Some inverse problems in Elastography

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Outline

1. Some general ideas
   - Elastography and inverse problems
   - Mathematical formulations
2. Uniformly bounded total variation
   - The one-dimensional case
   - The general N-dimensional case
3. Free total variation in one dimension
   - Formulation and non-existence
   - Relaxation and existence
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What is Elastography?

A technique to detect elastic properties of tissue
Applications in Medicine

Aspects:

- Three elements:
  Acoustic waves generator (Low frequency) mechanical excitation $\rightarrow$ waves
  Captor (mechanical waves detection and visualization; MR or ultrasound)
  Mathematical tool (solver $\rightarrow$ identification of tissue stiffness)
- Medical fields of application: detection and description of breast, liver, prostate and other cancers; arteriosclerosis (hardening of the arteries); fibrosis; deep vein thrombosis; treatment monitoring; ...
- At present: emerging techniques lead to the detection of internal waves through non-invasive techniques (a very precise description)

First works: [Ophir-et-al 1991], [Muthupillai-et-al 1995], [Sinkus-et-al 2000], [McKnight-et-al 2002], ...

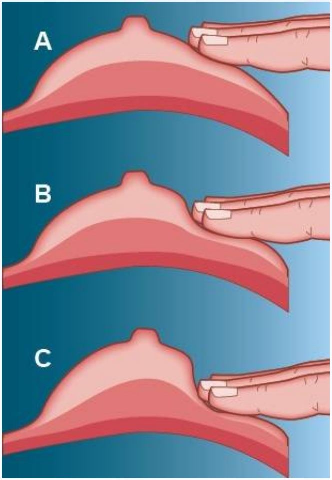


Figure: Classical detection methods in mammography (I): palpation

Figure: Classical detection methods in mammography (II): x-rays

Elastography is better suited than palpation and x-rays techniques:

— Tumors can be far from the surface
— or small
— or may have properties that become indistinguishable through palpation or x-rays

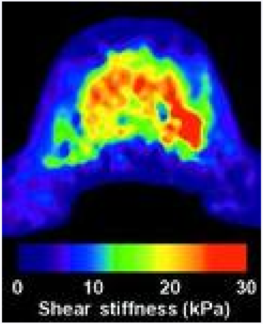
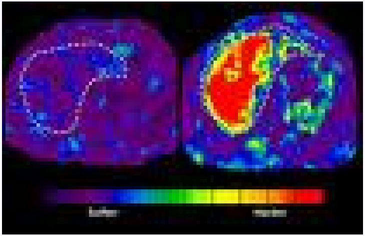


Figure: A breast elastogram. Identification of tissue stiffness



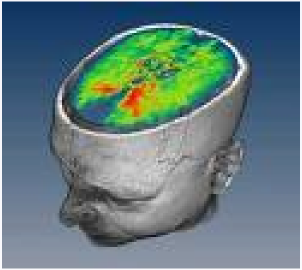


Figure: Applications of MR Elastography to tumor detection. Liver and brain elastograms

In mathematical terms: inverse problem governed by PDEs
Some words on inverse problems:

- General setting of a direct problem:
  Data $(\mathcal{D}_0 \cup \mathcal{D}_1) \to$ Results $(\mathcal{R}) \to$ Observation (additional information) $(\mathcal{I})$
- A related inverse problem:
  Some data $(\mathcal{D}_0)$ + Information $(\mathcal{I}) \to$ The other data $(\mathcal{D}_1)$
- Example: identification of the shape of a domain
  (a) Direct problem:
  Data: Ω, φ and D
  Result: the solution u to

(1)
$$\begin{cases} -\Delta u = 0, & x \in \Omega \setminus \overline{D} \\ u = 0, \ x \in \partial D; & u = \varphi, \ x \in \partial\Omega \end{cases}$$

Information:

(2)
$$\frac{\partial u}{\partial \nu} = \sigma, \quad x \in \gamma \subset \partial\Omega$$

(b) Inverse problem:
(Partial) data: Ω and φ
(Additional) information: σ (on γ)
Goal: Find D such that the solution to (1) satisfies (2)
[Andrieux-et-al 1993], [Alessandrini-et-al 2000 ...], [Kavian 2002], [Alvarez-et-al 2005], [Doubova-EFC-GlezBurgos-Ortega 2006], [Yan-Ma 2008]

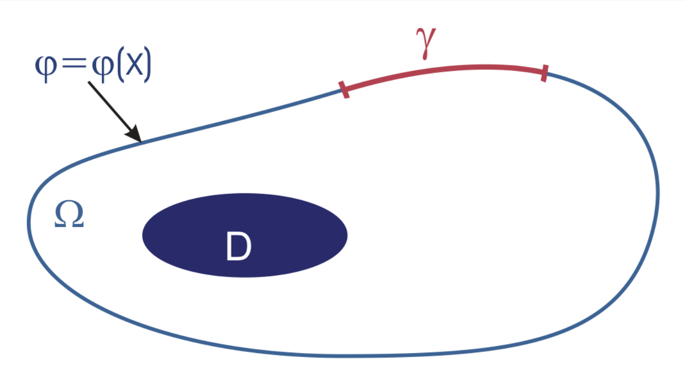


Figure: A geometrical inverse problem: identification of the open set D from Ω, φ and the additional information $\frac{\partial u}{\partial \nu} = \sigma$ on γ

Many interesting problems in Medicine, Biology, etc. lead to IPs for PDEs of this class: coefficient, source or shape identification

A SECOND EXAMPLE: identification of the conductivity of a dielectric body (Calderón)

(a) Direct problem:

Data: Ω, φ and $a = a(x)$

Result: the solution u to

$$\begin{cases} -\nabla \cdot (a(x)\nabla u) = 0, & x \in \Omega \\ u = \varphi, & x \in \partial\Omega \end{cases} \tag{1}$$

Information:

$$u|_{\omega} = z \tag{2}$$

(b) Inverse problem:

(Partial) data: Ω and φ

(Additional) information: z (in ω)

Goal: Find a such that the solution to (1) satisfies (2)

Applications to tomography . . .

[Calderón 1980], [Sylvester-Uhlman 1987], [Astala-Paavarinta 2003], . . .

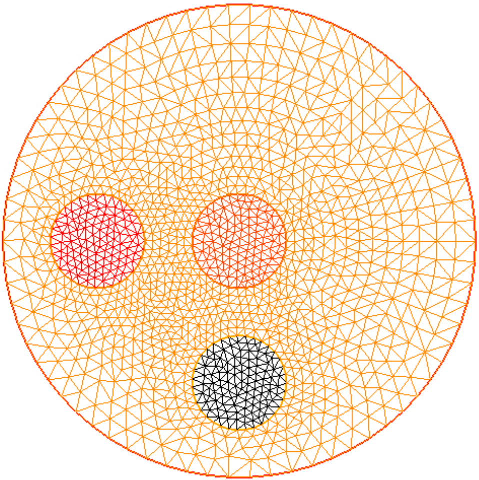


Figure: The domain and the mesh — The solution is given by $a = 0.01$ (resp. $a = 100$, $a = 1000$) in the central (resp. bottom, left) disk; $a = 1$ elsewhere — To solve the problem: FEM approach (P_1-Lagrange) — Nb of Triangles = 2638, Nb of Vertices = 1355

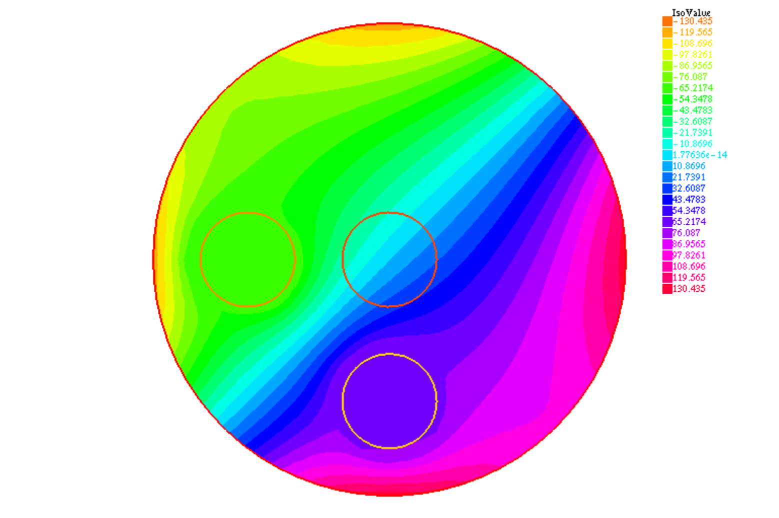


Figure: Computations with FreeFEM (these and those below; BFGS method) —- $\varphi \equiv x_1^3 - x_2^3$ — ω is the central disk — Reconstructed potential — $a = 1.01$ (resp. $a = 100$, $a = 1000$) in the central (resp. bottom, left) disk; $a = 1$ elsewhere

A THIRD SIMILAR EXAMPLE: identification of the viscosity of a Navier-Stokes fluid

(a) Direct problem:

Data: Ω, D, U and $\nu = \nu(x)$

Result: the solution (u, p) to

$$(1) \quad \begin{cases} (u \cdot \nabla) u - \nabla \cdot (\nu(x)(Du + Du^T)) + \nabla p = 0, \quad \nabla \cdot u = 0, \quad x \in \Omega \setminus \overline{D} \\ u = 0, \quad x \in \partial D; \quad u = U, \quad x \in \partial\Omega \end{cases}$$

Information:

$$(2) \qquad u|_\omega = z$$

(b) Inverse problem:

(Partial) data: Ω, D and U

(Additional) information: z (in ω)

Goal: Find ν such that the solution to (1) satisfies (2)

Applications to blood diseases description and therapy . . .
Thrombosis, detection of coagula in blood vessels . . .
[Nakamura-Uhlman 1994], [Yamamoto 2009], . . .

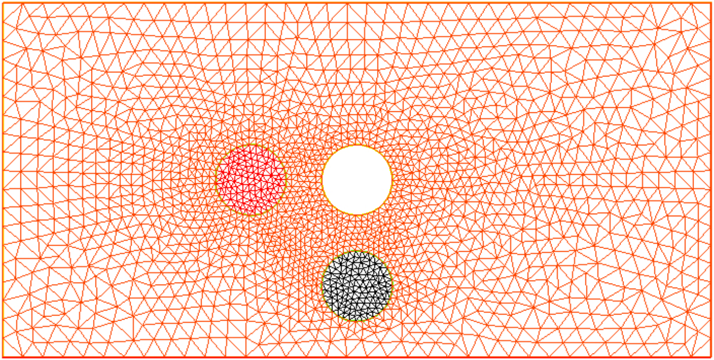


Figure: The domain and the mesh — The solution is given by $\nu = 10$ (resp. $\nu = 100$) in the left (resp. bottom) cylinder; $\nu = 0.1$ elsewhere — To solve the problem: FEM approach ($P_2 \otimes P_1$-Lagrange) — Nb of Triangles = 3799, Nb of Vertices = 1971

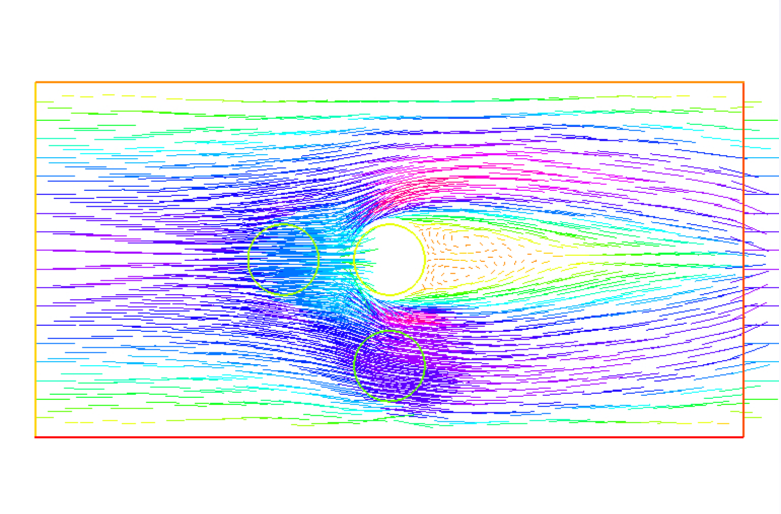


Figure: Computations with FreeFEM (BFGS method) —- $U = (1, 0)$ on the left and the right, $U = (0, 0)$ elsewhere — ω is the left cylinder — Reconstructed velocity field — $\nu = 10$ (resp. $\nu = 100$) in the left (resp. bottom) cylinder; $\nu = 0.1$ elsewhere

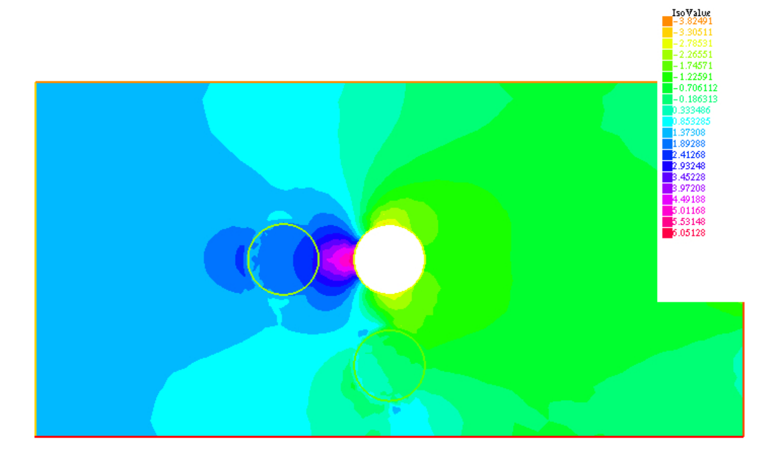


Figure: Computations with FreeFEM (BFGS method) —- $U = (1, 0)$ on the left and the right, $U = (0, 0)$ elsewhere — ω is the left cylinder — Reconstructed pressure — $\nu = 10$ (resp. $\nu = 100$) in the left (resp. bottom) cylinder; $\nu = 0.1$ elsewhere

A TYPICAL IP IN ELASTOGRAPHY

The data: $\Omega \subset \mathbb{R}^3$, F, (U^0, U^1) and B

The problem: Find $\lambda = \lambda(x)$ and $\mu = \mu(x)$ such that the solution $U = (U_1, U_2, U_3)$ to

$$\left\{ \begin{array}{ll} U_{tt} - \nabla \cdot \left(\mu(x)(\nabla U + \nabla U^T) + \lambda(x)(\nabla \cdot U)\mathbf{Id.} \right) = F, & (x,t) \in Q \\ U = 0, & (x,t) \in \Sigma \\ U(x,0) = U^0(x), \ \ U_t(x,0) = U^1(x), & x \in \Omega \end{array} \right.$$

satisfies

$$\sigma \cdot \nu := \left(\mu(x)(\nabla U + \nabla U^T) + \lambda(x)(\nabla \cdot U)\mathbf{Id.} \right) \cdot \nu = B \text{ on } S \times (0,T)$$

$\nu = \nu(x)$: outwards directed unit normal vector at $x \in \partial\Omega$ and $S \subset \partial\Omega$

[Sinkus-et-al 2000], [Barbone-et-al 2004], [Isaakov 2005], [Khaled-et-al 2006], [Perriñez 2009], [Imanuvilov-Yamamoto 2011]

Explanations:

- F: a given source, (U^0, U^1): an initial state (known)
- The tissue is described by λ and μ (under isotropy assumptions)
- The displacement $U = (U_1, U_2, U_3)$ is fixed on Σ
- $\sigma \cdot \nu$ is measured on $S \times (0,T)$

A SIMPLIFIED VERSION (for the axial displacement)

The data: $\Omega \subset \mathbb{R}^3$, f, (u^0, u^1) and σ

The inverse problem (IP): Find $\gamma \in L^\infty(\Omega; \{\alpha, \beta\}) \cap BV(\Omega)$ $(0 < \alpha < \beta)$ such that the solution to

$$\begin{cases} u_{tt} - \nabla \cdot (\gamma(x) \nabla u) = f(x,t), & (x,t) \in Q := \Omega \times (0,T) \\ u = 0, & (x,t) \in \Sigma := \partial\Omega \times (0,T) \\ u(x,0) = u^0(x), \; u_t(x,0) = u^1(x), & x \in \Omega \end{cases}$$

satisfies

$$\gamma \frac{\partial u}{\partial \nu} = \sigma \text{ on } S \times (0,T)$$

[Lurie 1999], [Allaire 2002], [Imanuvilov 2002], [Isaakov 2004], [Bellasoued-Yamamoto 2005], [Pedregal 2005], [Maestre-Pedregal 2006], [Maestre-Münch-Pedregal 2008], . . .

Explanations:

- Again: f is a given source, (u^0, u^1) is an initial state (known), $u|_\Sigma$ is fixed, $\frac{\partial u}{\partial \nu}|_S \times (0,T)$ is measured and γ is the unknown
- Sometimes it can be assumed that $u = U \cdot \nu$, $\gamma = \mu$

The inverse problem (IP): Find $\gamma \in L^\infty(\Omega; \{\alpha, \beta\}) \cap BV(\Omega)$ $(0 < \alpha < \beta)$ such that the solution to

$$\begin{cases} u_{tt} - \nabla \cdot (\gamma(x) \nabla u) = f(x,t) \\ u|_\Sigma = 0, \quad \ldots \end{cases}$$

satisfies

$$\gamma \frac{\partial u}{\partial \nu} = \sigma \text{ on } S \times (0,T)$$

Relevant questions:

- Uniqueness: γ and γ' solve (IP) $\Rightarrow \gamma = \gamma'$ [Barbonne 2004]
- Stability: γ (resp. γ') solve (IP) (resp. (IP) for σ')

$$\|\gamma' - \gamma\| \leq F(\sigma; \|\sigma' - \sigma\|) \text{ for small } \|\sigma' - \sigma\|?$$

- Reconstruction: given σ (and maybe some additional information), "compute" γ

For reconstruction: (a) Direct methods and (b) Iterative methods

THE INVERSE PROBLEM (IP): Find $\gamma \in L^\infty(\Omega; \{\alpha, \beta\}) \cap BV(\Omega)$ $(0 < \alpha < \beta)$ such that the solution to

$$\begin{cases} u_{tt} - \nabla \cdot (\gamma(x) \nabla u) = f(x,t) \\ u|_\Sigma = 0, \quad \dots \end{cases}$$

satisfies

$$\gamma \frac{\partial u}{\partial \nu} = \sigma \text{ on } S \times (0, T)$$

AN “ITERATIVE” METHOD: rewrite (IP) as an extremal problem
Cost function

$$I(\gamma) = \frac{1}{2} \int_0^T \|\gamma \frac{\partial u}{\partial \nu}\Big|_S - \sigma(t)\|^2 \, dt, \ \ \gamma \in L^\infty(\Omega; \{\alpha, \beta\}) \cap BV(\Omega)$$

($\| \cdot \|$ is an appropriate norm)
An extremal problem:

(EP)
$$\begin{cases} \text{Minimize } \ I(\gamma) \\ \text{Subject to } \ \gamma \in L^\infty(\Omega; \{\alpha, \beta\}) \cap BV(\Omega), \ \ u \text{ solves} \dots \end{cases}$$

Then: γ solves (IP) $\Leftrightarrow$ γ solves (EP), with $I(\gamma) = 0$
In the sequel: we analyze and try to “solve” (EP)

(EP) $\left\{ \begin{array}{l} \text{Minimize } I(\gamma) = \frac{1}{2} \int_0^T \| \gamma \frac{\partial u}{\partial \nu} |_S - \sigma(t) \|^2 \, dt \\ \text{Subject to } \gamma \in \ldots \quad u \text{ solves } \ldots \end{array} \right.$

$$\left\{ \begin{array}{l} u_{tt} - \nabla \cdot (\gamma(x) \nabla u) = f(x,t) \\ u|_\Sigma = 0, \quad \ldots \end{array} \right.$$

Results in collaboration with F. Maestre:

1st result:

The total variation of γ is uniformly bounded $\rightarrow$ Existence

2nd result:

$N = 1$, no a priori bound on the total variation of γ

- Non-existence
- Identification of the relaxed problem (and existence of "generalized γ")

(as in [Maestre-Münch-Pedregal 2008]; new proofs)

$N = 1$ and (EP) reads

(EP) $\begin{cases} \text{Minimize } \; I(\gamma) = \frac{1}{2} \int_0^T |\gamma(1) u_x(1,t) - \sigma(t)|^2 \, dt \\ \text{Subject to } \; \gamma \in L^\infty(\Omega; \{\alpha, \beta\}) \cap BV(\Omega), \quad u \text{ solves} \dots \end{cases}$

Necessarily: γ is piecewise constant in $[0, 1]$ and $\gamma(x)$ is a.e. equal to α or β, with a finite number of discontinuities

Consider

(EP-k) $\begin{cases} \text{Minimize } \; I(\gamma) = \frac{1}{2} \int_0^T |\gamma(1) u_x(1,t) - \sigma(t)|^2 \, dt \\ \text{Subject to } \; \gamma \in \Gamma_k, \quad u \text{ solves} \dots \end{cases}$

with $\Gamma_k = \{\gamma \in L^\infty(\Omega; \{\alpha, \beta\}) \cap BV(\Omega) : \gamma \text{ has, at most, } k \text{ discontinuities in } [0, 1]\}$

Theorem:

Existence for (EP-k)

A particular case of a result proved below

SOME COMMENTS:

- Since $N = 1$, γ has to be *simple.* For $N \geq 2$, more complex situations may appear
- Same argument $\Rightarrow$ existence for

$$\left\{ \begin{array}{l} \text{Minimize} \;\; I_\varepsilon(\gamma) = \frac{1}{2} \int_0^T |\gamma(1) u_x(1,t) - \sigma|^2 \, dt + \frac{\varepsilon}{2} TV(\gamma)^2 \\ \text{Subject to} \;\; \gamma \in L^\infty(\Omega; \{\alpha, \beta\}) \cap BV(\Omega), \;\; u \text{ solves} \ldots \end{array} \right.$$

What happens as $\varepsilon \to 0^+$?

$N \geq 2$, $\Omega \subset \mathbb{R}^N$ open, connected, regular and bounded
(at least, $\partial\Omega \in W^{2,\infty}$)

(EP) $\begin{cases} \text{Minimize } \ I(\gamma) = \frac{1}{2} \int_0^T \|\gamma \frac{\partial u}{\partial \nu}|_S - \sigma\|^2 \, dt \\ \text{Subject to } \ \gamma \in L^\infty(\Omega; \{\alpha, \beta\}) \cap BV(\Omega), \ \ u \text{ solves} \ldots \end{cases}$

Consider

(EP-C) $\begin{cases} \text{Minimize } \ I(\gamma) \\ \text{Subject to } \ \gamma \in \Lambda(C), \ \ u \text{ solves} \ldots \end{cases}$

with $\Lambda(C) = \{ \gamma \in L^\infty(\Omega; \{\alpha, \beta\}) \cap BV(\Omega) : TV(\gamma) \leq C \}$

Theorem:

Existence for (EP-C)

PROOF - FIRST PART:
$\forall \gamma \in \Lambda(C)$, $\gamma \frac{\partial u}{\partial \nu}$ is defined in $L^\infty(0,T;H^{-1/2}(\partial\Omega))$ by duality:

$$\langle \gamma \frac{\partial u}{\partial \nu}, z \rangle = \langle \nabla \cdot (\gamma \nabla u), z \rangle + \iint_Q \gamma \nabla u \cdot \nabla z \quad \forall z \in L^1(0,T;H^1(\Omega)),$$

$\{\gamma_n\}$: a minimizing sequence for I in $\Lambda(C)$. Then:

$$\begin{cases} \gamma^n \to \gamma^* & \text{weakly-}* \text{ in } BV(\Omega) \\ \gamma^n \to \gamma^* & \text{strongly in } L^p(\Omega) \text{ for all } p \in [1,+\infty) \text{ and a.e.} \end{cases}$$

with $\gamma^* \in \Lambda(C)$
u^n: the state associated to γ^n. Then:

$$\begin{cases} u^n \to u^* & \text{weakly-}* \text{ in } L^\infty(0,T;H^1_0(\Omega)) \\ u^n_t \to u^*_t & \text{weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)) \end{cases}$$

But: u^* is the state associated to γ^*, because

$$\gamma^n \nabla u^n \to \gamma^* \nabla u^* \quad \text{weakly in } L^{p_1}(0,T;L^{p_2}(\Omega)^N) \quad \forall p_1 \in [1,+\infty), \forall p_2 \in [1,2)$$

PROOF - SECOND PART:

$\liminf_{n\to+\infty} I(\gamma^n) \geq I(\gamma^*)$? Yes

Indeed:

- u^n is bounded in $C^0([0,T];X)$ for $X := [D(\Delta), H_0^1(\Omega)]_{\delta,\infty}$ (a Hilbert space compactly embedded in $H_0^1(\Omega)$) and u_t^n is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$
- Consequently, u^n is precompact in $L^2(0,T;H_0^1(\Omega))$ and

$$\gamma^n \frac{\partial u^n}{\partial \nu} \to \gamma^* \frac{\partial u^*}{\partial \nu} \quad \text{weakly in } L^2(0,T;H^{-1/2}(\partial\Omega))$$

To prove the first assertion: we write $-\nabla\cdot(\gamma^n \nabla u^n) = f - u_{tt}^n$ and we use

Lemma:

$\exists\delta$ such that, $\forall a \in L^\infty(\Omega)\cap BV(\Omega)$ with $\alpha \leq a \leq \beta$, $\forall h \in L^2(\Omega)$, the solution to

$$\begin{cases} -\nabla\cdot(a\nabla w) = h, & x\in\Omega \\ w = 0, & x \in \partial\Omega \end{cases}$$

satisfies:

$$\|w\|_X \leq C(N,\Omega,\alpha,\beta,\|a\|_{BV})\,\|h\|_{L^2}$$

Here, $X = [D(\Delta), H_0^1(\Omega)]_{\delta,\infty}$

For the proof: Meyers' Theorem, elliptic regularity and nonlinear interpolation (Tartar) $\square$

SKETCH OF THE PROOF OF THE LEMMA:

We use Meyers' Theorem, elliptic regularity and nonlinear interpolation (Tartar):

- If $a \in L^\infty(\Omega)$ and $\alpha \le a \le \beta$ a.e.,

$$\|w\|_{W^{1,p_M}} \le C(\Omega, N, \alpha, \beta)\,\|h\|_{L^2} \quad \forall h \in L^2(\Omega), \quad p_M > 2$$

  [Meyers 1963]

- If $a, a' \in L^\infty(\Omega)$, $\alpha \le a, a' \le \beta$ a.e. and $r = \frac{2p_M}{p_M - 2}$,

$$\|w' - w\|_{H_0^1} \le C(\Omega, N, \alpha, \beta)\,\|a' - a\|_{L^r}\,\|h\|_{L^2} \quad \forall h \in L^2(\Omega)$$

- If $a \in W^{1,r}(\Omega)$ and $\alpha \le a \le \beta$

$$\|w\|_{H^2} \le C(\Omega, N, \alpha, \beta)\,(1 + \|\nabla a\|_{L^r})\,\|h\|_{L^2} \quad \forall h \in L^2(\Omega)$$

  (elliptic regularity theory)

- $BV(\Omega) \cap L^\infty(\Omega) \subset [W^{1,r}(\Omega), L^r(\Omega)]_{1/r',\infty} \cap L^\infty(\Omega)$
- Finally, all this and a nonlinear interpolation result by [Tartar 1972] $\Rightarrow$

$$w \in [D(\Delta), H_0^1(\Omega)]_{\delta,\infty} \;\text{ for }\; \delta = \frac{1}{r'} = \frac{p_M - 2}{2p_M} \;+\; \text{estimates}$$

□

SOME COMMENTS:

- Again, same argument $\Rightarrow$ existence for

$$\begin{cases} \text{Minimize } \; I_\varepsilon(\gamma) = \frac{1}{2}\int_0^T \|\gamma \frac{\partial u}{\partial \nu}|_S - \sigma\|^2 \, dt + \frac{\varepsilon}{2} TV(\gamma)^2 \\ \text{Subject to } \; \gamma \in L^\infty(\Omega; \{\alpha, \beta\}) \cap BV(\Omega), \quad u \text{ solves } \dots \end{cases}$$

- Generalizations in several directions:

  (a) Lamé systems: Meyers-like estimates, elliptic regularity, . . .
  Applications in Elastography

  (b) Semilinear hyperbolic systems: global estimates

A RELATED BUT DIFFERENT PROBLEM

($N = 1$, $f \equiv 0$):

$$\begin{cases} \text{Minimize } \; J_\delta(\gamma) = \frac{1}{2\delta} \int_0^T \int_{1-\delta}^1 |u_x(x,t) - \sigma(t)|^2 \, dx \, dt \\ \text{Subject to } \; \gamma \in L^\infty(Q; \{\alpha, \beta\}), \;\; u \text{ solves } \dots \end{cases} \qquad \text{(EP-}\delta\text{)}$$

Attention: γ can now depend on x and t; u_x is observed for $x \in [1-\delta, 1]$

Rewritting the state equation, an idea from [Pedregal 2005]:

$$\nabla_{(x,t)} \cdot (-\gamma(x,t) u_x, u_t) = 0 \quad \Leftrightarrow \quad \exists v \in H^1(\Omega \times (0,T)) : u_t = v_x, \;\; -\gamma(x,t) u_x = v_t$$

Set

$$\Lambda_\eta = \{F \in \mathcal{M}^{2\times 2} : M_\eta F^{(1)} - F^{(2)} = 0\}, \quad M_\eta = \begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix} \text{ for } \eta = \alpha, \beta$$

and

$$W(t,F) = \begin{cases} |F_{11} - \sigma(t)|^2, & \text{if } F \in \Lambda_\alpha \cup \Lambda_\beta \\ +\infty, & \text{otherwise} \end{cases}$$

Then, (EP-δ) is equivalent to the variational problem

$$\begin{cases} \text{Minimize } K_\delta(U) = \frac{1}{\delta} \int_0^T \int_{1-\delta}^1 W(t, \nabla_{(x,t)} U(x,t)) \, dx \, dt \\ \text{Subject to } \; U = (u,v) \in H^1(Q; \mathbb{R}^2) \\ \qquad u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x) \;\text{ in }\; (0,1) \\ \qquad u(0,t) = u(1,t) = 0 \;\text{ in }\; (0,T) \end{cases}$$

In general, (EP-δ) possesses no solution

Theorem:

Let us introduce the function $\tilde{W}$, with

$$\tilde{W}(t,F) = \begin{cases} |F_{11} - \sigma(t)|^2 & \text{if } F \in Z_- \\ \left(\frac{\beta+\alpha}{\beta-\alpha}\right)^2 \left|F_{11} - \frac{2}{(\beta+\alpha)^2} F_{22}\right|^2 - 2\sigma(t)F_{11} + \sigma(t)^2 & \text{if } F \in Z_+ \\ +\infty & \text{otherwise} \end{cases}$$

where we have denoted by Z_- (resp. Z_+) the family of matrices $F \in \mathcal{M}^{2\times 2}$ satisfying $F_{12} - F_{21} = 0$ and $(\alpha F_{11} - F_{22})(\beta F_{11} - F_{22}) \leq 0$ (resp. $F_{12} - F_{21} = 0$ and $(\alpha F_{11} + F_{22})(\beta F_{11} + F_{22}) \geq 0$). Then $QW = \tilde{W}$.

SKETCH OF THE PROOF:
We can assume $\sigma \equiv 0$ and $W = W(F)$ and $\tilde{W} = \tilde{W}(F)$ respectively given by

$$W(F) = \begin{cases} |F_{11}|^2, \text{if } F \in \Lambda_\alpha \cup \Lambda_\beta \\ +\infty, \text{otherwise} \end{cases} \quad \tilde{W}(F) = \begin{cases} |F_{11}|^2, \text{if } F \in Z_- \\ \left(\frac{\beta+\alpha}{\beta-\alpha}\right)^2 \left|F_{11} - \frac{2}{(\beta+\alpha)^2} F_{22}\right|^2, \text{if } F \in Z_+ \\ +\infty, \text{otherwise} \end{cases}$$

CW, PW, QW and RW: the *convexification, poly-convexification, quasi-convexification* and *rank-one-convexification* of W. For instance:

$$CW(F) = \sup\{\, G(F) : G : \mathcal{M}^{2\times 2} \mapsto \overline{\mathbb{R}} \text{ is convex and } G \leq W \,\},$$

Then $CW \leq PW \leq QW \leq RW$, with possibly strict inequalities
But $RW \leq \tilde{W}$ and $\tilde{W} \leq CW$:

- If $G = G(F)$ is rank-one convex and $G \leq W$, then $G \leq \tilde{W}$ (a computation), whence $RW \leq \tilde{W}$
- $\tilde{W}$ is convex and $\tilde{W} \leq W$, whence $\tilde{W} \leq CW$ □

Other proof can be obtained from [Maestre-Münch-Pedregal 2008] (Young measures)
A consequence:

Corollary

The variational problem

(REP-δ)
$$\begin{cases} \text{Minimize } \tilde{K}_\delta(U) = \frac{1}{\delta} \int_0^T \int_{1-\delta}^1 \tilde{W}(t, \nabla_{(x,t)} U(x,t))\, dx\, dt \\ \text{Subject to } \; U = (u, v) \in H^1(Q; \mathbb{R}^2) \\ \qquad u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x) \;\text{ in }\; (0,1) \\ \qquad u(0,t) = u(1,t) = 0 \;\text{ in }\; (0,T) \end{cases}$$

is a relaxation of (EP-δ), i.e.

1. inf (EP-δ) $=$ inf (REP-δ)
2. (REP-δ) possesses optimal solutions
3. Optimal distributions of α and β in (REP-δ): given by the behavior in the limit of minimizing sequences of (EP-δ), i.e. codified by the related Young measure

SOME ADDITIONAL COMMENTS AND QUESTIONS:

- Interpretation: In the (optimal) minimizing sequence, α and β tissues are placed alternating “small” bars in proportions determined by the solution to (REP-δ)
- Forthcoming (theoretical and numerical) results for some variants:

$$\frac{1}{\delta}\int_0^T\int_{1-\delta}^1 |\gamma u_x - \sigma(t)|^2, \quad (\rho(x,t)u_t)_t - (\gamma(x,t)u_x)_x = 0, \ldots$$

- How to solve (and interpret) similar N-dimensional problems? Lamé versions?

JUST TO END: NUMERICAL SOLUTION OF A "SIMPLE" RELATED PROBLEM:
Identifying the wave speed coefficient

(a) Direct problem:

Data: Ω, D, T, ψ, (u^0, u^1) and $\gamma = \gamma(x)$

Result: the solution u to

$$(1) \qquad \begin{cases} u_{tt} - \nabla \cdot (\gamma(x)\nabla u) = 0, & (x,t) \in (\Omega \setminus \overline{D}) \times (0,T) \\ u = \psi, \ (x,t) \in \partial\Omega \times (0,T); & u = 0, \ (x,t) \in \partial D \times (0,T) \\ (u, u_t)|_{t=0} = (u^0, u^1) \end{cases}$$

Information:

$$(2) \qquad u(\cdot\,, T)|_\omega = \zeta$$

(b) Inverse problem:

(Partial) data: Ω, D, T, ψ and (u^0, u^1)

(Additional) information: ζ (in ω)

Goal: Find γ (piecewise constant) such that the solution to (1) satisfies (2)

Again: applications to thrombosis, detection of coagula in blood vessels . . .
This begins to look like an elastography problem

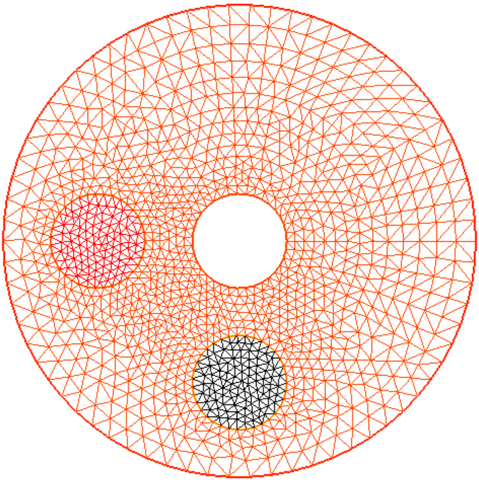


Figure: The domain and the mesh — The solution is given by $\gamma = 10$ (resp. $\gamma = 50$) in the left (resp. bottom) cylinder; $\gamma = 0.5$ elsewhere — To solve the problem: FEM approach (P_1-Lagrange) and BFGS method — Nb of Triangles = 2436, Nb of Vertices = 1271

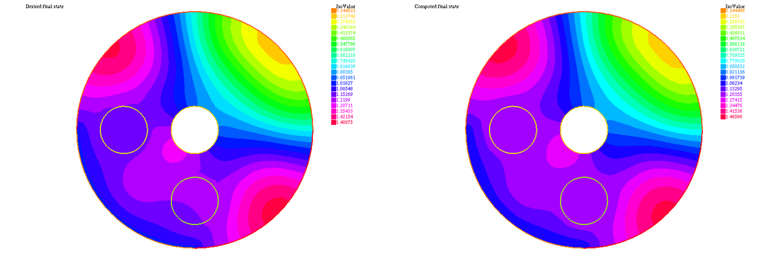


Figure: Computations with FreeFEM (Conjugate Gradient algorithm) — The desired and computed states — $\psi = 1$ on the left-bottom, $\psi = 0$ elsewhere — $f \equiv 0$ — ω is the left cylinder — The solution is given by $\gamma = 10$ (resp. $\gamma = 50$) in the left (resp. bottom) cylinder; $\gamma = 0.5$ elsewhere — To solve the problem: FEM approach (P_1-Lagrange) and BFGS method — Cost $= 0.000674199$ — Nb of Triangles = 2436, Nb of Vertices = 1271

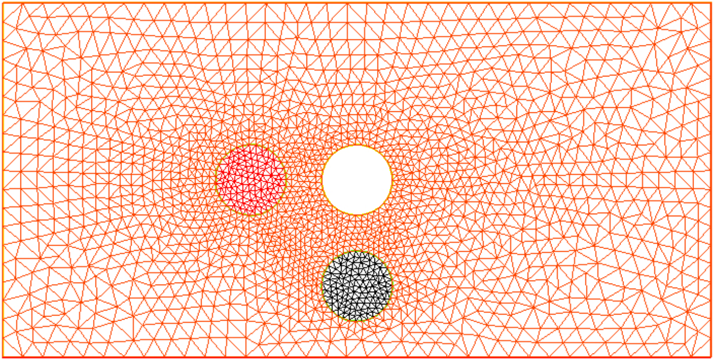


Figure: The domain and the mesh — The solution is given by $\gamma = 10$ (resp. $\gamma = 50$) in the left (resp. bottom) cylinder; $\gamma = 0.5$ elsewhere — To solve the problem: FEM approach (P_1-Lagrange) and BFGS method — Nb of Triangles = 3799, Nb of Vertices = 1971

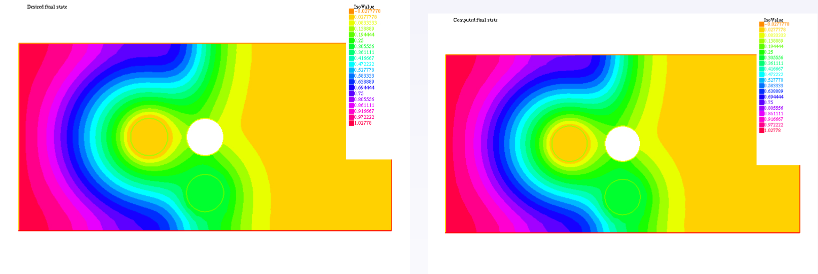


Figure: Computations with FreeFEM (Conjugate Gradient algorithm) — The desired and computed states — $\psi = 1$ on the left, $\psi = 0$ elsewhere — $f \equiv 0$ — ω is the left cylinder — The solution is given by $\gamma = 10$ (resp. $\gamma = 50$) in the left (resp. bottom) cylinder; $\gamma = 0.5$ elsewhere — To solve the problem: FEM approach (P_1-Lagrange) and BFGS method — Cost $= 0.00979552$ — Nb of Triangles = 3799, Nb of Vertices = 1971

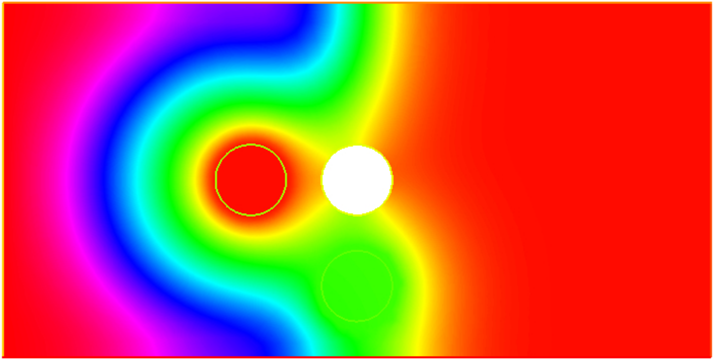


Figure: Computations with FreeFEM — The computed state

THANK YOU VERY MUCH . . .