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FREDHOLM MAPPINGS AND BANACH MANIFOLDS

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Abstract. Two C^1-mappings, whose domain is a connected compact C^1-Banach manifold modelled over a Banach space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and whose range is a Banach space Y over $\mathbb{K}$, are introduced. Sufficient conditions are given to assert they share only a value. The proof of the result, which is based upon continuation methods, is constructive.

1. Preliminaries

Scientific phenomena are locally described by parameters, whose choice is sometimes arbitrary. This implies the importance of the availability of a methodology for the comparison of results of measurements. Locally, a Banach manifold looks like a Banach space. For a local description, different Banach (or coordinate or parameter) spaces are allowed and transformation rules exist for these coordinates.

Let X, Y be two Banach spaces. Let $u : U \subset X \to Y$ be a continuous mapping. One way of solving the equation

$$u(x) = y \tag{1}$$

for any fixed $y \in Y$, is to embed (1) in a continuum of problems

$$H(x,t) = y, (0 \leq t \leq 1), \tag{2}$$

which is solved when $t = 0$. When $t = 1$, problem (2) becomes (1). If it is possible to continue the solution for all $t \in [0,1]$, then (1) is solved. This is the continuation method with respect to a parameter [1-25]. A continuation method was introduced to solve (1) when $u : M \to \mathbb{R}^n$, where M is a connected compact C^1-Banach manifold modelled on $\mathbb{R}^n$, and $H(\cdot,\cdot)$ is a C^1-mapping [25].

Here M is a Banach manifold modelled on an infinite-dimensional Banach space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and u ranges over an infinite-dimensional Banach space Y over $\mathbb{K}$.

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Sufficient conditions are given to prove that two C^1-mappings, of which one is Fredholm of index zero, share only one value on a Banach manifold by using continuation methods on charts. Other conditions, sufficient to guarantee the existence of zero points, have been given by the author in several other papers [7-25]. This shared value can be estimated following a curve. The proof supplies the existence of a curve which leads to the point whose image is the shared value. The keys are the use of the chart spaces [27], the compactness and connectness of M, together with the use of the Continuous Dependence Theorem (Theorem 2) [26] and Theorem 1 [28].

We briefly recall some theorems and notation to be used.

Definitions and Notation. [26-28]. Let $F : D(F) : X \to Y$, where X, Y are Banach spaces over $\mathbb{K}$. If $D(F)$ is open, then mapping F is said to be a *Fredholm* mapping if and only if both F is a C^1-mapping and $F'(x) : X \to Y$ is a Fredholm linear mapping for all $x \in D(F)$. That $L : X \to Y$ is a *linear Fredholm* mapping means that L is linear and continuous and both the numbers dim(ker(L)) and codim($R(L)$) are finite, where dim signifies dimension, codim codimension, ker kernel and R(L) stands for the range of mapping L. Therefore $\ker(L) = X_1$ is a Banach space and has topological complement X_2, since $\dim(X_1)$ is finite. The integer number ind(L)= dim(ker(L))-codim($R(L)$) is called the *index* of L. Let $\mathcal{F}(X, Y)$ denote the set of all *linear Fredholm* mappings $L : X \to Y$.

Let M be a topological space. A *chart* (U, φ) in M is a pair where the set U is open in M and $\varphi : U \to U_\varphi$ is a homeomorphism onto an open subset U_φ of a Banach space X_φ. We call φ a *chart map,* X_φ is called *chart space,* and U_φ *chart image.* For $x \in U$, $x_\varphi = \varphi(x)$ is called the *representative* of x in the chart (U, φ) or the *local coordinate* of x in the local coordinate system φ. The point $x \in M$ may have different local coordinates $x_\varphi = \varphi(x)$ and $x_\psi = \psi(x)$ for two different charts (U, φ) and (V, ψ), respectively. The transformation rules between them are $x_\varphi = \varphi(\psi^{-1}(x_\psi))$ and $x_\psi = \psi(\varphi^{-1}(x_\varphi))$.

Two charts, (U, φ) and (V, ψ) in M, are called C^k-*compatible* if and only if $U \cap V = \emptyset$, or both $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ and $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ are C^k-mappings, $k \geq 0$.

A C^k-*atlas* for $M, 0 \leq k \leq \infty$ is a collection of charts (U_i, φ_i), where $i \in I$, which satisfies the following conditions:

(i) the U_i cover M,

(ii) any two charts are C^k-compatible,

(iii) all chart spaces X_i are Banach spaces over $\mathbb{K}$.

If there is a C^k-atlas for M, then M is said to be a C^k-*Banach manifold.* If all chart spaces are equal to a fixed Banach space X, M is called a C^k-*Banach manifold modelled on* X. Here manifolds *without boundaries* will be considered, such as the surface of a ball in $\mathbb{R}^n$, an open set in a Banach space X, etc.

Let M and N be C^k-Banach manifolds with chart spaces over $\mathbb{K}, k \geq 1$. The mapping $f : M \to N$ is called a C^r-mapping, where $r \leq k$, if and only if f is C^r

at each point $x \in M$ in fixed admissible charts. This means the following: If (U, φ) and (V, ψ) are charts in M and N respectively, with $x \in U$ and $f(x) \in V$, then the mapping $\overline{f} = \psi \circ f \circ \varphi^{-1}$, which is well defined in a sufficiently small neighbourhood of x_φ, is C^r in the usual sense. $\overline{f}$ is called a *representative* of f.

Two C^1-curves in M, which pass through the point $x \in M$, are called *equivalent the point* x if and only if the representatives have the same tangent vector at x in some fixed admissible chart. A *tangent vector* v (otherwise known as $x'(t_0)$) to M at x consists of all C^1-curves which are equivalent at x to a fixed C^1-curve. The tangent "abstract" vector v of the previous definition to the curve $x(\cdot) : U(t_0) \subset \mathbb{R} \to M$, $x = x(t)$ at the point $x(t_0)$, has its *representative* or local coordinate $v_\varphi = x'_\varphi(t_0)$ in the chart (U, φ), where $x_\varphi = x_\varphi(t) = \varphi(x(t))$.

The *tangent space* TM_x to M at the point x is by definition the set of all tangent vectors. It is proven that this is a topological vector space which is linear homeomorphic to each chart space X_φ at the point x.

The map $f'(x) : TM_x \to TN_{f(x)}$ is called the *tangent map* of $f : M \to N$ at point x, which is clearly the normal F-derivative in local coordinates.

A mapping $f : M \to N$ is called a *Fredholm* operator at x if and only if the linearization $f'(x) : M_x \to N_{f(x)}$ is a Fredholm operator. Furthermore, f is a Fredholm operators at x if and only if the representatives of f in local charts are Fredholm operator at the corresponding points.

Let $f : M \to N$ be a C^k-mapping, $k \geq 1$, where M and N are C^k-Banach manifolds with chart space over $\mathbb{K}$. The mapping f is called a *submersion* at x if and only if $f'(x)$ is surjective and the null space $\ker(f'(x))$ splits the tangent space of M at point x (which is automatic when $\ker(f'(x)) = \{0\}$). A point $x \in M$ is called a *regular point of* f if and only if f is a submersion at x. A point $y \in N$ is called a *regular value* of f if and only if the set $f^{-1}(y)$ is empty or consists only of regular points.

If X, Y are Banach spaces, let $\mathcal{L}\ (X, Y)$ denote the set of all linear continuous mappings $L : X \to Y$. Let $\mathrm{Isom}(X, Y)$ denote the set of all the linear homeomorphisms $L : X \to Y$. Let $B(x_0, \rho)$ be the open ball of centre x_0 and radius ρ. If $u : X \to Y$ is a linear continuous bijective operator, then the inverse linear continuous operator will be denoted by u^{-1}.

Mapping $\overline{H}_x(x_\varphi, t)$ denotes the partial F-derivative of mapping $\overline{H}$ with respect to X at the point (x_φ, t), where $\overline{H} : U_\varphi \times [0, 1] \subset X \times \mathbb{R} \to Y$.

A representative point always has its corresponding chart map as subindex.

Theorem 1 ([28], p. 300)**.** *If $S \in \mathcal{F}(X, Y)$, where X, Y are Banach spaces over $\mathbb{K}$, then there is a number $\varepsilon > 0$ such that*

$$T \in \mathcal{F}(X, Y) \qquad \text{and} \qquad \operatorname{Ind} T = \operatorname{Ind} S$$

for all linear Fredholm mappings $T \in \mathcal{L}(X, Y)$ with $\| T - S \| < \varepsilon$.

Theorem 2 (Continuous Dependence Theorem [26], pp. 18–19)**.** *Let the following conditions be satisfied*:

(i) *P is a metric space, called the parameter space.*
(ii) *For each parameter $p \in P$, mapping T_p satisfies the following hypotheses*:
(1) *$T_p : M \to M$, i.e. M is mapped into itself by T_p.*
(2) *M is a closed non-empty set in a complete metric space (X, d).*
(3) *T_p is k-contractive for any fixed $k \in [0, 1)$.*
(iii) *For each $p_0 \in P$, and any $x \in M$, $\lim_{p \to p_0} T_p(x) = T_{p_0}(x)$.*

Then for each $p \in P$, the equation $x_p = T_p x_p$ has exactly one solution x_p, where $x_p \in M$ and $\lim_{p \to p_0} x_p = x_{p_0}$.

2. Fredholm mappings on compact Banach manifolds

Theorem 3. *Let $f, g : M \to Y$ be two C^1-mappings, where M is a compact, connected, C^1-Banach manifold modelled on X, where X, Y are two Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $(U_i, \varphi_i)_{i \in I}, I = 1, 2, \ldots, N$ be a C^1-atlas for M. Suppose that the following conditions hold*:

(i) *Mapping f has only one zero $x^\star$ in M, with $x^* \in U_j$, and f is a Fredholm mapping of index zero at x^*.*

(ii) *For any fixed $t \in [0, 1]$, zero is a regular value of any mapping*

$$H(\cdot, t) : M \times \{t\} \to Y, \qquad H(\cdot, t) := f(\cdot) - tg(\cdot).$$

Hence the following statement holds true:
(a) *The mappings f and g share only one value x^{**} on M.*
(b) *There is a C^1-mapping $\alpha(\cdot) : [0, 1] \to M$, with*

$$H(\alpha(t), t) = 0, \forall t \in [0, 1], \alpha(0) = x^*, \alpha(1) = x^{**}, \text{ where}$$

$$H : M \times [0, 1] \to Y,\ H(x, t) = f(x) - tg(x).$$

Proof. $\mathcal{L}(X, Y)$ is provided by the topology given by its operator norm and $X \times \mathbb{R}$ is provided by a product topology.

The point x^* belongs to $U_j, j \in I$, and $x^*_{\varphi_j} = \varphi_j(x^*)$ is its representative point in the chart (U_j, φ_j). The representative mapping of the mapping H in this chart (U_j, φ_j) is the C^1-mapping

$$\overline{H} : U_{j_{\varphi_j}} \times [0, 1] \subset X \times \mathbb{R} \to Y,\ \overline{H}(x_{\varphi_j}, t) := (H \circ (\varphi_j^{-1}, Id))(x_{\varphi_j}, t),$$

with $Id(t) := t$. This mapping verifies that $\overline{H}(x^*_{\varphi_j}, 0) = 0$.

The representative mapping of mapping H on any chart is also written as $\overline{H}$ for simplicity, and the same criterium will be used for any mapping. Any extended mapping will be denoted as the original mapping.

(**a**) By hypothesis (i), f is a Fredholm mapping of index zero at x^*, i.e., the representative of f in local charts $\overline{f}$ are Fredholm mappings of index zero at the corresponding representative points. Since f is a C^1 mapping and since index is an integer, therefore Theorem 1 implies that $\operatorname{Ind} f'(x)$ is locally constant.

Hence, since M is connected, f is a Fredholm mapping of index zero on M. Hence

(3) $$f'(x) \in \mathcal{F}(X,Y), \text{ and } \operatorname{Ind} f'(x) = 0, \ \forall x \in M.$$

Since H is a C^1mapping, therefore

$$\forall \varepsilon > 0, \| \overline{f}'(x_{\varphi_i}) - (\overline{f}'(x_{\varphi_i}) - t\overline{g}'(x_{\varphi_i})) \| = | t | \| \overline{g}'(x_{\varphi_i}) \| < \varepsilon, \text{when } | t | < \delta(\varepsilon),$$

and since the index is an integer and $[0,1]$ is connected, Theorem 1 and Equation (3) imply for any fixed $x \in M$ in local charts, that

$$\overline{f}'(x_{\varphi_i}) - t\overline{g}'(x_{\varphi_i}) \in \mathcal{F}(X,Y)$$

and

$$\operatorname{Ind}(\overline{f}'(x_{\varphi_i}) - t\overline{g}'(x_{\varphi_i})) = 0, \forall i \in I, \ \forall t \in [0,1].$$

Hence for any fixed $t \in [0,1]$, $H(\cdot,t)$ is a Fredholm mapping and $\operatorname{Ind}(H_x(x,t)) = \operatorname{Ind}(f'(x)) = 0$, and

(4) $$\operatorname{Ind}(\overline{H}_x(x_{\varphi_i},t)) = 0, \ i \in I, \ x \in U_i, \ t \in [0,1].$$

By hypothesis (ii), since zero is a regular value of the mapping $H(\cdot,t)$ for any fixed $t \in [0,1]$, therefore the mapping

$$\overline{H}_x(x_{\varphi_i},t)(\cdot) \in \mathcal{L}(X,Y)$$

maps X onto Y for any fixed $(x_{\varphi_i},t) \in (U_{\varphi_i} \times [0,1]) \cap \overline{H}^{-1}(0)$, therefore

(5) $$\operatorname{codim}(\operatorname{range}(\overline{H}_x(x_{\varphi_i},t))) = 0,$$

and hence

$$\operatorname{Ind}(\overline{H}_x(x_{\varphi_i},t)) = \dim(\ker(\overline{H}_x(x_{\varphi_i},t))).$$

Equations (4) and (5) imply that

$$\operatorname{Ind}(\overline{H}_x(x_{\varphi_i},t)) = 0 = \dim(\ker(\overline{H}_x(x_{\varphi_i},t))),$$

i.e., $(\overline{H}_x(x_{\varphi_i},t))$ is injective. Therefore $\overline{H}_x(x_{\varphi_i},t)(\cdot)$ is a linear continuous bijective mapping, and since X, Y are Banach spaces, the linear mapping $\overline{H}_x(x_{\varphi_i},t)^{-1}(\cdot)$ is continuous. Hence $\overline{H}_x(x_{\varphi_i},t)(\cdot)$ is a linear homeomorphism, i.e.,

$$\overline{H}_x(x_{\varphi_i},t)(\cdot) \in \operatorname{Isom}(X,Y).$$

(b) Let us fix any chart of the atlas $(U_j,\varphi_j), j \in I$, which will be called (U,φ), whose corresponding chart image is U_φ. Let us suppose that $(x_a,t_a) \in H^{-1}(0)$ with $x_a \in U$. Such a point (x_a,t_a) will be call a "starting point".

The representative mapping of H in the chart (U,φ)

$$\overline{H} : U_\varphi \times [0,1] \subset X \times \mathbb{R} \to Y,$$

clearly has as zero (x_{a_φ},t_a) with $x_{a_\varphi}, = \varphi(x_a)$.

The representative point of (x_a,t_a) in the chart $(U \times [0,1], (\varphi, Id))$ verifies $(x_{a_\varphi},t_a) \in \overline{H}^{-1}(0) \cap (U_\varphi \times [0,1])$. Such a point (x_{a_φ},t_a) will be called a "representative starting point" in $U_\varphi \times [0,1]$, and there is a positive number R such that the ball $B(x_{a_\varphi}, R) \subset U_\varphi$, and furthermore

$$\| \overline{H}_x(x_{a_\varphi}, t_a)^{-1} \| = C.$$

(b1) Two positive numbers r_a, r_{0_a} must be found for later use.
Let us construct the set

$$B = B(x_{a_\varphi}, R) \times [0,1],$$

and define the mapping

$$\overline{h} : B \subset U_\varphi \times \mathbb{R} \to Y,$$

$$\overline{h}((x_{a_\varphi} + x_\varphi), t) := \overline{H}_x(x_{a_\varphi}, t_a)((x_{a_\varphi} + x_\varphi) - x_{a_\varphi}) - \overline{H}((x_{a_\varphi} + x_\varphi), t).$$

Since $\overline{h}$ is a continuous mapping as a composition of continuous mappings, therefore for any $r > 0$ and for C given above, there is a

$$\delta\left(\frac{r}{2C}\right) > 0, \tag{6}$$

such that, if $((x_{a_\varphi} + x_\varphi), t), \in B$ with

$$\| x_\varphi, t - t_a \| < \delta\left(\frac{r}{2C}\right) \text{ then}$$

$$\| \overline{h}((x_{a_\varphi} + x_\varphi), t) - \overline{h}(x_{a_\varphi}, t_a) \| < \frac{r}{2C}. \tag{7}$$

On the other hand, another mapping can be defined as

$$\overline{h}_x : B \to \mathcal{L}(X, Y)$$

$$\overline{h}_x((x_{a_\varphi} + x_\varphi), t) := \overline{H}_x(x_{a_\varphi}, t_a) - \overline{H}_x(x_{a_\varphi} + x_\varphi, t_a)$$

and is also continuous in the set B, and therefore there is an

$$r := \delta\left(\frac{1}{2C}\right) > 0 \tag{8}$$

such that, if $((x_{a_\varphi} + x_\varphi), t) \in B$ with

$$\| (x_\varphi, t - t_a) \| < \delta\left(\frac{1}{2C}\right), \text{ then}$$

$$\| \overline{h}_x((x_{a_\varphi} + x_\varphi), t) - \overline{h}_x(x_{a_\varphi}, t_a) \| < \frac{1}{2C}. \tag{9}$$

By taking r given by Equation (8) and fixing $r_0' := \delta(\frac{r}{2C})$, given by Equation (6), the number $r_0 := \min\{r, r_0'\}$ can be defined. We select $r_a := \min\{R, r\}$ and $r_{0_a} = \min\{R, r_0\}$.

(b2) The sets

$$I_a := \{t \in [0,1] : | t - t_a | \leq r_{0a}\}, \; A_a := \{x_\varphi \in X : \| x_\varphi \| \leq r_a\},$$

will be associated to the "representative starting point" (x_{a_φ}, t_a). Since $\| x_\varphi \| \le R, \forall x_\varphi \in A_a$, therefore

$$(x_\varphi + x_{a_\varphi}) \in U_\varphi \subset X_\varphi.$$

Given a "starting point" $(x_a,\, t_a)$ and its "representative starting point" $(x_{a_\varphi},\, t_a)$, the existence of two continuous mappings are proved:

$$\overline{\alpha}(\cdot) : I_a \subset \mathbb{R} {\rightarrow} A_a {+} x_{a_\varphi} \subset U_\varphi \subset X, \text{ such that } \overline{H}(\overline{\alpha}(t), t) = 0,\ \forall t \in I_a$$

and

$$\alpha(\cdot) : I_a \subset \mathbb{R} \rightarrow U \subset M$$

such that

$$H(\alpha(t), t) = 0,\ \forall t \in I_a.$$

Let us solve the equation

$$\overline{H}((x_{a_\varphi} + x_\varphi), t) = 0, \tag{10}$$

for fixed $t \in I_a$ when x_φ is in A_a. Obviously, $\overline{H}(x_{a_\varphi}, t_a) = 0$. Equation (10) is equivalent to the following equation

$$\overline{H}_x(x_{a_\varphi}, t_a)^{-1}[\overline{H}_x(x_{a_\varphi}, t_a)(x_\varphi) - \overline{H}((x_{a_\varphi} + x_\varphi), t)] = x_\varphi, \tag{11}$$

which leads us to define the mappings

$$\overline{h}_t : A_a \times \{t\} \rightarrow Y \text{ for fixed } t \in I_a,$$

$$\overline{h}_t(x_\varphi) := \overline{H}_x(x_{a_\varphi}, t_a)(x_\varphi) - \overline{H}((x_{a_\varphi} + x_\varphi), t) = \overline{h}((x_{a_\varphi} + x_\varphi), t),$$

and

$$\overline{T}_t : A_a \rightarrow X, \overline{T}_t(x_\varphi) := \overline{H}_x(x_{a_\varphi}, t_a)^{-1}\overline{h}_t(x_\varphi).$$

Observe that $\overline{h}_t(x_\varphi)$ is $\overline{h}((x_{a_\varphi} + x_\varphi), t)$ defined in (b1) when t is fixed and belongs to I_a. Let us also observe that t in the definitions of $\overline{h}_t$ and $\overline{T}_t$ is an index to highlight that t is fixed.

Evidently

$$\overline{h}_{t_a}(0) = 0, \tag{12}$$

and

$$\overline{h}'_{t_a}(0) = 0. \tag{13}$$

Equation (11) is equivalent to the following key Fixed Point Equation

$$\overline{T}_t(x_\varphi) = x_\varphi, \tag{14}$$

which is studied below.

Let $x_\varphi, x'_\varphi \in A_a, t \in I_a$, and hence the Taylor Theorem together with Equations (9) and (13) imply that

$$\begin{aligned} &\| \overline{h}_t(x_\varphi) - \overline{h}_t(x'_\varphi) \| \\ &\le \sup\left\{\| \overline{h}'_t(x'_\varphi + \theta(x_\varphi - x'_\varphi)) \| : \theta \in [0,1]\right\} \cdot \| x_\varphi - x'_\varphi \| \le \frac{1}{2C} r_a. \end{aligned} \tag{15}$$

Equations (12) and (15) imply that

$$\| \overline{h}_t(x_\varphi) \| \leq \| \overline{h}_t(x_\varphi) - \overline{h}_{t_a}(0) \| + \| \overline{h}_{t_a}(0) \| \leq \frac{r_a}{2C},$$

hence

$$\| \overline{T}_t(x_\varphi) \| \leq \| \overline{H}_x(x_{a_\varphi}, t_a)^{-1} \| \| \overline{h}(x_\varphi, t) \| \leq r_a. \tag{16}$$

We will apply Theorem 2 to the sets and mappings which have just been defined. The metric space $(I_a, |\cdot|)$ is the parameter space of hypothesis (i) needed in Theorem 2. The set A_a is considered as the closed and non-empty set and X as the complete metric space of hypothesis (ii), which is verified below:

From Equation (16), for any fixed $t \in I_a$, and for all $x_\varphi \in A_a$, we have $\| \overline{T}_t x_\varphi \| \leq r_a$, therefore $\overline{T}_t x_\varphi \in A_a$, and hence $\overline{T}_t : A_a \to A_a$, i.e., $\overline{T}_t$ maps the closed and non-empty set A_a of the Banach space X into itself.

From Equations (9), (13), and the Taylor Theorem, for any $x_\varphi, x'_\varphi \in A_a$, and for any $t \in I_a$, the following holds:

$$\begin{aligned} \| \overline{T}_t(x_\varphi) - \overline{T}_t(x'_\varphi) \| &\leq \| \overline{H}_x(x_{a_\varphi}, t_a)^{-1} \| \| \overline{h}'_t(x'_\varphi + \theta(x_\varphi - x'_\varphi)) \| \| x_\varphi - x'_\varphi \| \\ &\leq C \| \overline{h}'_t(x'_\varphi + \theta(x_\varphi - x'_\varphi)) - \overline{h}'_{t_a}(0) \| \| x_\varphi - x'_\varphi \| \\ &\leq \frac{1}{2} \| x_\varphi - x'_\varphi \|, (\theta \in [0,1]). \end{aligned}$$

Therefore $\overline{T}_t$ is half-contractive for any fixed $t \in I_a$. Hence hypothesis (ii) of Theorem 2 is verified.

For any fixed $t_0 \in I_a$ and for all $x_\varphi \in A_a$, the following holds:

$$\begin{aligned} \overline{T}_t(x_\varphi) &= \overline{H}_x(x_{a_\varphi}, t_a)^{-1}(\overline{H}_x(x_{a_\varphi}, t_a)(x_\varphi) - \overline{H}(x_{a_\varphi} + x_\varphi, t)) \\ &\to \overline{H}_x(x_{a_\varphi}, t_a)^{-1}(\overline{H}_x(x_{a_\varphi}, t_a)(x_\varphi) - \overline{H}(x_{a_\varphi} + x_\varphi, t_0)) \\ &= \overline{T}_{t_0}(x_\varphi) \text{ as } t \to t_0, t \in I_a, \end{aligned}$$

therefore hypothesis (iii) of Theorem 2 is also verified. Hence Theorem 2 implies, for any $t \in I_a$, that $\overline{T}_t$ has a unique fixed point $x_\varphi \in A_a, \overline{T}_t(x_\varphi) = x_\varphi := x_\varphi(t)$, and $x_\varphi(t) \to x_\varphi(t_0)$ while $t \to t_0; t, t_0 \in I_a$, i.e., $x_\varphi(\cdot)$ is a continuous mapping. Thus for any $t \in I_a$ there is only one $x_\varphi \in A_a$, i.e.,

$$\overline{T}_t(x_\varphi) = x_\varphi := x_\varphi(t)$$

and the mapping, we have just defined, $x_\varphi(\cdot)$ verifies

$$x_\varphi(t) \to x_\varphi(t_0), \text{ while } t \to t_0, \ \forall t, t_0 \in I_a,$$

which implies that $x_\varphi(\cdot)$ is a continuous mapping. Thus for any $t \in I_a$ there is one $x_\varphi(t)$ such that

$$\overline{H}(x_{a_\varphi} + x_\varphi(t), t) = 0, \tag{17}$$

and furthermore

$$\overline{H}(x_{a_\varphi} + x_\varphi(t), t) \to \overline{H}(x_{a_\varphi} + x_\varphi(t_0), t_0) = 0 \text{ while } t \to t_0, t, t_0 \in I_a.$$

Let us observe that $\overline{T}_{t_a}(0) = 0, x_\varphi(t_a) = 0$.

Equation (17) can be written as $\overline{H}(\overline{\alpha}(t), t) = 0$, which is verified for all $t \in I_a$, where $\overline{\alpha}$ is the following curve on the chart space (U, φ),

$$\overline{\alpha} : I_a \to U_\varphi \subset X, \overline{\alpha}(t) := x_{a_\varphi} + x_\varphi(t), \text{ where } \overline{\alpha}(t_a) = x_{a_\varphi}, \tag{18}$$

which is one of the goals of this section.

The continuity of both $\overline{\alpha}$ and φ^{-1} lets us construct the following curve α on the topological space M.

$$\alpha : I_a \subset \mathbb{R} \to U \subset M, \alpha(t) := (\varphi^{-1} \circ \overline{\alpha}). \tag{19}$$

Equations (18) and (19) implies that

$$\alpha(t_a) = x_a \tag{20}$$

to be used in the next section.

Equation (17) implies that

$$\overline{H}(\overline{\alpha}(t), t) = (H \circ (\varphi^{-1}, Id))(\overline{\alpha}(t), t) = H(\alpha(t), t) = 0, \ \forall t \in Ia,$$

which is the another goal of this section.

(c) Conclusions (a) and (b) will be proved here.

Since $f^{-1}(0) = x^*$ from hypothesis (i), then there exists U_j such that $H^{-1}(0) \cap (U_j \times [0,1]) \neq \emptyset$, therefore there is a point $(x^*, 0) \in H^{-1}(0), x^* \in U_j$, i.e., $(x^*, 0)$ is a "starting point".

Since $(x^*_{\varphi_j}, 0) \in \overline{H}^{-1}(0) \cap (U_{\varphi_j} \times [0,1])$, i.e., $(x^*_{\varphi_j}, 0)$ is a "representative starting point", therefore from (b2) there exist a set I_0, and two continuous mappings $\overline{\alpha}$ and α such that

$$\overline{\alpha} : I_0 \to U_{\varphi_j} \subset X, \text{ which verifies } \overline{H}(\overline{\alpha}(t), t) = 0, \ \forall t \in I_0,$$

and

$$\alpha : I_0 \to U_j \subset M, \text{ which verifies } H(\alpha(t), t) = 0, \ \forall t \in I_0. \tag{21}$$

We want to extend the continuous mapping $\alpha : I_0 \to M$ to be a continuous mapping $\alpha : [0,1] \to M$, and to extend Equation (21) to become

$$H(\alpha(t), t) = 0, \ \forall t \in [0,1].$$

Let us suppose that $\alpha(t) \in M, \forall t \in [0,b], b \in U_i$. Mapping α is extended to the right of b by taking $(\alpha(b), b)$, which belongs to $H^{-1}(0) \cap (U_i \times [0,1]), i \in I$, as the following "starting point". Equation (20) enables the continuous extension of the continuous mapping α to the right. The continuous extended mapping is also known as α.

Mapping α is successively extended to the right in the same way by using its representative in the different charts of the atlas. Now we consider all intervals

$[0, b] = I_\alpha$ such that $H(\alpha(t), t) = 0$, has a solution, $\forall t \in I_\alpha$. This solution is unique with maximal interval of existence equal to

$$J = \bigcup_\alpha I_\alpha.$$

Since M is compact if $J = [0, b), b < 1$, it follows from the *compactness of* M that $\alpha(t) \to x \in M$, while $t \to b^-$, with $x \in U_j, j \in I$. From the continuity of H, we know that $H(x, b) = 0$, therefore the point $(x, b) \in U_j \times [0, 1]$, is a "starting point" and the point (x_{φ_j}, b) is a "representative starting point" in $U_{j_{\varphi_j}} \times [0, 1]$ and thus the solution of $H(\alpha(t), t) = 0$ can be continued beyond b, which is a contradiction. Hence we can continue α until $t = 1$, which leads to

$$H(\alpha(1), 1) = f(\alpha(1)) - g(\alpha(1)) = 0.$$

This provides an x^{**}such that $f(x^{**}) = g(x^{**})$ is reached by the continuous mapping $\alpha : [0, 1] \to M$, with $H(\alpha(t), t) = 0, \forall t \in [0, 1]$.

If f and g shared more than one value, then the previous construction from two different zeros of $f - g$, would lead us to conclude that there are two zeros for f, which contradicts one of the hypotheses, and therefore it can be concluded that there is only one single shared value. □

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