Low-dimensional Filiform Lie Algebras Over Finite Fields

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Abstract: - In this paper we use some objects of Graph Theory to classify low-dimensional filiform Lie algebras over finite fields. The idea lies in the representation of each Lie algebra by a certain type of graphs. Then, some properties on Graph Theory make easier to classify the algebras. As results, which can be applied in several branches of Physics or Engineering, for instance, we find out that there exist, up to isomorphism, six 6-dimensional filiform Lie algebras over $\mathbb{Z}/p\mathbb{Z}$, for $p = 2, 3, 5$.

Key–Words: - Bipartite graph; adjacency matrix; classification; filiform Lie algebra; finite fields; applications to Physics.

1 Introduction

At present, the most important interest in dealing with Lie algebras lies in the applications of such algebras to many other scientific subjects, like Physics or Engineering, for instance. Indeed, the theory of General Relativity describes our spacetime as a four-dimensional smooth manifold $\mathcal{M}$ endowed with a Riemannian metric field g of Lorentzian signature $(- + ++)$. Once a matter distribution and appropriate boundary conditions are specified, the curvature of this metric field is dictated by the Einstein equations, a set of ten highly nonlinear partial differential equations. The group of symmetries of the metric is a Lie group.

Solvable Lie algebras are also used to describe the scalar field sector of supergravity in relation with its non-perturbative structure encoded in a particular duality group. In this sense, Frè and other authors [6] used solvable Lie algebras as a decisive and useful tool for studying some problems relative to black holes and giving an alternative description of the scalar manifold in a broad class of supergravity theories.

Then, by bearing in mind all of these arguments and focusing on this paper, our goal is firstly to deal with filiform Lie algebras, due to the two following reasons.

In the first place, filiform Lie algebras, introduced by M. Vergne in the late 60's of the past century [11], constitute the most structured subset of nilpotent Lie algebras, which allows us to study and classify them easier than the set of nilpotent Lie algebras. Remember that in earlier papers, some of authors have deeply studied these algebras and obtained quite a lot results about them. Indeed, we already got the classification of those over real or complex field having dimensions $10, 11$ and 12 (see [1], [5] and [2]), for instance).

*Partially supported by P.A.I. Group FQM-326
†Partially supported by P.A.I. Group FQM-296
‡Partially supported by P.A.I. Group FQM-164

ISBN: 978-1-61804-002-2

Secondly, with respect to Lie algebras over finite fields, we have already got several results related to classifications of a low-dimensional family of Lie algebras that in a certain sense can be considered a precursor of the filiform Lie algebras: the n-dimensional family $\mathcal{F}_p$ defined over the field $\mathbb{Z}/p\mathbb{Z}$, with p prime, and having a basis $\{u_1, \ldots, u_n\}$, such that: if $r, s < n$, then $[u_r, u_s] = 0$; and $[u_r, u_n]$ is a linear combination of some basis elements $u_1, \ldots, u_{n-1}$ with coefficients over the field $\mathbb{Z}/p\mathbb{Z}$ (note that u_n does not appear in this last linear combination).

To get those results we develop a novel research which consists in using Discrete Mathematics as a tool to classify these families (the main results obtained are indicated in Section 3). In any case, it is convenient to say that the main interest of our work is not the computations itself but for one hand to provide new strategies to find out properties of Lie algebras and for another, to exemplify a suitable technique to be used in classifications for larger dimensions. Some previous papers related to the using of Graph Theory to deal with Lie algebras are [3, 4, 8, 9], for instance.

2 Basic concepts

This section is devoted to recall some concepts on both Lie and Graph Theories. In any case, the paper is not self-contained due to reasons of length. For a more general review of both theories, the reader can consult [10, 7], respectively, for instance.

A nilpotent Lie algebra $\mathbf{g}$ is said to be *filiform*, if it is verified that

$$dim\, \mathbf{g}^2 = n - 2; \ldots dim\, \mathbf{g}^k = n - k; \ldots dim\, \mathbf{g}^n = 0,$$

with $dim\, \mathbf{g} = n$, and $\mathbf{g}^k = [\mathbf{g}, \mathbf{g}^{k-1}]$, $2 \le k \le n$. A basis $\{e_1, \ldots, e_n\}$ of $\mathbf{g}$ is called an *adapted basis* if: $[e_1, e_2] = 0$; $[e_1, e_h] = e_{h-1}$ $(h = 3, \ldots, n)$, $[e_3, e_n] = 0$.

A filiform Lie algebra $\mathbf{g}$ is said to be *model*, if the only non-null brackets between the elements of an adapted basis are the following: $[e_1, e_h] = e_{h-1}$ $(h = 3, \ldots, n)$. These brackets are called *brackets due to the filiformity of the algebra* and they are not usually indicated in the law of the filiform Lie algebra because they are supposed.

From now on, we will suppose that all the Lie algebras appearing in this paper are filiform over finite fields $\mathbb{Z}/p\mathbb{Z}$, with p prime and that all bases are adapted. We denote by $J(a, b, c) = 0$ Jacobi identity associated with vectors a, b and c.

An *isomorphism* between Lie algebras is a vector space isomorphism ϕ such that $\phi([u, v]) = [\phi(u), \phi(v)]$ for each pair of vectors u and v in the algebra.

Let us now recall some concepts on Graph Theory.

A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets U and V such that every edge links a vertex in U to one in V. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles.

The *adjacency matrix* of a graph with vertices $\{v_1, \cdots, v_n\}$ is a $n \times n$ matrix, where the element in the i-th row and j-th column is the number of edges between the vertices v_i and v_j. Note that in the case of simple graphs the matrix is binary.

3 Associating graphs to 6-dimensional filiform Lie algebras over $\mathbb{Z}/p\mathbb{Z}$, with p prime

In this section we show the procedure which links the family $\mathcal{F}_6^p$ of 6-dimensional filiform Lie algebras over $\mathbb{Z}/p\mathbb{Z}$ $(p$ prime) with bipartite simple or multiple graphs.

3.1 The procedure

Let $\mathcal{F}_6^p$ be the family of filiform Lie algebras of dimension 6 defined over the finite field $\mathbb{Z}/p\mathbb{Z}$, having a basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$. The law of an algebra $\mathfrak{g} \in \mathcal{F}_6^p$ with respect to the basis $\{e_k\}_{k=1}^6$ is given by

ISBN: 978-1-61804-002-2

the following brackets (different of the already supposed), some of them can be null

$$[e_i, e_j] = \sum_{h=2}^{6} c_{ij}^h e_h \quad \text{for} \quad 4 \le i < j \le 6$$

Each algebra $\mathfrak{g} \in \mathcal{F}_6^p$ is associated with a matrix of order 3×3 determined by the structure constants of $\mathfrak{g}$, that is

$$M_{\mathfrak{g}} = \begin{pmatrix} c_{45}^2 & c_{45}^3 & c_{45}^4 \\ c_{46}^2 & c_{46}^3 & c_{46}^4 \\ c_{56}^2 & c_{56}^3 & c_{56}^4 \end{pmatrix}$$

where $c_{45}^3 = c_{45}^4 = c_{46}^4 = 0$ due to filiformity reasons (see [2], for instance).

Let $\mathcal{B}_6$ be the set of bipartite graphs with three and three vertices. We define the map $F : \mathcal{F}_6^p \to \mathcal{B}_6$ such that the image of an algebra $\mathfrak{g} \in \mathcal{F}_6^p$ is a bipartite graph $G \in \mathcal{B}_6$ which satisfies

1. The vertices of G are the elements e_2, e_3, e_4, and the brackets $[e_4, e_5], [e_4, e_6], [e_5, e_6]$.
2. The adjacency matrix of G, M_G, is a symmetric and square matrix of dimension 6 formed by four submatrices of order 3×3 verifying
   (a) Two of the submatrices of M_G are null.
   (b) The two non-null submatrices of M_G are transpose each of another. Moreover, one of them corresponds to the matrix associated with the algebra $\mathfrak{g}$,

$$M_G = \left(\begin{array}{c|c} \mathcal{O} & M_{\mathfrak{g}}^t \\ \hline M_{\mathfrak{g}} & \mathcal{O} \end{array} \right)$$

   where $\mathcal{O}$ denotes the null matrix of dimension 3.

Note that although the dimension of the matrix coincides with the number of vertices of the graph in the case of dimension 6, it is not true in general, as can be observed in dimension 7, where the bipartite associated graph has 4 and 6 vertices.

As examples of the application of this procedure, let us see the following ones in which a simple graph and a multi-graph are involved, respectively.

Let $\mathfrak{g} \in \mathcal{F}_6^2$ be an algebra whose law is given by $[e_4, e_5] = e_2, \quad [e_4, e_6] = e_3, \quad [e_5, e_6] = e_2 + e_4$.

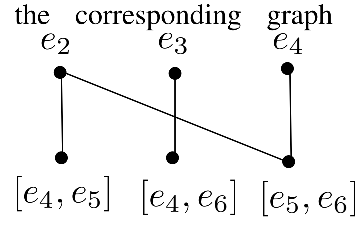
The matrix associated with $\mathfrak{g}$ is

$$M_{\mathfrak{g}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

The image of $\mathfrak{g}$ by the map F is a bipartite graph $G \in \mathcal{B}_6$ of vertices $e_2, e_3, e_4, [e_4, e_5], [e_4, e_6], [e_5, e_6]$ and adjacency matrix M_G.

$$M_G = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{array} \right)$$

Then the corresponding graph G is



Let now $\mathfrak{h} \in \mathcal{F}_6^3$ be another algebra whose law is given by $[e_4, e_5] = e_2$, $[e_4, e_6] = e_3$, $[e_5, e_6] = e_2 + 2e_4$. The matrix associated with $\mathfrak{h}$ is

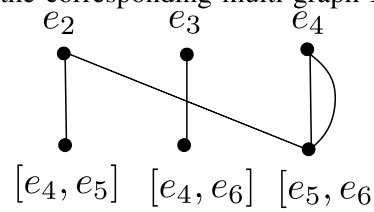
ISBN: 978-1-61804-002-2

$$M_{\mathfrak{h}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

The image of $\mathfrak{h}$ by the map F is a bipartite multi-graph $H \in \mathcal{B}_6$ of vertices $e_2, e_3, e_4, [e_4, e_5], [e_4, e_6], [e_5, e_6]$ and adjacency matrix

$$M_H = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

Then the corresponding multi-graph H is



Next, to classify filiform Lie algebras we study the matrices $M_{\mathfrak{g}}$ and compute the possible values of coefficients c_{ij}^k by using Jacobi identities.

Proposition 1 *The matrix $M_{\mathfrak{g}}$ is the following, where $a, b, c \in \mathbb{Z}/p\mathbb{Z}$*

$$M_{\mathfrak{g}} = \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & b & a \end{pmatrix}$$

Proof:
It is straightforward by using $J(e_1, e_4, e_6) = 0$ and $J(e_1, e_5, e_6) = 0$. $\square$

Note that this result allows us to set the following

Corollary 1 *There exists a bijective map between matrices $M_{\mathfrak{g}}$ and the set of three-dimensional vectors (a, b, c), with $a, b, c \in \mathbb{Z}/p\mathbb{Z}$.*

In turn Corollary 1 permits us to justify the following relationship between three-dimensional vectors: *two three-dimensional vectors with component in $\mathbb{Z}/p\mathbb{Z}$ are equivalent (denoted by $\sim$) if their corresponding matrices are associated with isomorphic filiform Lie algebras*.

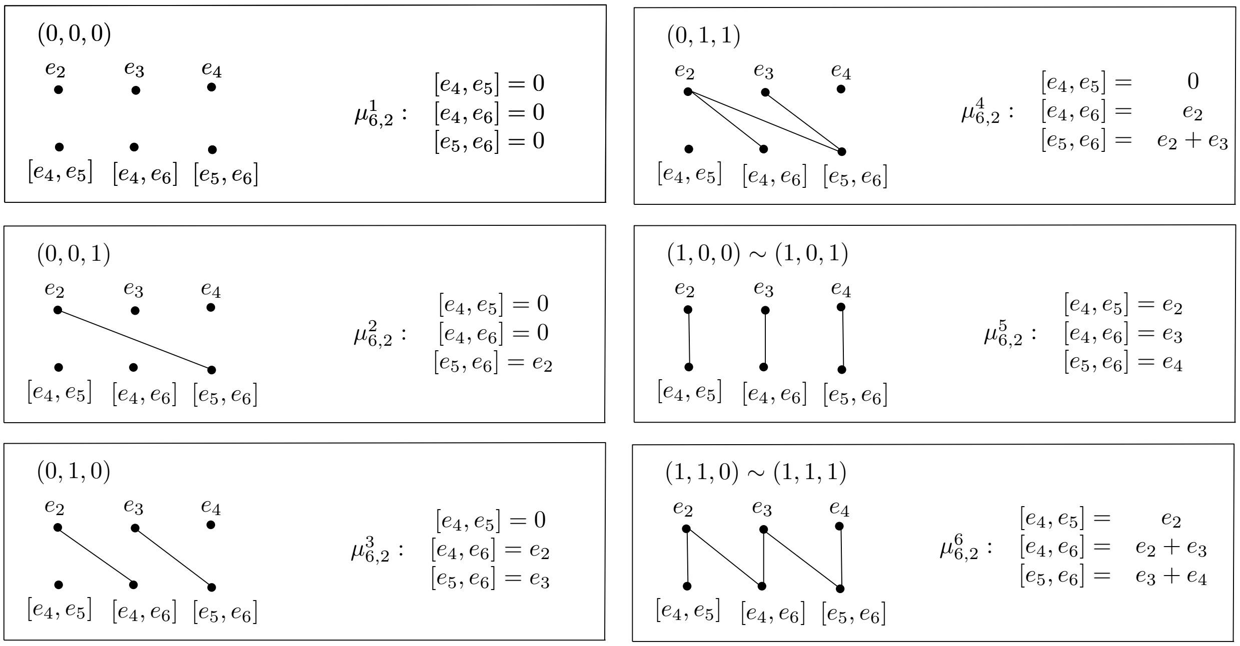
As a consequence of Proposition 1 and Corollary 1 it follows that the vector (a, b, c) is related with the initial filiform Lie algebra $\mathfrak{g}$ if the matrix associated with $\mathfrak{g}$ is $M_{\mathfrak{g}}$.

Now, considering these results, 6-dimensional filiform Lie algebras over $\mathbb{Z}/p\mathbb{Z}$ can be classified into equivalence classes according to the field of definition. For computations we have used the symbolic computation package *Singular*, due to that it is very suitable to deal with Gröbner basis which could be used in related works in future. Three different cases were studied.

3.2 The case $p = 2$. In this case the following result is held

Theorem 1 *Up to isomorphism, there exist six 6-dimensional filiform Lie algebras over $\mathbb{Z}/2\mathbb{Z}$. They are shown in the following table, in which the corresponding vector, the graph and the law of each algebra are indicated*

ISBN: 978-1-61804-002-2



3.3 The case $p = 3$. Similarly, we have obtained the following result in $\mathbb{Z}/3\mathbb{Z}$ (only the corresponding vectors of the algebras are indicated).

Theorem 2 *Up to isomorphism, there exist six* 6*-dimensional filiform Lie algebras over* $\mathbb{Z}/3\mathbb{Z}$*. They are*

1. *(0,0,0)* **2.** *(0,0,1) ∼ (0,0,2)* **3.** *(0,1,0) ∼ (0,1,1) ∼ (0,1,2)* **4.** *(0,2,0) ∼ (0,2,1) ∼ (0,2,2)*
5. *(1,0,0) ∼ (1,0,1) ∼ (1,0,2) ∼ (2,0,0) ∼ (2,0,1) ∼ (2,0,2)*
6. *(1,1,0) ∼ (1,1,1) ∼ (1,1,2) ∼ (1,2,0) ∼ (1,2,1) ∼ (1,2,2) ∼ (2,1,0) ∼ (2,1,1) ∼ (2,1,2) ∼ (2,2,0) ∼ (2,2,1) ∼ (2,2,2)*

Using a similar procedure we have also obtained the classification of 6-dimensional filiform Lie algebras over $\mathbb{Z}/5\mathbb{Z}$.

3.4 The case $p = 5$. In this case it is deduced

Theorem 3 *Up to isomorphism, there exist six* 6*-dimensional filiform Lie algebras over* $\mathbb{Z}/5\mathbb{Z}$*. They are*

1. *(0,0,0)* **2.** *(0,0,1) ∼ (0,0,2) ∼ (0,0,3) ∼ (0,0,4)*
3. *(0,1,0) ∼ (0,1,1) ∼ (0,1,2) ∼ (0,1,3) ∼ (0,1,4) ∼ (0,4,0) ∼ (0,4,1) ∼ (0,4,2) ∼ (0,4,3) ∼ (0,4,4)*
4. *(0,2,0) ∼ (0,2,1) ∼ (0,2,2) ∼ (0,2,3) ∼ (0,2,4) ∼ (0,3,0) ∼ (0,3,1) ∼ (0,3,2) ∼ (0,3,3) ∼ (0,3,4)*
5. *(1,0,0) ∼ (1,0,1) ∼ (1,0,2) ∼ (1,0,3) ∼ (1,0,4) ∼ (2,0,0) ∼ (2,0,1) ∼ (2,0,2) ∼ (2,0,3) ∼ (2,0,4) ∼ (3,0,0) ∼ (3,0,1) ∼ (3,0,2) ∼ (3,0,3) ∼ (3,0,4) ∼ (4,0,0) ∼ (4,0,1) ∼ (4,0,2) ∼ (4,0,3) ∼ (4,0,4)*
6. *(110) ∼ (111) ∼ (112) ∼ (113) ∼ (114) ∼ (120) ∼ (121) ∼ (122) ∼ (123) ∼ (124) ∼ (130) ∼ (131) ∼ (132) ∼ (133) ∼ (134) ∼ (140) ∼ (141) ∼ (142) ∼ (143) ∼ (144) ∼ (210) ∼ (211) ∼ (212) ∼ (213) ∼ (214) ∼ (220) ∼ (221) ∼ (222) ∼ (223) ∼ (224) ∼ (230) ∼ (231) ∼ (232) ∼ (233) ∼ (234) ∼ (240) ∼ (241) ∼ (242) ∼ (243) ∼ (244) ∼ (310) ∼ (311) ∼ (312) ∼ (313) ∼ (314) ∼ (320) ∼ (321) ∼ (322) ∼ (323) ∼ (324) ∼ (330) ∼ (331) ∼ (332) ∼ (333) ∼ (334) ∼ (340) ∼ (341) ∼ (342) ∼ (343) ∼ (344) ∼ (410) ∼ (411) ∼ (412) ∼ (413) ∼ (414) ∼ (420) ∼ (421) ∼ (422) ∼ (423) ∼ (424) ∼ (430) ∼ (431) ∼ (432) ∼ (433) ∼ (434) ∼ (440) ∼ (441) ∼ (442) ∼ (443) ∼ (444) (commas have been suppressed in this last item to save space).*

ISBN: 978-1-61804-002-2

4 Some Conclusions and Open Problems

In this paper we have dealt with 6-dimensional filiform Lie algebras over $\mathbb{Z}/p\mathbb{Z}$, for $p = 2, 3, 5$. Our intention is continue with this topic by increasing both the value of p prime and the dimension of the algebras. As main results, we have obtained that there exist six 6-dimensional filiform Lie algebras in the cases of $p = 2, 3, 5$. Remember that there exist six filiform Lie algebras of dimension 6 over $\mathbb{C}$.

By considering the set of vectors belonging to the same equivalence class, we formulate the following conjectures, all of them related to filiform Lie algebras of dimension 6 and whatever p prime is.

1. In $\mathbb{Z}/p\mathbb{Z}$ there would be six equivalence classes whose representatives should be $(0,0,0), (0,0,1), (0,1,0), (0,2,0), (1,0,0), (1,1,0)$.
2. Let $\mathfrak{g} \in \mathcal{F}_6^p$ with $p > 2$ be a non-model filiform Lie algebra and let (a, b, c) be its related vector. The vector $(a, b, c+1)$ (module p) will relate with another Lie algebra $\mathfrak{h} \in \mathcal{F}_6^p$, which would be isomorphic with $\mathfrak{g}$. In terms of graphs, it means to add an edge (module p) from vertex $[e_5, e_6]$ to e_2 in the graph associated with $\mathfrak{g}$.
3. Given a vector (a, b, c) corresponding with a Lie algebra $\mathfrak{g} \in \mathcal{F}_6^p$ with $c \neq 0$, an isomorphic filiform Lie algebra $\mathfrak{h}$ would be obtained changing a by another $d \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$. The graph associated with $\mathfrak{g}$ has the same number of edges from vertex $[e_4, e_5]$ to e_2, from $[e_4, e_6]$ to e_3, and from $[e_5, e_6]$ to e_4. Then by adding the same number of edges (module p) between each pair of these vertices, a new graph whose associated algebra would be isomorphic to $\mathfrak{g}$ will be obtained. Similarly, it would be satisfied $(a, b, c) \sim (a, d, c)$ for $a, b, d \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$. It would be reflected by a natural operation between the associated graphs.

We will hope to advance in the study of these conjectures in future works, as well as to increase the dimension of the algebras.

References:

[1] L. Boza Prieto, F.J. Echarte Reula and J. Núñez Valdés, *Classification of complex filiform Lie algebras of dimension 10*, Algebras, Groups and Geometries **11** (1994), 253–276.

[2] L. Boza Prieto, E. M. Fedriani Martel and J. Núñez Valdés, *A new method for classifying complex filiform Lie algebras*, Applied Mathematics and Computation **121**: 2-3 (2001), 169–175.

[3] L. Boza, E.M. Fedriani y J. Núñez, Una relación entre los pseudo-grafos dirigidos sin aristas repetidas y algunas álgebras de Lie, *Actas del IV Encuentro Andaluz de Matemática Discreta* (2005), 99–104.

[4] A. Carriazo, L.M. Fernández and J. Núñez, Combinatorial structures associated with Lie algebras of finite dimension, *Linear Algebra and its Applications* **389** (2004), 43–61.

[5] E. M. Fedriani, *Clasificación de las álgebras de Lie filiformes complejas de dimensión 12*. Tesis de Licenciatura. Universidad de Sevilla (España), 1997.

[6] P. Frè, V. Gili, F. Gargiulo, A. Sorin, H. Rulik, M. Trigiante, Cosmological backgrounds of superstring theory and solvable algebras: Oxidation and branes, Nucl. Phys. B 685 (2004) 3–64.

[7] F. Harary, Graph Theory, Addison Wesley, Reading, Mass., 1969.

[8] J. Núñez, A. Pacheco y M.T. Villar, Matemática Discreta aplicada al tratamiento de algunos problemas de la Teoría de Lie, Actas de las *VI Jornadas de Matemática Discreta y Algorítmica (VIJMDA)*, (2008) 485–492.

[9] J. Núñez, A. Pacheco y M. T. Villar, Study of a family of Lie algebras over $Z/3Z$, *International Journal of Applied Mathematics and Statistics*, Special volume **7**:W10 (2010), 40-45.

[10] V.S. Varadarajan. Lie Groups, Lie Algebras and their Representations. Springer, 1984.

[11] M. Vergne, *Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algebres de Lie nilpotentes*, Bull. Soc. Math. France **98** (1970), 81–116.