}_{\mathbb{D}'}$.
- (b) \mathbb{D} and \mathbb{D}' are called *strongly* (or *coherently*) *equivalent* if there is a strong equivalence $\phi : \mathbb{D} \rightarrow \mathbb{D}'$.

Remark 3.3.3. A strong isomorphism Φ from ϕ_0 to ϕ_1 induces a natural isomorphism

$$\text{dia}_{[1], -}(0 \rightarrow 1) : \phi_0 \Longrightarrow \phi_1.$$

From this follows that strong equivalences of prederivators are also equivalences in the 2-categorical sense of the previous section.

Example 3.3.4. For every prederivator \mathbb{D} and any X in $\mathcal{D}ia$ with an initial object $x_0 \in \text{Ob } X$, the prederivator $\mathbb{D}(X \times -)_{\text{eq}}$ is strongly equivalent to \mathbb{D} . Indeed, consider the morphisms

$$(p \times -)^* : \mathbb{D}\mathbb{D}(X \times -)_{\text{eq}}, \quad (i_{X, x_0} \times -)^* : \mathbb{D}(X \times -)_{\text{eq}} \rightarrow \mathbb{D},$$

where $p : X \rightarrow e$ is the unique functor in this direction. Note that the underlying X -diagrams of elements in the image of $(p \times -)^*$ are constant functors (cf. Remark 2.1.4). Since pi_{X, x_0} is the identity functor on e , we have

$$(i_{X, x_0} \times -)^*(p \times -)^* = \text{id}_{\mathbb{D}}.$$

Moreover, $i_{X, x_0}p : X \rightarrow X$ is the constant functor $x \mapsto x_0$, and since x_0 is initial, there is a unique functor $H : [1] \times X \rightarrow X$ with $H(0, -) = i_{X, x_0}p$ and $H(1, -) = \text{id}_X$.

The induced functor

$$(H \times -)^* : \mathbb{D}(X \times -)_{\text{eq}} \rightarrow \mathbb{D}([1] \times X \times -)_{\text{eq}} = \mathbb{D}(X \times -)_{\text{eq},1}$$

is a strong isomorphism from $(p \times -)^*(i_{X,x_0} \times -)^*$ to $\text{id}_{\mathbb{D}(X \times -)_{\text{eq}}}$. One can argue similarly if X has a final object. This shows, in particular, that the face and degeneracy operators in $\mathbb{D}_{\text{eq},\bullet}$ are strong equivalences.

The notion of strong equivalence differs from the standard notion of equivalence defined in terms of the 2-categorical structure of PDer . This observation will be crucial in connection with the definitions of K -theory that follow in the next sections.

Example 3.3.5. Let \mathbb{D} be a prederivator and $\text{iso}_n \mathbb{D}$ denote the prederivator for which $(\text{iso}_n \mathbb{D})(X)$ is the full subcategory spanned by the strings of n composable isomorphisms in the diagram category of $\text{Cat}([n], \mathbb{D}(X))$. Then the canonical “inclusion of identities” morphism $\mathbb{D} \rightarrow \text{iso}_n \mathbb{D}$ is clearly an equivalence of prederivators, but not a strong equivalence in general. This assertion is a consequence of the invariance properties of Waldhausen K -theory and will be justified in Remark 5.1.5 below.

Remark 3.3.6. In connection with the examples of Section 2.3, a natural transformation $\alpha : F \Rightarrow F'$ between homotopical functors $F, F' : (\mathcal{C}, \mathcal{W}) \rightarrow (\mathcal{C}', \mathcal{W}')$ induces a 1-simplex in $\underline{\text{PDer}}^{\text{str}}$,

$$\alpha_* : \mathbb{D}(\mathcal{C}', \mathcal{W}') \rightarrow \mathbb{D}(\mathcal{C}, \mathcal{W})([1] \times -),$$

which upgrades the 2-morphism $\mathbb{D}(\alpha) : \mathbb{D}(F) \Rightarrow \mathbb{D}(F')$. If α takes values in \mathcal{W}' then α_* is a 1-simplex in $\underline{\text{PDer}}_{\text{eq}}^{\text{str}}$. This implies that for every homotopical functor $F : (\mathcal{C}, \mathcal{W}) \rightarrow (\mathcal{C}', \mathcal{W}')$ which admits a “homotopy inverse” (i.e., there is a homotopical functor $G : \mathcal{C}' \rightarrow \mathcal{C}$ such that the composites FG and GF can be connected to the respective identity functors via zigzags of natural weak equivalences), the associated morphism $\mathbb{D}(F)$ is a strong equivalence.

A 2-category \mathcal{C} will be regarded as a simplicial category $N_\bullet \mathcal{C}$ via the nerve functor $N_\bullet : \text{Cat} \rightarrow \mathcal{S}\text{Set}$ from small categories to simplicial sets, which preserves products. We have a simplicial functor

$$\underline{\text{PDer}}^{\text{str}} \rightarrow N_\bullet \text{PDer}^{\text{str}},$$

which is the identity on objects, and is given on morphisms by the simplicial maps

$$\underline{\text{PDer}}^{\text{str}}(\mathbb{D}, \mathbb{D}') \rightarrow N_\bullet \text{PDer}^{\text{str}}(\mathbb{D}, \mathbb{D}') \tag{3.3.7}$$

defined using the functors $\text{dia}_{[\bullet], \dots}$. These simplicial maps also restrict to simplicial maps

$$\underline{\text{PDer}}_{\text{eq}}^{\text{str}}(\mathbb{D}, \mathbb{D}') \rightarrow N_\bullet \text{PDer}_{\text{eq}}^{\text{str}}(\mathbb{D}, \mathbb{D}')$$

which assemble to a simplicial functor

$$\rho : \underline{\text{PDer}}_{\text{eq}}^{\text{str}} \rightarrow N_{\bullet}\text{PDer}_{\text{eq}}^{\text{str}}$$

given by the identity on objects.

Consider the following adjoint pairs

$$\mathcal{S}\text{Set} \begin{array}{c} \xleftarrow{\tau_1} \\ \xrightarrow{N_{\bullet}} \end{array} \text{Cat} \begin{array}{c} \xleftarrow{\text{incl.}} \\ \xrightarrow{\quad} \end{array} \mathcal{G}\text{rd}.$$

Here $\mathcal{G}\text{rd}$ is the category of groupoids, $\mathcal{G}\text{rd} \rightarrow \text{Cat}$ is the inclusion, the lower arrows are the right adjoints, τ_1 is the fundamental category functor, and the composite $\mathcal{S}\text{Set} \rightarrow \mathcal{G}\text{rd}$ is the fundamental groupoid functor, denoted by Π_1 . All these functors preserve products, hence, for example, we can apply them to a simplicial category \mathcal{S} to obtain a 2-category $\tau_1\mathcal{S}$, or a category enriched in groupoids $\Pi_1\mathcal{S}$. In particular, the simplicial functors (3.3.7) and ρ above also define 2-functors

$$\tau_1\underline{\text{PDer}}^{\text{str}} \rightarrow \text{PDer}^{\text{str}}, \quad \Pi_1\underline{\text{PDer}}_{\text{eq}}^{\text{str}} \rightarrow \text{PDer}_{\text{eq}}^{\text{str}},$$

by adjunction. These functors are not 2-equivalences of 2-categories. This means that the simplicial enrichment of the category of prederivators encodes more structure than the 2-category of prederivators.

Similarly let $\underline{\text{Der}}^{\text{str}}$ and $\underline{\text{Der}}_{\text{eq}}^{\text{str}}$ denote the corresponding simplicial subcategories of $\underline{\text{PDer}}^{\text{str}}$ and $\underline{\text{PDer}}_{\text{eq}}^{\text{str}}$, respectively. In both cases the objects are pointed right derivators, and for a pair of pointed right derivators \mathbb{D} and \mathbb{D}' we have

$$\begin{aligned} \underline{\text{Der}}^{\text{str}}(\mathbb{D}, \mathbb{D}')_{\bullet} &= \text{Ob Der}^{\text{str}}(\mathbb{D}, \mathbb{D}'), \\ \underline{\text{Der}}_{\text{eq}}^{\text{str}}(\mathbb{D}, \mathbb{D}')_{\bullet} &= \text{Ob Der}^{\text{str}}(\mathbb{D}, \mathbb{D}'_{\text{eq}, \bullet}). \end{aligned}$$

This is well-defined because if X is in $\mathcal{D}\text{ia}$ and \mathbb{D} is a right (or left, pointed, stable/triangulated) derivator, then so are $\mathbb{D}(X \times -)$ and $\mathbb{D}(X \times -)_{\text{eq}}$.

Specializing the discussion above to pointed right derivators, we define similarly a simplicial functor

$$\rho : \underline{\text{Der}}_{\text{eq}}^{\text{str}} \rightarrow N_{\bullet}\text{Der}_{\text{eq}}^{\text{str}}.$$

Again, the associated 2-functor

$$\Pi_1\underline{\text{Der}}_{\text{eq}}^{\text{str}} \rightarrow \text{Der}_{\text{eq}}^{\text{str}}$$

is not a 2-equivalence.

4. Waldhausen K -theory of derivators

In this section, we define the Waldhausen K -theory of a pointed right derivator and show that it agrees with the Waldhausen K -theory of a strongly saturated derivable Waldhausen category.

4.1. The $\mathcal{S}_{\bullet\bullet}$ -construction. First we recall the analogue of Waldhausen's \mathcal{S}_\bullet -construction in the setting of derivators due to Garkusha [2005; 2006]. Let \mathbb{D} be a pointed right derivator. We denote by $\text{Ar}[n]$ the category (finite poset) of arrows of the poset $[n]$. Let $\mathcal{S}_n\mathbb{D}$ denote the full subcategory of $\mathbb{D}(\text{Ar}[n])$ spanned by objects F that satisfy the following conditions:

- (i) for every $0 \leq i \leq n$, the object $i_{\text{Ar}[n], i \rightarrow i}^* F \in \text{Ob } \mathbb{D}(e)$ is a zero object;
- (ii) for every $0 \leq i \leq j \leq k \leq n$, the restriction of F along the inclusion of the subcategory

$$\begin{array}{ccc} (i \rightarrow j) & \longrightarrow & (i \rightarrow k) \\ \downarrow & & \downarrow \\ (j \rightarrow j) & \longrightarrow & (j \rightarrow k) \end{array}$$

of $\text{Ar}[n]$, isomorphic to \square , is a cocartesian object of $\mathbb{D}(\square)$.

This defines a simplicial category $\mathcal{S}_\bullet\mathbb{D}$ where the simplicial operators are defined by the structure of \mathbb{D} as a prederivator. Since morphisms in Der^{str} preserve cocartesian squares, it follows easily that the correspondence $\mathbb{D} \mapsto \mathcal{S}_\bullet\mathbb{D}$ defines a functor from the underlying 1-category $\text{Der}_{\text{eq}, 0}^{\text{str}}$ of $\text{Der}_{\text{eq}}^{\text{str}}$ (or Der^{str}) — which can also be obtained by forgetting the simplices of positive dimension in $\underline{\text{Der}}_{\text{eq}}^{\text{str}}$ — to the (1-)category of simplicial categories.

For the definition of Waldhausen K -theory, we need to consider a more refined version of this construction. Let $\mathcal{S}_{\bullet\bullet}\mathbb{D}$ be the bisimplicial set whose set of (n, m) -simplices $\mathcal{S}_{n, m}\mathbb{D}$ is the set of objects

$$F \in \text{Ob } \mathbb{D}([m] \times \text{Ar}[n])_{\text{eq}}$$

such that

- (*) for every $j : [0] \rightarrow [m]$ the object $(j \times \text{Ar}[n])^*(F) \in \text{Ob } \mathbb{D}(\text{Ar}[n])$ is in $\mathcal{S}_n\mathbb{D}$.

Note that if this condition holds for *some* $j : [0] \rightarrow [m]$ then it holds for all j . The bisimplicial operators are again defined using the structure of the underlying prederivator. Moreover, it is easy to see that the construction is natural in \mathbb{D} ; that is, we obtain a functor $\mathbb{D} \mapsto \mathcal{S}_{\bullet\bullet}\mathbb{D}$ from the underlying 1-category $\text{Der}_{\text{eq}, 0}^{\text{str}}$ of $\text{Der}_{\text{eq}}^{\text{str}}$ to the category of bisimplicial sets.

Definition 4.1.1. The Waldhausen K -theory of a pointed right derivator \mathbb{D} is defined to be the space $K^W(\mathbb{D}) := \Omega|\mathcal{S}_{\bullet\bullet}\mathbb{D}|$.

Our next goal is to show that the functor K^W can be extended to a simplicial functor from $\underline{\text{Der}}_{\text{eq}}^{\text{str}}$ to the simplicially enriched category of topological spaces $\underline{\text{Top}}$. Here the n -simplices of the simplicial mapping space $\underline{\text{Top}}(X, Y)$ between topological spaces X and Y are the continuous maps $X \times \Delta^n \rightarrow Y$, where Δ^n denotes the

geometric n -simplex. Since both the geometric realization functor and the loop space functor are simplicial, it is enough to show that the functor

$$\mathbb{D} \mapsto \text{diag } \mathcal{S}_{\bullet\bullet} \mathbb{D}$$

can be extended to a simplicial functor from $\underline{\text{Der}}_{\text{eq}}^{\text{str}}$ to the standard simplicially enriched category of simplicial sets $\underline{\text{SSet}}$. We recall that for simplicial sets X and Y , the n -simplices of $\underline{\text{SSet}}(X, Y)$ are the simplicial maps $X \times \Delta[n] \rightarrow Y$, where $\Delta[n]$ denotes the n -simplex and $|\Delta[n]| \cong \Delta^n$. A useful way of describing an n -simplex of $\underline{\text{SSet}}(X, Y)$ is by giving a natural transformation as follows (cf. [Waldhausen 1985, Section 1.4]):

$$\begin{array}{ccccc}
 & \text{source} & \rightarrow & \Delta^{\text{op}} & \xrightarrow{X} \\
 & \searrow & & \Downarrow \alpha & \searrow \\
 (\Delta \downarrow [n])^{\text{op}} & & & & \text{Set} \\
 & \swarrow & & \Downarrow & \swarrow \\
 & \text{source} & \rightarrow & \Delta^{\text{op}} & \xrightarrow{Y}
 \end{array}$$

Such a natural transformation α produces a simplicial map $\phi : X \times \Delta[n] \rightarrow Y$ which is defined by

$$\phi(x, [k] \xrightarrow{\sigma} [n]) = \alpha(\sigma)(x).$$

Conversely, a simplicial map $\phi : X \times \Delta[n] \rightarrow Y$ defines the components of such a natural transformation by setting $\alpha([k] \xrightarrow{\sigma} [n]) = \phi(-, \sigma) : X_k \rightarrow Y_k$.

Proposition 4.1.2. *Waldhausen K-theory extends to a simplicial functor*

$$K^W : \underline{\text{Der}}_{\text{eq}}^{\text{str}} \rightarrow \underline{\text{Top}}.$$

Proof. As remarked above, it suffices to show that the (1-)functor

$$\begin{aligned}
 \text{Der}_{\text{eq},0}^{\text{str}} &\rightarrow \text{SSet}, \\
 \mathbb{D} &\mapsto \text{diag } \mathcal{S}_{\bullet\bullet} \mathbb{D},
 \end{aligned}$$

extends to a simplicial functor from $\underline{\text{Der}}_{\text{eq}}^{\text{str}}$ to $\underline{\text{SSet}}$. This extension is defined as follows: given an n -simplex $\phi : \mathbb{D} \rightarrow \mathbb{D}'([n] \times -)_{\text{eq}}$ in $\underline{\text{Der}}_{\text{eq}}^{\text{str}}$, its image in $\underline{\text{SSet}}$ is an n -simplex, which we specify by giving the associated natural transformation

$$\begin{array}{ccccc}
 & \text{source} & \rightarrow & \Delta^{\text{op}} & \xrightarrow{\text{diag } \mathcal{S}_{\bullet\bullet} \mathbb{D}} \\
 & \searrow & & \Downarrow \phi_* & \searrow \\
 (\Delta \downarrow [n])^{\text{op}} & & & & \text{Set} \\
 & \swarrow & & \Downarrow & \swarrow \\
 & \text{source} & \rightarrow & \Delta^{\text{op}} & \xrightarrow{\text{diag } \mathcal{S}_{\bullet\bullet} \mathbb{D}'}
 \end{array}$$

The component of ϕ_* at an object $\sigma : [k] \rightarrow [n]$ in $\Delta \downarrow [n]$ is the map

$$\phi_*(\sigma) : \mathcal{S}_{k,k} \mathbb{D} \rightarrow \mathcal{S}_{k,k} \mathbb{D}'$$

defined as the (co)restriction of the map on objects that comes from the following functor:

$$\begin{array}{c}
 \mathbb{D}([k] \times \text{Ar}[k])_{\text{eq}} \\
 \downarrow \phi([k] \times \text{Ar}[k]) \\
 \mathbb{D}'([n] \times [k]) \times \text{Ar}[k]_{\text{eq}} \\
 \downarrow \mathbb{D}'(\sigma \times [k] \times \text{Ar}[k]) \\
 \mathbb{D}'([k] \times [k]) \times \text{Ar}[k]_{\text{eq}} \\
 \downarrow \mathbb{D}'(\Delta \times \text{Ar}[k]) \\
 \mathbb{D}'([k] \times \text{Ar}[k])_{\text{eq}}
 \end{array}$$

It is straightforward to check that ϕ_* is a natural transformation. Moreover, it is easy to check that the correspondence $\phi \mapsto \phi_*$ respects the composition (by arguments analogous to those in 3.1). □

As an immediate consequence, we have the following invariance property of Waldhausen K -theory.

Corollary 4.1.3. *Let $\phi : \mathbb{D} \rightarrow \mathbb{D}'$ be a strong equivalence of pointed right derivators. Then the induced map $K^W(\phi) : K^W(\mathbb{D}) \rightarrow K^W(\mathbb{D}')$ is a homotopy equivalence.*

4.2. The s_\bullet -construction. We mention a variant of the $S_{\bullet\bullet}$ -construction, which is actually the analogue of Waldhausen’s s_\bullet -construction in this context (cf. [Waldhausen 1985, Section 1.4]). Let $s_\bullet \mathbb{D}$ denote the simplicial set with n -simplices

$$s_n \mathbb{D} := \text{Ob } S_n \mathbb{D} = S_{n,0} \mathbb{D},$$

and define

$$K^{W,\text{ob}}(\mathbb{D}) := \Omega |s_\bullet \mathbb{D}|.$$

The inclusion of the 0-simplices defines a canonical comparison map

$$\iota : K^{W,\text{ob}}(\mathbb{D}) \rightarrow K^W(\mathbb{D}).$$

Proposition 4.2.1. *The comparison map ι is a weak equivalence.*

Proposition 4.2.1 is a consequence of the following lemma (cf. [Waldhausen 1985, Lemma 1.4.1]).

Lemma 4.2.2. *Let $\phi, \phi' : \mathbb{D} \rightarrow \mathbb{D}'$ be two cocontinuous strict morphisms of pointed right derivators. Then a 1-simplex Ψ in $\underline{\text{Der}}_{\text{eq}}^{\text{str}}$ with $d_1 \Psi = \phi$ and $d_0 \Psi = \phi'$ induces a simplicial homotopy $s_\bullet \phi \simeq s_\bullet \phi' : s_\bullet \mathbb{D} \rightarrow s_\bullet \mathbb{D}'$.*

Proof. The idea is analogous to the definition of the simplicial enhancement in Proposition 4.1.2. The required homotopy $s_{\bullet}\phi \simeq s_{\bullet}\phi'$ is a map $s_{\bullet}\mathbb{D} \times \Delta[1] \rightarrow s_{\bullet}\mathbb{D}'$, which we will specify by defining a natural transformation as follows:

$$\begin{array}{ccccc}
 & \text{source} & \rightarrow & \Delta^{\text{op}} & \xrightarrow{s_{\bullet}\mathbb{D}} \\
 (\Delta \downarrow [1])^{\text{op}} & & & \Downarrow \alpha & \\
 & \text{source} & \rightarrow & \Delta^{\text{op}} & \xrightarrow{s_{\bullet}\mathbb{D}'} \\
 & & & & \text{Set}
 \end{array}$$

Recall that Ψ is a strict cocontinuous morphism $\Psi : \mathbb{D} \rightarrow \mathbb{D}'([1] \times -)_{\text{eq}}$. Given an object $\sigma : [k] \rightarrow [1]$ in $\Delta \downarrow [1]$, we define $\alpha(\sigma)$ to be the (co)restriction of the map on objects that comes from the following functor:

$$\begin{array}{c}
 \mathbb{D}(\text{Ar}[k]) \\
 \downarrow \Psi_{(\text{Ar}[k])} \\
 \mathbb{D}'([1] \times \text{Ar}[k])_{\text{eq}} \\
 \downarrow (p \times \text{Ar}[k])^* \\
 \mathbb{D}'(\text{Ar}[1] \times \text{Ar}[k])_{\text{eq}} \\
 \downarrow (\text{Ar}(\sigma), \text{id}_{\text{Ar}[k]})^* \\
 \mathbb{D}'(\text{Ar}[k])
 \end{array}$$

Here $p : \text{Ar}[1] \rightarrow [1]$ is the functor defined by $p(0, 0) = 0$, $p(0, 1) = 1$, $p(1, 1) = 1$. The restriction of the composite functor to the s_{\bullet} -construction is well-defined because the 1-simplex Ψ is in $\underline{\text{Der}}_{\text{eq}}^{\text{str}}$ (and not merely in $\underline{\text{Der}}^{\text{str}}$). The naturality of α is straightforward to check. \square

Remark 4.2.3. Moreover, the functor $\mathbb{D} \mapsto s_{\bullet}\mathbb{D}$ extends to a simplicial functor from $\underline{\text{Der}}_{\text{eq}}^{\text{str}}$ to $\underline{\text{SSet}}$. The same argument works with $[1]$ replaced more generally by $[n]$ and p by the functor $\text{Ar}[n] \rightarrow [n]$, $(i \rightarrow j) \mapsto j$.

An immediate consequence is the following invariance under strong equivalences.

Corollary 4.2.4. *Let $\phi : \mathbb{D} \rightarrow \mathbb{D}'$ be a strong equivalence of pointed right derivators. Then the induced maps $|s_{\bullet}\phi| : |s_{\bullet}\mathbb{D}| \rightarrow |s_{\bullet}\mathbb{D}'|$ and $K^{W, \text{ob}}(\phi) : K^{W, \text{ob}}(\mathbb{D}) \rightarrow K^{W, \text{ob}}(\mathbb{D}')$ are homotopy equivalences.*

We can now return to the proof of Proposition 4.2.1.

Proof of Proposition 4.2.1. Since

$$|([n], [m]) \mapsto \mathcal{S}_{n,m}\mathbb{D}| \cong |[m] \mapsto |[n] \mapsto s_n\mathbb{D}([m] \times -)_{\text{eq}}|,$$

it suffices to show that every simplicial operator in the m -direction is a weak equivalence of simplicial sets after realizing in the n -direction. This follows from Corollary 4.2.4 and Example 3.3.4. \square

4.3. Agreement with Waldhausen K -theory. The agreement of K^W with the Waldhausen K -theory of well-behaved Waldhausen categories is based on results about the homotopically flexible variations of the \mathcal{S}_\bullet -construction by Blumberg and Mandell [2008] and Cisinski [2010b]. We recall that Waldhausen’s original \mathcal{S}_\bullet -construction [1985] of a Waldhausen category \mathcal{C} is a simplicial Waldhausen category $[n] \mapsto \mathcal{S}_n\mathcal{C}$, where the objects of $\mathcal{S}_n\mathcal{C}$ are given by diagrams $F : \text{Ar}[n] \rightarrow \mathcal{C}$ such that $F(i \rightarrow i)$ is the zero object for all $i \in [n]$, and for every $i \leq j \leq k$, the square

$$\begin{array}{ccc} F(i \rightarrow j) & \longrightarrow & F(i \rightarrow k) \\ \downarrow & & \downarrow \\ F(j \rightarrow j) & \longrightarrow & F(j \rightarrow k) \end{array}$$

has cofibrations as horizontal maps and is required to be a pushout. Restricting degreewise to the subcategory of (pointwise) weak equivalences gives a simplicial category $[n] \mapsto w\mathcal{S}_n\mathcal{C}$. We denote by $N_\bullet w\mathcal{S}_n\mathcal{C}$ the nerve of $w\mathcal{S}_n\mathcal{C}$. Then the Waldhausen K -theory of \mathcal{C} is defined to be the space $K(\mathcal{C}) := \Omega|N_\bullet w\mathcal{S}_\bullet\mathcal{C}|$.

Theorem 4.3.1. *Let \mathcal{C} be a strongly saturated derivable Waldhausen category. Then there is a natural weak equivalence*

$$K(\mathcal{C}) \xrightarrow{\sim} K^W(\mathbb{D}(\mathcal{C})).$$

Proof. The map is induced by a bisimplicial map $N_m w\mathcal{S}_n\mathcal{C} \rightarrow \mathcal{S}_{n,m}\mathbb{D}(\mathcal{C})$ which sends an element $[m] \times \text{Ar}[n] \rightarrow \mathcal{C}$ in $N_m w\mathcal{S}_n\mathcal{C}$ to the corresponding object of $\mathcal{S}_{n,m}\mathbb{D}$. Since \mathcal{C} is strongly saturated, so too are the Waldhausen categories $\mathcal{S}_n\mathcal{C}$ for every n . It follows that the bisimplicial set $\mathcal{S}_{*,\bullet}\mathbb{D}(\mathcal{C})$ is isomorphic to the bisimplicial set $N_\bullet w\mathcal{S}_*^h\mathcal{C}$ of [Cisinski 2010b] (and also to the bisimplicial set $N_\bullet w\mathcal{S}'_\bullet\mathcal{C}$ of [Blumberg and Mandell 2008], since every map can be replaced by a cofibration). Then the result follows from the agreement of the \mathcal{S}_\bullet^h -construction with the \mathcal{S}_\bullet -construction; see [Cisinski 2010b, Proposition 4.3] (cf. [Blumberg and Mandell 2008, Theorem 2.9] under the assumption that factorizations are functorial). \square

5. Derivator K -theory

5.1. Recollections and 2-categorical properties. Derivator K -theory was first defined for triangulated derivators by Maltsiniotis [2007]. The definition, however, applies similarly to all pointed right derivators. Here we consider the explicit model defined in terms of the \mathcal{S}_\bullet -construction which was introduced by Garkusha [2005;

2006], who also showed that it is equivalent to Maltsiniotis’s in the triangulated setting.

Definition 5.1.1. The *derivator K-theory* of a pointed right derivator \mathbb{D} is defined to be the space $K(\mathbb{D}) := \Omega|N_{\bullet} \text{ iso } \mathcal{S}_{\bullet}\mathbb{D}|$.

Since a cocontinuous strict morphism $\phi : \mathbb{D} \rightarrow \mathbb{D}'$ preserves cocartesian squares and zero objects, it can be easily checked that derivator K-theory defines a functor from $\text{Der}_{\text{eq},0}^{\text{str}}$ to the category of topological spaces. Moreover, it is invariant under equivalences of derivators.

Proposition 5.1.2. *If the strict morphism $\phi : \mathbb{D} \rightarrow \mathbb{D}'$ is an equivalence of pointed right derivators, then the induced map $K(\phi) : K(\mathbb{D}) \rightarrow K(\mathbb{D}')$ is a weak equivalence.*

Proof. This is an immediate consequence of the fact that the geometric realization of simplicial categories sends pointwise (weak) equivalences to weak equivalences of spaces. □

We emphasize that an equivalence of right pointed derivators does not necessarily admit a *strict* inverse. This means that an equivalence in Der is not in general a (2-categorical) equivalence in Der^{str} . However, the two concepts are closely related as morphisms of prederivators can be made strict up to a strict equivalence in PDer ; see [Cisinski and Neeman 2008, Proposition 10.14].

Derivator K-theory is compatible with the 2-categorical structure of $\text{Der}_{\text{eq}}^{\text{str}}$. We show next how to enhance derivator K-theory to a simplicial functor from $N_{\bullet}\text{Der}_{\text{eq}}^{\text{str}}$ to Top .

Proposition 5.1.3. *Derivator K-theory extends to a simplicial functor*

$$K : N_{\bullet}\text{Der}_{\text{eq}}^{\text{str}} \rightarrow \text{Top}.$$

Proof. It suffices to construct a simplicial enhancement for the (1-)functor

$$\begin{aligned} \text{Der}_{\text{eq},0}^{\text{str}} &\rightarrow \text{SSet}, \\ \mathbb{D} &\mapsto \text{diag } N_{\bullet} \text{ iso } \mathcal{S}_{\bullet}\mathbb{D}. \end{aligned}$$

Suppose we are given an n -simplex

$$\alpha = (\phi_0 \xrightarrow{\alpha_1} \phi_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} \phi_n)$$

in $\text{Der}_{\text{eq}}^{\text{str}}(\mathbb{D}, \mathbb{D}')$, where

$$\begin{array}{ccc} & \phi_{k-1} & \\ \curvearrowright & \downarrow \alpha_k & \curvearrowleft \\ \mathbb{D} & & \mathbb{D}' \\ \curvearrowleft & \downarrow \alpha_k & \curvearrowright \\ & \phi_k & \end{array}$$

are invertible modifications. We construct a simplicial map

$$(\text{diag } N_\bullet \text{ iso } \mathcal{S}_\bullet)(\alpha) : (\text{diag } N_\bullet \text{ iso } \mathcal{S}_\bullet \mathbb{D}) \times \Delta[n] \rightarrow (\text{diag } N_\bullet \text{ iso } \mathcal{S}_\bullet \mathbb{D}')$$

by defining a natural transformation as follows:

$$\begin{array}{ccc}
 & \text{source} \rightarrow \Delta^{\text{op}} & \xrightarrow{\text{diag } N_\bullet \text{ iso } \mathcal{S}_\bullet \mathbb{D}} \\
 (\Delta \downarrow [n])^{\text{op}} & \searrow & \downarrow \alpha_* \\
 & \Delta^{\text{op}} & \xrightarrow{\text{diag } N_\bullet \text{ iso } \mathcal{S}_\bullet \mathbb{D}'} \\
 & \text{source} \rightarrow & \text{Set}
 \end{array}$$

Given $\sigma : [k] \rightarrow [n]$ in Δ , the map

$$\alpha_*(\sigma) : N_k \text{ iso } \mathcal{S}_k \mathbb{D} \rightarrow N_k \text{ iso } \mathcal{S}_k \mathbb{D}'$$

is defined as follows. Let

$$\beta = \sigma^*(\alpha) = (\psi_0 \xrightarrow{\beta_1} \psi_1 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_k} \psi_k).$$

Consider an element in the domain of $\alpha_*(\sigma)$, denoted by (f_1, \dots, f_k) , which is a chain of k composable isomorphisms in $\mathcal{S}_k \mathbb{D} \subset \mathbb{D}(\text{Ar}[k])$:

$$X_0 \rightarrow \dots \rightarrow X_{r-1} \xrightarrow{f_r} X_r \rightarrow \dots \rightarrow X_k.$$

The k -simplex β gives rise to a $k \times k$ grid of commutative squares of solid arrows in $\mathcal{S}_k \mathbb{D}' \subset \mathbb{D}'(\text{Ar}[k])$:

$$\begin{array}{ccccccc}
 \psi_0(X_0) & \longrightarrow & \dots & \longrightarrow & \psi_0(X_{r-1}) & \xrightarrow{\psi_0(f_r)} & \psi_0(X_r) & \longrightarrow & \dots & \longrightarrow & \psi_0(X_k) \\
 \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
 \vdots & & & & \vdots & & \vdots & & & & \vdots \\
 \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
 \psi_{r-1}(X_0) & \longrightarrow & \dots & \longrightarrow & \psi_{r-1}(X_{r-1}) & \xrightarrow{\psi_{r-1}(f_r)} & \psi_{r-1}(X_r) & \longrightarrow & \dots & \longrightarrow & \psi_{r-1}(X_k) \\
 \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
 \beta_r(X_0) & & & & \beta_r(X_{r-1}) & & \beta_r(X_r) & & & & \beta_r(X_k) \\
 \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
 \psi_r(X_0) & \longrightarrow & \dots & \longrightarrow & \psi_r(X_{r-1}) & \xrightarrow{\psi_r(f_r)} & \psi_r(X_r) & \longrightarrow & \dots & \longrightarrow & \psi_r(X_k) \\
 \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
 \vdots & & & & \vdots & & \vdots & & & & \vdots \\
 \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
 \psi_k(X_0) & \longrightarrow & \dots & \longrightarrow & \psi_k(X_{r-1}) & \xrightarrow{\psi_k(f_r)} & \psi_k(X_r) & \longrightarrow & \dots & \longrightarrow & \psi_k(X_k)
 \end{array}$$

We set $\alpha_*(\sigma)(f_1, \dots, f_k)$ to be the sequence of k diagonal morphisms, depicted above as dashed arrows,

$$\psi_0(X_0) \longrightarrow \cdots \longrightarrow \psi_{r-1}(X_{r-1}) \xrightarrow[\psi_r(f_r)\beta_r(X_{r-1})]{\beta_r(X_r)\psi_{r-1}(f_r)} \psi_r(X_r) \longrightarrow \cdots \longrightarrow \psi_k(X_k).$$

The naturality of α_* in σ is straightforward. For the compatibility with composition, we consider an n -simplex in $N_{\bullet}\text{Der}_{\text{eq}}^{\text{str}}(\mathbb{D}', \mathbb{D}'')$,

$$\alpha' = (\phi'_0 \xrightarrow{\alpha'_1} \phi'_1 \xrightarrow{\alpha'_2} \cdots \xrightarrow{\alpha'_n} \phi'_n),$$

and then it suffices to check that for all $\sigma : [k] \rightarrow [n]$ in $\Delta \downarrow [n]$, we have

$$(\alpha'\alpha)_*(\sigma) = \alpha'_*(\sigma)\alpha_*(\sigma).$$

Indeed, if

$$\beta' = \sigma^*(\alpha') = (\psi'_0 \xrightarrow{\beta'_1} \psi'_1 \xrightarrow{\beta'_2} \cdots \xrightarrow{\beta'_k} \psi'_k)$$

then each of the maps above, when applied to an element $(f_1, \dots, f_k) \in N_k \text{ iso } \mathcal{S}_k \mathbb{D}$, gives

$$\begin{array}{ccc} \psi'_0\psi_0(X_0) \longrightarrow \cdots \longrightarrow \psi'_{r-1}\psi_{r-1}(X_{r-1}) & & \\ & \downarrow (\beta'_r\beta_r)(X_r)(\psi'_r\psi_{r-1})(f_r) & \\ & \psi'_r\psi_r(X_r) \longrightarrow \cdots \longrightarrow \psi'_k\psi_k(X_k). & \end{array}$$

This k -simplex can be obtained as a diagonal in a 3-dimensional cube, in the same way that $\alpha_*(\sigma)(f_1, \dots, f_k)$ is a diagonal in a square. Therefore, the vertical map can be written in six different ways. We have just chosen two of them. \square

Remark 5.1.4. The proposition shows that the homotopy class of the morphism $K(\phi) : K(\mathbb{D}) \rightarrow K(\mathbb{D}')$ depends only on the isomorphism class of $\phi : \mathbb{D} \rightarrow \mathbb{D}'$ in the 2-category $\text{Der}_{\text{eq}}^{\text{str}}$. This together with the invariance of derivator K -theory under equivalences implies that derivator K -theory is in fact functorial in the homotopy category of spaces with respect to *all* morphisms of derivators. More precisely, if for a category \mathcal{G} which is enriched in groupoids, we denote by $\pi_0\mathcal{G}$ the 1-category obtained by identifying isomorphic morphisms, then there exists a unique factorization

$$\begin{array}{ccc} \pi_0\text{Der}_{\text{eq}}^{\text{str}} & \xrightarrow{K} & \text{Top}/\simeq \\ \pi_0\rho \downarrow & \nearrow & \\ \pi_0\text{Der}_{\text{eq}} & & \end{array}$$

Here Top/\simeq is the homotopy category of topological spaces. Compare [Cisinski and Neeman 2008, Corollary 10.19].

Remark 5.1.5. Derivator K -theory $K(\mathbb{D})$ is weakly equivalent to the geometric realization of Waldhausen K -theories

$$|[n] \mapsto K^{W, \text{ob}}(\text{iso}_n \mathbb{D})|.$$

Note that the derivators $\{\text{iso}_n \mathbb{D}\}_{n \geq 0}$ are equivalent in Der and the simplicial operators are equivalences of derivators. Therefore, given that Waldhausen and derivator K -theory are different in general [Muro and Raptis 2011], it follows that Waldhausen K -theory is not invariant under equivalences of derivators. In particular, it follows that there are equivalences of (pre)derivators which are not strong, and more specifically, there are derivators \mathbb{D} such that the canonical “degeneracy” equivalence $\mathbb{D} \rightarrow \text{iso}_n \mathbb{D}$ is not a strong equivalence.

In the case where $\mathbb{D} = \mathbb{D}(\mathcal{C})$ for some derivable Waldhausen category $(\mathcal{C}, \text{co}\mathcal{C}, w\mathcal{C})$, the following variant of derivator K -theory is available. Passing to the homotopy categories of the \mathcal{S}_\bullet -construction, we obtain a new simplicial category $[n] \mapsto \text{Ho } \mathcal{S}_n \mathcal{C}$, and a canonical morphism of simplicial categories $\text{Ho } \mathcal{S}_\bullet \mathcal{C} \rightarrow \mathcal{S}_\bullet \mathbb{D}(\mathcal{C})$. This is degreewise an equivalence of categories, and therefore the induced map

$$\Omega |N_\bullet \text{ iso Ho } \mathcal{S}_\bullet \mathcal{C}| \xrightarrow{\sim} K(\mathbb{D}(\mathcal{C}))$$

is a weak equivalence.

5.2. Comparison with Waldhausen K -theory. There is a natural comparison map from Waldhausen to derivator K -theory. For $n, m \geq 0$, the functors

$$\text{dia}_{[m], \text{Ar}[n]} : \mathbb{D}([m] \times \text{Ar}[n]) \rightarrow \text{Cat}([m], \mathbb{D}(\text{Ar}[n]))$$

assemble to define a bisimplicial map

$$\mathcal{S}_{\bullet\bullet} \mathbb{D} \rightarrow N_\bullet \text{ iso } \mathcal{S}_\bullet \mathbb{D}$$

which then induces the comparison map

$$\mu : K^W(\mathbb{D}) \rightarrow K(\mathbb{D})$$

from Waldhausen K -theory to derivator K -theory (cf. [Maltsiniotis 2007; Garkusha 2005; Muro and Raptis 2011]). We note that composing with the weak equivalence $\iota : K^{W, \text{ob}}(\mathbb{D}) \rightarrow K^W(\mathbb{D})$, we obtain

$$\begin{array}{ccc} K^{W, \text{ob}}(\mathbb{D}) & \xrightarrow[\sim]{\iota} & K^W(\mathbb{D}) \\ & \searrow \mu^{\text{ob}} & \downarrow \mu \\ & & K(\mathbb{D}) \end{array}$$

where the map μ^{ob} is given degreewise simply by the inclusion of objects. The comparison maps μ and μ^{ob} define natural transformations. Moreover, μ defines

a natural transformation of simplicially enriched functors

$$\begin{array}{ccc}
 \underline{\text{Der}}_{\text{eq}}^{\text{str}} & \xrightarrow{K^W} & \text{Top} \\
 \rho \downarrow & \mu \Downarrow & \nearrow \\
 N_{\bullet}\text{Der}_{\text{eq}}^{\text{str}} & \xrightarrow{K} & \text{Top}
 \end{array}$$

and the same holds for μ^{ob} (cf. Lemma 4.2.2 and Remark 4.2.3). However, making use of these simplicial enrichments will not be required in what follows since it is possible to think of them, in a homotopical fashion, only as asserting certain invariance properties. We will concentrate instead on the natural transformation

$$\begin{array}{ccc}
 \text{Der}_{\text{eq},0}^{\text{str}} & \xrightarrow{K^{W,\text{ob}}} & \text{Top} \\
 \mu^{\text{ob}} \Downarrow & & \nearrow \\
 & \xrightarrow{K} & \text{Top}
 \end{array}$$

because this is technically a more convenient model of the comparison map for the statement of our results. Here Top is the ordinary category of topological spaces.

In connection with the diagram above, it is interesting to mention that Toën and Vezzosi [2004] gave a neat abstract argument, based only on functoriality, to show that Waldhausen K -theory cannot factor through $N_{\bullet}\text{Der}_{\text{eq}}^{\text{str}}$ by a functor which is invariant under equivalences of derivators.

Maltsiniotis [2007] conjectured that μ is a weak equivalence when \mathbb{D} is the triangulated derivator associated with an exact category [Keller 2007]. This conjecture remains open, but several relevant results are known. Garkusha [2006], based on previous results by Neeman on the K -theory of triangulated categories, showed that μ admits a retraction when \mathbb{D} arises from an abelian category. Maltsiniotis [2007] and Muro [2008] showed that μ induces an isomorphism on π_0 and π_1 , respectively, for any \mathbb{D} that arises from a strongly saturated derivable Waldhausen category. In [Muro and Raptis 2011], we showed that μ fails to be a weak equivalence in general for triangulated derivators that arise from differential graded algebras (or stable module categories). Moreover, we showed that the conjecture fails if derivator K -theory satisfies *localization*, a property also conjectured by Maltsiniotis [2007].

However, the pair (K, μ) turns out to be the best approximation to Waldhausen K -theory by a functor which sends equivalences of derivators to weak equivalences. We choose a rather ad hoc but direct way of formulating this property precisely as follows.

First, in order to ensure that our categories remain locally small and so to avoid set-theoretical troubles, we fix a (small) set S of pointed right derivators \mathbb{D} closed

under taking $\text{iso } \mathbb{D}$, and restrict to the full subcategory of $\text{Der}_{\text{eq},0}^{\text{str}}$ spanned by S , which we still denote by $\text{Der}_{\text{eq},0}^{\text{str}}$. Second, it will be more convenient to work here with simplicial techniques and the delooped versions of Waldhausen and derivator K -theory. Thus we set

$$\begin{aligned} \Omega^{-1}K^W(\mathbb{D}) &:= \text{diag } S_{\bullet\bullet}\mathbb{D}, \\ \Omega^{-1}K^{W,\text{ob}}(\mathbb{D}) &:= s_{\bullet}\mathbb{D}, \\ \Omega^{-1}K(\mathbb{D}) &:= \text{diag } \text{iso}_{\bullet} S_{\bullet}\mathbb{D}, \end{aligned}$$

and we have natural transformations

$$\begin{aligned} \mu &: \Omega^{-1}K^W \Rightarrow \Omega^{-1}K, \\ \mu^{\text{ob}} &: \Omega^{-1}K^{W,\text{ob}} \Rightarrow \Omega^{-1}K, \end{aligned}$$

and a natural weak equivalence $\iota : \Omega^{-1}K^{W,\text{ob}} \Rightarrow \Omega^{-1}K^W$ with $\mu^{\text{ob}} = \mu\iota$.

Definition 5.2.1. Let $\mathcal{S}\text{Set}^{\text{Der}_{\text{eq},0}^{\text{str}}}$ be the functor category. The *category of invariant approximations* to Waldhausen K -theory $\Omega^{-1}K^{W,\text{ob}}$ is the full subcategory of the comma category $\Omega^{-1}K^{W,\text{ob}} \downarrow \mathcal{S}\text{Set}^{\text{Der}_{\text{eq},0}^{\text{str}}}$ spanned by the objects $\eta : \Omega^{-1}K^{W,\text{ob}} \Rightarrow F$ such that $F : \text{Der}_{\text{eq},0}^{\text{str}} \rightarrow \mathcal{S}\text{Set}$ sends equivalences of derivators to weak equivalences. A morphism

$$\begin{array}{ccc} & \Omega^{-1}K^{W,\text{ob}} & \\ \eta \swarrow & & \searrow \eta' \\ F & \xrightarrow{\quad u \quad} & F' \end{array}$$

in App is a *weak equivalence* if the components of u are weak equivalences of simplicial sets.

Note that $\mu^{\text{ob}} : \Omega^{-1}K^{W,\text{ob}} \Rightarrow \Omega^{-1}K$ is an object of App . Following [Dwyer et al. 2004], we say that an object X of a category with weak equivalences $(\mathcal{C}, \mathcal{W})$ (satisfying in addition the “2-out-of-6” property) is *homotopically initial* if there are homotopical functors $F_0, F_1 : \mathcal{C} \rightarrow \mathcal{C}$ and a natural transformation $f : F_0 \Rightarrow F_1$ such that

- (i) F_0 is naturally weakly equivalent to the constant functor at X ,
- (ii) F_1 is naturally weakly equivalent to the identity functor on \mathcal{C} , and
- (iii) $f_X : F_0(X) \rightarrow F_1(X)$ is a weak equivalence.

If X is initial in \mathcal{C} , then it is also homotopically initial in this sense. If X is homotopically initial in \mathcal{C} , then X is initial in $\text{Ho } \mathcal{C}$. Finally, the category of homotopically initial objects in $(\mathcal{C}, \mathcal{W})$ is either empty or homotopically contractible. We refer the reader to [Dwyer et al. 2004] for more details.

Theorem 5.2.2. *The object $\mu^{\text{ob}} : \Omega^{-1}K^{W,\text{ob}} \Rightarrow \Omega^{-1}K$ is homotopically initial in the category with weak equivalences App .*

Proof. Let $F : \text{Der}_{\text{eq},0}^{\text{str}} \rightarrow \mathcal{S}\text{Set}$ be a functor. Then there is a canonical way of associating to F a new functor

$$\begin{aligned} \mathbb{H}F : \text{Der}_{\text{eq},0}^{\text{str}} &\rightarrow \mathcal{S}\text{Set}, \\ \mathbb{D} &\mapsto \text{diag } F(\text{iso}_{\bullet} \mathbb{D}). \end{aligned}$$

The inclusion of 0-simplices defines a natural transformation

$$\iota_F : F \Longrightarrow \mathbb{H}F.$$

By definition, we have

$$\iota_{\Omega^{-1}K^{W,\text{ob}}} = \mu^{\text{ob}} : \Omega^{-1}K^{W,\text{ob}} \Longrightarrow \mathbb{H}\Omega^{-1}K^{W,\text{ob}} = \Omega^{-1}K.$$

If F sends equivalences of derivators to weak equivalences then the simplicial operators of $F(\text{iso}_{\bullet} \mathbb{D})$ in the iso_{\bullet} -direction are weak equivalences, so ι_F is a natural weak equivalence. In this case, it follows that $\mathbb{H}F$ also sends equivalences of derivators to weak equivalences. Using this fact, we can view the \mathbb{H} -construction as an endofunctor, denoted $\tilde{\mathbb{H}} : \text{App} \rightarrow \text{App}$, which sends an object $\eta : \Omega^{-1}K^{W,\text{ob}} \Rightarrow F$ in App to the natural transformation $\tilde{\mathbb{H}}(\eta)$ given by the diagonal in the following commutative square:

$$\begin{array}{ccc} \Omega^{-1}K^{W,\text{ob}} & \xrightarrow{\eta} & F \\ \mu^{\text{ob}} \downarrow & & \sim \downarrow \iota_F \\ \Omega^{-1}K & \xrightarrow{\mathbb{H}\eta} & \mathbb{H}F \end{array}$$

The natural transformation ι induces a natural weak equivalence $\iota' : \text{id}_{\text{App}} \Rightarrow \tilde{\mathbb{H}}$ given by the right vertical arrow, and the bottom horizontal arrow defines a natural transformation from the constant functor at $\mu^{\text{ob}} : \Omega^{-1}K^{W,\text{ob}} \Rightarrow \Omega^{-1}K$ to $\tilde{\mathbb{H}}$. Hence, the result follows. \square

We wish to remark that we could have worked entirely with simplicially enriched categories and functors in this section. More specifically, the construction $\mathbb{H}F$ in the proof of the last theorem has a simplicial enhancement which can be constructed as in the proof of Proposition 5.1.3. We decided to work with 1-categories in order to avoid the ensuing technicalities.

6. Some open questions

6.1. Derivators and the homotopy theory of homotopy theories. The simplicial enrichment of the category of derivators leads to a homotopy theory of derivators which is more discerning than the 2-categorical one and is closer to the homotopy theory of categories with weak equivalences. An interesting problem is to

understand exactly how close this relationship is, and find out whether this homotopy theory of derivators is rich and structured enough to be (or contain a part of) a model for the homotopy theory of homotopy theories. In the case of the 2-category of derivators, a theorem of Renaudin [2009] specified the relationship between combinatorial model categories and their associated derivators (see also Appendix A for a review). In this context, the question would be whether this result can be improved in view of the simplicial enrichment of derivators. The results of Appendix A may be a first step in this direction.

6.2. *Derived equivalences vs. strong equivalences.* We do not know whether an exact functor of well-behaved Waldhausen categories which is a derived equivalence also induces a strong equivalence between the associated pointed right derivators. This is clear in the case where the derived equivalence admits a homotopy inverse (cf. Remark 3.3.6), but such an inverse may not exist strictly at the level of models in general. If the statement is true, then we will be able to deduce the invariance of Waldhausen K -theory (of Waldhausen categories) under derived equivalences also from the invariance of Waldhausen K -theory of pointed right derivators under strong equivalences.

6.3. *Additivity for derivator K -theory.* The additivity of derivator K -theory was proved by Cisinski and Neeman [2008] for triangulated derivators. However, the more general case of additivity for derivator K -theory of pointed right derivators seems to remain an open problem. We emphasize that this seems to be so also in the case where the derivator admits a model. In this case, we tried to apply Waldhausen's original proof and generalize the approach in [Garkusha 2005], but we discovered a gap in the proof of [Garkusha 2005, Theorem 3.1] which we could not fix. (Namely, in diagram (7), at the bottom of page 655, the arrow $\varphi_{X_i}^* v_i c_i : V_i'' \rightarrow \bar{X}_i$ need not be a weak equivalence.) In particular, we do not know whether derivator K -theory of pointed right derivators is invariant under an appropriately defined notion of stabilization which would produce a triangulated derivator.

A related problem is to show that additivity holds for the Waldhausen K -theory of pointed right derivators. Of course, this is true if the derivator admits a model. However it would still be interesting to establish the general case as it is through this generality that the concept of derivator can also be tested.

Appendix A. Combinatorial model categories and derivators

The purpose of this appendix is to highlight some results on the connections between combinatorial model categories and derivators. Since the discussion is heavily based on [Renaudin 2009], we will give a very concise review of his results while providing precise references where necessary. Then we will record some

minor strengthenings of Renaudin’s main theorem with a view to addressing the questions of Section 6.1.

Let $\mathcal{M}\mathcal{O}\mathcal{D}$ denote the 2-category of left proper combinatorial model categories, Quillen adjunctions and natural transformations between left Quillen functors. Following [Renaudin 2009], we view the morphism categories as categories with weak equivalences where the weak equivalences are given by *Quillen homotopies*. We recall that a natural transformation of left Quillen functors is a Quillen homotopy if it is pointwise a weak equivalence at the cofibrant objects; see [Renaudin 2009, Définition 2.1.2]. Passing to the homotopy categories of all morphism categories yields a new 2-category $\underline{\mathcal{M}\mathcal{O}\mathcal{D}}$. We note that $\underline{\mathcal{M}\mathcal{O}\mathcal{D}}$ is enriched in the category of all categories $\mathcal{C}\mathcal{A}\mathcal{T}$.

The class of Quillen equivalences in $\underline{\mathcal{M}\mathcal{O}\mathcal{D}}$ admits a calculus of right fractions [Renaudin 2009, Proposition 2.3.2]. Thus the bilocalization of $\underline{\mathcal{M}\mathcal{O}\mathcal{D}}$ at the class of Quillen equivalences exists, denoted here by $\underline{\mathcal{M}\mathcal{O}\mathcal{D}}[Q^{-1}]$, and is actually equivalent to the bilocalization of $\mathcal{M}\mathcal{O}\mathcal{D}$ at the class of Quillen equivalences [Renaudin 2009, Théorème 2.3.3].

Let $\mathcal{M}\mathcal{O}\mathcal{D}^p$ be the 1- and 2-full subcategory of presentable model categories — that is, combinatorial model categories that arise from a left Bousfield localization of the projective model category of C -diagrams in $\mathcal{S}\mathcal{S}et$, for some small category C , at a set of morphisms S . Every combinatorial model category is equivalent to a presentable one [Dugger 2001]. Presentable model categories have certain nice “cofibrancy” properties which can in particular be used to show that the 2-functor

$$\underline{\mathcal{M}\mathcal{O}\mathcal{D}}^p \rightarrow \underline{\mathcal{M}\mathcal{O}\mathcal{D}}[Q^{-1}]$$

is a biequivalence [Renaudin 2009, Proposition 2.3.4]. Here $\underline{\mathcal{M}\mathcal{O}\mathcal{D}}^p$ denotes the corresponding 1- and 2-full subcategory of $\underline{\mathcal{M}\mathcal{O}\mathcal{D}}$. The restriction to presentable model categories in what follows is mainly a technical matter and is due essentially to the rigidity of $\mathcal{M}\mathcal{O}\mathcal{D}$ compared, say, to the essentially equivalent context of presentable ∞ -categories.

Let $\mathcal{D}\mathcal{E}\mathcal{R}$ (resp. $\mathcal{D}\mathcal{E}\mathcal{R}_1$, $\mathcal{D}\mathcal{E}\mathcal{R}_{ad}$) denote the 2-category of derivators with domain $Dia = Cat$ and values in the 2-category $\mathcal{C}\mathcal{A}\mathcal{T}$ together with pseudonatural transformations (resp. cocontinuous morphisms, adjunctions between derivators) as 1-morphisms, and modifications as 2-morphisms. Cisinski [2003] constructed a pseudofunctor

$$\mathbb{D}(-) : \mathcal{M}\mathcal{O}\mathcal{D} \rightarrow \mathcal{D}\mathcal{E}\mathcal{R}_{ad}$$

which is defined on objects by $\mathcal{M} \mapsto \mathbb{D}(\mathcal{M})$ (cf. Section 2.3) and sends Quillen equivalences to equivalences of derivators. We note that $\mathbb{D}(\mathcal{M})$ takes values in locally small categories. There is an induced pseudofunctor of 2-categories

$$\underline{\mathbb{D}}(-) : \underline{\mathcal{M}\mathcal{O}\mathcal{D}}[Q^{-1}] \rightarrow \mathcal{D}\mathcal{E}\mathcal{R}_{ad}.$$

Renaudin [2009, Théorème 3.3.2] showed that this functor is a *local equivalence*, i.e., it induces equivalences between the morphism categories. This could be interpreted as identifying a part of \mathcal{DER}_{ad} with a truncation of the homotopy theory of homotopy theories as modeled by \mathcal{MOD} . For our purposes, it will be necessary to reformulate this result in terms of the larger 2-category $\mathcal{DER}_!$ (cf. [Renaudin 2009, Remarque 3.3.3]).

Theorem A.1. *The canonical pseudofunctor*

$$\mathbb{D}(-) : \mathcal{MOD}^p \rightarrow \mathcal{DER}_!$$

is a local equivalence.

Proof. Since the composition $\mathcal{MOD}^p \rightarrow \mathcal{MOD}[Q^{-1}] \rightarrow \mathcal{DER}_{ad}$ is a local equivalence, it suffices to show that for all \mathcal{M} and \mathcal{N} in \mathcal{MOD}^p , the fully faithful inclusion functor

$$\mathcal{DER}_{ad}(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N})) \hookrightarrow \mathcal{DER}_!(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N})) \tag{A.2}$$

is also essentially surjective. Let $F : \mathbb{D}(\mathcal{M}) \rightarrow \mathbb{D}(\mathcal{N})$ be a cocontinuous morphism. Suppose that $\mathcal{M} = L_S SSet^C$. The Quillen adjunction $\text{Id} : SSet^C \rightleftarrows \mathcal{M} : \text{Id}$ induces a morphism in \mathcal{DER}_{ad} , denoted as

$$\mathbb{L}_S(\text{Id}) : \mathbb{D}(SSet^C) \rightleftarrows \mathbb{D}(\mathcal{M}) : \mathbb{R}_S(\text{Id}).$$

The composite $F' = F \circ \mathbb{L}_S(\text{Id}) : \mathbb{D}(SSet^C) \rightarrow \mathbb{D}(\mathcal{N})$ is a cocontinuous morphism. By [Renaudin 2009, Remarque 3.3.3] (or more directly, by using the universal property of $SSet^C$ due to Dugger [Renaudin 2009, Proposition 2.2.7], and that of $\mathbb{D}(SSet^C)$ due to Cisinski [Renaudin 2009, Théorème 3.3.1]), there is a Quillen adjunction

$$\tilde{F}' : SSet^C \rightleftarrows \mathcal{N} : \tilde{G}'$$

such that $\mathbb{D}(\tilde{F}')$ is isomorphic to F' in $\mathcal{DER}_!(\mathbb{D}(SSet^C), \mathbb{D}(\mathcal{N}))$. Then the universal property of Bousfield localization shows that (\tilde{F}', \tilde{G}') descends to a Quillen adjunction

$$\tilde{F}'' : L_S SSet^C \rightleftarrows \mathcal{N} : \tilde{G}''$$

such that $\mathbb{D}(\tilde{F}'') \circ \mathbb{L}_S(\text{Id})$ is isomorphic to $F \circ \mathbb{L}_S(\text{Id})$. Then

$$\mathbb{D}(\tilde{F}'') : \mathbb{D}(\mathcal{M}) \rightleftarrows \mathbb{D}(\mathcal{N}) : \mathbb{D}(\tilde{G}'')$$

is an adjunction of derivators and the left adjoint $\mathbb{D}(\tilde{F}'')$ is isomorphic to F since the functor

$$\mathbb{L}_S(\text{Id})^* : \mathcal{DER}_!(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N})) \rightarrow \mathcal{DER}_!(\mathbb{D}(SSet^C), \mathbb{D}(\mathcal{N}))$$

is fully faithful [Tabuada 2008, Definition 4.2, Lemma 4.3 and Theorem 4.4]. \square

We would like to emphasize that the equivalence of categories (A.2) in the last proof can be regarded as an adjoint functor theorem for derivators that arise from combinatorial model categories.

We recall from [Groth 2012] the construction of internal hom-objects in the 2-category of derivators. Given prederivators $\mathbb{D}, \mathbb{D}' : \mathit{Cat} \rightarrow \mathcal{CAT}$ there is a prederivator $\mathit{HOM}(\mathbb{D}, \mathbb{D}') : \mathit{Cat}^{\text{op}} \rightarrow \mathcal{CAT}$ which is defined explicitly by

$$\mathit{HOM}(\mathbb{D}, \mathbb{D}')(X) = \mathcal{DER}(\mathbb{D}, \mathbb{D}'_X).$$

Moreover, if \mathbb{D}' is a derivator, then so is $\mathit{HOM}(\mathbb{D}, \mathbb{D}')$; see [Groth 2012, Proposition 1.20]. The simplicial enrichments of the previous sections are obtained from this by setting $X = [n]$ and restricting to the objects. If \mathbb{D} and \mathbb{D}' are derivators we also consider the following closely related prederivator

$$\begin{aligned} \mathit{HOM}_!(\mathbb{D}, \mathbb{D}') : \mathit{Cat}^{\text{op}} &\rightarrow \mathcal{CAT}, \\ X &\mapsto \mathcal{DER}_!(\mathbb{D}, \mathbb{D}'_X). \end{aligned}$$

To see that this is again a prederivator, it suffices to consider $u : X \rightarrow Y$ in Cat and a cocontinuous morphism $\phi : \mathbb{D} \rightarrow \mathbb{D}'_Y$, and then note that the morphism

$$\mathit{HOM}(\mathbb{D}, \mathbb{D}')(u)(\phi) := u^* \phi : \mathbb{D} \rightarrow \mathbb{D}'_Y \rightarrow \mathbb{D}'_X$$

is again cocontinuous because $u^* : \mathbb{D}'_Y \rightarrow \mathbb{D}'_X$ is cocontinuous (in fact, it admits a right adjoint $u_* : \mathbb{D}'_X \rightarrow \mathbb{D}'_Y$).

Similarly, it is easy to check that $\mathit{HOM}_!(\mathbb{D}, \mathbb{D}')$ is in fact a right derivator. For every $u : X \rightarrow Y$, the pullback functor

$$u^* : \mathcal{DER}_!(\mathbb{D}, \mathbb{D}'_Y) \rightarrow \mathcal{DER}_!(\mathbb{D}, \mathbb{D}'_X)$$

defined above admits a left adjoint

$$u_! : \mathcal{DER}_!(\mathbb{D}, \mathbb{D}'_X) \rightarrow \mathcal{DER}_!(\mathbb{D}, \mathbb{D}'_Y)$$

which is defined as for the derivator $\mathit{HOM}(\mathbb{D}, \mathbb{D}')$: given a cocontinuous morphism $\phi : \mathbb{D} \rightarrow \mathbb{D}'_X$, set

$$u_!(\phi) := u_! \phi : \mathbb{D} \rightarrow \mathbb{D}'_X \rightarrow \mathbb{D}'_Y.$$

We refer the reader to [Groth 2013, Propositions 2.5 and 2.9, Example 2.10] for more details about adjunctions.

The purpose of this appendix is to show that the functor $\mathbb{D}(-)$ also preserves hom-objects in the sense of the following theorem. For \mathcal{M} and \mathcal{N} in \mathcal{MOD} , let $\mathcal{MOD}_1(\mathcal{M}, \mathcal{N})$ denote the category of left Quillen functors $\mathcal{M} \rightarrow \mathcal{N}$ and natural transformations. This is again a category with weak equivalences, the Quillen homotopies, and the forgetful functor $\mathcal{MOD}(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{MOD}_1(\mathcal{M}, \mathcal{N})$ is an equivalence.

Theorem A.3. *Let \mathcal{M} and \mathcal{N} be presentable model categories. Then there is an equivalence of prederivators*

$$\Phi(\mathcal{M}, \mathcal{N}) : \mathbb{D}(\mathcal{M}\mathcal{O}\mathcal{D}_1(\mathcal{M}, \mathcal{N})) \simeq \text{HOM}_1(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N})).$$

Proof. For every small category X , there is a natural equivalence of categories

$$\mathbb{D}(\mathcal{M}\mathcal{O}\mathcal{D}_1(\mathcal{M}, \mathcal{N}))(X) \simeq \text{Ho}(\mathcal{M}\mathcal{O}\mathcal{D}_1(\mathcal{M}, (\mathcal{N}^X)_{\text{inj}})),$$

since an X -diagram of left Quillen functors $\mathcal{M} \rightarrow \mathcal{N}$ is the same as a left Quillen functor $\mathcal{M} \rightarrow (\mathcal{N}^X)_{\text{inj}}$ where the target is given the injective model structure. The latter model category is strictly speaking no longer presentable, but we can find a natural replacement for it by a presentable one \mathcal{N}^X simply by a change to the projective (co)fibrations. Then by [Renaudin 2009, Proposition 2.2.9], we have an equivalence of categories

$$\text{Ho}(\mathcal{M}\mathcal{O}\mathcal{D}_1(\mathcal{M}, (\mathcal{N}^X)_{\text{inj}})) \xleftarrow{\simeq} \text{Ho}(\mathcal{M}\mathcal{O}\mathcal{D}_1(\mathcal{M}, \mathcal{N}^X)).$$

There is a morphism of prederivators

$$\Phi(\mathcal{M}, \mathcal{N}) : \text{Ho}(\mathcal{M}\mathcal{O}\mathcal{D}_1(\mathcal{M}, (\mathcal{N}^?)_{\text{inj}})) \rightarrow \text{HOM}_1(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N})),$$

induced by $\mathbb{D}(-)$, whose components are equivalences of categories, because we have commutative diagrams

$$\begin{array}{ccc} \text{Ho}(\mathcal{M}\mathcal{O}\mathcal{D}_1(\mathcal{M}, (\mathcal{N}^X)_{\text{inj}})) & \xrightarrow{\Phi(\mathcal{M}, \mathcal{N})_X} & \mathcal{D}\mathcal{E}\mathcal{R}_1(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N}^X)) \\ & \swarrow \simeq & \nearrow \simeq \\ & \text{Ho}(\mathcal{M}\mathcal{O}\mathcal{D}_1(\mathcal{M}, \mathcal{N}^X)) & \end{array}$$

where the indicated equivalence on the right is a consequence of Theorem A.1. \square

Appendix B. A remark on the approximation theorem

The original approximation theorem of Waldhausen [1985] states sufficient conditions for an exact functor of Waldhausen categories to induce an equivalence in K -theory. Although Waldhausen did not analyze the meaning of these conditions from the viewpoint of homotopical algebra, various authors have later studied connections between abstract homotopy theory and Waldhausen K -theory and have shown more general and refined versions of the approximation theorem; see [Thomason and Trobaugh 1990; Sagave 2004; Dugger and Shipley 2004; Cisinski 2010b; Blumberg and Mandell 2011]. These results ultimately say that Waldhausen K -theory is an invariant of homotopy theories and allow definitions of the theory via ∞ -categories or simplicial categories; see also [Toën and Vezzosi 2004].

Theorem B.1 [Cisinski 2010b; Blumberg and Mandell 2011]. *Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be an exact functor of strongly saturated derivable Waldhausen categories. If the induced functor $\text{Ho } F : \text{Ho } \mathcal{C} \rightarrow \text{Ho } \mathcal{C}'$ is an equivalence of categories, then the map $w\mathcal{S}_n F : w\mathcal{S}_n \mathcal{C} \rightarrow w\mathcal{S}_n \mathcal{C}'$ is a weak equivalence for all $n \geq 0$. In particular, the map $K(F) : K(\mathcal{C}) \rightarrow K(\mathcal{C}')$ is also a weak equivalence.*

The purpose of this appendix is to note the following result, which may be regarded as a partial converse to the approximation theorem. The proof is based on ideas of Dwyer and Kan for modeling mapping spaces in homotopical algebra via zigzag diagrams (see, e.g., [Dwyer and Kan 1980]) and related results from [Blumberg and Mandell 2011].

Theorem B.2. *Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be an exact functor of derivable Waldhausen categories. Suppose that*

- (i) $wF : w\mathcal{C} \rightarrow w\mathcal{C}'$ induces isomorphisms on π_0 and π_1 for all basepoints,
- (ii) $w\mathcal{S}_2 F : w\mathcal{S}_2 \mathcal{C} \rightarrow w\mathcal{S}_2 \mathcal{C}'$ is 1-connected (i.e., it induces an isomorphism on π_0 and an epimorphism on π_1 for all basepoints).

Then $\text{Ho } F : \text{Ho } \mathcal{C} \rightarrow \text{Ho } \mathcal{C}'$ is an equivalence of categories.

Proof. Consider the commutative square

$$\begin{array}{ccc}
 w\mathcal{S}_2 \mathcal{C} & \xrightarrow{w\mathcal{S}_2 F} & w\mathcal{S}_2 \mathcal{C}' \\
 \downarrow (d_1, d_2) & & \downarrow (d_1, d_2) \\
 w\mathcal{C} \times w\mathcal{C} & \xrightarrow{wF \times wF} & w\mathcal{C}' \times w\mathcal{C}'
 \end{array}$$

Using the properties of the long exact sequence of homotopy groups and assumptions (i) and (ii), it follows that the induced map between the homotopy fibers of the vertical maps (at any basepoint) induces an isomorphism on π_0 ; see, e.g., [May and Ponto 2012, Lemma 1.4.7]. Applying [Blumberg and Mandell 2011, Theorem 1.2], the map between the homotopy fibers at the points defined by $(X_1, X_2) \in \text{Ob } \mathcal{C} \times \text{Ob } \mathcal{C}$ and $(F(X_1), F(X_2)) \in \text{Ob } \mathcal{C}' \times \text{Ob } \mathcal{C}'$ can be identified with the map induced by F between the corresponding mapping spaces in the respective hammock localizations

$$L^H(\mathcal{C})(X_1, X_2) \rightarrow L^H(\mathcal{C}')(F(X_1), F(X_2)).$$

Thus, applying π_0 to this map gives an isomorphism

$$\text{Ho } F : \text{Ho } \mathcal{C}(X_1, X_2) \cong \text{Ho } \mathcal{C}'(F(X_1), F(X_2)),$$

and therefore $\text{Ho } F$ is fully faithful. It is also essentially surjective because wF is an epimorphism on π_0 . □

To sum up, we have the following corollary.

Corollary B.3. *Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be an exact functor of strongly saturated derivable Waldhausen categories. If*

- (i) $wF : w\mathcal{C} \rightarrow w\mathcal{C}'$ induces isomorphisms on π_0 and π_1 , and
- (ii) $wS_2F : wS_2\mathcal{C} \rightarrow wS_2\mathcal{C}'$ is 1-connected,

then $wS_nF : wS_n\mathcal{C} \rightarrow wS_n\mathcal{C}'$ is a weak equivalence for all $n \geq 0$. In particular, the induced map $K(F) : K(\mathcal{C}) \rightarrow K(\mathcal{C}')$ is also a weak equivalence.

These results show that being a derived equivalence is much stronger than being a K -equivalence. More specifically, the property of being a derived equivalence does not take into account the “group completion” process that takes place in the definition of K -theory. To obtain an ideal approximation theorem that encodes this group completion process, one would need to “localize \mathcal{C} and \mathcal{C}' ” at all the relations which are derived from the additivity property, and then ask for the weaker property that these localized objects are equivalent. This localization is accomplished using ∞ -categories with the construction of the universal additive invariant in [Blumberg et al. 2013], and it is essentially shown that it is equivalent to Waldhausen K -theory.

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