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A note on the numerical solution of high-order differential equations

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Abstract

Numerical solution of high-order differential equations with multi-boundary conditions is discussed in this paper. Motivated by the discrete singular convolution algorithm, the use of fictitious points as additional unknowns is proposed in the implementation of locally supported Lagrange polynomials. The proposed method can be regarded as a local adaptive differential quadrature method. Two examples, an eigenvalue problem and a boundary-value problem, which are governed by a sixth-order differential equation and an eighth-order differential equation, respectively, are employed to illustrate the proposed method.

Keywords: High-order differential equation; Multi-boundary conditions; Local adaptive differential quadrature method

1. Introduction

High-order differential equation arises in many fields. For example, the free vibration analysis of beam structures is governed by a fourth-order differential equation, and that of ring structures by a sixth-order differential equation. Moreover, when considering the instability setting in an infinite horizontal layer of fluid, which is heated from below and is subject to the action of rotation, we model the instability as ordinary convection and overstability by a sixth-order ordinary differential equation (ODE) and an eighth-order ODE, respectively. Even higher-order ODEs can be involved when a uniform magnetic field is applied across the fluid in the same direction as gravity. Ordinary convection and overstability yield a 10th-order and a 12th-order ODE, respectively. Such problems

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modeled by high-order differential equations are often associated with multi-boundary conditions, so that the problem is well posed. For example, the beam vibration problem, which is governed by a fourth-order differential equation, has four boundary conditions. It is noted that for well-posed problems, the number of boundary conditions is the same as the order of the differential equation, and in general, they are evenly given at each boundary.

When interpolating polynomials are employed as approximation kernels, three types of implementations of multi-boundary conditions have been proposed in the literature [1–8,11,13,15,16]. The first type builds the boundary conditions into the approximation kernel, i.e., interpolating polynomials. The discretization of a one-dimensional computational domain with N nodes results in N unknowns. The satisfaction of the governing equation at each node leads to N linear equations. These linear equations have to be linearly independent for a boundary value problem, but not necessary for an eigenvalue problem, so that a solution can be obtained. Huang and Sloan [7] presented the pseudospectral method of this kind to solve differential eigenvalue problems. One example is the fourth-order eigenvalue problem

$$u'''' + Ru''' = su'', \quad -1 < x < 1, \tag{1}$$

$$u(-1) = u'(-1) = u(1) = u'(1) = 0, \tag{2}$$

where u is the unknown function, R is a real parameter, s is the eigenparameter, and a prime denotes differentiation with respect to x. After discretization, N grid points are yielded. A polynomial expression is sought to satisfy all the boundary conditions. Hence, it is represented as the summation of some interpolating polynomials,

$$u(x) \approx u^N(x) = \sum_{k=2}^{N-1} u_k h_k(x), \tag{3}$$

where $u_k = u(x_k)$ and

$$h_k(x) = \frac{(1-x^2)}{(1-x_k^2)}\, l_k(x). \tag{4}$$

Here $l_k(x)$ is the Lagrange interpolating polynomial given by

$$l_k(x) = \prod_{i=1, i\neq k}^{N} \frac{x - x_i}{x_k - x_i}. \tag{5}$$

It is seen that, the boundary conditions have been accounted into the construction of the polynomial, i.e.,

$$u^N(\pm 1) = (u^N)'(\pm 1) = 0. \tag{6}$$

However, this technique is hardly useful when the boundary conditions become complicated as in the higher-order differential equation.

The second and the third types of implementation separate the approximation kernel from boundary conditions. Therefore, $N + p$ linear equations are obtained, where N is the number of nodes (or unknowns) and p is the number of boundary conditions (or order of the differential equation), i.e., N linear equations are from the governing equation, p linear equations are from boundary conditions. For the second type of methods, it is noted that the N equations obtained from discretizing the governing equation are always linearly dependent. The rank of the coefficient matrix is usually $N-p$, for a pth-order differential equation. Note that the number of multi-boundary conditions is usually the same as the order of the differential equation, we can take out a number of $N-p$ linear equations and join them with p linear equations yielded from boundary conditions and proceed for a solution. The differential quadrature method (DQM) and the generalized differential quadrature (GDQ) method all belong to this type. Typical examples of this kind may refer to Refs. [6,11].

For the third type of methods, it is noted that there is a difference between the number of unknowns and the number of linear equations. Alternatively, one can introduce additional unknowns. For example, the generalized differential quadrature rule (GDQR) method introduces higher-order derivative values of the boundary nodes. The involvement of these higher-order derivative values requires the GDQR to use the Hermite interpolating polynomials, instead of the Lagrange ones. The finite difference method developed by Boutayeb and Twizell [1–3,5,13] can be regarded as a special case of this kind, in which a local formulation with low-order approximation is considered. Liu and his co-workers have presented some results on the GDQR method for solving high-order differential equations, such as Refs. [8,16,15]. The GDQR formulations for two-dimensional problems are quite different from that of one-dimensional ones [16], while the conventional DQM formulations have the same form for both two-dimensional and one-dimensional problems. In this paper, we propose an alternative solution method of the third type, called local adaptive differential quadrature method (La-DQM). Stimulated by using fictitious points in our discrete singular convolution (DSC) algorithm [14], we construct fictitious points outside the boundary as additional unknowns. The La-DQM has its root in the finite difference approach. Chen [4] mentioned the fictitious point technique when he presented his differential quadrature finite difference method. However, how to construct La-DQM for the solution of high-order differential equations has not been explored yet. Using the Lagrange polynomials, the proposed method is found to have much simpler formulations than the GDQR method, while its accuracy is close to that of the GDQR method. Furthermore, it is noticed that the use of fictitious points outside the boundary guarantees the stability of the approximation kernel. This can be observed when it is compared with spline solutions [12] for some eighth-order boundary-value problems, which fail to converge at some nodes next to the boundary.

The paper is organized as follows. Section 2 presents the formulation of the proposed solution method, including the explicit weighting coefficient formula, and the details of the implementation of the multi-boundary conditions. A local adaptive GDQ method is thus developed. In Section 3, two examples are used, including an eigenvalue problem governed by a six-order differential equation and a boundary value problem governed by an eighth-order differential equation. Results obtained by the proposed method are compared with some of the existing ones, such as the DQM [6], spline [12] and GDQR [8] solutions. This paper ends with a conclusion.

2. Formulation

In this section, a local adaptive differential quadrature method (La-DQM) is developed in association with the implementation of multi-boundary conditions by using fictitious points. The problem under consideration is assumed to be governed by a one-dimensional pth ($p \geqslant 4$)-order differential equation, with p boundary conditions, including two Dirichlet type of boundary conditions (one at each boundary).

2.1. Weighting coefficients

The La-DQM states that, to approximate the derivative of a function with respect to a space variable at a given discrete point, a weighted linear combination of the function values at some of the neighboring discrete points in the direction of the space variable within the computational domain is employed. For example, $u_x^{(m)}(x_i)$, the mth derivative of a function $u(x)$ at the ith point, x_i, is approximated as

$$u^{(m)}(x_i) = \sum_{j=-L_i}^{R_i} c_{i,j}^{(m)} u(x_{i+j}), \tag{7}$$

where $c_{i,j}^{(m)}$, $(j = -L_i, \ldots, R_i)$, are the weighting coefficients for the mth derivative approximation of the ith point, L_i and R_i denote the number of function values of the neighboring points on the left and right, respectively. These weighting coefficients have to be pre-determined. When L_i and R_i are specified, formulation similar to the explicit GDQ [10] for the weighting coefficients can be used to determine the weighting coefficients for the ith node. For each node, a system of Lagrange interpolating polynomials are used. For example, for the ith point, one has [10]

$$g_{i,j}(x) = \prod_{k=i-L_i, k\neq i+j}^{i+R_i} \frac{x - x_k}{x_{i+j} - x_k}, \quad j = -L_i, \ldots, R_i. \tag{8}$$

The weighting coefficients for the first derivative can be obtained from the exact differentiation of the above polynomial systems. Consequently, one has

$$c_{i,j}^{(1)} = g_{i,j}^{(1)}(x_i) \quad \text{for } j = -L_i, \ldots, R_i; \ \ j \neq 0 \tag{9}$$

and as for $j = 0$,

$$c_{i,0}^{(1)} = -\sum_{j=-L_i, j\neq 0}^{R_i} c_{i,j}^{(1)}. \tag{10}$$

The weighting coefficients for higher-order derivatives can be obtained using a recurrence formula:

$$c_{i,j}^{(m)} = m\left(c_{i,j}^{(1)} c_{i,i}^{(m-1)} - \frac{c_{i,j}^{(m-1)}}{x_i - x_{i+j}}\right), \quad j = -L_i, \ldots, R_i; \ \ j \neq 0 \tag{11}$$

and

$$c_{i,0}^{(m)} = -\sum_{j=-L_i, j\neq 0}^{R_i} c_{i,j}^{(m)}. \tag{12}$$

2.2. Implementation of multi-boundary conditions by fictitious points

In view of the difference between a boundary value problem and an eigenvalue problem, the implementation of boundary conditions may change accordingly. First, we consider a boundary value problem. Assume that the one-dimensional pth-order differential equation is defined in an interval $[a,b]$, with $p(=p_\mathrm{l} + p_\mathrm{r})$ boundary conditions, where p_l and p_r represent the number of boundary conditions on the left and right boundaries, respectively. We also assume that $p_l \geqslant 2$ and $p_r \geqslant 2$, and there is one Dirichlet type of boundary condition on each boundary. The proposed method introduces $p_l - 1$ and $p_r - 1$ fictitious points outside the left and the right boundaries, respectively. Therefore, we have

$$x_{2-p_l} < \cdots < x_1 = a < x_2 < \cdots < x_N = b < \cdots < x_{N+p_r-1}. \tag{13}$$

In view of the existence of the two Dirichlet type of boundary conditions, we have the following solution procedure. For a pth-order differential equation under the discretization of $N + p - 2$ nodes, the satisfaction of the governing equation on each point yields $N + p - 2$ linear equations. As mentioned early, we choose $(N + p - 2) - p$, i.e., $N - 2$ equations from them. Here, we use those equations obtained from the satisfaction of the governing equation on the inner points of the interval $[a, b]$, i.e., $x_2, \ldots, x_{N-2}$. Together with the p linear equations obtained from the boundary conditions, we have $N + p - 2$ linear equations to solve the $N + p - 2$ unknowns, including the values for the $p - 2$ fictitious points.

For an eigenvalue problem, the solution procedure slightly differs from the boundary value problem. The fictitious points' values and the boundary points' values are first solved in terms of those of inner points, using the linear equations obtained from the boundary conditions, i.e. p equations are used to solve p unknowns. Subsequently, substituting the solution of these p variables into the $N-2$ linear equations obtained from the governing equations, we obtain the eigenvalue problem in the form

$$Ax = \lambda x, \tag{14}$$

where A is an $(N - 2) \times (N - 2)$ square matrix, x is a column vector consisting of the $N - 2$ inner points' values, i.e., values on $x_2, \ldots, x_{N-2}$.

3. Application

Two examples are used to illustrate the results of the proposed method. Both a boundary value problem and an eigenvalue problem are presented, which are governed by an eighth-order and a sixth-order differential equations, respectively.

3.1. Example 1: a sixth-order eigenvalue problem

In this example, a circular ring structure with constraints (see Refs. [6,15]), which has rectangular cross-sections of constant width and parabolic variable thickness is studied. Considering half of the ring structure, this problem is an eigenvalue one formulated by the following sixth-order differential equation:

$$\beta_1 w^{(6)} + \beta_2 w^{(5)} + \beta_3 w^{(4)} + \beta_4 w^{(3)} + \beta_5 w^{(2)} + \beta_6 w^{(1)} = \Omega^2 (f w^{(2)} + f^{(1)} w^{(1)} - \pi^2 f w), \tag{15}$$

where $w^{(r)} = \mathrm{d}^r w/\mathrm{d}x^r$, Ω is the dimensionless frequency, w is the tangential displacement, and

$$\beta_1 = \varphi/\pi^4,$$

$$\beta_2 = 3\varphi^{(1)}/\pi^4,$$

$$\beta_3 = (2\varphi/\pi^2) + (3\varphi^{(2)}/\pi^4),$$

$$\beta_4 = (4\varphi^{(1)}/\pi^2) + (\varphi^{(3)}/\pi^4),$$

$$\beta_5 = \varphi + 3\varphi^{(2)}/\pi^2,$$

$$\beta_6 = \varphi^{(1)} + \varphi^{(3)}/\pi^2, \tag{16}$$

in which

$$\varphi = [f(x)]^3,$$

$$f = f(x) = -4(r-1)x^2 + 4(r-1)x + 1 \tag{17}$$

for $x \in [0,1]$ and $\varphi^{(i)} = \mathrm{d}^i\varphi/\mathrm{d}x^i$, $f^{(i)} = \mathrm{d}^i f/\mathrm{d}x^i$, and r is the variable related to the thickness of the cross-section of the ring.

Rewriting Eq. (15), we have

$$\left(\beta_1 \frac{\mathrm{d}^6}{\mathrm{d}x^6} + \beta_2 \frac{\mathrm{d}^5}{\mathrm{d}x^5} + \beta_3 \frac{\mathrm{d}^4}{\mathrm{d}x^4} + \beta_4 \frac{\mathrm{d}^3}{\mathrm{d}x^3} + \beta_5 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta_6 \frac{\mathrm{d}}{\mathrm{d}x}\right) w$$

$$= \Omega^2 \left(f \frac{\mathrm{d}^2}{\mathrm{d}x^2} + f^{(1)} \frac{d}{\mathrm{d}x} - \pi^2 f\right) w. \tag{18}$$

The discretization of operators yields a generalized eigenvalue problem.

The ring structure has three boundary conditions at each end. Altogether, it has six boundary conditions. Two different settings of boundary conditions are considered.

Example 1.1.

$$w(0) = w^{(1)}(0) = w^{(3)}(0) = 0, \quad w(1) = w^{(1)}(1) = w^{(3)}(1) = 0. \tag{19}$$

Example 1.2. This example is also taken from Refs. [6,15]. Instead of considering a half ring structure, a quarter of the ring structure without constraints is considered. The governing equation is quite similar to (15), but the last term is changed from $\pi^2 fw$ to $\pi^2 fw/4$. Other changes include

$$\beta_1 = 16\varphi/\pi^4,$$

$$\beta_2 = 48\varphi^{(1)}/\pi^4,$$

$$\beta_3 = (8\varphi/\pi^2) + (48\varphi^{(2)}/\pi^4),$$

$$\beta_4 = (16\varphi^{(1)}/\pi^2) + (16\varphi^{(3)}/\pi^4),$$

$$\beta_5 = \varphi + 12\varphi^{(2)}/\pi^2,$$

$$\beta_6 = \varphi^{(1)} + 4\varphi^{(3)}/\pi^2,$$

$$\varphi = [f(x)]^3,$$

$$f(x) = -(r-1)x^2 + 2(r-1)x + 1 \tag{20}$$

and

$$w(0) = w^{(2)}(0) = 0, \quad \varphi^{(1)}(0)[w^{(1)}(0) + 4w^{(3)}(0)/\pi^2] + 4\varphi(0)w^{(4)}(0)/\pi^2 = 0,$$
$$w(1) = w^{(2)}(1) = 0, \quad \varphi^{(1)}(1)[w^{(1)}(1) + 4w^{(3)}(1)/\pi^2] + 4\varphi(1)w^{(4)}(1)/\pi^2 = 0. \tag{21}$$

A standard eigenvalue solver can be used to solve the eigenvalue problem and to obtain the frequency Ω.

3.2. Example 2: an eighth-order boundary value problem

Here, an eighth-order boundary-value problem with four different settings solved in [12,8] are dealt with again to obtain accurate results in the entire domain. They have the form of

$$y^{(8)} + \phi(x)y = \psi(x), \quad -\infty < a \leqslant x \leqslant b < \infty, \tag{22}$$

$$y(a) = A_0,$$
$$y^{(2)}(a) = A_2,$$
$$y^{(4)}(a) = A_4,$$
$$y^{(6)}(a) = A_6,$$
$$y(b) = B_0,$$
$$y^{(2)}(b) = B_2,$$
$$y^{(4)}(b) = B_4,$$
$$y^{(6)}(b) = B_6, \tag{23}$$

where $y = y(x)$ and $\phi(x)$ and $\psi(x)$ are continuous functions defined in the interval $x \in [a, b]$. A_i and B_i, $(i = 0, 2, 4, 6)$, are finite real constants. Analytical solutions with various constants and functions for all four examples are listed in Table 1.

3.3. Results and discussion

For simplicity, a uniform grid is considered in all the calculations. Moreover, the L_i and R_i are given by the following criterion:

$$L_i = \min\{M, i - 2 + p_l\}, \qquad R_i = \min\{M, N + p_r - 1 - i\}. \tag{24}$$

In other words, the derivative at a point is approximated by $2M + 1$ neighboring points or all the points, depending on which is smaller. To compete with the GDQR solutions, a large M value is chosen in all the calculations (see Tables 2–7).

The frequencies of the ring structure, i.e., Examples 1.1 and 1.2, are calculated and listed in Tables 2 and 3. In Table 2, it is noted that the present method converges to the same frequency parameter

Table 1
Variables for differential equations and boundary conditions in four examples

Example	2.1	2.2	2.3	2.4
$[a,b]$	[0,1]	$[-1,1]$	$[-1,1]$	$[-1,1]$
$\phi(x)$	x	$-x$	-1	-1
$\psi(x)$	$-(48+15x+x^3)e^x$	$-(55+17x+x^2-x^3)e^x$	$-8[2x\cos(x)+7\sin(x)]$	$8[2x\sin(x)-7\cos(x)]$
A_0	0	0	0	0
A_2	0	$2/e$	$-4\cos(1)-2\sin(1)$	$-4\sin(1)+2\cos(1)$
A_4	-8	$-4/e$	$8\cos(1)+12\sin(1)$	$8\sin(1)-12\cos(1)$
A_6	-24	$-18/e$	$-12\cos(1)-30\sin(1)$	$-12\sin(1)+30\cos(1)$
B_0	0	0	0	0
B_2	$-4e$	$-6e$	$4\cos(1)+2\sin(1)$	$-4\sin(1)+2\cos(1)$
B_4	$-16e$	$-20e$	$-8\cos(1)-12\sin(1)$	$8\sin(1)-12\cos(1)$
B_6	$-36e$	$-42e$	$12\cos(1)+30\sin(1)$	$-12\sin(1)+30\cos(1)$
Analytical Solution $y(x)$	$x(1-x)e^x$	$(1-x^2)e^x$	$(x^2-1)\sin(x)$	$(x^2-1)\cos(x)$

Table 2
Comparison of fundamental frequencies for Example 1.1 of the sixth-order eigenvalue problem

r	DQM [6]	Rayleigh–Ritz [6]	GDQR					Present				
			7[a]	8[a]	9[a]	10[a]	11[a]	7[a]	8[a]	9[a]	10[a]	11[a]
1.0	2.268(12[a])	2.274	2.2631	2.2669	2.2667	2.2667	2.2667	2.2624	2.2647	2.2669	2.2668	2.2667
1.1	2.417(12[a])	2.416	2.4133	2.4137	2.4137	2.4137	2.4137	2.4135	2.4136	2.4137	2.4137	2.4137
1.2	2.561(12[a])	2.557	2.5597	2.5565	2.5567	2.5568	2.5568	2.5583	2.5576	2.5570	2.5569	2.5568
1.3	2.701(12[a])	2.697	2.7139	2.6944	2.6962	2.6966	2.6966	2.7019	2.6995	2.6976	2.6972	2.6968
1.4	2.839(14[a])	2.834	2.8946	2.8242	2.8318	2.8336	2.8335	2.8452	2.8400	2.8364	2.8353	2.8341
1.5	2.976(14[a])	2.970	3.1297	2.9407	2.9623	2.9681	2.9678	2.9878	2.9791	2.9738	2.9715	2.9694

[a]Number of the grid points, N; $M=N+1$.

Table 3
Comparison of fundamental frequencies for Example 1.2 of the sixth-order eigenvalue problem

r	DQM [6]	Rayleigh–Ritz [6]	GDQR					Present				
			6[a]	7[a]	8[a]	9[a]	10[a]	6[a]	7[a]	8[a]	9[a]	10[a]
1.0	2.686(12[a])	2.687	2.6828	2.6833	2.6833	2.6833	2.6833	2.6956	2.6828	2.6830	2.6833	2.6833
1.1	2.849(12[a])	2.846	2.8452	2.8452	2.8452	2.8452	2.8452	2.8523	2.8488	2.8488	2.8489	2.8489
1.2	3.010(12[a])	3.006	3.0062	3.0062	3.0062	3.0062	3.0062	3.0199	3.0181	3.0182	3.0181	3.0181
1.3	3.171(12[a])	3.167	3.1666	3.1665	3.1665	3.1665	3.1665	3.1917	3.1884	3.1887	3.1884	3.1884
1.4	3.332(12[a])	3.326	3.3267	3.3263	3.3262	3.3263	3.3263	3.3595	3.3577	3.3579	3.3578	3.3578
1.5	3.493(12[a])	3.486	3.4861	3.4858	3.4857	3.4858	3.4858	3.5150	3.5251	3.5230	3.5252	3.5248

[a]Number of the grid points, N; $M=N+1$.

Table 4
Maximum absolute errors of Example 2.1 of the eighth-order boundary-value problem

$y_i^{(k)}$	Ref. [12] $[x_4, x_{N-4}]$	$N = 32$[a] Otherwise	GDQR		Present	
			$N = 7$[a]	$N = 11$[a]	$N = 7$[a]	$N = 11$[a]
$k = 0$	8.113E − 8	1.820E3	2.92E − 10	4.95E − 14	7.47E − 9	5.27E − 10
$k = 1$	2.461E − 7	4.302E4	9.90E − 10	2.65E − 13	2.77E − 8	1.79E − 9
$k = 2$	8.451E − 7	3.667E6	2.80E − 9	7.93E − 13	8.56E − 8	5.41E − 9
$k = 3$	2.738E − 6	3.865E9	1.20E − 8	7.97E − 13	4.11E − 7	2.18E − 8
$k = 4$	1.153E − 5	9.013E13	5.04E − 8	3.07E − 14	1.41E − 6	6.99E − 8
$k = 5$	4.870E − 5	8.844E15	1.18E − 6	6.25E − 13	1.20E − 5	4.41E − 7
$k = 6$	5.307E − 4	4.430E17	4.74E − 6	2.41E − 12	5.09E − 5	1.75E − 6
$k = 7$	1.142E − 2	9.450E18	2.00E − 4	2.53E − 10	1.79E − 3	5.42E − 5

[a]Number of the grid points, N; $M = N + 2$.

Table 5
Maximum absolute errors of Example 2.2 of the eighth-order boundary-value problem

$y_i^{(k)}$	Ref. [12] $[x_4, x_{N-4}]$	$N = 64$[a] Otherwise	GDQR		Present	
			$N = 7$[a]	$N = 11$[a]	$N = 7$[a]	$N = 11$[a]
$k = 0$	9.443E − 5	4.948E3	3.13E − 6	1.54E − 11	7.58E − 5	2.71E − 9
$k = 1$	1.454E − 4	1.169E5	5.09E − 6	2.75E − 11	1.30E − 4	4.83E − 9
$k = 2$	2.335E − 4	4.984E7	7.52E − 6	4.07E − 11	1.90E − 4	7.31E − 9
$k = 3$	3.578E − 4	1.050E10	1.50E − 5	9.28E − 11	4.10E − 4	1.66E − 8
$k = 4$	5.867E − 4	2.450E12	2.20E − 5	1.94E − 10	6.48E − 4	2.82E − 8
$k = 5$	8.612E − 4	2.404E14	1.83E − 4	1.58E − 9	2.34E − 3	1.18E − 7
$k = 6$	1.729E − 3	1.204E16	4.09E − 4	3.21E − 9	4.53E − 3	2.69E − 7
$k = 7$	1.206E − 2	2.569E17	8.81E − 3	1.70E − 7	7.77E − 2	6.93E − 6

[a]Number of the grid points, N; $M = N + 2$.

as that of the GDQR at $N = 11$ for smaller r values, $r = 1.0$, 1.1 and 1.2. For slightly higher r values, the present method starts to deviate from the GDQR solution. However, the difference is acceptable. The same trend is observed in Table 3. Since there is no exact solution for this problem, the results obtained by the DQM and Rayleigh–Ritz methods are used as a reference.

For the eighth-order boundary-value problems, i.e., Examples 2.1–2.4, we present only maximum absolute errors since no other error is available in the literature [12,8]. Detailed comparisons are shown in Tables 4–7. The accuracy of the present method is slightly lower than that of the GDQR method. This is because the use of a simple uniform grid in the present work, whereas a Chebyshev grid was used in the GDQR. The Chebyshev gird does improve the accuracy of slow-varying solutions but can cause additional errors for high-order modes. Both the La-DQM and GDQR work well at boundaries, whereas, the spline method failed to converge near the boundary.

Table 6
Maximum absolute errors of Example 2.3 of the eighth-order boundary-value problem

$y_i^{(k)}$	Ref. [12] $[x_4, x_{N-4}]$	$N = 64^a$ Otherwise	GDQR		Present	
			$N = 7^a$	$N = 11^a$	$N = 7^a$	$N = 11^a$
$k = 0$	1.203E − 6	1.616E3	9.58E − 8	9.09E − 13	2.72E − 6	1.65E − 9
$k = 1$	2.936E − 6	3.819E4	3.97E − 7	2.99E − 12	9.94E − 6	3.23E − 9
$k = 2$	7.182E − 6	1.628E7	9.32E − 7	9.72E − 12	2.74E − 5	5.26E − 9
$k = 3$	2.016E − 5	3.430E9	6.78E − 6	3.08E − 11	1.04E − 4	1.38E − 8
$k = 4$	5.920E − 5	8.000E11	1.08E − 5	1.28E − 10	2.92E − 4	2.78E − 8
$k = 5$	1.854E − 4	7.851E13	1.83E − 4	1.52E − 9	1.37E − 3	1.22E − 7
$k = 6$	5.749E − 4	3.932E15	3.21E − 4	2.73E − 9	3.34E − 3	3.05E − 7
$k = 7$	4.715E − 3	8.388E16	6.73E − 3	1.45E − 7	6.03E − 2	7.79E − 6

[a]Number of the grid points, N; $M = N + 2$.

Table 7
Maximum absolute errors of Example 2.4 of the eighth-order boundary-value problem

$y_i^{(k)}$	Ref. [12] $[x_4, x_{N-4}]$	$N = 64^a$ Otherwise	GDQR		Present	
			$N = 7^a$	$N = 11^a$	$N = 7^a$	$N = 11^a$
$k = 0$	1.313E − 4	2.239E3	2.61E − 6	1.49E − 11	7.38E − 5	4.05E − 9
$k = 1$	2.024E − 4	5.504E4	4.16E − 6	2.34E − 11	1.16E − 4	6.38E − 9
$k = 2$	3.238E − 4	4.692E6	6.23E − 6	3.64E − 11	1.81E − 4	9.97E − 9
$k = 3$	5.008E − 4	4.945E9	1.13E − 5	5.85E − 11	2.90E − 4	1.60E − 8
$k = 4$	7.937E − 4	1.153E12	1.47E − 5	8.49E − 11	4.36E − 4	2.41E − 8
$k = 5$	1.258E − 3	1.132E14	4.75E − 5	1.82E − 10	8.29E − 4	4.62E − 8
$k = 6$	1.853E − 3	5.668E15	9.11E − 5	4.66E − 10	8.94E − 4	5.21E − 8
$k = 7$	2.150E − 3	1.209E17	1.30E − 3	1.62E − 8	1.19E − 2	1.22E − 6

[a]Number of the grid points, N; $M = N + 2$.

4. Conclusion

This paper presents a La-DQM for the implementation of multi-boundary conditions in solving high-order differential equations. In the principle of DSC, fictitious points are used as additional unknowns in association with the La-GDQ method. Both the formulation for weighting coefficients and the details of implementing multi-boundary conditions are presented. Furthermore, two examples, one sixth-order eigenvalue problem and one eighth-order boundary value problem, are utilized to illustrate the performance of the proposed method.

Numerical results are compared with those of the generalized differential quadrature rule (GDQR) [8,15,16] and the spline method [12]. The La-DQM gives much better results than the spline method. It is found that the present results agree very well with those of the GDQR. For most of cases, the GDQR slightly out performs the proposed method. However, in view of the complexity of the

GDQR method, which involves the Hermite interpolating polynomials, the proposed method is much simpler. Moreover, the La-DQM can be easily extended into two-dimensional problems, which are relatively difficult for the GDQR method. Therefore, we believe that the proposed method is of great merit for solving high-order differential equations.

It should be noted that, apart from the implementation of multi-boundary conditions, the grid distribution used is another key factor that determines the performance, e.g., accuracy and stability, of a numerical method. For example, the DQM and the GDQR methods mentioned in this paper respectively employed the δ technique and Chebyshev grid, which enhance the accuracy of slowly varying solutions but are unfavorable for high-frequency vibration modes. For simplicity, the proposed method is illustrated by a uniform grid. Of course, we can also use the Chebyshev points in the proposed approach if a better accuracy is required for the lower-order eigenmodes. An extensive analysis of accuracy and stability of the present La-DQM will be accounted elsewhere [9].

Note added in proof

After the paper was accepted, the authors learned that Fornberg discussed a similar treatment of boundary conditions for pseudospectral methods [17].

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