Pullback, forward and chaotic dynamics in 1-D non-autonomous linear-dissipative equations

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Abstract. The global attractor of a skew product semiflow for a non-autonomous differential equation describes the asymptotic behaviour of the model. This attractor is usually characterized as the union, for all the parameters in the base space, of the associated cocycle attractors in the product space. The continuity of the cocycle attractor in the parameter is usually a difficult question. In this paper we develop in detail a 1D non-autonomous linear differential equation and show the richness of non-autonomous dynamics by focusing on the continuity, characterization and chaotic dynamics of the cocycle attractors. In particular, we analyse the sets of continuity and discontinuity for the parameter of the attractors, and relate them with the eventually forward behaviour of the processes. We will also find chaotic behaviour on the attractors in the Li-Yorke and Auslander-Yorke senses. Note that they hold for linear 1D equations, which shows a crucial difference with respect to the presence of chaotic dynamics in autonomous systems.

Keywords: global attractor, pullback attractor, forward attractor, 1D non-autonomous linear differential equation, chaotic behavior in Li-Yorke sense, chaotic behavior in Auslander-Yorke sense

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1. Introduction

We are interested in the asymptotic dynamics of initial value problems of the form

$$\begin{cases} \dot{x} = f(t,x), \ t > s \\ x(s) = x_0 \in X, \end{cases} \tag{1}$$

where $f : \mathbb{R} \times D \subset \mathbb{R} \times X \to X$ is a map belonging to some metric space $\mathcal{C}$, and X a Banach space. Assume that, for each $f \in \mathcal{C}$ and $x_0 \in X$, the solution of (1) is defined for all $t \geq s$; that is, for each $x_0 \in X$, there is a unique continuous function $[s, \infty) \ni t \mapsto x(t, s, f, x_0) \in X$ satisfying (1). For each t, $f(t, \cdot)$ is the vector field that drives the solution at time t. Hence, the path described by the solution in X between s and $s + \tau$ will depend on both the initial time s and the elapsed time τ.

In this paper we assume some kind of recurrence in the temporal variation of the vector fields. In particular, we pay special attention to the almost periodic case.

There is a general method to consider the family of non-linearities as a base flow driven by the time shift applied to the non-linearity $f(t, \cdot)$ of the original equation. We consider $f \in C_b(\mathbb{R}, X)$, the set of bounded and uniformly continuous functions from $\mathbb{R}$ into X with the metric ρ of the uniform convergence. Denote by P_0 the set of all translates of f,

$$P_0(f) = \{f(s + \cdot) : \ s \in \mathbb{R}\},$$

and define the shift operator $\theta_t : C_b(\mathbb{R}, X) \to C_b(\mathbb{R}, X)$ by

$$\theta_t f(\cdot) = f(\cdot + t).$$

For autonomous and periodic time dependence this construction yields a closed base space P_0. However, for more general almost-periodic terms it is convenient to consider the closure of P_0 with respect to ρ:

$$P := P_\rho(f) = \text{closure of } P_0(f) \text{ in } C_b(\mathbb{R}, X) \text{ with respect to } \rho,$$

known as the hull of the function f in the space $(C_b(\mathbb{R}, X); \rho)$, see [41]. Continuity of θ_t on P_0 then extends to continuity of θ_t on P.

In this paper we consider the 1D linear and dissipative differential equation

$$x' = h(\theta_t p)x + g(x),\ p \in P, x \in \mathbb{R}, \tag{2}$$

with h a real almost-periodic function with null mean value and unbounded primitive and

$$P = \overline{\{\theta_t h,\ t \in \mathbb{R}\}}$$

the hull of h. Note that (P, θ) is a continuous flow in a compact metric space. $g : \mathbb{R} \to \mathbb{R}$ is a smooth function with $xg(x) \leq 0$ for all $x \in \mathbb{R}$, $\lim_{x\to\infty} g(x) = -\infty$ and $\lim_{|x|\to\infty} \frac{g(x)}{x} = -\infty$. We denote by $C(P)$ the set of real continuous functions on P and thus $C_0(P)$ will denote the subset of $C(P)$ with null mean value. $B(P)$ will represent the subset of $C_0(P)$ with continuous primitive, and $U(P)$ its complementary, i.e., the subset of $C_0(P)$ of functions with unbounded primitive (see Section 3).

In this framework, two asymptotic behaviours give rise to completely different scenarios. Indeed, asymptotics with respect to time t (uniformly, see Chepyzhov and Vishik [13], or not in s) or with respect to s (when $s \to -\infty$ and t is arbitrary but fixed). These are called, respectively, forward and pullback dynamics and are in general unrelated.

Thus, during the last twenty years two main approaches have been developed in order to study attractors for (1): on the one hand, the pullback attractor (Carvalho et al. [11], Kloeden and Rasmussen [27]), an invariant set for the evolution process which is pullback (but, in general, not forward) attracting; on the other hand, the global attractor for the associated skew-product flow, an invariant compact set attracting forward in time (Sell [42], Kloeden and Rasmussen [27]).

The cocycle attractor $A(p)$ (see Definition 3) for (2) is described by an interval $[a(p), b(p)]$, for all $p \in P$. The aim of this paper is to study in detail the structure and internal dynamics on this family of attractors. Another definition of attractor for non-autonomous dynamical systems is that of the uniform attractor (see Chepyzhov and Vishik [13]) which is then described as the union of all the associated cocycle attractor (see Kloeden and Rasmussen [27], Bortolan et al. [9, 8]) so that, by studying the structure of cocycle attractor, in fact we are also going in detail into the characterization of uniform attractors. A recent related work is Hoang et al. [20], where the authors prove that, given a family of parameterized processes, continuity points (with respect to the parameter) for pullback and uniform attractors is a residual set, so dense in the set of parameters. Our results are different in the sense that our family is given by the driven space of functions $p \in P$, and not by perturbation of dynamical systems.

An important result in Cheban et al. [12] proves that, if the function $p \to A(p)$ is upper and lower semicontinuous, then, uniform pullback and uniform forward attraction are equivalent. The results in this paper will confirm that the property of continuity of this set-valued map cannot be weakened. Indeed, in Section 3 we study, for a particular $h \in U(P)$, the set $P_s \subset P$ of continuity and non-continuity $P_f \subset P$ of function $p \to A(p)$, showing that our attractor is a pinched set (see Definition 1), described as $A(p) = 0$ for all $p \in P_s$ and $A(p) = [-b(p), b(p)]$ with $b(p) > 0$ for all $p \in P_f$.

For a residual set in P_s, we prove (see Proposition 24 and Corollary 26) that there is no forward attraction to $A(p)$, i.e., we lose forward attraction specifically in the continuity points of the cocycle attractor. In some cases this residual set is all P_s. In Section 5 we prove that, generically, this is the situation we find, i.e., if we define

$$R_s(P) = \{h \in C_0(P) : \nu(P_s(h)) = 1\}$$

and

$$R_f(P) = \{h \in C_0(P) : \nu(P_f(h)) = 1\}$$

with ν the Haar measure on P, we deduce (see Theorem 29) that $R_s(P)$ is a residual set in $C_0(P)$. Although topologically more unusual, in Section 5.2 we concentrate in the case when $R_f(P) \neq \emptyset$, so that we can deal with $h \in U(P)$, with $\nu(P_f(h)) = 1$. Theorem 31 proves that we obtain forward attraction in P_f, i.e., we have forward attraction in full measure precisely in the set of non-continuity of the map $p \to A(p)$. A recent discussion on forward nonautonomous attractors can be found in Kloeden and Lorenz [28].

In Section 6 we find chaos inside the cocycle attractor. To our knowledge this is the first time in the literature where chaos is studied related to this kind of attractors. Indeed, Theorem 36 shows that, in the previous case with $h \in U(P)$ and $\nu(P_f(h)) = 1$ the sets $[-b(p), b(p)]$ are scrambled (see Definition 33) for a.a. $p \in P$, leading to Li-Yorke chaotic dynamics in measure (see Blanchard et al. [7]). In addition, we also obtain explicit examples in the case $\nu(P_s(h)) = 1$ where the cocycle attractor is Li-Yorke chaotic. Finally, in Section 6.2 we can also find sensitive dependence on the set $\mathbb{A}_0 = \cup_{p\in P}\{p\} \times \{0\}$, so that we also find chaotic dynamics in the Auslander-Yorke sense (see [5]).

2. Basic notions

We start with some preliminary concepts and results on topological dynamics and ergodic theory that can be found in Ellis [15], Nemytskii and Stepanov [35], Sell [42] and Shen and Yi [43].

Let (P, d_P) be a compact metric space and $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ a real continuous flow on P. Given $p \in P$, the set $\{\theta_t p\}_{t\in\mathbb{R}}$ is called the orbit of p. We say that a subset $P_1 \subset P$ is θ-invariant if $\theta_t(P_1) = P_1$ for all $t \in \mathbb{R}$. A subset P_1 is minimal if it is compact invariant and it does not contain properly any other compact invariant set. We say that the continuous flow (P, θ) is recurrent or minimal if P is minimal.

A normalized regular measure ν defined on the Borel sets of P is invariant if $\nu(\theta_t(P_1)) = \nu(P_1)$ for every Borel subset $P_1 \subset P$ and every $t \in \mathbb{R}$. It is ergodic if, in addition, $\nu(P_1) = 1$ or $\nu(P_1) = 0$ for every invariant subset P_1. The set of normalized invariant measures is not void. We say that (P, d_P) is uniquely ergodic if it has a unique normalized invariant measure which is necessarily ergodic.

We say that the flow (P, d_P) is almost-periodic if the family $\{\theta_t\}_{t\in\mathbb{R}}$ of section maps is equicontinuous, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $p_1, p_2 \in P$ and $\mathrm{d}_P(p_1, p_2) < \delta$ then $\mathrm{d}_P(\theta_t p_1, \theta_t p_2) < \varepsilon$ for every $t \in \mathbb{R}$.

A subset $L \subset \mathbb{R}$ is said to be relatively dense if there exists a number $l > 0$ such that every interval $[r, r + l]$ contains at least a point of L. We say that $f \in C_b(\mathbb{R}, \mathbb{R})$ is almost periodic if for every $\varepsilon > 0$ there exists a relatively dense subset $L_\varepsilon(f)$ such that $\sup_{t\in\mathbb{R}} |f(t + r) - f(t)| \le \varepsilon$ for every $r \in L_\varepsilon(f)$. If $f \in C_b(\mathbb{R}, \mathbb{R})$ is almost-periodic then the hull $P = P(f)$ of f is a compact metric space and if $\{\theta_t\}_{t\in\mathbb{R}}$ denotes the shift operator, then the flow (P, θ) is almost-periodic, minimal and uniquely ergodic. In fact P is an abelian topological group and the Haar measure is its only invariant measure.

We introduce two types of almost-periodic functions that will play a relevant role in what follows. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ be a vector with rational independent components. The Kronecker flow of vector α is defined on the $m-$dimensional torus $\mathbb{T}^m$ by the map $\theta_\alpha : \mathbb{R} \times \mathbb{T}^m \to \mathbb{T}^m$, $(t, x_1, \ldots, x_m) \to (x_1 + t\alpha_1, \ldots, x_m + t\alpha_m)$, and it is almost periodic and minimal (see [15, 43]). We say that a function $f \in C(\mathbb{R}, \mathbb{R})$ is quasi-periodic if there exists a Kronecker flow $(\mathbb{T}^m, \theta_\alpha)$ and a function $h \in C(\mathbb{T}^m)$ with $f(t) = h(\alpha_1 t, \ldots, \alpha_m t)$ for every $t \in \mathbb{R}$. Under this condition the hull of f is isomorphic to a k-dimensional torus of $(\mathbb{T}^m, \theta_\alpha)$.

We say that a function $h \in C(\mathbb{R}, \mathbb{R})$ is limit-periodic if is the uniform limit of a sequence of continuous and periodic functions. In this case the hull of h has frequently a more complicated structure: in simple cases it provides a solenoid. Many relevant examples in the literature considered in this paper have been developed by quasi-periodic or limit-periodic functions.

We can try to analyse non-autonomous differential equations (1) as the combination of a base flow $\{\theta_t\}_{t\in\mathbb{R}}$ on P and, for each $p \in P$, the semiflow $\mathbb{R}^+ \times X \ni (t, x_0) \mapsto \varphi(t,p)x_0 \in X$ where, for each $x_0 \in X$, $\mathbb{R}^+ \ni t \mapsto \varphi(t,p)x_0 \in X$ is the solution of the initial value problem

$$\begin{cases} \dot{x} = p(t,x), \ t > 0, \\ x(0) = x_0 \in X. \end{cases} \tag{3}$$

Then, the family of mappings

$$(t,p) \in \mathbb{R}^+ \times p \mapsto \varphi(t,p) \in \mathcal{C}(X),$$

satisfies

- $\varphi(0,p) = \mathrm{Id}_X$ for all $p \in P$,
- $x \mapsto \varphi(t,p)x \in X$ is continuous, and
- for all $t \geq s$, $s \in \mathbb{R}$, and $p \in P$,

$$\varphi(t+s,p) = \varphi(t,\theta_s p)\varphi(s,p),$$

  the 'cocycle property'.

One interprets $\varphi(t,p)x$ as the solution at time t that has started in the state x at time zero subjected to the non-autonomous driving term $p \in P$.

The pair $(\varphi,\theta)_{(X,P)}$ will be called a *non-autonomous dynamical system* on (X,P) (see Kloeden and Rasmussen [27]). Now, given a non-autonomous dynamical system $(\varphi,\theta)_{(X,P)}$, one can also define an associated autonomous dynamical system (see [41, 42]) $\Pi(\cdot)$ on $\mathbb{X} = P \times X$ (with the metric $\mathrm{d}_{\mathbb{X}}((x,p),(\bar{x},\bar{p})) = \mathrm{d}(x,\bar{x}) + \mathrm{d}_P(p,\bar{p})$) by setting

$$\Pi(t)(p,x) = (\theta_t p, \varphi(t,p)x)), \ t \geq 0.$$

The semigroup property of θ_t and the cocycle property of φ ensure that $\Pi(\cdot)$ satisfies the semigroup property.

Thus, given a non-autonomous differential equation such as (1), we need to deal with four different dynamical systems:

(a) The *driving semigroup* $\{\theta_t : t \geq 0\}$ on P associated to the dynamics of the time-dependent nonlinearities appearing in the equation.

(b) the *skew-product semiflow* $\{\Pi(t) : \ t \geq 0\}$ defined on the product space $P \times X$,

(c) the associated *non-autonomous dynamical system* $(\varphi,\theta)_{(X,P)}$ with $\varphi(t,\theta_s f)x_0 = x(t+s,f,x_0)$,

(d) and the evolution process $S(t,s)x_0 = u(t-s,\theta_s f)x_0$.

Observe that these dynamical systems can possess an associated attractor:

(i) A global attractor $\mathbb{A}$ for the skew-product semiflow $\Pi(t)$,

(ii) a cocycle attractor $\{A(p)\}_{p\in p}$ for the cocycle semiflow φ, (see Kloeden and Rasmussen [27])

(iii) a pullback attractor $\{A(t)\}_{t\in\mathbb{R}}$ for the evolution process $S(t,s)$ (see Carvalho et al. [11]).

We next introduce and compare some concepts of the topological and random theory of dynamical systems. In this paper we always assume that the base flow $(P,\theta,\mathbb{R})$ is mininal. We first consider some topological notions. The concepts of minimal and invariant measure admit natural extensions for semiflows.

Definition 1. *(i) A minimal set $K\subset P\times X$ is called an automorphic extension of the base P if, for some $p\in P$, $K\cap\Pi_P^{-1}(p)$ is singleton, with Π_P the projection on the first component of $P\times X$. In these conditions we say that the minimal set K is almost-automorphic when the flow on the base P is almost-periodic.*

(ii) A compact invariant set $K\subset P\times X$ is called a pinched set if there exists a residual set $P_0\subsetneq P$ such that $K\cap\Pi_P^{-1}(p)$ is a singleton for all $p\in P_0$ and $K\cap\Pi_P^{-1}(p)$ is not a singleton for all $p\notin P_0$.

Note that an invariant compact set $K\subset P\times X$ is almost automorphic if it is pinched and minimal.

Given a NDS $(\varphi,\theta)_{(X,P)}$, suppose that the associated skew–product semiflow semigroup $\{\Pi(t):t\geqslant 0\}$ possesses a global attractor $\mathbb{A}$ on $P\times X$. We know that $\{\Pi(t):t\geqslant 0\}$ has a global attractor if and only if there exists a compact set $\mathbb{K}\subset P\times X$ such that

$$\lim_{t\to\infty}\operatorname{dist}(\Pi(t)\mathbb{B},\mathbb{K})=0, \tag{4}$$

for any bounded subset $\mathbb{B}$ of $P\times X$, where dist denotes the Hausdorff semidistance between sets defined as

$$\operatorname{dist}(A,B)=\sup_{a\in A}\inf_{b\in b}\mathrm{d}(a,b).$$

Definition 2. *(i) A* non-autonomous set *is a family $\{D(p)\}_{p\in P}$ of subsets of X indexed in p. We say that $\{D(p)\}_{p\in P}$ is an open (closed, compact) non-autonomous set if each fiber $D(p)$ is an open (closed, compact) subset of X.*

(ii) A non-autonomous set $\{D(p)\}_{p\in P}$ is invariant *under the NDS $(\varphi,\theta)_{(X,P)}$ if*

$$\varphi(t,p)D(p)=D(\theta_t p),$$

for all $t\geqslant 0$ and each $p\in P$.

It is immediate that a non-autonomous set $\{D(p)\}_{p\in P}$ is invariant for $(\varphi,\theta)_{(X,P)}$ if and only if the corresponding subset $\mathbb{D}$ of $P\times X$, given by

$$\mathbb{D}=\bigcup_{p\in P}\{p\}\times D(p),$$

is invariant for the semigroup $\{\Pi(t):t\geqslant 0\}$.

Given a subset $\mathbb{E}$ of $P\times X$ we denote by $E(p)=\{x\in X:(x,p)\in\mathbb{E}\}$ the $p-$section of $\mathbb{E}$; hence

$$\mathbb{E}=\bigcup_{p\in p}\{p\}\times E(p) \tag{5}$$

Given a non-autonomous set $\{E(p)\}_{p\in P}$ we denote by $\mathbb{E}$ the set defined by (5).

Note that

$$\bigcup_{p\in p}E(p)=\Pi_X\mathbb{E},$$

where we denote by Π_X the projection on the second component in $P\times X$.

We can now relate the concept of cocycle attractors for $(\varphi,\theta)_{(X,P)}$ with the global attractor for the associated skew–product semiflow $\{\Pi(t):t\geqslant 0\}$.

Definition 3. *Suppose P is compact and invariant and that $\{\theta_t:t\in\mathbb{R}\}$ is a group over P and $\theta_t^{-1}=\theta_{-t}$, for all $t>0$. A compact non-autonomous set $\{A(p)\}_{p\in P}$ is called a* cocycle attractor *of $(\varphi,\theta)_{(X,P)}$ if*

- **(i)** *$\{A(p)\}_{p\in P}$ is invariant under the NDS $(\varphi,\theta)_{(X,P)}$; i.e., $\varphi(t,p)A(p)=A(\theta_t p)$, for all $t\geqslant 0$.*
- **(ii)** *$\{A(p)\}_{p\in P}$ pullback attracts all bounded subsets $B\subset X$, i.e., for all $p\in P$,*

$$\lim_{t\to+\infty}\operatorname{dist}(\varphi(t,\theta_{-t}p)B,A(p))=0.$$

The following result can be found, for instance, in Propositions 3.30 and 3.31 in Kloeden and Rasmussen [27], or Theorem 3.4 in Caraballo et al. [10].

Theorem 4. *Let $(\varphi,\theta)_{(X,P)}$ be a non-autonomous dynamical system, where P is compact, and let $\{\Pi(t):t\geqslant 0\}$ be the associated skew–product semiflow on $P\times X$ with a global attractor $\mathbb{A}$. Then $\{A(p)\}_{p\in P}$ with $A(p)=\{x\in X:(x,p)\in\mathbb{A}\}$ is the cocycle attractor of $(\varphi,\theta)_{(X,P)}$.*

The following result offers a converse (see Proposition 3.31 in [27], or Lemma 16.5 in [11]).

Theorem 5. *Suppose that $\{A(p)\}_{p\in P}$ is the cocycle attractor of $(\varphi,\theta)_{(X,P)}$, and $\{\Pi(t):t\geqslant 0\}$ is the associated skew–product semiflow. Assume that $\{A(p)\}_{p\in P}$ is uniformly attracting, i.e., there exists $K\subset X$ compact such that, for all $B\subset X$ bounded,*

$$\lim_{t\to+\infty}\sup_{p\in P}\operatorname{dist}(\varphi(t,\theta_{-t}p)B,K)=0,$$

and that $\bigcup_{p\in P}A(p)$ is precompact in X. Then the set $\mathbb{A}$ associated with $\{A(p)\}_{p\in P}$, given by

$$\mathbb{A}=\bigcup_{p\in P}\{p\}\times A(p),$$

is the global attractor of the semigroup $\{\Pi(t):t\geqslant 0\}$.

3. Non-uniform cocycle attractors

Let $(P, \theta, \mathbb{R})$ be a minimal flow on a compact metric space P. For a given Banach space X we consider a skew-product semiflow $\{\Pi(t)\}_{t\in\mathbb{R}^+}$ on $P \times X$. Suppose $\Pi(t)$ admits a global attractor $\mathbb{A}$ described by

$$\mathbb{A} = \bigcup_{p\in P} \{p\} \times A(p).$$

In Cheban et al. [12] it is proved that the continuity of the set-function $p \to A(p)$ implies the uniform pullback, and therefore uniform forward, attraction to the cocycle attractor $A(p)$ given by Theorem 4.

The aim of the following sections is to develop some non-trivial models in which the above function is not continuous in the whole P, and, by a careful study of its sets of continuity, to give a detailed description on the dynamics and the structure of the attractors.

3.1. Attractors for order preserving non-autonomous systems

In what follows we suposse X is a partially ordered Banach space, i.e. there exists a closed convex positive cone $X^+ \subset X$, which is also a vectorial subspace of X, such that $X^+ \cap (-X^+) = \{0\}$.

This set X^+ defines a partial order relation on X in the way $x \leq y$ if $y - x \in X^+$; we write $x < y$ if $x \leq y$ and $x \neq y$. If in addition $\mathrm{int}(X^+) \neq \emptyset$ we say that X is strongly ordered.

Definition 6. *Let $(\varphi, \theta)_{(X,P)}$ be a non-autonomous dynamical system. We say that φ is order-preserving for the order relation '$\leq$' in X if $u_0 \leq v_0$ implies that $\varphi(t, p)u_0 \leq \varphi(t, p)v_0$, for all $p \in P$ and $t \geq 0$.*

In this section we assume that the non-autonomous dynamical system (φ, θ) generated by equation (1) is order preserving for the order in X.

We introduce the concepts of sub–, super– and equilibrium given by Arnold and Chueshov [3] in the stochastic setting (see also Chueshov [14]) and by Novo et al. [34] in the topological framework.

Definition 7. *A Borel map $a : P \to X$ such that $\varphi(t, p)a(p)$ is defined for any $t \geq 0$ is said to be*

a) an equilibrium if $a(\theta_t p) = \varphi(t, p)a(p)$, for any $p \in P$ and $t \geq 0$,

b) a super-equilibrium if $a(\theta_t p) \geq \varphi(t, p)a(p)$, for any $p \in P$ and $t \geq 0$,

c) a sub-equilibrium if $a(\theta_t p) \leq \varphi(t, p)a(p)$, for any $p \in P$ and $t \geq 0$.

Definition 8. *A super-equilibrium (resp. sub-equilibrium) $a : P \to X$ is semi-continuous if the following holds*

i) $\Gamma_a = \mathrm{closure}_X\{a(p) : p \in P\}$ is a compact subset in X;

ii) $C_a = \{(p,x) :\ x \le a(p)\}$ *(resp.* $C_a = \{(p,x) :\ x \ge a(p)\}$*) is a closed subset of* $P \times X$.

An equilibrium is semi–continuous if it holds i) and ii) above. We name a semi-equilibrium to a sub-equilibrium or a super-equilibrium.

The following result, that will be relevant in the topological version of the semi-equilibria, was proved in Proposition 3.4 of Novo et al. [34], following classical arguments from Aubin and Frankowska [4].

Proposition 9. *Assume that* $a : P \to X$ *is a semi-continuous semi-equilibrium. Then it possesses a residual invariant set* P_c *of continuity points.*

We assume that φ admits a cocycle attractor. The following result provides sufficient conditions for the existence of upper and lower asymptotically stable semi-equilibria, giving some useful information on the structure of this invariant set. The proof was given by Arnold and Chueshov [3] in the random context and generalized to the topological formulation in Novo et al. [34].

Theorem 10. *Let* φ *be an order-preserving process and* $A(p)$ *be its associated cocycle attractor. Suppose there exist Borel maps* $\alpha, \beta : P \to X$ *such that the cocycle attractor is in the "interval"* $[\alpha(p), \beta(p)]$, *i.e.*

$$A(p) \subset I_\alpha^\beta(p) = [\alpha(p), \beta(p)] = \{x \in X :\ \alpha(p) \le x \le \beta(p)\}.$$

Then, there exist two equilibria $a, b : P \to X$ *with* $a(p), b(p) \in A(p)$ *such that*

i) $\alpha(p) \le a(p) \le b(p) \le \beta(p)$, *and* $A(p) \subset I_a^b(p)$, *for all* $p \in P$.

ii) a is minimal (b is maximal) in the sense that it does not exist any complete trajectory in the interval $I_\alpha^a(I_b^\beta)$.

iii) $a(p)$ *is pullback asymptotically stable from below, that is, for all* $v(\cdot)$ *with* $\alpha(p) \le v(p) \le a(p)$, *for all* $p \in P$, *we have that*

$$\lim_{t\to+\infty} d(\varphi(t, \theta_{-t}p)v(\theta_{-t}p), a(p)) = 0.$$

$b(p)$ *is pullback asymptotically stable from above, that is, for all* $v(\cdot)$ *with* $\beta(p) \ge v(p) \ge b(p)$, *for all* $p \in P$, *we have that*

$$\lim_{t\to+\infty} d(\varphi(t, \theta_{-t}p)v(\theta_{-t}p), b(p)) = 0.$$

iv) If $\mathbb{A} = \bigcup_{p\in P}\{p\} \times A(p)$ *is compact and the maps* α, β *are continuous, then the functions* $p \to a(p)$, $p \to b(p)$ *are semi-continuous and admits a residual set* $P_c \subset P$ *of points of continuity.*

v) Assume condition in iv), and take $p_0 \in P_c$. *Then the sets*

$$\mathbb{K}_a = \overline{\{(\theta_t p_0, a(\theta_t p_0),\ t \in \mathbb{R}\}}$$

and

$$\mathbb{K}_b = \overline{\{(\theta_t p_0, b(\theta_t p_0),\ t \in \mathbb{R}\}}$$

define semiflows in $P \times X$, with $\mathbb{K}_a, \mathbb{K}_b \subset \mathbb{A}$. Moreover

$$\mathrm{card}(\mathbb{K}_a \cap \Pi_P^{-1}(p)) = \mathrm{card}(\mathbb{K}_b \cap \Pi_P^{-1}(p)) = 1,$$

for all $p \in P_c$, i.e., $\mathbb{K}_a, \mathbb{K}_b$ are almost automorphic extensions of (P, θ).

Proof. Items i), ii) and iii) can be found in Arnold and Chueshov [3]

Items iv) and v) are proved in Theorem 3.6 of Novo et al. [34]. We repeat the argument here, for completeness. Note that $\Gamma_a = \overline{\{a(p) : p \in P\}}$, $\Gamma_b = \overline{\{b(p) : p \in P\}} \subset \Pi_X \mathbb{A}$ are compact sets in X.

From $a_T(p) = \varphi(T, \theta_{-T}p)\alpha(\theta_{-T}p)$, $b_T(p) = \varphi(T, \theta_{-T}p)\beta(\theta_{-T}p)$, we deduce that these functions are continuous semi-equilibria. If $T_1 < T_2$ then $a_{T_1} \leq a_{T_2}$, $b_{T_2} \leq b_{T_1}$, and $a(p) = \lim_{T\to\infty} a_T(p)$, $b(p) = \lim_{T\to\infty} b_T(p)$ for every $p \in P$, showing that these functions are equilibria. Thus,

$$\{(p, x) : x \leq b(p)\} = \bigcap_{T \geq 0} \{(p, x) : x \leq b_T(p)\}$$

$$\{(p, x) : x \geq a(p)\} = \bigcap_{T \geq 0} \{(p, x) : x \geq a_T(p)\}$$

are closed. Moreover, $\overline{\{a(p) : p \in P\}}$, $\overline{\{b(p) : p \in P\}} \subset \Pi_X(\mathbb{A}) \subset X$ and both are compact sets. Consequently, the equilibria a, b are semi-continuous, so that, by Proposition 9 they admit a residual invariant set $P_c \subset P$ of continuity points.

For v), suppose $p_0 \in P_c$ and $p_1 \in P_c$. Let t_n such that $\theta_{t_n} p_0 \to p_1$. Then, by continuity, we also have that $a(\theta_{t_n} p_0) \to a(p_1)$ and $b(\theta_{t_n} p_0) \to b(p_1)$. Thus, $\mathbb{K}_a \cap \Pi_P^{-1}(p_1) = \{(p_1, a(p_1))\}$ and $\mathbb{K}_b \cap \Pi_P^{-1}(p_1) = \{(p_1, b(p_1))\}$. This implies that $\mathbb{K}_a, \mathbb{K}_b$ are minimal semiflows and sections (in p) are singleton if $p \in P_c$, so that they are almost automorphic extension of (P, θ). □

Remark 11. *We want to study the continuity of the cocycle attractor $A(p)$. Note that, in this framework, the continuity of $A(p)$ requires continuity of functions $a(\cdot), b(\cdot)$.*

3.2. Oscillatory functions on an almost periodic base

In the following we consider (P, θ) minimal and almost periodic. Then, P is ergodic with a unique invariant measure ν given by Haar measure. Let

$$C_0(P) = \{h \in C(\mathrm{P}) : \int_P h d\nu = 0\}.$$

The following result is classical and can be found in Gottschalk and Hedlund [19].

Proposition 12. *Let $h \in C_0(P)$. The following items are equivalent*

i) There exists $k \in C(P)$ satisfying

$$k(\theta_t p) - k(p) = \int_0^t h(\theta_s p) ds \tag{6}$$

for all $p \in P, t \in \mathbb{R}$.

ii) For all $p \in P$ it holds

$$\sup\left\{\left|\int_0^t h(\theta_s p)ds\right|,\ t \in \mathbb{R}\right\} < \infty.$$

iii) There exists $p_0 \in P$ such that

$$\sup\left\{\left|\int_0^t h(\theta_s p_0)ds\right|,\ t \in \mathbb{R}\right\} < \infty.$$

iv) There exists $p_0 \in P$ such that

$$\sup\left\{\left|\int_0^t h(\theta_s p_0)ds\right|,\ t \geq 0\right\} < \infty \text{ or } \sup\left\{\left|\int_0^t h(\theta_s p_0)ds\right|,\ t \leq 0\right\} < \infty.$$

We denote by $B(P) = \{h \in C_0(P)$ satisfying (6)$\}$, i.e., the set of functions in $C_0(P)$ with bounded primitive. It is known that if P is almost-periodic but not periodic then $C_0(P) \setminus B(P) \neq \emptyset$. Moreover, it is easy to see that

i) $B(P)$ is dense in $C_0(P)$.

ii) $U(P) = C_0(P) \setminus B(P)$ is residual in $C_0(P)$.

The following theorem comes from Johnson [22] (see also Jorba et al. [26]):

Theorem 13. *Let $h \in U(P)$. Then there exists a residual invariant set $P_o \subset P$ such that for all $p_0 \in P_o$ there exist sequences $\{t_n^i\}_{n\in\mathbb{N}}$, $i = 1, 2, 3, 4$ with*

$$\lim_{n\to\infty} t_n^i = \infty, \quad i = 1, 2, \quad \lim_{n\to\infty} t_n^i = -\infty, \quad i = 3, 4,$$

and

$$\lim_{n\to\infty} \int_0^{t_n^i} h(\theta_s p_0)ds = \infty, \quad i = 1, 3,$$

$$\lim_{n\to\infty} \int_0^{t_n^i} h(\theta_s p_0)ds = -\infty, \quad i = 2, 4.$$

Thus, the set P_o contains points where the functions $H(t, p) = \int_0^t h(\theta_s p)ds$ exhibits a strong oscillation when t goes to $\pm\infty$.

3.3. A 1-D linear model for $h \in U(P)$

Consider the linear equation

$$y'(t) = h(\theta_t p)y(t), \quad p \in P,\ t, y \in \mathbb{R}. \tag{7}$$

with $h \in U(P)$. For each $p \in P$ and $y_0 \in \mathbb{R}$ we denote by $y(t, p; y_0)$ the solution through p with initial value y_0, i.e, $y(0, p; y_0) = y_0$. It is easy to check that equation (7) has no exponential dichotomy in $C_0(P)$ (see, for instance, Sacker and Sell [39] for a precise definition of this concept and some of its consequences). Thus, there exists a nontrivial bounded solution (see Selgrade [40]), i.e., there exists $p_0 \in P \setminus P_o$, $y_0 \neq 0$ with

$$y(t,p_0;y_0) = y_0 e^{\int_0^t h(\theta_s p_0)ds} \text{ bounded,} \tag{8}$$

so that for $c_1 \in \mathbb{R}$

$$\int_0^t h(\theta_s p_0)ds \leq c_1, \text{ for all } t \in \mathbb{R}.$$

For p_0 satisfying (8), we define

$$M_0 = \overline{\{(\theta_t p_0, \pm y(t,p_0;1)),\ t \in \mathbb{R}\}}$$

It is clear that M_0 is an invariant compact set in $P \times X$.

Lemma 14. *Then*

a) If $(p,x) \in M_0$ *then* $(p,-x) \in M_0$.

b) $(p_0, \pm 1) \in M_0$.

c) $\{p\} \times \{0\} \in M_0$ *for all* $p \in P$.

d) $M_0 \cap \Pi_P^{-1}(p) = \{p\} \times \{0\}$ *for all* $p \in P_o$, *where* P_o *comes from Theorem 13.*

Proof. We only need to prove d). If d) is not true, let $p_1 \in P_o$ and $y_1 \in \mathbb{R}^+ \setminus \{0\}$ with $(p_1, y_1) \in M_0$. Then $\{(\theta_t p_1, y(t,p_1;y_1),\ t \in \mathbb{R}\} \subset M_0$, as it is a compact invariant set, but $y(t,p_1;y_1) = y_1 \exp(\int_0^t h(\theta_s p_1)ds)$ is unbounded in t, which is a contradiction. □

The above lemma is showing that the set M_0 is pinched, since it is the singleton $p \times \{0\}$ for $p \in P_o$ and strictly bigger (containing $(p_0, \pm 1)$) outside P_o. In what follows we will take advantage of this fact.

3.4. A 1-D nonlinear equation for $h \in U(P)$

Let

$$r_0 = 2 \sup\{x \in \mathbb{R} : \text{ such that } (p,x) \in M_0\}. \tag{9}$$

In the following model we will find a cocycle attractor which is a pinched set containing M_0. We define the family of linear-dissipative differential equations given by

$$x' = h(\theta_t p)x + g(x), \tag{10}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function with $g(x) = 0$ if $x \in [-r_0, r_0]$, $xg(x) \leq 0$ for all $x \in \mathbb{R}$, $\lim_{x\to\infty} g(x) = -\infty$ and $\lim_{|x|\to\infty} \frac{g(x)}{x} = -\infty$. For simplicity we take in what follows

$$g(x) = \begin{cases} -(x-r_0)^2, & x \geq r_0 \\ 0, & -r_0 \leq x \leq r_0 \\ (x+r_0)^2, & x \leq -r_0. \end{cases} \tag{11}$$

An alternative study of the structure of the set of bounded solutions for a convex or concave scalar ODE was given in Alonso and Obaya [1]. For each $p \in P$ and $x_0 \in \mathbb{R}$ we denote by $x(t)$ the solution to (10) through p with initial value x_0, i.e, $x(0, p; x_0) = x_0$. Note that if $r \gg r_0$ then $h(p)r + g(r) < 0$ and $-h(p)r + g(-r) > 0$, i.e., the functions $\beta(p) = r$ and $\alpha(p) = -r$ are continuous super and sub-equilibria respectively, i.e, if $x(t, p; x_0)$ is solution of (10)

$$x(t, p; r) \leq r, \text{ for all } t \geq 0, p \in P,$$

$$x(t, p; r) \geq -r, \text{ for all } t \geq 0, p \in P.$$

We define, for $T > 0$,

$$b_T(p) = x(T, \theta_{-T}p; r)$$

and

$$a_T(p) = x(T, \theta_{-T}p; -r).$$

Then b_T, a_T are respectively super and sub-equilibria satisfying

$$0 \leq b_{T_1}(p) \leq b_{T_2}(p) \leq r,$$

$$-r \leq a_{T_2}(p) \leq a_{T_1}(p) \leq 0,$$

for all $p \in P$, $0 < T_2 < T_1$.

From now on we fix r, b_T and a_T. Define

$$b(p) = \lim_{T\to\infty} b_T(p); \quad a(p) = \lim_{T\to\infty} a_T(p). \tag{12}$$

Proposition 15. *The following items hold:*

a) $a, b : P \to [-r, r]$ *are equilibria for (10), i.e., for all* $p \in P$ *and* $t \in \mathbb{R}$

$$x(t, p; a(p)) = a(\theta_t p), \quad x(t, p; b(p)) = b(\theta_t p).$$

b) $a(p) = -b(p)$, *for all* $p \in P$.

c) $M_0 \subset \bigcup_{p\in P}\{p\} \times [a(p), b(p)]$. *In particular,* $a(p_0) < 0, b(p_0) > 0$.

d) There exists a residual set P_s *such that, for all* $p \in P_s$ *it holds* $a(p) = 0 = b(p)$.

e) For all $p \in P \setminus P_s$, $\sup_{t\in\mathbb{R}} b(\theta_t p) \geq r_0$.

Proof. a) is a consequence of Theorem 10. Note that $a_T(p) = -b_T(p)$ for all $T > 0, p \in P$, which implies b).

For c), define

$$b_0(p) = \sup\{x \in X : (p, x) \in M_0\}; \quad a_0(p) = \inf\{x \in X : (p, x) \in M_0\}.$$

It is clear that $b_0(p) = -a_0(p)$. Since $x(T, \theta_{-T}p, b_0(\theta_{-T}p) = b_0(p)$, we have that

$$b(p) = \lim_{T\to\infty} b_T((p) = \lim_{T\to\infty} x(T, \theta_{-T}p, r) \geq b_0(p),$$

and, similarly,

$$a(p) = \lim_{T\to\infty} a_T(p) = \lim_{T\to\infty} x(T, \theta_{-T}p, -r) \le a_0(p).$$

In particular, $b(p_0) > 0$ and $a(p_0) < 0$. Note that, if $p_0 \in P$, then $b(p_0) = 0$ implies $b(\theta_t p_0) = 0$ for all $t \ge 0$, i.e., the set $P_s = \{p \in P : \; b(p) = 0\}$ is invariant.
For d), it follows from Theorem 3.4 that a, b possess a subset P_c of points of continuity. We will prove that $a(p) = b(p) = 0$ for all $p \in P_c$, i.e. $P_c = P_s$. Indeed, if there exists $p_1 \in P_s$ with $b(p_1) = 2\delta > 0$ for some $\delta > 0$ there exists $\tilde{r} > 0$ such that, for all $p \in P$ with $\mathrm{d}_P(p, p_1) \le \tilde{r}$ we have $b(p) > \delta$. From the minimality of (P, θ) there exists $T > 0$ such that if $p \in P$ we can find $0 \le t \le t(p) \le T$ with $\theta_t p \in \overline{B}(p_1, \tilde{r})$. Moreover, $b(p) = x(-t, \theta_t p; b(\theta_t p))$.
Thus, the mapping

$$\begin{aligned} x : [-T, 0] \times \overline{B}(p_1, \tilde{r}) \times [\delta, r] &\longrightarrow \mathbb{R}^+ \\ (t, p; x_0) &\longrightarrow x(t, p; x_0) > 0 \end{aligned}$$

is continuous and strictly positive on a compact set, so that there exists $\delta_1 > 0$ with $x(t, p; x_0) > \delta_1$ for all $(t, p; x_0) \in [-T, 0] \times \overline{B}(p_1, \tilde{r}) \times [\delta, r]$. In particular, as for all $p \in P$ there exists $t \in [0, T]$ with $b(p) = x(-t, \theta_t p; b(\theta_t p))$, $\mathrm{d}_P(\theta_t p, p_1) \le \tilde{r}$, then $b(p) \ge \delta_1 > 0$, for all $p \in P$. Moreover,

$$b'(\theta_t p) = h(\theta_t p) b(\theta_t p) + g(b(\theta_t p)) \le h(tp) b(\theta_t p),$$

Thus, a standard comparison argument provides

$$y(t, p; b(p)) \ge b(\theta_t p) \ge \delta_1 \text{ for all } p \in P, t \ge 0.$$

But, if $p_0 \in P_o$ (P_o from Theorem 13) there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to \infty$ and

$$\lim_{n\to\infty} y(t_n, p_0; b(p_0)) = \lim_{n\to\infty} b(p_0) e^{\int_0^{t_n} h(\theta_s p_0) ds} = 0,$$

which implies $\lim_{n\to\infty} b(\theta_{t_n} p_0) = 0$, a contradiction. As a consequence, $b(p) = 0$ for all $p \in P_c = P_s$. For the proof of e) we again argue by contradiction. Suppose $p \in P$ with $0 < \sup_{t\in\mathbb{R}} b(\theta_t p) \le \delta < r_0$. This means that the function $t \to \frac{r_0}{\delta} b(\theta_t p)$ is simultaneously a bounded solution of the linear equation (7) and of the nonlinear one (10). The same argument in c) shows that it is bounded by $b(\theta_t p)$, i.e., $b(p) \ge \frac{r_0}{\delta} b(\theta_t p)$, which is impossible, and proves e). □

We can now prove the main result for the attractor associated to (10).

Theorem 16. *In the above conditions,*

a) $\{A(p) = [a(p), b(p)]\}_{p\in P}$ *is the cocycle attractor for (10).*

b) The maps

$$\begin{aligned} P &\longrightarrow \mathbb{R} \\ p &\longrightarrow b(p) \\ p &\longrightarrow -b(p) = a(p) \end{aligned}$$

are continuous in the invariant residual set $P_s = \{p \in P : \; b(p) = 0\}$ *and discontinuous in the first category invariant set* $P_f = P \setminus P_s$.

c) $\mathbb{A} = \cup_{p\in P}\{p\} \times [a(p), b(p)]$ *is the global attractor for (10) with respect to the associated skew-product semiflow* Π.

Proof. a) and b) follows directly from Proposition 15, and c) from Theorem 5. □

As a consequence, our cocycle attractor $\mathbb{A}$ is a compact pinched set with a complex dynamical structure (see Glendinning et al. [18] and references therein); this question will be analysed in Section 6.

4. Recurrent and asymptotic points. forward versus pullback attraction

Consider $h \in U(P)$ and the function $H(t,p) = \displaystyle\int_0^t h(\theta_s p)ds$, $p \in P, t \in \mathbb{R}$. We next introduce different possible properties of H with important dynamical consequences on the corresponding cocycle attractors. Precise examples of all these situations appear in the work of Poincaré (see [37] and the references therein); such examples have been frequently constructed in the quasi-periodic and limit-periodic cases.

Definition 17. *a) A point $p \in P$ is said to be (Poincaré) recurrent at ∞ for h if there exists a sequence $t_n \to \infty$ with $\displaystyle\int_0^{t_n} h(\theta_s p)ds \to 0$. Analogously, a point $p \in P$ is said to be (Poincaré) recurrent at $-\infty$ for h if there exists a sequence $t_n \to -\infty$ with $\displaystyle\int_0^{t_n} h(\theta_s p)ds \to 0$.*

b) A point $p \in P$ is said to be asymptotic for h if $\displaystyle\int_0^{t} h(\theta_s p)ds \to -\infty$ as $t \to \infty$.

Note that if $h \in B(P)$, then every $p \in P$ is recurrent. We will denote by P_r^+ the set of recurrent points at ∞, by P_r^- the set of recurrent points at $-\infty$, and by $P_r = P_r^+ \cap P_r^-$.

The following result comes from Shneiberg [44].

Theorem 18. *Let $h \in C_0(P)$. The set $P_r \subset P$ of recurrent points is invariant and of full measure, i.e. $\nu(P_r) = 1$.*

It is immediate that the set of oscillatory points P_o satisfies $P_o \subset P_r$. As a consequence, P_r is residual and has full measure. Actually the argument of Shneiberg [44] proves that the set P_r has full measure. The invariance in the present conditions is a simple application of Fubini's theorem.

Moreover, for the n-dimensional torus, we have that all the points are recurrent in the quasi periodic case if enough regularity is required (Kozlov [30], Konyagin [29], Moschevitin [32]):

Theorem 19. *Let $n \geq 2$. There exists $k_n \in \mathbb{N}$ such that, if $h \in C^k(\mathbb{T}^n) \cap C_0(\mathbb{T}^n)$ then every $p \in \mathbb{T}^n$ is recurrent for h.*

This result was deduced by Kozlov [30] for $n = 2$ and conjectured for the general case. It has been proved by Konyagin [29] for n odd and by Moshchevitin [32] for general $n \geq 2$. Last result leads us to the following definition

Definition 20. *A function $h \in C_0(P)$ is Kozlov if every $p \in P$ is recurrent for h.*

We consider $h \in C_0(p)$ and the above framework for (10). Then there exists P_f, invariant and of first category, and its complementary, the residual set P_s, such that the cocycle attractor $A(p) = [-b(p), b(p)]$ with $b(p) > 0$ if $p \in P_f$ and $b(p) = 0$ if $p \in P_s$. Let P_r be the recurrent points and P_a the asymptotic points. Recall that we denote by P_o the oscillatory points in P. We firstly have the following result

Proposition 21. *Let $p_0 \in P$.*

i)

$$\sup_{t\le 0}\int_0^t h(\theta_s p_0)ds = \infty \text{ if and only if } b(p_0) = 0, \text{ i.e., } p_0 \in P_s,$$

and

ii)

$$\sup_{t\le 0}\int_0^t h(\theta_s p_0)ds < \infty \text{ if and only if } b(p_0) > 0, \text{ i.e., } p_0 \in P_f.$$

iii) If

$$\limsup_{t\to-\infty}\int_0^t h(\theta_s p_0)ds < \sup_{t\in\mathbb{R}}\int_0^t h(\theta_s p_0)ds$$

then $p_0 \in P_s$ and there exists $t \in \mathbb{R}$ with $b(\theta_t p_0) > r_0$.

Proof. i) Let $y_{p_0}(t) = \exp(\int_0^t h(\theta_s p_0)ds)$. Note that $y_{p_0}(t) = y(t, p_0; 1)$ is the solution of (7) with $y_{p_0}(0) = 1$. Then there exists $t_n \to \infty$ with $\lim_{n\to\infty}\displaystyle\int_0^{-t_n} h(\theta_s p_0)ds \to \infty$. Suppose r big enough. For $T > 0$, $b_T(p_0) \le y(T, p_0(-T), r)$, and $b(p_0) = \lim_{T\to\infty} b_T(p_0)$. We have

$$y(t_n, p_0(-t_n); r) = \frac{r}{y_{p_0}(-t_n)} y(t_n, p_0(-t_n); y_{p_0}(-t_n)) = \frac{r}{y_{p_0}(-t_n)}$$

converges to zero as $n \to \infty$, which implies the equivalence with $b(p_0) = 0$, i.e. $p_0 \in P_s$.

For ii), let $\rho > 0$ with

$$\sup_{t\le 0}\rho\, e^{\int_0^t h(\theta_s p_0)ds} \le r_0.$$

Then, if $x(t; p_0; \rho)$ is the solution of (10) with $x(0) = \rho$ it holds

$$x(t, p_0; \rho) = \rho\, e^{\int_0^t h(\theta_s p_0)ds} \text{ for all } t \le 0.$$

On the other hand, since $\{x(t, p_0; \rho) : t \in R\}$ is bounded, it is on the cocycle attractor, i.e. $[0, \rho] \subset [0, b(p_0)]$ and then $b(p_0) > 0$.

For iii), it is clear that $\sup_{t\le 0}\displaystyle\int_0^t h(\theta_s p_0)ds < \infty$ and hence $p_0 \in P_f$. There are $t_1 < t_2$ with $\sup_{t\le t_1}\displaystyle\int_0^t h(\theta_s p_0)ds < \int_0^{t_2} h(\theta_s p_0)ds = \rho$ for every $t \le t_1$. Let $y(t) = y(t, p_0; \frac{r_0}{\rho}) = \frac{r_0}{\rho}\exp(\int_0^t h(\theta_s p_0)ds)$ be the solution of (7). Let $t_3 \in (-\infty, t_2]$

be the first point with $y(t_3) = r_0$. There exists $\gamma > 1$ with $y(t, \theta_{t_1} p_0; \gamma y(t_1)) \leq r_0$ for every $t \leq t_1$. Then the solution of the nonlinear equation (10) satisfies

$$x(t, \theta_{t_1} p_0; \gamma y(t_1)) = y(t, \theta_{t_1} p_0; \gamma y(t_1))$$

for every $t \leq t_1$ and

$$x(t_3 - t_1, \theta_{t_1} p_0; \gamma y(t_1)) > x(t_3 - t_1, \theta_{t_1} p_0, y(t_1)) = y(t_3 - t_1, \theta_{t_1} p_0, y(t_1)) = y(t_3, p_0; \frac{r_0}{\rho}) = r_0.$$

Hence

$$b(\theta_{t_3} p_0) \geq x(t_3 - t_1, \theta_{t_1} p_0; \gamma y(t_1)) > r_0.$$

Note that in this case the function $t \to b(\theta_t p_0)$ is not solution of the linear equation (7) as its affected by the dissipation term in (10). □

From this last result we deduce

Corollary 22. $P_o \subset P_s$.

The following result characterizes the forward attraction in the cocycle attractor.

Proposition 23. *Let $p_0 \in P$ and $x_0 \in \mathbb{R}$. Then it holds*

$$\lim_{t \to \infty} x(t, p_0; x_0) = 0 \text{ if and only if } p_0 \in P_a.$$

Proof. Suppose there exists t_0 such that if $t \geq t_0$ then $x(t, p_0; x_0) \leq r_0$. Then

$$x(t, p_0; x_0) = x(t - t_0, \theta_{t_0} p_0; x(t_0, p_0; x_0)) = x(t_0, p_0; x_0) e^{\int_0^{t-t_0} h(p_0(t_0+s)) ds},$$

which implies

$$\lim_{t \to \infty} \int_0^t h(\theta_s p_0) ds = -\infty,$$

so that $p_0 \in P_a$.

On the other hand,

$$0 < x(t, p_0; x_0) \leq x_0 e^{\int_0^t h(\theta_s p_0) ds}$$

which tends to zero for every $x_0 \in \mathbb{R}$ if $p_0 \in P_a$. □

Proposition 24. *Let $p_0 \in P$.*

i) If $p_0 \in P_a$, the process $\varphi(t, p_0)$ has a forward attractor defined by $\{0\}$.

ii) If $p_0 \in P_s \cap P_r^+$, the process $\varphi(t, p_0)$ has no forward attractor.

Proof. i) and ii) are clear from Proposition 23. Indeed, if we have forward attraction to zero we have that $p_0 \in P_a$. □

Remark 25. *Note that if $p_0 \in P_a \cap P_f$ we have proved that $b(p_0) > 0$ and $\lim_{t\to\infty} b(\theta_t p_0) = 0$. Thus, a proper definition of a forward attractor $\{A(p)\}_{p\in P}$ for the cocycle should consider minimality of the family $A(p)$, in the sense that there is no proper invariant forward attracting family included in $A(p)$. Thus, for $p_0 \in P_a \cap P_f$ the forward attractor should be defined as the constant family $A(p) = 0$.*

The following results are immediate consequences of Proposition 24.

Corollary 26. *If h is Kozlov then there is no forward attractor in P_s.*

Corollary 27. *If $h \in U(P)$ and $\nu(P_s) = 1$ then there exists a residual set of full measure, P_s^* such that, if $p_0 \in P_s^*$ the process $\varphi(t, p_0)$ has no forward attractor.*

5. The sets $R_s(P), R_f(P)$. Genericity of $\nu(P_s) = 1$.

In the rest of the paper we will represent by $P_s(h), P_f(h)$ the invariant subsets defined in the previous section to emphasize their dependence with respect to $h \in C_0(P)$. We define the sets

$$R_s(P) = \{h \in C_0(P) : \nu(P_s(h)) = 1\}$$

and

$$R_f(P) = \{h \in C_0(P) : \nu(P_f(h)) = 1\}.$$

It is clear that $B(P) \subset R_s(P)$ and that $R_s(P) \cup R_f(P) = C_0(P)$. In this section we analyse the topological size of these sets in $C_0(P)$.

5.1. $R_s(P)$ is residual in $C_0(P)$.

We consider the time reversed flow $\hat{\sigma}$ on $\mathbb{R} \times P$ defined as

$$\hat{\sigma}(t, p) = \theta_{-t} p.$$

If $y(t)$ is a solution of (7) through $p_0 \in P$ then $\hat{y}(t) = y(-t)$ satisfies $\hat{y}'(t) = -h(\theta_{-t}p_0)\hat{y}(t)$. For simplicity we denote by $\hat{P}$ the base space with time reversed flow, i.e. $\hat{P} = (P, \hat{\sigma}, \mathbb{R})$. Note that the reverse of the flow $\hat{\sigma}$ is again σ.

Proposition 28. *It holds*

- *i) For any $h \in C_0(P)$, either $h \in R_s(P)$ or $-h \in R_s(P)$.*
- *ii) For the time-reversed flow, $R_s(\hat{P}) = -R_s(P)$, $R_f(\hat{P}) = -R_f(P)$.*

Proof. Let $h \in U(P)$ and fix $p_0 \in P$. If $\sup_{t\le 0} \int_0^t h(\theta_s p_0)ds \le \infty$, then it follows from Proposition 12 that $\inf_{t\le 0} \int_0^t h(\theta_s p_0)ds = -\infty$, so that $\sup_{t\le 0} \int_0^t -h(\theta_s p_0)ds = \infty$ and $p_0 \in P_s(-h)$. As a consequence of the ergodicity of ν we conclude that at least either

$\nu(P_s(h)) = 1$ or $\nu(P_s(-h)) = 1$.
For ii), suppose that we are in the case with $\nu(P_s(h)) = 1$ and take $p_0 \in P_s(h)$, so that

$$\sup_{t\leq 0} \int_0^t h(\theta_s p_0) ds = \infty.$$

Then there exists a sequence $t_n^1 \to -\infty$ with

$$\int_0^{t_n^1} h(\theta_s p_0) ds = \infty.$$

As $\nu(P_r) = 1$ we can suppose that all the points of the sets $\{\theta_t p_0 : t \in \mathbb{R}\}$ are recurrent points. For each $n \in \mathbb{N}$ there exists $t_n^2 > 0$ such that the sequence

$$\int_{t_n^1}^{t_n^2} h(\theta_s p_0) ds = \int_0^{t_n^2 - t_n^1} h(\theta_{(t_n^1+s))} p_0 ds \to 0,$$

this last property by the recurrence of path. But note that

$$\int_{t_n^1}^{t_n^2} h(\theta_s p_0) ds = -\int_0^{t_n^1} h(\theta_s p_0) ds + \int_0^{t_n^2} h(\theta_s p_0) ds$$

so that, as $\displaystyle\int_0^{t_n^1} h(\theta_{(t_n^1+s))} p_0 ds \to -\infty$, then $\sup_{t\geq 0} \int_0^t h(\theta_s p_0) ds = \infty$.

We consider the time reversed flow $\hat{\sigma}$ on $\mathbb{R} \times P$. Since, for $t > 0$

$$\int_0^t h(\theta_s p_0) ds = \int_0^{-t} -h(\theta_{-s} p_0) ds,$$

it holds

$$\sup_{t\leq 0} \int_0^t -h(\theta_{-s} p_0) ds = \sup_{t\geq 0} \int_0^t h(\theta_s p_0) ds = \infty$$

hence $p_0 \in \hat{P}_s(-h)$ and then we have that $\nu(\hat{P}_s(-h)) = 1$ and $-h \in R_s(\hat{P})$. As a consequence $R_s(P) \subset -R_s(\hat{P})$. A symmetric argument proves that $R_s(\hat{P}) \subset R_s(P)$ and thus equality. Now it is also straightforward that $R_f(\hat{P}) = -R_f(P)$ □

Observe that this last result shows how big $R_s(P)$ is on $C_0(P)$, since it shows that $C_0(P) = R_s(P) \bigcup (-R_s(P))$.

The references Johnson [25] and Novo and Obaya [33] provide criteria for existence and precise examples of functions $h \in U(P)$ and $L : P \to \mathbb{R}$ measurable with $L(\theta_t p) - L(p) = \int_0^t h(\theta_s p) ds$ for almost every $p \in P$ and $t \in \mathbb{R}$. We say that L is a measurable primitive along the flow on h. The examples are based in previous construction given in Furstenberg [16].

The condition $h \in R_f(P)$ requires in addition that $e^L \in L^\infty(P)$. The example 3.2.1 in Johnson [23] uses methods, already suggested in Anosov [2], to construct quasi period flows in the 2D torus $\mathbb{T}^2$ and a function $h \in C_0(\mathbb{T}^2)$ with

$$L(p) = \sup_{t\in\mathbb{R}} \int_0^t h(\theta_s p) ds \leq L_0 < \infty \quad a.e.$$

In this case $h \in R_f(P)$ and, moreover, for a.a. $p \in P$

$$L(\theta_t p) - L(p) = \int_0^t h(\theta_s p) ds \text{ for all } t \in \mathbb{R}.$$

This method was improved in the Appendix of Ortega and Tarallo [36], which in particular implies that this kind of function h exists for every quasi-periodic flow.

Theorem 29. *i) $R_s(P)$ is a residual set in $C_0(P)$.*
ii) The set $R_o(P) = \{h \in C_0(P) : \nu(P_o(h)) = 1)\}$ is also residual in $C_0(P)$.

Proof. For $h \in C_0(P)$, $k \in \mathbb{N}, k \geq 1$ we define

$$N_k(h) = \{p \in P : \limsup_{T\to\infty} \frac{1}{2T} l(\{t \in [-T,T] : \left|\int_0^t h(\theta_s p) ds\right| \leq k\}) = 0\},$$

with l the Lebesgue measure in $\mathbb{R}$. It is clear that $N_{k+1}(h) \subset N_k(h)$ for all $k \in \mathbb{N}$. In Johnson [24] it is proved that the set

$$C_0^* = \{h \in C_0(P) : \nu(N_k(h) = 1 \text{ for all } k \in \mathbb{N}\}$$

is residual in $C_0(P)$. If $h \in R_f(P)$ then $\nu(P_f(h)) = 1$, i.e.

$$L(p) = \sup_{t\in\mathbb{R}} \int_0^t h(\theta_s p) ds < \infty \quad a.e.$$

and hence for a.e. $p \in P$ and all $t \in \mathbb{R}$

$$L(\theta_t p) - L(p) = \int_0^t h(\theta_s p) ds.$$

For k big enough it is easy to prove that

$$\limsup_{T\to\infty} \frac{1}{2T} l(\{t \in [-T,T] : \left|\int_0^t h(\theta_s p) ds\right| \leq k\}) > 0$$

for a subset of P with positive measure. Thus, for k large enough, $\nu(N_k(h)) = 0$, so that $h \in C_0(P) \setminus C_0^*$, i.e., $C_0^* \subset R_s(P)$ and this set is residual.

For ii), let R^* be the set of functions $h \in U(P)$ satisfying

$$\sup_{t\leq 0} \int_0^t h(\theta_s p_0) ds = \sup_{t\geq 0} \int_0^t h(\theta_s p_0) ds = \infty \text{ for a.a. } p \in P,$$

$$\inf_{t\leq 0} \int_0^t h(\theta_s p_0) ds = \inf_{t\geq 0} \int_0^t h(\theta_s p_0) ds = -\infty \text{ for a.a. } p \in P.$$

Then $R^* = R_s(P) \bigcap (-R_s(P))$ is residual, which implies that

$$R_o(P) = \{h \in U(P) : \nu(P_0(h)) = 1\} \text{ is a residual set in } C_0(P).$$

□

5.2. The case $R_f(h)) \neq \emptyset$.

In this section suppose there exists $h \in U(P)$ with $\nu(P_f(h)) = 1$. Then it holds

Proposition 30. *$R_f(P)$ is a dense first category set in $C_0(P)$.*

Proof. From the last result, it is clear that $R_f(P)$ is of first category. Fix $h^* \in R_f(P)$. Then

$$\{h + \rho h^* : \ h \in B(P), \ \rho > 0\} \subset R_f(P).$$

Fix $h \in C_0(P)$ and $\varepsilon > 0$. For $||\cdot||$ the supreme norm on $C_0(P)$. There exist $h_0 \in B(P)$, $\rho_0 > 0$ with $||h - h_0|| < \varepsilon/2$, and $\rho_0 ||h^*|| < \varepsilon/2$. Then $h_0 + \rho_0 h^* \in R_f(P)$ and $||h - h_0 - \rho_0 h^*|| < \varepsilon$. □

In Section 3.4 we have shown the existence of a cocycle attractor defined by a pinched set, which is continuous in parameter p whenever $p \in P_s$, and which is not forward attracting in the residual set P_s.

The following result gives a forward attraction to the cocycle attractor in a set of non-continuity and of full measure. Note that, from the result in Cheban et al [12] one could tend to think that the forward attraction in a cocycle attractor is related to the continuity in the parameter for the cocycle attractor. The following result shows that the uniformity condition for the continuity in [12] is necessary.

Theorem 31. *Let $h \in C_0(P) \setminus B(P)$, with $\nu(P_s(h)) = 0$. Then there exists an invariant set $P_{f_a} \subset P_f(h)$ with $\nu(P_{f_a}) = 1$ such that if $p \in P_{f_a}$ then $A(p)$ is the forward attractor of the process $\varphi(t, \theta_s p)x_0 = x(t - s, \theta_s p; x_0)$ associated to (10).*

Proof. For $p \in P_f(h)$ we have that $A(p) = [a(p), b(p)]$ with $a(p) < 0 < b(p)$. Moreover, for r big enough,

$$b(p) = \lim_{T\to\infty} b_T(p) = \lim_{T\to\infty} x(T, \theta_{-T}p; r); \quad a(p) = \lim_{T\to\infty} a_T(p) = \lim_{T\to\infty} x(T, \theta_{-T}p; -r).$$

By Egorov's theorem (Rudin [38]) there exists a compact set $P_{f_0} \subset P_f(h)$ with $\nu(P_{f_0}) > 0$ (as close to one as desired) such that

$$b(p) = \lim_{T\to\infty} b_T(p), \quad \text{uniformly in } P_{f_0}.$$

Thus, b is continuous in the compact set P_{f_0} and then there exists $\delta > 0$ with $b_{|P_{f_0}} \geq \delta > 0$. Let $\lambda \geq 1$. We now prove that $\lambda g(b(p)) \geq g(\lambda b(p))$. Indeed, if $g(b(p)) = 0$ is clear. If $g(b(p)) = -(b(p) - r_0)^2$ then $g(\lambda b(p)) = -(\lambda b(p) - r_0)^2 \leq -\lambda^2 (b(p) - r_0)^2 \leq -\lambda(b(p) - r_0)^2 = \lambda g(b(p))$. Thus,

$$(\lambda b(p))' = \lambda b(p)h(p) + \lambda g(b(p)) \geq h(p)\lambda b(p) + g(\lambda b(p)),$$

which means that $\lambda b(p)$ is a super-equilibrium for (10). Thus, if $\lambda > 1$ and $p \in P$

$$b(\theta_t p) \leq x(t, p; \lambda b(p)) \leq \lambda b(\theta_t p), \text{ for all } t \geq 0.$$

By Birkhoff's Ergodic Theorem (Nemytskii and Stepanov [35]) there exists an invariant set P_{f_a} with $\nu(P_{f_a}) = 1$ such that for all $p \in P_{f_a}$ there exists a sequence $\{t_n^*\}_{n\in\mathbb{N}}$ with $t_n^* \to \infty$ and $\theta_t p_n^* \in P_{f_0}$. We will prove that for $p \in P_{f_a}$ and $r > r_0$ big enough we have that

$$\lim_{t\to\infty} (x(t, p; r) - b(\theta_t p)) = 0.$$

Let $\varepsilon > 0$ and $\lambda > 1$ with $b(p)(\lambda - 1) \leq \varepsilon$ for all $p \in P$. For $p \in P_{f_a}$, there exists a t_n^* with $\theta_t p_n^* \in P_{f_0}$ satisfying $b_{t_n^*}(\theta_t p_n^*) \leq \lambda b(\theta_t p_n^*)$ by the uniform convergence in P_{f_0}, hence

$$0 \leq b_{t_n^*}(\theta_t p_n^*) - b(\theta_t p_n^*) \leq (\lambda - 1) b(\theta_t p_n^*) \leq \varepsilon.$$

Then, if $t \geq t_n^*$

$$\begin{aligned} &x(t, p; r) = x(t - t_n^*, \theta_t p_n^*; x(t_n^*, p; r)) = x(t - t_n^*, \theta_t p_n^*; b_{t_n^*}(\theta_t p_n^*)) \\ &\leq x(t - t_n^*, \theta_t p_n^*; \lambda b(\theta_t p_n^*)) \\ &\leq \lambda b(t - t_n^*, \theta_t p_n^*; b(\theta_t p_n^*)) = \lambda b(\theta_t p). \end{aligned}$$

Then, for all $t \geq t_n^*$,

$$0 \leq x(t, p; r) - b(\theta_t p) \leq (\lambda - 1) b(\theta_t p) \leq \varepsilon,$$

which implies the forward convergence in P_{f_a}.

□

Remark 32. *Note that in this case we have obtained that the cocycle attractor $A(p) \neq \{0\}$ with full measure (as $\nu(P_{f_a}) = 1$) in a subset of no continuity points for the cocycle attractor, in which we also find forward attraction. We see that is a natural fact not to obtain forward convergence where the cocycle attractor is continuous (see also [28] for a related example on this fact).*

6. Chaotic dynamics on the attractor

In this last section we study in detail the dynamical complexity of cocycle attractors. We show the presence of different types of chaotic behaviour in our cocycle attractor. In particular, we prove that the attractor possesses chaotic dynamics in the Li-Yorke sense, and that there exists sensitive dependence on initial conditions.

6.1. Chaotic cocycle attractors in the Li-Yorke sense

In this final section we will study chaotic dynamics in the Li-Yorke sense on our cocycle attractors, introduced in [31]. Important consequences of this chaotic behaviour can be found in Blanchard et al. [7].

Definition 33. *Given (K, σ, d) a continuous flow in a compact metric space, a pair $\{x, y\} \subset K$ is called a Li-Yorke pair if it holds*

$$\limsup_{t\to\infty} d(\sigma(t; x), \sigma(t; y) > 0, \quad \liminf_{t\to\infty} d(\sigma(t; x), \sigma(t; y)) = 0.$$

A set $S \subset K$ is said to be scrambled if every $\{x, y\} \subset S$ is a Li-Yorke pair. Finally, we say that the flow (K, σ, d) is chaotic in the Li-Yorke sense if there exists an uncountable scrambled $S \subset K$.

We will now consider our cocycle attractor $A(p) = [a(p), b(p)]$ associated to (10) and consider

$$K = \mathbb{A} = \bigcup_{p \in P} \{p\} \times [a(p), b(p)].$$

Since our flow on the base $(P, \sigma, \mathbb{R})$ is almost-periodic it is obvious that if $(p_1, x_1) \in P \times \mathbb{R}$, $(p_2, x_2) \in P \times \mathbb{R}$ are a Li-Yorke pair then $p_1 = p_2$. Thus, if $S_0 \subset P \times \mathbb{R}$ is a scrambled set there exists $p_0 \in P$ such that $S_0 \subset \{p_0\} \times A(p_0)$. This motivates the following definition:

Definition 34. *We say that $\mathbb{A}$ is fiber-chaotic in measure in the Li-Yorke sense if there exists an invariant set $P_{ch} \subset P$ with $\nu(P_{ch}) = 1$ such that $\{p\} \times A(p)$ is scrambled for all $p \in P_{ch}$.*

Note that $P_{ch} \subset P_f$ and it is a set of first category. Thus, our set is different from the residually Li-Yorke chaotic sets analysed in Bjerklov and Johnson [6] and Huand and Yi [21]. The arguments of these papers also shows that our fiber-chaotic compact set has zero topological entropy.

6.1.1. Chaotic dynamics with full measure We consider the framework of the previous section, that is, we have $\nu(P_f) = 1$ being $b(p) > 0$ for all $p \in P_f$.

We first need the following important result which guarantees that, with full measure, the cocycle attractor is described from complete bounded trajectories of the linear system (7).

Theorem 35. *There exists $P_l \subset P_f$ invariant and with $\nu(P_l) = 1$ such that $0 < b(p) \leq r_0$ for all $p \in P_l$.*

Proof. Let us define $D_0 = \{p \in P : \text{ there exists } t \in \mathbb{R} \text{ with } b(\theta_t p) > r_0\}$. It is clear that D_0 is measurable and invariant. We argue by contradiction and assume that $\nu(D_0) = 1$. Take $m \in \mathbb{N}$ and $D_m = \{p \in P : \text{ there exists } t \in \mathbb{R} \text{ with } b(\theta_t p) > r_0 + \frac{1}{m}\}$. Note that $D_0 = \bigcup_{m=1}^{\infty} D_m$. Then there exists $m_0 \in \mathbb{N}$ with $\nu(D_{m_0}) > 0$. Define

$$D_m^+ = \{p \in P : \text{ there exists } t > 0 \text{ with } b(\theta_t p) > r_0 + \frac{1}{m}\}.$$

Let $E_0 \subset D_{m_0}$ compact with $\nu(E_0) > 0$, and consider the restriction of b to E_0, $b_{|E_0}$ continuous. Birkhoff ergodic theorem assures the existence of a compact set $E_1 \subset E_0$ with $0 < \nu(E_1) < 1$ such that for all $p \in E_1$ there exist sequences $s_n^1 \to \infty$, $s_n^2 \to -\infty$ (depending on p) such that $\theta_s p_n^1, \theta_s p_n^2 \in E_0$ for all $n \in \mathbb{N}$. Note that for all $p \in E_0$ there exists $t(p)$ with $b(\theta_t p(p)) > r_0 + \frac{1}{m_0}$. Since $b(\theta_t p) = x(t, p; b(p))$ for every $p \in P$ and b is continuous on E_0, then $b(\theta_t p_1(p)) > r_0 + \frac{1}{m_0}$ for all $p_1 \in B(p, \delta(p)) \cap E_0$ for suitable $\delta(p) > 0$.

Finally, $E_0 \subset \cup_{p\in E_0} B(p,\delta(p))$ admits, by compactness, a recovering by a finite number of sets, so that there exists $T_0 > 0$ such that, for all $p \in E_0$ we find $t(p)$ with $|t(p)| \leq T_0$ satisfying $b(\theta_t p) > r_0 + \frac{1}{m_0}$.

From here it is easy to prove that $E_1 \subset D_m^+$. If $p \in E_1$ there are $s^1 = s(p) > T_0$ and $t^1 = t(\theta_s p^1)$ with $|t_1| < T_0$, $\theta_s p^1 \in E_0$ and such that $b(\theta_{s^1+t^1} p) > r_0 + \frac{1}{m}$. This implies that $p \in D_m^+$.

If we now denote as usual by $y(t,p;x_0)$ the solution of the linear equation (7) through p with $y(0) = x_0$, we will prove that, for each fixed $p_1 \in E_1$, then $\lim_{n\to\infty} y(s_n^1, p_1; b(p_1)) = \infty$, where $s_n^1 = s_n^1(p_1)$. Denote $s_n = s_n^1$. We can suppose that $s_{n+1} - s_n \geq T_0 + 1$ for every $n \in \mathbb{N}$. We argue by contradiction and suppose also that $\theta_{s_n} p_1 \in E_0$ tends to $p_* \in E_0$ and $\lim_{n\to\infty} y(s_n, p_1; b(p_1) = \gamma_0 b(p_*) < \infty$.

Note that if $\lambda \geq 0$

$$(\lambda b(\theta_t p))' = h(\theta_t p)\lambda b(\theta_t p) + \lambda g(b(\theta_t p)) \leq h(\theta_t p)\lambda b(\theta_t p), \text{ for all } t \in \mathbb{R}, p \in P$$

i.e., $\gamma_0 b(p)$ is a sub-equilibrium for (10) and, for all $t \geq 0$, $p \in P$

$$y(t,p;\gamma_0 b(p)) \geq \gamma_0 b(\theta_t p).$$

Moreover, for p_* there exists $t(p_*) > 0$ with $b(\theta_{t(p_*)} p_*) > r_0 + \frac{1}{m}$ and hence $\gamma_0 g(b(\theta_{t(p_*)} p_*)) > 0$, then

$$\frac{d}{dt}(\gamma_0 b(\theta_t p_*))_{|t=t(p_*)} < h(\theta_t p_*)\gamma_0 b(\theta_t p_*)_{|t=t(p_*)},$$

implying that the sub-equilibrium is strong in the sense we next explain.

There exist $\gamma_2 > \gamma_1 > \gamma_0$ and $t_0 > t_*(p_*) > 0$ with

$$y(t,p_*;\gamma_0 b(p_*)) \geq \gamma_2 b(\theta_t p_*)$$

for every $t \geq t_0$. Moreover

$$b(\theta_{t_0} p_*) = \lim_{n\to\infty} b(\theta_{s_n+t_0} p_1)$$

hence there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} y(s_{n_0} + t_0, p_1; b(p_1)) &\geq y(t_0, \theta_{s_{n_0}} p_1; y(s_{n_0}, p_1; b(p_1)) \\ &\geq y(t_0, \theta_{s_{n_0}} p_1; \gamma_0 b(\theta_{s_{n_0}} p_1)) > \gamma_1 b(\theta_{s_{n_0}+t_0} p_1) \end{aligned}$$

and thus

$$y(s_n, p_1; b(p_1)) \geq \gamma_1 b(p_*)$$

if $s_n \geq s_{n_0} + t_0$. Finally

$$\lim_{n\to\infty} y(s_n, p_1; b(p_1)) \geq \gamma_1 b(p_*),$$

which contradicts the definition of γ_0. As a consequence

$$\lim_{n\to\infty} \int_0^{s_n} h(\theta_s b(p)) ds = \infty.$$

But, application of Theorem 18 shows that $\sup_{t\leq 0} \int_0^t h(\theta_s b(p)) ds = \infty$ for a.e. $p \in E_1$, which is impossible as $\nu(P_f(h)) = 1$. Thus, $\nu(D_0) = 0$ and the result is proved. □

Let $E \subset P_l$ be a compact set such that $\nu(E) > 0$ and the restriction $b_{|E}$ continuous. Let

$$E_\infty = \{p \in P_l : \text{ there exists } t_n \to \infty \text{ with } \theta_t p_n \in E\}.$$

We know that $\nu(E_\infty) = 1$. We will prove that

Theorem 36. *For all $p \in E_\infty$, the sets $\{p\} \times [-b(p), b(p)]$ is scrambled.*

Proof. Note that it is enough to prove it for $\{p\} \times [0, b(p)]$. Take $p \in E_\infty$ and $p_0 \in P_s$. Then there exist sequences t_n^1, t_n^2 with $\theta_t p_n^1 \in E$ for all $n \in \mathbb{N}$ and $\theta_t p_n^2 \to p_0 \in P_s$. Then, if $x_1, x_2 \in (0, b(p)]$ there exist $\gamma_1, \gamma_2 \in (0,1)$, $\gamma_1 \neq \gamma_2$, such that $x_1 = \gamma_1 b(p)$ and $x_2 = \gamma_2 b(p)$. Then

$$|x(t_n^1, p; \gamma_1 b(p)) - x(t_n^1, p; \gamma_2 b(p))| = |\gamma_1 - \gamma_2| b(p) e^{\int_0^{t_n^1} h(\theta_s p) ds} = |\gamma_1 - \gamma_2| b(\theta_t p_n^1)$$

which is between $\delta|\gamma_1 - \gamma_2|$ and $\gamma|\gamma_1 - \gamma_2|$ for some $\delta, \gamma > 0$ by the continuity of b on the compact set E.

In the same way

$$|x(t_n^2, p; \gamma_1 b(p)) - x(t_n^2, p; \gamma_2 b(p))| \leq b(\theta_t p_n^2) \to 0.$$

Note that the result is also true if $\gamma_1 = 0$.

□

The compact $\overline{\bigcup_{p \in P_l} \{p\} \times [-b(p), b(p)]} \subset P \times \mathbb{R}$ is also invariant for the linear flow defined by (7), so that the above result shows the restriction of the flow is Li-Yorke chaotic.

6.1.2. Chaotic dynamics in a fiber In this final section we prove the existence of chaotic dynamics in the Li-Yorke sense in some cases where $\nu(P_f) = 0$.

Theorem 37. *Let $h \in C_0(P)$ be a function of Kozlov. Then the cocycle attractor associated to (10) is chaotic in the Li-Yorke sense.*

Proof. There exists $p_0 \in P_f$ with

$$\sup_{t \in \mathbb{R}} \int_0^t h(\theta_s p_0) ds = \rho < \infty.$$

If $0 \leq x_0 \leq \dfrac{r_0}{\rho}$ then

$$x(t, p_0; x_0) = x_0 e^{\int_0^t h(\theta_s p_0) ds}.$$

It then holds that $[0, \frac{r_0}{\rho}] \subset A(p_0) = [0, b(p_0)]$. We see that $\{p_0\} \times [0, \frac{r_0}{\rho}]$ is scrambled. Let $0 < \lambda_1 < \lambda_2 < \frac{r_0}{\rho}$. We have that

$$x(t, p_0; \lambda_2) - x(t, p_0; \lambda_1) = (\lambda_2 - \lambda_1)e^{\int_0^t h(\theta_s p_0) ds}.$$

As p_0 is recurrent, there exists a sequence $t_n^1 \to \infty$ with $\displaystyle\int_0^{t_n^1} h(\theta_s p_0) ds \to 0$. Then

$$\lim_{n\to\infty} (x(t_n^1, p_0; \lambda_2) - x(t_n^1, p_0; \lambda_1)) = (\lambda_2 - \lambda_1) > 0.$$

On the other hand, as P is minimal and P_s dense, given $p_1 \in P_s$ there exists $t_n^2 \to \infty$ such that $\displaystyle\int_0^{t_n^2} h(\theta_s p_0) ds \to -\infty$ and

$$\lim_{n\to\infty} (x(t_n^2, p_0; \lambda_2) - x(t_n^2, p_0; \lambda_1)) = 0.$$

□

Remark 38. *Observe that if h is a function of Kozlov, we have proved the non-existence of forward attractor with full measure. Now we have proved the Li-Yorke chaotic motion in this framework.*

6.2. Sensitive dependence on initial conditions

In the description of a chaotic behaviour it is usual to consider sensitive dependence on initial conditions on the compact set. In this section we analyse this property on the cocycle attractor.

Let (K, d) be a compact metric space with continuous flow σ and $M \subset K$ compact and invariant.

Definition 39. *We say that M is sensitive with respect to initial conditions (sensitive for brevity) in K if there exists $\rho > 0$ such that for all $x \in M, \delta > 0$ there exists $y \in K$ and $t > 0$ with*

$$d(x, y) \leq \delta \text{ and } d(\sigma(t, x), \sigma(t, y)) \geq \rho.$$

If $M = K$ we say that K is sensitive with respect to initial conditions.

Definition 40. *A dynamical system $(K, \sigma, \mathbb{R})$ is called transitive if there exists a point $x \in K$ with semiorbit $\{xt : t \geq 0\}$ dense in K. Any such point is called a transitive point.*

Definition 41. *We call dynamical system $(K, \sigma, \mathbb{R})$ chaotic in the Auslander-Yorke sense if it is both sensitive and transitive (see Auslander and Yorke [5]).*

Now we consider the cocycle attractor for (10).

Proposition 42. *The minimal $\mathbb{A}_0 = \cup_{p\in P}\{p\} \times 0$ is sensitive in $\mathbb{A}$.*

Proof. Let $p_0 \in P$ and $p_1 \in P_f$ with $b(p_1) = r_0$. Fix $\delta > 0$. Then there exists $p_2 \in P_s$ with $d(p_0, p_2) < \delta/2$ and a sequence $t_n \to -\infty$ such that $\lim_{n\to\infty}(\theta_{t_n}p_1, b(\theta_{t_n}p_1)) = (p_2, 0)$.

We consider the distance $\tilde{d}((p_1, x_1), (p_2, x_2)) = d(p_1, p_2) + |x_1 - x_2|$. Then there exists n_0 with $\tilde{d}((\theta_{t_n}p_1, b(\theta_{t_n}p_1)), (p_2, 0)) \le \delta/2$ for all $n \ge n_0$. Thus,

$$\tilde{d}((\theta_{t_n}p_1, b(\theta_{t_n}p_1)), (p_0, 0)) \le \delta$$

and

$$\tilde{d}((p_1, b(p_1)), (\theta_{-t_n}p_0, 0)) \ge r_0,$$

which completes the proof. □

We now consider the case in which $\nu(P_f) = 1$.

We know that $A(p) = [-b(p), b(p)]$ with $b(p) > 0$ if $p \in P_f$. For each $\lambda \in [0, 1]$ we define the measure μ_λ on $\mathbb{A}$ by Riesz theorem by

$$\int_{\mathbb{A}} f d\mu_\lambda = \int_P f(p, \lambda b(p))d\nu, \text{ for all } f \in C(\mathbb{A}).$$

By Proposition 35 we know that $b(p) \le r_0$ for all $p \in P_l$ with $\nu(P_l) = 1$. Then, since $x(t, p; \lambda b(p)) = \lambda b(\theta_t p)$ for all $p \in P_l, t \in \mathbb{R}$ then for each $t \in \mathbb{R}$ and $f \in C(\mathbb{A})$

$$\int_{\mathbb{A}} f \circ \Pi(t) d\mu_\lambda = \int_P f(\theta_t p, \lambda b(\theta_t p))d\nu = \int_P f(p, \lambda b(p))d\nu = \int_{\mathbb{A}} f d\mu_\lambda$$

so that μ_λ is an invariant measure on $\mathbb{A}$ with $\mu(\mathbb{A}) = 1$, which is also ergodic. We now denote by $\mathbb{A}_\lambda = \sup \mu_\lambda$ the support of μ_λ, which is a compact invariant set. It is clear that $\mathbb{A}_\lambda = \{(p, \lambda x) : (p, x) \in \mathbb{A}_1\}$.

Theorem 43. *Suppose $\nu(P_f) = 1$. Then the compact invariant set $\mathbb{A}_\lambda$ is sensitive and chaotic in the Auslander-Yorke sense.*

Proof. Since μ_λ is ergodic there exists an invariant set $T_\lambda \subset \mathbb{A}_\lambda$ of transitive points with $\mu_\lambda(T_\lambda) = 1$. Thus, $\mathbb{A}_\lambda$ is topologically transitive. Clearly, $(p, \lambda b(p)) \in \mathbb{A}_\lambda$ for a.e. $p \in P_f$ and $\mathbb{A}_0 \subset \mathbb{A}_\lambda$. It is obvious that the flow $\Pi(t)$ on $\mathbb{A}_\lambda$ is not equicontinuous. Thus, by Theorem 1.3 in Glasner and Weiss [17], $\mathbb{A}_\lambda$ is sensitive, which finishes the proof. □

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