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# ON COMPLETE CONVERGENCE FOR ARRAYS OF ROWWISE INDEPENDENT RANDOM ELEMENTS

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# ON COMPLETE CONVERGENCE FOR ARRAYS OF ROWWISE INDEPENDENT RANDOM ELEMENTS

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Abstract. A complete convergence theorem for arrays of rowwise independent random variables was proved by Sung, Volodin, and Hu [14]. In this paper, we extend this theorem to the Banach space without any geometric assumptions on the underlying Banach space. Our theorem also improves some known results from the literature.

## 1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [5] as follows. A sequence $\{U_n, n \geq 1\}$ of random variables *converges completely* to the constant $\theta$ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \epsilon) < \infty \text{ for all } \epsilon > 0.$$

In view of the Borel-Cantelli lemma, this implies that $U_n \to \theta$ almost surely. The converse is true if $\{U_n, n \geq 1\}$ are independent random variables. Hsu and Robbins [5] proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. Erdös [2] proved the converse. We refer to Gut [3] for a survey on results on complete convergence related to strong laws and published before the nineties.

The result of Hsu-Robbins-Erdös has been generalized and extended in several directions. Some of these generalizations are in a Banach space setting. A sequence of Banach space valued random elements is said to *converge completely* to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

Recently, Sung et al. [14] proved the following complete convergence theorem for arrays of rowwise independent random variables.

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**Theorem 1.** *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise independent random variables, $\{k_n, n \ge 1\}$ a sequence of positive integers and $\{a_n, n \ge 1\}$ a sequence of positive constants. Suppose that for every $\epsilon > 0$ and some $\delta > 0$:*

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|X_{ni}| > \epsilon) < \infty$,

(ii) *there exists $J \ge 2$ such that*

$$\sum_{n=1}^{\infty} a_n \left( \sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}| \le \delta) \right)^J < \infty,$$

(iii) $\sum_{i=1}^{k_n} EX_{ni} I(|X_{ni}| \le \delta) \to 0$ *as* $n \to \infty$.

*Then $\sum_{n=1}^{\infty} a_n P(|\sum_{i=1}^{k_n} X_{ni}| > \epsilon) < \infty$ for all $\epsilon > 0$.*

Theorem 1 was first presented by Hu et al. [8]. Hu and Volodin [9] imposed one additional condition in addendum to the paper. Many people tried to prove Theorem 1 without the additional condition (for random variables, see Hu et al. [6] and Kuczmaszewska [11], and for random elements, see Hu et al. [7]).

The following theorem is a version of Banach space setting of Theorem 1 and is due to Hu et al. [7]. No assumptions are made concerning the geometry of the underlying Banach space.

**Theorem 2.** *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise independent random elements, $\{k_n, n \ge 1\}$ a sequence of positive integers and $\{a_n, n \ge 1\}$ a sequence of positive constants. Suppose that for every $\epsilon > 0$ and some $\delta > 0$:*

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(||X_{ni}|| > \epsilon) < \infty$,

(ii) *there exists $J \ge 2$ such that*

$$\sum_{n=1}^{\infty} a_n \left( \sum_{i=1}^{k_n} E||X_{ni}||^2 I(||X_{ni}|| \le \delta) \right)^J < \infty,$$

(iii) $\sum_{i=1}^{k_n} P(||X_{ni}|| > \delta) = o(1)$,

(iv) $||\sum_{i=1}^{k_n} X_{ni}|| \to 0$ *in probability.*

*Then $\sum_{n=1}^{\infty} a_n P(||\sum_{i=1}^{k_n} X_{ni}|| > \epsilon) < \infty$ for all $\epsilon > 0$.*

In this paper, we extend Theorem 1 to the Banach space without any geometric assumptions on the underlying Banach space. Our result also improves Theorem 2. More precisely, Theorem 2 holds without condition (iii).

We state our first theorem which shows that $o(1)$ in condition (iii) of Theorem 2 can be replaced by $O(1)$. The proof will appear in Section 3.

**Theorem 3.** *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise independent random elements, $\{k_n, n \ge 1\}$ a sequence of positive integers and $\{a_n, n \ge 1\}$ a sequence of positive constants. Suppose that for every $\epsilon > 0$ and some $\delta > 0$:*

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(||X_{ni}|| > \epsilon) < \infty$,

(ii) *there exists* $J \geq 2$ *such that*

$$\sum_{n=1}^{\infty} a_n \Big( \sum_{i=1}^{k_n} E||X_{ni}||^2 I(||X_{ni}|| \leq \delta) \Big)^J < \infty,$$

(iii) $\sum_{i=1}^{k_n} P(||X_{ni}|| > \delta) = O(1)$,

(iv) $||\sum_{i=1}^{k_n} X_{ni}|| \to 0$ *in probability.*

*Then* $\sum_{n=1}^{\infty} a_n P(||\sum_{i=1}^{k_n} X_{ni}|| > \epsilon) < \infty$ *for all* $\epsilon > 0$.

The following theorem is our main result which shows that condition (iii) of Theorem 2 can be removed. The proof will appear in Section 3.

**Theorem 4.** *Let* $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ *be an array of rowwise independent random elements,* $\{k_n, n \geq 1\}$ *a sequence of positive integers and* $\{a_n, n \geq 1\}$ *a sequence of positive constants. Suppose that for every* $\epsilon > 0$ *and some* $\delta > 0$*:*

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(||X_{ni}|| > \epsilon) < \infty$,

(ii) *there exists* $J \geq 2$ *such that*

$$\sum_{n=1}^{\infty} a_n \Big( \sum_{i=1}^{k_n} E||X_{ni}||^2 I(||X_{ni}|| \leq \delta) \Big)^J < \infty,$$

(iii) $||\sum_{i=1}^{k_n} X_{ni}|| \to 0$ *in probability.*

*Then* $\sum_{n=1}^{\infty} a_n P(||\sum_{i=1}^{k_n} X_{ni}|| > \epsilon) < \infty$ *for all* $\epsilon > 0$.

As an application of Theorem 4, we have the following corollary, which will be proved in Section 3.

**Corollary 1.** *Let* $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be an array of rowwise independent random elements which are weakly mean dominated by a random variable* $X$*, that is, there exists a constant* $C > 0$ *such that* $\frac{1}{n}\sum_{i=1}^{n} P(||X_{ni}|| > x) \leq CP(|X| > x)$ *for all* $x \geq 0$ *and* $n \geq 1$*. Let* $\alpha$ *and* $\phi$ *be nondecreasing functions defined on* $(0, \infty)$ *satisfying*

$$0 < \alpha(x) \uparrow \infty \text{ and } 0 < \phi(2x) \leq D\phi(x) \text{ for all } x > 0,$$

*where* $D > 0$ *is a constant. Suppose that* $E\phi(|X|) < \infty, E|X|^s < \infty$ *for some* $1 \leq s \leq 2$, $||\sum_{i=1}^{n} X_{ni}||/\alpha(n) \to 0$ *in probability, and there exists* $J \geq 2$ *such that*

$$\sum_{n=1}^{\infty} \frac{\phi(\alpha(n)) - \phi(\alpha(n-1))}{n} \Big( \frac{n}{\alpha^s(n)} \Big)^J < \infty.$$

*Then*

$$\sum_{n=1}^{\infty} \frac{\phi(\alpha(n)) - \phi(\alpha(n-1))}{n} P(||\sum_{i=1}^{n} X_{ni}|| > \epsilon\alpha(n)) < \infty \text{ for all } \epsilon > 0.$$

## 2. Preliminary lemmas

Let $B$ be a real separable Banach space with norm $||\cdot||$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A random element (or $B$-valued random element) is defined to be a $\mathcal{F}$-measurable mapping from $\Omega$ to $B$ equipped with the Borel $\sigma$-algebra (the $\sigma$-algebra generated by the open sets determined by $||\cdot||$). The expected value of a random element $X$ is defined to be Bochner integral (when $E||X|| < \infty$) and is denoted by $EX$.

The following lemma is an iterated form of Hoffmann-J$\phi$rgensen [4] inequality and is due to Jain [10].

**Lemma 1.** *If $X_1, \ldots, X_n$ are independent symmetric random elements, then for every integer $j \geq 1$ and every $t > 0$*

$$P(||S_n|| > 3^j t) \leq C_j P(\max_{1\leq i\leq n} ||X_i|| > t) + D_j \biggl(P(||S_n|| > t)\biggr)^{2^j},$$

*where $C_j$ and $D_j$ are positive constants depending only on $j$, and $S_n = \sum_{i=1}^n X_i$.*

The following lemma gives us a useful contraction principle and can be found in Lemma 6.5 of Ledoux and Talagrand [13].

**Lemma 2.** *Let $\{X_i, i \geq 1\}$ be a sequence of symmetric random elements. Let further $\{\xi_i, i \geq 1\}$ and $\{\zeta_i, i \geq 1\}$ be real random variables such that $\xi_i = \phi_i(X_i)$, where $\phi_i : B \to R$ is symmetric (even), and similarly for $\zeta_i$. Then, if $|\xi_i| \leq |\zeta_i|$ almost surely for every $i$, for every $t > 0$*

$$P(||\sum_i \xi_i X_i|| > t) \leq 2P(||\sum_i \zeta_i X_i|| > t).$$

*In particular, this inequality applies when $\xi_i = I_{\{X_i \in A_i\}} \leq 1 \equiv \zeta_i$ where the sets $A_i$ are symmetric in $B$ (in particular $A_i = \{||x|| \leq a_i\}$).*

The next lemma is a modification of a result of Kuelbs and Zinn [12] concerning the relationship between convergence in probability and mean convergence for sums of independent bounded random variables. We refer to Lemma 2.1 of Hu el al. [7] for the proof.

**Lemma 3.** *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent symmetric random elements. Suppose there exists $\delta > 0$ such that $||X_{ni}|| \leq \delta$ almost surely for all $1 \leq i \leq k_n, n \geq 1$. Put $S_n = \sum_{i=1}^{k_n} X_{ni}$. If $S_n \to 0$ in probability, then $E||S_n|| \to 0$ as $n \to \infty$.*

The following inequality is a Banach space analogue of the classical Marcinkiewicz-Zygmund inequality and is due to de Acosta [1]. When $p = 2$, $C_2$ can be taken to be 4.

**Lemma 4.** *Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of independent random elements. Then for $1 < p \leq 2$, there is a positive constant $C_p$ depending only on $p$ such* *that*

$$E\big|||S_n|| - E||S_n||\big|^p \le C_p \sum_{i=1}^{n} E||X_i||^p,$$

*where* $S_n = \sum_{i=1}^{n} X_i$.

Finally, we need the following lemma. The proof is standard and is omitted.

**Lemma 5.** *If $X$ and $Y$ have the same distribution, then for every $t > 0$*

$$E||X-Y||^2 I(||X-Y|| \le t) \le 8E||X||^2 I(||X|| \le \frac{t}{2}) + 2t^2 P(||X|| > \frac{t}{2}).$$

## 3. Proofs

*Proof of Theorem 3.* Let $\{X_{ni}^s, 1 \le i \le k_n, n \ge 1\}$ be an array of the symmetrized version of $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$, i.e., $X_{ni}^s = X_{ni} - X_{ni}^*$, where $X_{ni}$ and $X_{ni}^*$ are independent and have the same distribution. Let $\mu_n$ be a median of $||\sum_{i=1}^{k_n} X_{ni}||$. By (iv), $\mu_n \to 0$ as $n \to \infty$. Then we have by the weak symmetrization inequality that for all large $n$

$$\begin{aligned}
&P(||\sum_{i=1}^{k_n} X_{ni}|| > \epsilon) \le P(||\sum_{i=1}^{k_n} X_{ni}|| - \mu_n > \frac{\epsilon}{2}) \le 2P(||\sum_{i=1}^{k_n} X_{ni}^s|| > \frac{\epsilon}{2}) \\
&\le 2P(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \le 2\delta)|| > \frac{\epsilon}{4}) + 2P(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| > 2\delta)|| > \frac{\epsilon}{4}) \\
&\le 2P(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \le 2\delta)|| > \frac{\epsilon}{4}) + 2\sum_{i=1}^{k_n} P(||X_{ni}^s|| > 2\delta) \\
&\le 2P(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \le 2\delta)|| > \frac{\epsilon}{4}) + 4\sum_{i=1}^{k_n} P(||X_{ni}|| > \delta).
\end{aligned}$$

By (i), it is enough to prove that

$$\sum_{n=1}^{\infty} a_n P(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \le 2\delta)|| > \frac{\epsilon}{4}) < \infty.$$

By Lemma 2 and (iv), we have that

$$\begin{aligned}
&P(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \le 2\delta)|| > \frac{\epsilon}{4}) \\
&\le 2P(||\sum_{i=1}^{k_n} X_{ni}^s|| > \frac{\epsilon}{4})
\end{aligned}$$

$$\leq 2P(||\sum_{i=1}^{k_n} X_{ni}|| + ||\sum_{i=1}^{k_n} X_{ni}^*|| > \frac{\epsilon}{4})$$
$$\leq 4P(||\sum_{i=1}^{k_n} X_{ni}|| > \frac{\epsilon}{8}) \to 0.$$

Noting that $||X_{ni}^s I(||X_{ni}^s|| \leq 2\delta)|| \leq 2\delta$, it follows by Lemma 3 that

$$E||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \leq 2\delta)|| \to 0$$

as $n \to \infty$. Take an integer $j$ such that $2^j \geq J$. Then we have by Lemma 1 that

$$\begin{aligned}
&P(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \leq 2\delta)|| > \frac{\epsilon}{4}) \\
&\leq C_j P(\max_{1\leq i\leq k_n} ||X_{ni}^s I(||X_{ni}^s|| \leq 2\delta)|| > \frac{\epsilon}{4\cdot 3^j}) \\
&\quad + D_j P\left(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \leq 2\delta)|| > \frac{\epsilon}{4\cdot 3^j}\right)^{2^j} \\
&\leq C_j P(\max_{1\leq i\leq k_n} ||X_{ni}^s|| > \frac{\epsilon}{4\cdot 3^j}) \\
&\quad + D_j P\left(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \leq 2\delta)|| > \frac{\epsilon}{4\cdot 3^j}\right)^{J} \\
&\leq 2C_j \sum_{i=1}^{k_n} P(||X_{ni}|| > \frac{\epsilon}{8\cdot 3^j}) \\
&\quad + D_j P\left(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \leq 2\delta)|| > \frac{\epsilon}{4\cdot 3^j}\right)^{J}.
\end{aligned}$$

Thus, by (i), it suffices to prove that

$$\sum_{n=1}^{\infty} a_n P\left(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \leq 2\delta)|| > \frac{\epsilon}{4\cdot 3^j}\right)^{J} < \infty.$$

Since $E||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \leq 2\delta)|| \to 0$, it follows by Lemma 4 and Lemma 5 that for all large $n$

$$\begin{aligned}
&P(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \leq 2\delta)|| > \frac{\epsilon}{4\cdot 3^j}) \\
&\leq P(\big| ||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \leq 2\delta)|| - E||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \leq 2\delta)|| \big| > \frac{\epsilon}{8\cdot 3^j})
\end{aligned}$$

$$\le (\frac{8 \cdot 3^j}{\epsilon})^2 E\Big| ||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \le 2\delta)|| - E||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \le 2\delta)||\Big|^2$$

$$\le 4(\frac{8 \cdot 3^j}{\epsilon})^2 \sum_{i=1}^{k_n} E||X_{ni}^s||^2 I(||X_{ni}^s|| \le 2\delta)$$

$$\le 4(\frac{8 \cdot 3^j}{\epsilon})^2 \sum_{i=1}^{k_n} \Big\{8E||X_{ni}||^2 I(||X_{ni}|| \le \delta) + 8\delta^2 P(||X_{ni}|| > \delta)\Big\}.$$

Noting that $\Big(\sum_{i=1}^{k_n} P(||X_{ni}|| > \delta)\Big)^J \le O(1)\sum_{i=1}^{k_n} P(||X_{ni}|| > \delta)$ by (iii), the $c_r$-inequality implies that $\sum_{n=1}^{\infty} a_n P\Big(||\sum_{i=1}^{k_n} X_{ni}^s I(||X_{ni}^s|| \le 2\delta)|| > \frac{\epsilon}{4\cdot 3^j}\Big)^J <$ $\infty$ by (i) and (ii). Thus the proof is complete. □

*Proof of Theorem 4.* Let $A = \{n| \sum_{i=1}^{k_n} P(||X_{ni}|| > \delta) \le 1\}$. Define a sequence $\{u_n, n \ge 1\}$ of positive integers by

$$u_n = \begin{cases} k_n, & \text{if } n \in A, \\ 1, & \text{if } n \notin A. \end{cases}$$

Define an array $\{Y_{ni}, 1 \le i \le u_n, n \ge 1\}$ of random elements by

$$Y_{ni} = \begin{cases} X_{ni}, & \text{if } n \in A, \\ 0, & \text{if } n \notin A. \end{cases}$$

Then $\{Y_{ni}, 1 \le i \le u_n, n \ge 1\}$ satisfies all conditions of Theorem 3 and so we have that

$$\sum_{n\in A} a_n P(||\sum_{i=1}^{k_n} X_{ni}|| > \epsilon) = \sum_{n=1}^{\infty} a_n P(||\sum_{i=1}^{u_n} Y_{ni}|| > \epsilon) < \infty.$$

Observe that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(||X_{ni}|| > \delta) &= \sum_{n\in A} a_n \sum_{i=1}^{k_n} P(||X_{ni}|| > \delta) \\ &\quad + \sum_{n\notin A} a_n \sum_{i=1}^{k_n} P(||X_{ni}|| > \delta) \\ &\ge \sum_{n\in A} a_n \sum_{i=1}^{k_n} P(||X_{ni}|| > \delta) + \sum_{n\notin A} a_n.\end{aligned}$$

It follows by (i) that $\sum_{n\notin A} a_n < \infty$. Thus we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n P(||\sum_{i=1}^{k_n} X_{ni}|| > \epsilon) = & \sum_{n\in A} a_n P(||\sum_{i=1}^{k_n} X_{ni}|| > \epsilon) \\ & + \sum_{n\notin A} a_n P(||\sum_{i=1}^{k_n} X_{ni}|| > \epsilon) \\ \le & \sum_{n\in A} a_n P(||\sum_{i=1}^{k_n} X_{ni}|| > \epsilon) + \sum_{n\notin A} a_n < \infty.\end{aligned}$$

□

*Proof of Corollary 1.* We will apply Theorem 4 with $a_n = (\phi(\alpha(n)) - \phi(\alpha(n-1)))/n, n \ge 1$ and $X_{ni}$ replaced by $X_{ni}/\alpha(n), 1 \le i \le n, n \ge 1$. We only need to verify that conditions (i) and (ii) of Theorem 4 hold. By the weak mean domination hypothesis, we have that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \sum_{i=1}^{n} P(||\frac{X_{ni}}{\alpha(n)}|| > \epsilon) \le & C \sum_{n=1}^{\infty} a_n n P(\frac{|X|}{\epsilon} > \alpha(n)) \\ = & C \sum_{i=1}^{\infty} P(\alpha(i) < \frac{|X|}{\epsilon} \le \alpha(i+1)) \sum_{n=1}^{i} n a_n \\ \le & CE\phi(\frac{|X|}{\epsilon}) < \infty,\end{aligned}$$

since $E\phi(|X|) < \infty$ and $\phi(2x) \le D\phi(x)$. Hence condition (i) holds.

To establish condition (ii), note that

$$\begin{aligned}& \sum_{n=1}^{\infty} a_n \left( \sum_{i=1}^{n} E||\frac{X_{ni}}{\alpha(n)}||^2 I(||\frac{X_{ni}}{\alpha(n)}|| \le \delta) \right)^J \\ \le & \delta^{J(2-s)} \sum_{n=1}^{\infty} a_n \left( \sum_{i=1}^{n} E||\frac{X_{ni}}{\alpha(n)}||^s \right)^J \\ \le & \delta^{J(2-s)} \sum_{n=1}^{\infty} a_n \left( \frac{nCE|X|^s}{\alpha^s(n)} \right)^J,\end{aligned}$$

since $\sum_{i=1}^{n} E||X_{ni}||^s \le nCE|X|^s$ by the weak mean domination. Hence condition (ii) holds. □

A sequence $\{U_n, n \ge 1\}$ of random variables is *bounded in probability* if for every $\epsilon > 0$ there exists a constant $C > 0$ such that $P(|U_n| > C) < \epsilon$ for all $n \ge 1$.

*Remark* 1. Let $\{U_n, n \ge 1\}$ be a bounded in probability. Let $\{\beta_n, n \ge 1\}$ be a sequence of positive real numbers such that $\beta_n \to 0$ as $n \to \infty$. Then $\beta_n U_n \to 0$ in probability.

*Proof.* Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\{U_n, n \geq 1\}$ is bounded in probability, there exists a constant $C > 0$ such that $P(|U_n| > C) < \epsilon$ for all $n \geq 1$. Since $\beta_n \to 0$ as $n \to \infty$, there exists a positive integer $N$ such that $\beta_n < \delta/C$ if $n > N$. For $n > N$, $P(|\beta_n U_n| > \delta) \leq P(|U_n| > C) < \epsilon$. Hence the proof is complete. $\square$

*Remark* 2. Let $\{\gamma_n, n \geq 1\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \alpha(n)/\gamma_n = \infty$, and $\{||\sum_{i=1}^n X_{ni}||/\gamma_n, n \geq 1\}$ is bounded in probability. By Remark 1, $||\sum_{i=1}^n X_{ni}||/\alpha(n) \to 0$ in probability. When $\theta(x) = |x|^s$ for some $s > 0$,

$$\frac{n}{\alpha^s(n)} < \frac{1}{r}\left(\frac{rn + \theta(\gamma_n)}{\theta(\alpha(n))}\right) \text{ for any } r > 0.$$

It follows that Corollary 1 improves Theorem 3.1 of Tómács [15] when $\theta(x) = |x|^s$ for some $1 \leq s \leq 2$.

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