Relation between $E(5)$ models and the interacting boson model

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(Received 21 February 2008; published 9 May 2008)

The connections between the E(5) models [the original E(5) using an infinite square well, E(5)-β^4, E(5)-β^6, and E(5)-β^8], based on particular solutions of the geometrical Bohr Hamiltonian with γ-unstable potentials, and the interacting boson model (IBM) are explored. For that purpose, the general IBM Hamiltonian for the U(5)-O(6) transition line is used and a numerical fit to the different E(5) models energies is performed, later on the obtained wave functions are used to calculate $B(E2)$ transition rates. It is shown that within the IBM one can reproduce very well all these E(5) models. The agreement is the best for E(5)-β^4 and reduces when passing through E(5)-β^6, E(5)-β^8, and E(5), where the worst agreement is obtained (although still very good for a restricted set of lowest-lying states). The fitted IBM Hamiltonians correspond to energy surfaces close to those expected for the critical point. A phenomenon similar to the quasidynamical symmetry is observed.

DOI: 10.1103/PhysRevC.77.054307 PACS number(s): 21.60.Fw, 21.60.Ev

I. INTRODUCTION

Both the Bohr-Mottelson (BM) collective model [1–3] and the interacting boson model (IBM) [4–7] have thoroughly been used to study the same kind of nuclear structure problems. Although very different in their formulation, both models present clear relationships. In an approximate way, the IBM can be interpreted as the second quantization of the BM shape variables [8]. More detailed connections between both models were studied during the 1980s by several authors [9–14] and, more recently, by Rowe and collaborators [15]. Both models have three particular cases that can be easily solved and for which a clear correspondence can be done. These three cases are (i) the BM anharmonic vibrator and the dynamical symmetry U(5) IBM limit, (ii) the BM γ-unstable deformed rotor and the dynamical O(6) IBM limit, and (iii) the BM axial rotor and the dynamical symmetry O(6) IBM limit including $Q \cdot Q \cdot Q$ interactions [15,16]. Note that although it is traditionally accepted the correspondence of the dynamical symmetry SU(3) IBM limit to a submodel of the BM, this fact has never been explicitly probed [15]. Each of these cases are assigned to a particular shape using the Hill-Wheeler variables (β, γ) [17]: spherical, deformed with γ-instability, and axially deformed, respectively. For transitional situations the correspondence between the two models is difficult, as Rowe stated, "what is simple in one model will be complicated when expressed in terms of the observables of the other" [15]. This situation suggests, for the case of transitional Hamiltonians, to look for the connection between BM and IBM through numerical studies.

Among the transitional Hamiltonians, an especially interesting case occurs when it describes a critical point in the transition from a given shape to another. In general, for such a situation, where the structure of the system can change abruptly by applying a small perturbation, both the BM and the IBM have to be solved numerically. However, recently Iachello has proposed schematic Bohr Hamiltonians that intend to describe different critical points and that can be solved exactly in terms of the zeros of Bessel functions. The first of these models is known as E(5) [18]. E(5) is designed to describe the critical point at the transition from spherical to deformed γ-unstable shapes. The potential to be used in the differential Bohr equation is assumed to be γ independent and, for the β degree of freedom an infinite square well is taken. Similar models were proposed later on by Iachello, called X(5) and Y(5) [19,20], to describe the critical points between spherical and axially deformed shapes and between axial and triaxial deformed shapes, respectively. All these models give rise to spectra and electromagnetic transition rates that are parameter free, up to a scale. In spite of their simplicity, some experimental examples were found [21,22] just after the appearance of these models.

In this work, we concentrate on E(5) and related models. The corresponding study for X(5) models will be published elsewhere [23]. The formulation of E(5) immediately attracted attention both experimentally and theoretically. Soon after the introduction of the E(5) model, the nucleus ^{134}Ba was proposed by Casten and Zamfir [21] as a realization of it. Other experimental examples proposed are ^{104}Ru [24], ^{102}Pd [25], and ^{108}Pd [26]. Concerning theoretical extensions of E(5), first, Arias [27] proposed a generalization of the E2 operator to be used with the E(5) model, and then Caprio [28] checked that a substitution of the original infinite well in the β variable by a finite one, which makes the model not exactly solvable anymore, provides similar results. It showed that the E(5) description is "robust in nature," i.e., the main features of the model remain almost unchanged under strong modification of the depth of the potential. Arias and collaborators [29,30] were the first authors who tried to analyze in a quantitative way the connection between the U(5)-O(6) IBM critical point and the E(5) model. In particular, they established, looking to few observables that the IBM, at the critical point, gives results close to E(5) for a small ($N \approx 5$) number of bosons. However, the IBM results for large N nicely reproduce the spectra and

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electromagnetic transition rates of a Bohr Hamiltonian with a β^4 potential (in the following E(5)-β^4). Once more, the model is no longer analytically solvable. Lévai and Arias [31] solved the Bohr equation with a sextic potential with a centrifugal barrier [32], arriving to almost closed analytical formulas for the energies and wave functions. Immediately after, Bonatsos and collaborators explored the possibility of getting numerical solutions for the γ-independent Bohr Hamiltonian with potentials of the type β^{2n}, with $n \geqslant 1$ [33]. These sequences of potentials allow to go from the vibrational limit, $n = 1$, to E(5), $n \to \infty$. In particular, in Ref. [33] spectra and transition rates for the potentials β^4, β^6, and β^8 are given explicitly and compared with the original E(5) (infinite square well potential) case. As mentioned above, all these models are produced in the BM scheme and a natural question is to ask for the corresponding equivalence in the IBM. Is the IBM able to produce the same spectra and transition rates? If yes, does the IBM Hamiltonian correspond to a critical point? This work is intended to answer these questions for the E(5) and related models ($-\beta^4$, $-\beta^6$, and $-\beta^8$ potentials) and analyze the convergence as a function of the boson number.

For that purpose, a large set of E(5) and related models results for excitation energies and transition rates are taken as reference for numerical fits of the general U(5)-O(6) IBM transitional Hamiltonian. This procedure will allow to establish the IBM Hamiltonian that best fits the different E(5) models and their relation with the critical points.

The article is organized as follows: in Sec. II the fitting procedure is described and the obtained results are presented. Section III explains the energy surfaces of the fitted IBM Hamiltonians and analyzes these in relation to the critical point. In Sec. IV the connection between the present results and the concept of quasidynamical symmetry is discussed. Finally, in Sec. V the summary and conclusions of this work are presented.

II. THE IBM FIT TO E(5) MODELS

A. The model

The most general, including up to two-body terms, IBM Hamiltonian can be written in multipolar form as

$$\hat{H} = \varepsilon_d \hat{n}_d + \kappa_0 \hat{P}^\dagger \hat{P} + \kappa_1 \hat{L}\cdot\hat{L} + \kappa_2 \hat{Q}\cdot\hat{Q} + \kappa_3 \hat{T}_3\cdot\hat{T}_3 + \kappa_4 \hat{T}_4\cdot\hat{T}_4, \tag{1}$$

where $\hat{n}_d$ is the d boson number operator, and

$$\hat{P}^\dagger = \frac{1}{2}(d^\dagger\cdot d^\dagger - s^\dagger\cdot s^\dagger), \tag{2}$$

$$\hat{L} = \sqrt{10}(d^\dagger \times \tilde{d})^{(1)}, \tag{3}$$

$$\hat{Q} = (s^\dagger \times \tilde{d} + d^\dagger \times \tilde{s})^{(2)} - \frac{\sqrt{7}}{2}(d^\dagger \times \tilde{d})^{(2)}, \tag{4}$$

$$\hat{T}_3 = (d^\dagger \times \tilde{d})^{(3)}, \tag{5}$$

$$\hat{T}_4 = (d^\dagger \times \tilde{d})^{(4)}. \tag{6}$$

The symbol $\cdot$ stands for the scalar product, defined as $\hat{T}_L\cdot\hat{T}_L = \sum_M (-1)^M \hat{T}_{LM}\hat{T}_{L-M}$, where $\hat{T}_{LM}$ corresponds to the M component of the operator $\hat{T}_L$. The operator $\tilde{\gamma}_{\ell m} = (-1)^m \gamma_{\ell -m}$ (where γ refers to s and d bosons) is introduced to ensure the correct tensorial character under spatial rotations.

The electromagnetic transitions can also be analyzed in the framework of the IBM. In particular, in this work we will focus on the $E2$ transitions. The most general $E2$ transition operator including up to one-body terms can be written as

$$\hat{T}^{E2}_M = e_{\text{eff}}\big[(s^\dagger \times \tilde{d} + d^\dagger \times \tilde{s})^{(2)}_M + \chi(d^\dagger \times \tilde{d})^{(2)}_M\big], \tag{7}$$

where e_{eff} is the boson effective charge and χ is a structure parameter.

The E(5) models are intended to be of use for γ-unstable nuclei having O(5) as symmetry algebra. For the construction of an IBM γ-unstable transitional Hamiltonian it is sufficient to impose in Eq. (1) $\kappa_2 = 0$ (this implies that no Casimir operator from the SU(3) algebra is included) as can be observed if the Hamiltonian (1) is rewritten in terms of Casimir operators (the definition for the Casimir operators have been taken from Ref. [34]):

$$\begin{aligned}\hat{H} = {} & \frac{\kappa_0}{4}N(N+4) + \left(\varepsilon_d + \frac{18}{35}\kappa_4\right)\hat{C}_1[\text{U}(5)] \\ & + \frac{18}{35}\kappa_4\,\hat{C}_2[\text{U}(5)] \\ & + \left(\kappa_1 - \frac{\kappa_3}{10} - \frac{\kappa_4}{14}\right)\hat{C}_2[\text{O}(3)] \\ & + \left(\frac{\kappa_3}{2} - \frac{3}{14}\kappa_4\right)\hat{C}_2[\text{O}(5)] - \frac{\kappa_0}{4}\,\hat{C}_2[\text{O}(6)].\end{aligned} \tag{8}$$

If, additionally, we want to construct an IBM transitional Hamiltonian that preserves the O(5) symmetry, Casimir operators for U(5), O(6), and O(5) can be included but not the quadratic O(3) Casimir operator. This condition, translated to the multipolar form language used in Eq. (1), leads to the constraint $\kappa_1 - \kappa_3/10 - \kappa_4/14 = 0$ [see Eq. (8)]. In addition, the structure parameter, χ, in the T^{E2} operator is usually taken as zero in the standard IBM calculations for γ-flat Hamiltonians. In our calculations we will impose $\kappa_2 = 0$, i.e., the γ flatness. To simplify the latter analysis, we will restrict ourselves to the case $\kappa_4 = 0$, leaving $\kappa_0, \kappa_1, \kappa_3$ (plus ε_d that fixes the energy scale) as free parameters. In practice, we do not impose the constraint $\kappa_1 - \kappa_3/10 = 0$ but, as it will be shown, the condition will be fulfilled in every fit.

B. The fitting procedure

In this section we describe the procedure for getting the IBM Hamiltonian that best fits the different E(5) models.

The χ^2 test is used to perform the fitting. The χ^2 function is defined in the standard way,

$$\chi^2 = \frac{1}{N_{\text{data}} - N_{\text{par}}} \sum_{i=1}^{N_{\text{data}}} \frac{[X_i(\text{data}) - \text{X}_\text{i}(\text{IBM})]^2}{\sigma_i^2}, \tag{9}$$

where N_{data} is the number of data, from a specific E(5) model, to be fitted, N_{par} is the number of parameters used in the IBM fit, X_i(data) is an energy level [or a $B(E2)$ value] taken from a particular E(5) model, X_i(IBM) is the corresponding

TABLE I. States included in the energy fit.

Band	Error	τ	States
$\xi = 1$	$\sigma = 0.001$	$\tau = 0$	0_1^+
	$\sigma = 0.0001$	$\tau = 1$	2_1^+
	$\sigma = 0.001$	$\tau = 2$	$4_1^+, 2_2^+$
	$\sigma = 0.001$	$\tau = 3$	$6_1^+, 4_2^+, 3_1^+, 0_3^+$
	$\sigma = 0.001$	$\tau = 4$	$6_2^+, 5_1^+, 4_3^+, 2_4^+$
$\xi = 2$	$\sigma = 0.01$	$\tau = 0$	0_2^+
	$\sigma = 0.01$	$\tau = 1$	2_3^+
	$\sigma = 0.01$	$\tau = 2$	$4_4^+, 2_5^+$
$\xi = 3$	$\sigma = 1$	$\tau = 0$	0_4^+
	$\sigma = 1$	$\tau = 1$	2_7^+

calculated IBM value, and σ_i is an arbitrary error assigned to each X_i(data).

To perform the fit, we minimize the χ^2 function for the energies, using ε_d, κ_0, κ_1, and κ_3 as free parameters and κ_2 and κ_4 fixed to zero. For doing this task we use MINUIT [35], which allows us to minimize any multivariable function.

The labels for the energy levels follow the usual notation introduced for the E(5) model: ξ enumerates the zeros of the β part of the wave function and τ is the label for the O(5) algebra, i.e., the O(5) seniority quantum number, which is a good quantum number along all the transition from U(5) to O(6). The selected set of levels included in the fit for the different E(5) models are:

(i) For the $\xi = 1$ band, all the states with angular momentum lower than 8 and $\tau < 5$. An arbitrary $\sigma = 0.001$ is used for these states except for the 2_1^+ state for which $\sigma = 0.0001$ is used. This latter value allows to normalize all the IBM energies to $E(2_1^+) = 1$. Note that the energy of the state 2_1^+ is fixed arbitrarily to 1 (the spectrum is calculated up to a global scale factor).

(ii) For the $\xi = 2$ band, all the states with angular momentum lower than 5 and $\tau < 3$. An arbitrary $\sigma = 0.01$ is used for these states.

(iii) For the $\xi = 3$ band, just the states with $(L = 0, \tau = 0)$ and $(L = 2, \tau = 1)$ are included. An arbitrary $\sigma = 1$ is used for these states.

With this selection, the number of energy levels included in the fit, N_{data}, is equal to 17. Note that the state 0_1^+ is not a real datum to be reproduced because we are interested just in excitation energies and therefore the ground state is naturally fixed to zero in both E(5) models and IBM. In Table I the states included in the fit are explicitly given.

Once the IBM Hamiltonian is fixed for each E(5) model by fitting the energy levels, the χ^2 function for the $B(E2)$ values is constructed without any additional fitting. The only parameter in the $E2$ operator (7), e_{eff}, is a global scale and is fixed to give $B(E2; 2_1^+ \rightarrow 0_1^+) = 100$ in all cases [the structure parameter $\chi = 0$ in the $E2$ operator for the transitional class going from U(5) to O(6) studied here]. The transitions calculated are enlisted in Table II.

C. The results

We have done fits of the IBM Hamiltonian (1) parameters to reproduce as well as possible the energies of the states given in Table I and generated by the different E(5) models: E(5)-β^4, E(5)-β^6, E(5)-β^8, and E(5). Calculations for the four cited E(5) models as a function of the boson number, N, were performed.

As mentioned, κ_2 and κ_4 are set to zero in Eq. (1), whereas ε_d, κ_0, κ_1, and κ_3 are free parameters in a χ^2 fit to the energy levels produced by the different E(5) models. In Fig. 1 the value of the χ^2 for a best fit to the different E(5) models as a function of N is shown. Different type of lines in Fig. 1 represent the fit to a different E(5) model as stated in the legend box. It is clearly observed that for any N the agreement between the fitted IBM and the E(5)-β^4 model is excellent and is getting worse for E(5)-β^6, E(5)-β^8, and E(5), which is the worst case. In particular $\chi^2[\text{E(5)} - \beta^4] \approx \chi^2[\text{E(5)}]/1000$. It is worth noting that these results change slowly with the boson number and in all cases, except for E(5)-β^4, for which the agreement is always excellent, the χ^2 value is an increasing function of N.

In Fig. 2 the variation of the parameters fitted in the Hamiltonian are shown. Note that the best fit parameters give rise approximately to the cancellation of the quadratic Casimir operator for O(3), i.e., $\kappa_1 \approx \kappa_3/10$. This can be quantitatively observed in Table III.

To have a clearer idea of the degree of agreement between the fitted IBM results with the data from the E(5) models, numerical comparisons are shown in Table IV for $N = 60$. This table includes not only the states used in the fit, but also an extra set of states not included in it. These allow us to control

TABLE II. $B(E2)$ transitions to be calculated.

	ξ_i	ξ_f	τ_i	τ_f		ξ_i	ξ_f	τ_i	τ_f
$B(E2: 2_1^+ \rightarrow 0_1^+)$	1	1	1	0	$B(E2: 3_1^+ \rightarrow 4_1^+)$	1	1	3	2
$B(E2: 4_1^+ \rightarrow 2_1^+)$	1	1	2	1	$B(E2: 0_3^+ \rightarrow 2_2^+)$	1	1	3	2
$B(E2: 6_1^+ \rightarrow 4_1^+)$	1	1	3	2	$B(E2: 0_3^+ \rightarrow 2_1^+)$	1	1	3	1
$B(E2: 2_2^+ \rightarrow 2_1^+)$	1	1	2	1	$B(E2: 2_3^+ \rightarrow 0_2^+)$	2	2	1	0
$B(E2: 2_2^+ \rightarrow 0_1^+)$	1	1	2	0	$B(E2: 4_4^+ \rightarrow 2_3^+)$	2	2	2	1
$B(E2: 4_2^+ \rightarrow 2_1^+)$	1	1	3	1	$B(E2: 2_7^+ \rightarrow 0_4^+)$	3	3	1	0
$B(E2: 4_2^+ \rightarrow 2_2^+)$	1	1	3	2	$B(E2: 0_2^+ \rightarrow 2_1^+)$	2	1	0	1
$B(E2: 4_2^+ \rightarrow 4_1^+)$	1	1	3	2	$B(E2: 0_4^+ \rightarrow 2_3^+)$	3	2	0	1
$B(E2: 3_1^+ \rightarrow 2_2^+)$	1	1	3	2					

TABLE III. Parameters (in arbitrary units) of the IBM Hamiltonians used in table IV.

	ε_d	κ_0	κ_1	κ_3
E(5)	3780.90	69.74	2.4308	24.4520
E(5)-β^8	3319.20	58.26	1.4028	14.0770
E(5)-β^6	3061.10	52.06	1.0753	10.7760
E(5)-β^4	2561.50	40.24	0.6218	6.2157

the goodness of the obtained fit because they are predicted states that, as we can see, have their counterpart in the E(5) models. The agreement for E(5)-β^4, E(5)-β^6, and E(5)-β^8 is really remarkable for all the states. In the case of E(5), only the $\xi = 1$ band is perfectly reproduced, whereas for the bands with $\xi = 2$ and $\xi = 3$ the agreement is poor.

The IBM calculations presented are done with the usual IBM codes and consequently are restricted, due to numerical limitations, to N around 100. However, one should note that for the transitional class studied in this work the O(5) seniority is a good quantum number all along the transition. This allows us to diagonalize easily matrices corresponding to large number of bosons using the procedure described in Ref. [30] and explore the quality of the fits in the large N limit. The results of the χ^2 fitting for such calculations are presented in Fig. 3 as a function of N. Note that the curves presented in this figure do not match exactly with the corresponding ones in Fig. 1 in the common N range. This is because in the case in which the O(5) symmetry is imposed the χ^2 function is constructed with only one state, of those appearing in Table I, per seniority. Then, the number of states included in the fit is different, which results in slightly different values for the χ^2 fitted function. The main conclusion to be extracted from Fig. 3 is that only the model E(5)-β^4 is exactly (at least for the states considered in this work) reproduced by IBM Hamiltonians with O(5) symmetry in the large N limit. For the rest of models the discrepancy in the IBM fit slowly increases as a function of N.

As a test for the produced wave functions with the fitted IBM Hamiltonian, they are used for calculating $E2$ transition probabilities, $B(E2)$. The effective charge (scale parameter) in the $E2$ operator (7), is fixed so as to give $B(E2; 2_1^+ \rightarrow 0_1^+) = 100$, thus no free parameters are left in this calculation. For the $B(E2)$'s calculated (not a fit) a χ^2 value has been obtained for each E(5) model with an arbitrary $\sigma = 10$. In Fig. 4 the corresponding χ^2 value is plotted as a function of N for all the E(5) models considered. Figure 4 shows a clear dependence of χ^2 on N. The χ^2 value decreases monotonically as N increases for all the E(5) models, except for E(5). In this last case, χ^2 start increasing for $N \approx 20$. For $N < 20$, E(5) provides the

TABLE IV. Comparison of energy levels for fitted IBM Hamiltonians, with $N = 60$, compared with those provided by the E(5) models (see text). The asterisk marks states not included in the fitting procedure. In the states labeled with two subindexes, the first one corresponds to E(5), whereas the second to the rest of models.

	ξ, τ	E(5)	IBM	E(5)-β^8	IBM	E(5)-β^6	IBM	E(5)-β^4	IBM
0_1^+	1,0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2_1^+	1,1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
4_1^+	1,2	2.199	2.214	2.157	2.164	2.135	2.139	2.093	2.092
2_2^+	1,2	2.199	2.214	2.157	2.164	2.135	2.139	2.093	2.092
0_2^+	2,0	3.031	3.051	2.756	2.763	2.619	2.622	2.390	2.390
6_1^+	1,3	3.590	3.608	3.459	3.467	3.391	3.395	3.265	3.265
4_2^+	1,3	3.590	3.608	3.459	3.467	3.391	3.395	3.265	3.265
3_1^+	1,3	3.590	3.609	3.459	3.467	3.391	3.395	3.265	3.265
0_3^+	1,3	3.590	3.609	3.459	3.467	3.391	3.395	3.265	3.265
2_3^+	2,1	4.800	4.509	4.255	4.148	4.012	3.961	3.625	3.632
6_2^+	1,4	5.169	5.159	4.894	4.890	4.757	4.755	4.508	4.508
5_1^+	1,4	5.169	5.159	4.894	4.890	4.757	4.755	4.508	4.508
4_3^+	1,4	5.169	5.159	4.894	4.890	4.757	4.755	4.508	4.508
2_4^+	1,4	5.169	5.160	4.894	4.890	4.757	4.755	4.508	4.508
4_4^+	2,2	6.780	6.108	5.874	5.636	5.499	5.387	4.918	4.934
2_5^+	2,2	6.780	6.109	5.874	5.636	5.499	5.387	4.918	4.934
0_4^+	3,0	7.577	6.682	6.364	6.073	5.887	5.752	5.153	5.175
2_7^+	3,1	10.107	8.511	8.269	7.754	7.588	7.348	6.563	6.604
6_3^+*	1,5	6.930	6.850	6.456	6.421	6.225	6.207	5.813	5.817
5_2^+*	1,5	6.930	6.850	6.456	6.421	6.225	6.207	5.813	5.817
4_5^+*	1,5	6.930	6.850	6.456	6.421	6.225	6.207	5.813	5.817
2_6^+*	1,5	6.930	6.850	6.456	6.421	6.225	6.207	5.813	5.817
$6_{6,4}^+*$	2,3	8.967	8.669	7.607	7.222	7.075	6.895	6.266	6.295
$4_{7,6}^+*$	2,3	8.967	8.669	7.607	7.222	7.075	6.895	6.266	6.295
$3_{3,2}^+*$	2,3	8.967	8.669	7.607	7.222	7.075	6.895	6.266	6.295
$0_{6,5}^+*$	2,3	8.967	8.669	7.607	7.222	7.075	6.895	6.266	6.295
4_9^+*	3,2	12.854	10.437	10.274	9.509	9.363	9.007	8.015	8.078
2_9^+*	3,2	12.854	10.437	10.274	9.509	9.363	9.007	8.015	8.078

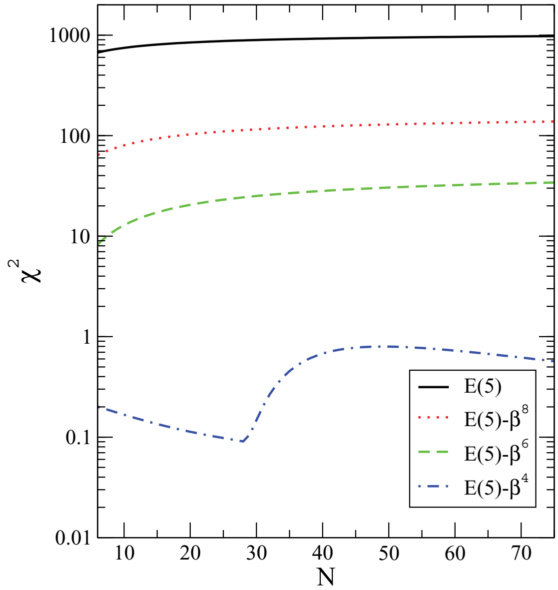

FIG. 1. (Color online) χ^2 for the IBM fit to the energy levels of the different E(5) models as a function of N.

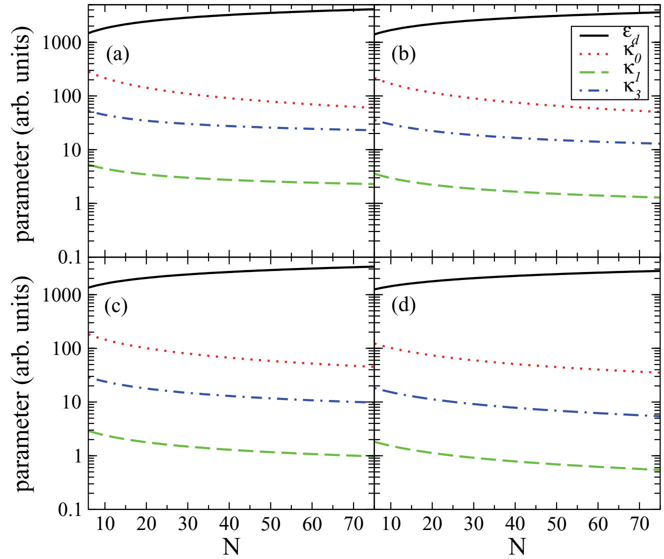

FIG. 2. (Color online) Values of the fitted IBM parameters (see text) as a function of N. Different panels correspond to the fit to the different E(5) models: (a) E(5), (b) E(5)-β^8, (c) E(5)-β^6, and (d) E(5)-β^4.

best agreement while E(5)-β^4 is the worst. This fact changes when N increases, and for $N \approx 75$ already E(5)-β^4, E(5)-β^6, and E(5)-β^8 provide a similar (excellent) agreement while the χ^2 value for E(5) is clearly larger.

For a quantitative comparison, the $B(E2)$ values for the selected transitions with $N = 60$ are shown in Table V. In this table, it is clear the remarkable agreement between the IBM calculations and E(5) models. Note the $\Delta\tau = \pm 1$ selection rule. Thus, the wave functions produced by the fit to the energy levels are giving roughly the correct $B(E2)$'s. However, it should be noted that the calculated IBM $B(E2)$ values always increase as a function of N. Therefore, looking at the transition rates $B(E2: 2_3^+ \to 0_2^+)$, $B(E2: 4_4^+ \to 2_3^+)$, $B(E2: 2_7^+ \to 0_4^+)$, $B(E2: 0_2^+ \to 2_1^+)$, and $B(E2: 0_4^+ \to 2_3^+)$ in Table V one observes that, already for $N = 60$, the IBM values are larger than those provided by E(5) and E(5)-β^8. This is also observed for some transitions in the E(5)-β^6 model, but for none in the E(5)-β^4 model. Thus, one expects for these models to start giving larger χ^2 values from a given N value on. The IBM results are always lower that the E(5)-β^4 ones and both are approaching as N increases.

In view of the excellent agreement between E(5) models and the IBM, we can state that it is impossible to discriminate,

TABLE V. $B(E2)$ values obtained, for $N = 60$, for fitted IBM Hamiltonians (see text) compared with those provided by the different E(5) models.

	E(5)	IBM	E(5)-β^8	IBM	E(5)-β^6	IBM	E(5)-β^4	IBM
$B(E2: 2_1^+ \to 0_1^+)$	100	100	100	100	100	100	100	100
$B(E2: 4_1^+ \to 2_1^+)$	167.4	165.2	173.3	170.7	176.6	173.9	183.2	180.5
$B(E2: 6_1^+ \to 4_1^+)$	216.9	215.4	231.6	227.0	239.8	233.9	256.4	248.7
$B(E2: 2_2^+ \to 2_1^+)$	167.4	165.2	173.3	170.7	176.6	173.9	183.2	180.5
$B(E2: 2_2^+ \to 0_1^+)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$B(E2: 4_2^+ \to 2_1^+)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$B(E2: 4_2^+ \to 2_2^+)$	113.6	112.8	121.3	118.9	125.6	122.5	134.3	130.3
$B(E2: 4_2^+ \to 4_1^+)$	103.3	102.6	110.3	108.1	114.2	111.4	122.1	118.4
$B(E2: 3_1^+ \to 2_2^+)$	154.9	153.8	165.5	162.1	171.3	167.2	183.1	177.6
$B(E2: 3_1^+ \to 4_1^+)$	62.0	61.5	66.2	64.9	68.5	66.8	73.3	71.1
$B(E2: 0_3^+ \to 2_2^+)$	216.9	215.4	231.6	227.0	239.8	233.9	256.4	248.7
$B(E2: 0_3^+ \to 2_1^+)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$B(E2: 2_3^+ \to 0_2^+)$	75.2	90.2	91.2	95.9	99.0	99.6	112.6	107.9
$B(E2: 4_4^+ \to 2_3^+)$	124.3	152.3	156.1	163.5	172.0	170.5	197.9	186.5
$B(E2: 2_7^+ \to 0_4^+)$	65.7	89.3	91.6	97.9	103.7	103.4	126.6	115.9
$B(E2: 0_2^+ \to 2_1^+)$	86.8	81.6	107.6	100.8	119.0	112.1	141.8	135.4
$B(E2: 0_4^+ \to 2_3^+)$	123.2	155.0	178.5	182.0	205.3	198.5	257.9	235.6

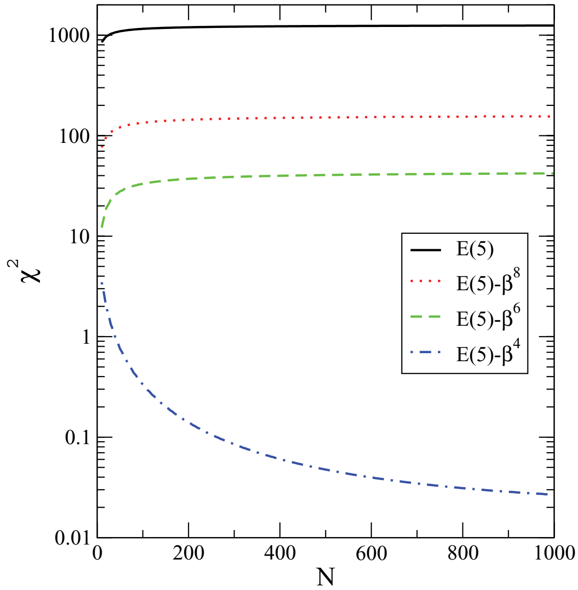

FIG. 3. (Color online) χ^2 value for the IBM fit to the energy levels of the different E(5) models, as a function of N (large N limit), for an IBM Hamiltonian with O(5) symmetry (see text).

from a experimental point of view, between a E(5) model and its IBM counterpart.

III. THE CRITICAL HAMILTONIAN

One of the most attractive features of the E(5) models treated in this work is that they are supposed to describe, at different approximation levels, the critical point in the transition from spherical to deformed γ-unstable shapes. Because they are connected to a given IBM Hamiltonian, as shown in the preceding section, this should correspond to the critical point in the transition from U(5) to O(6) IBM limits, i.e., this Hamiltonian should produce an energy surface with $(d^2E/d\beta^2)_{\beta=0} = 0$. Is this the case for the fitted IBM Hamiltonians obtained in the preceding section? Before starting with the discussion it is necessary to establish a measure on how close is a given IBM Hamiltonian to the critical point.

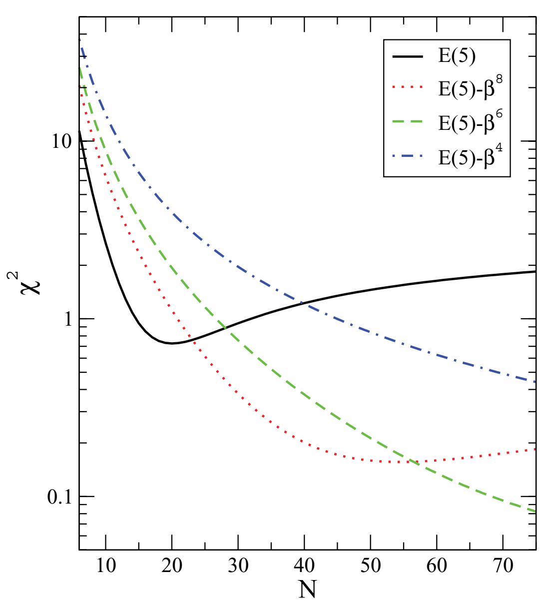

FIG. 4. (Color online) χ^2 values for the $E2$ transition rates for the different E(5) models, as a function of N, and an IBM electromagnetic operator $T(E2) = e_{\rm eff}(s^\dagger\tilde{d} + d^\dagger\tilde{s})$.

An energy surface can be associated to a given IBM Hamiltonian by using the intrinsic state formalism [9,10,12] that introduces the shape variables (β,γ) in the IBM. To define the intrinsic state one has to consider that the dynamical behavior of the system can be approximately described in terms of independent bosons moving in an average field [36]. The ground state of the system is written as a condensate, $|c\rangle$, of bosons that occupy the lowest-energy phonon state, $\Gamma_c^\dagger$:

$$|c\rangle = \frac{1}{\sqrt{N!}}(\Gamma_c^\dagger)^N|0\rangle, \tag{10}$$

where

$$\Gamma_c^\dagger = \frac{1}{\sqrt{1+\beta^2}}\left[s^\dagger + \beta\cos\gamma\, d_0^\dagger + \frac{1}{\sqrt{2}}\beta\sin\gamma\,(d_2^\dagger + d_{-2}^\dagger)\right]. \tag{11}$$

β and γ are variational parameters related with the shape variables in the geometrical collective model [12]. The expectation value of the Hamiltonian (1) in the intrinsic state (10) provides the energy surface of the system, $E(N,\beta,\gamma) = \langle c|\hat{H}|c\rangle$. This energy surface in terms of the parameters of the Hamiltonian (1) and the shape variables can be readily obtained [37],

$$\begin{aligned}\langle c|\hat{H}|c\rangle = {} & \frac{N\beta^2}{(1+\beta^2)}\left(\varepsilon_d + 6\kappa_1 - \frac{9}{4}\kappa_2 + \frac{7}{5}\kappa_3 + \frac{9}{5}\kappa_4\right)\\ & + \frac{N(N-1)}{(1+\beta^2)^2}\left[\frac{\kappa_0}{4} + \beta^2\left(-\frac{\kappa_0}{2} + 4\kappa_2\right)\right.\\ & + 2\sqrt{2}\beta^3\kappa_2\cos(3\gamma)\\ & \left. + \beta^4\left(\frac{\kappa_0}{4} + \frac{\kappa_2}{2} + \frac{18}{35}\kappa_4\right)\right]. \end{aligned}\tag{12}$$

The shape of the nucleus is defined through the equilibrium value of the deformation parameters, β and γ, which are obtained minimizing the ground-state energy, $\langle c|\hat{H}|c\rangle$. A spherical nucleus has a minimum in the energy surface at $\beta = 0$, whereas a deformed one presents the minimum at a finite value of β. The parameter γ represents the departure from axial symmetry, i.e., $\gamma = 0$ and $\gamma = \pi/6$ stand for an axially deformed nucleus, prolate and oblate, respectively, whereas any other value corresponds to a triaxial shape. An additional situation appears when the energy surface is independent on γ but presents a minimum in β, with the nucleus being γ unstable. It should be noted that for a general IBM Hamiltonian including up to two-body terms the shape is either axially symmetric or γ unstable. Moreover, the Hamiltonians considered in this work correspond always to the γ-unstable situation.

With the tools described above one can study phase transitions in the IBM [9]. First, the parameters that define the Hamiltonian are the control parameters and normally are chosen in such a way that only one of them is a variable, whereas the rest remain constant. The deformation parameters β and γ become the order parameters, although in our case the only order parameter is β. Roughly speaking, a phase transition appears when there exists an abrupt change in the shape of the system when changing smoothly the control parameter. The phase transitions can be classified according to the Ehrenfest classification [38]. First-order phase transitions appear when there exists a discontinuity in the first derivative of the energy with respect to the control parameter. This discontinuity appears when two degenerate minima exist in the energy surface for two values of the order parameter β. Second-order phase transitions appear when the second derivative of the energy with respect to the control parameter displays a discontinuity. This happens when the energy surface presents a single minimum for $\beta = 0$ and the surface satisfies the condition $(d^2E/d\beta^2)_{\beta=0} = 0$. In a more modern classification, second-order phase transitions belongs to the high-order or continuous phase transitions [38].

To determine whether a given Hamiltonian corresponds to a critical point, the flatness or the existence of two degenerate minima in the energy surface should be investigated. For the case of one parameter IBM Hamiltonian, e.g., consistent Q (CQF) Hamiltonians [39], it is simple to find an analytical expression for the critical control parameter in the Hamiltonian. However, for a general IBM Hamiltonian it is necessary to rewrite the energy surface in a special way, as the one presented in Ref. [40]. There, the authors manage to write the energy surface of a general IBM Hamiltonian in terms of two parameters. The authors make use of some concepts from the catastrophe theory [41] to define the two essential parameters, (r_1, r_2). In terms of these they find expressions for the locus, in the essential parameter space, that give a critical point at the origin in β, called bifurcation set, and for the locus that gives rise to two degenerate minima, called the Maxwell set. For the Hamiltonians considered in Sec. II, $\kappa_2 = 0$ and $\kappa_4 = 0$, in these cases $r_2 = 0$ and r_1 can be written as,

$$r_1 = \frac{a_3 - u_0 + \tilde{\varepsilon}/(N-1)}{2a_1 + \tilde{\varepsilon}/(N-1) - a_3}, \tag{13}$$

where

$$\tilde{\varepsilon} = \varepsilon_d + 6\kappa_1 + \frac{7}{5}\kappa_3,$$
$$a_1 = \frac{1}{4}\kappa_0,$$
$$a_3 = -\frac{1}{2}\kappa_0,$$
$$u_0 = \frac{\kappa_0}{2}. \tag{14}$$

Note that in the large N limit, ε_d is proportional to N (see Fig. 2) and therefore Eq. (13) can be approached by

$$r_1 \approx \frac{\varepsilon_d/N - \kappa_0}{\varepsilon_d/N + \kappa_0}. \tag{15}$$

This expression agrees with the use of an energy surface derived through a Holstein-Primakoff expansion [42].

In this language, a critical Hamiltonian corresponds to $r_1 = 0$. In Fig. 5 the values of r_1 as a function of N for the IBM Hamiltonians obtained from the fit are presented for the different E(5) models studied. For the E(5) model, the fitted IBM Hamiltonian produces $r_1 = 0$ for $N \approx 7$. In the case of the E(5)-β^8 the value $r_1 = 0$ is obtained for $N \approx 25$, whereas for E(5)-β^6 it is obtained for $N \approx 70$. For the E(5)-β^4 model it is known that $r_1 = 0$ is reached for very large number of bosons [29,30].

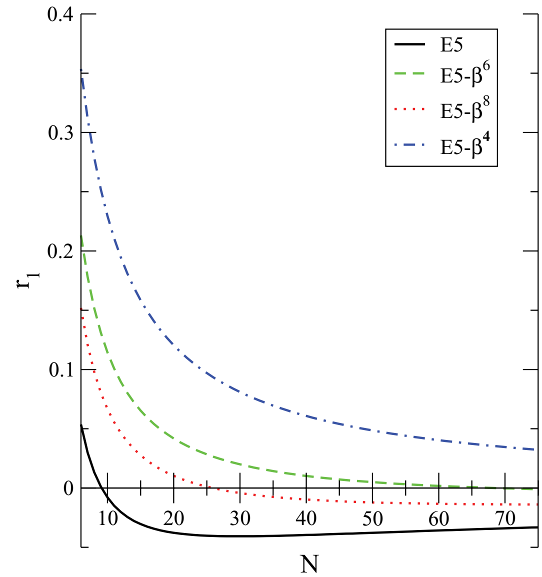

FIG. 5. (Color online) Values of r_1 (see text for definition) as a function of N for the fitted IBM Hamiltonians.

It is worth to show (see Fig. 6) that for all the fitted IBM Hamiltonians, the resulting energy surfaces are quite flat in a large interval of N values and, therefore, it is justified to say that the fitted IBM Hamiltonians are very close to the critical

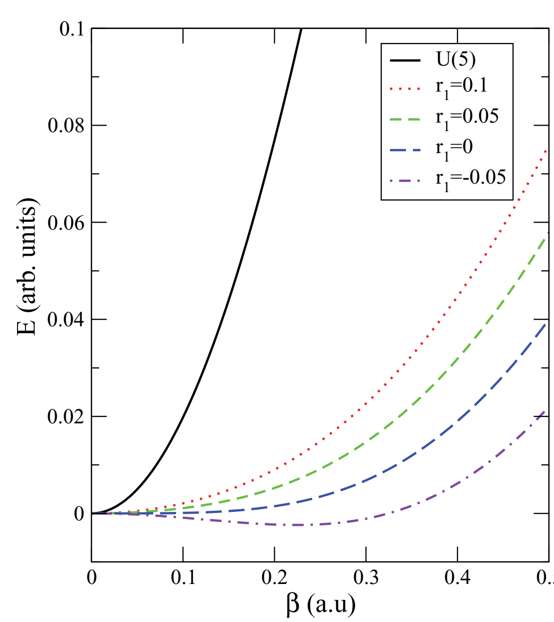

FIG. 6. (Color online) IBM energy surfaces as a function of β, for selected values of r_1 (see text for definition).

area. As a consequence, the E(5) models will be appropriated to describe phase transition regions close to the critical point.

IV. QUASIDYNAMICAL SYMMETRIES

The concept of quasidynamical symmetry (QDS) was introduced in Refs. [15,43–46] and has been used in the study of phase transitions. This concept is very useful for working with Hamiltonians that present two dynamical symmetries (depending on the value of a control parameter) as limits. In this situation, the system shows the tendency to hold onto a given symmetry until the control parameter reaches a critical value, passing the system, at this moment, onto the other symmetry. The remarkable feature is that the system can present a set of states that behave as belonging to irreducible representations (irreps) of the corresponding symmetry group, although in fact, they do not belong to a given irrep but to a mixture of them.

In mathematical terms, a QDS can be defined through the *embedded representations* [45]: "If a subset of states of a system are in one-to-one correspondence with the states of an irrep of a group G and if all the properties of the subset of states associated with observables in the Lie algebra of G (including their relationships to one another but not necessarily their relationships to states outside of the subset) are as they would be if the states actually belonged to an irrep of the group G, then the subset of states is said to span an embedded representation of G." Therefore, in the case of a QDS, there exist a set of states that behave as belonging to a unique irrep of G, although that is only apparent, because they correspond to a superposition of irreps, but all their observables (up to certain degree of accuracy) are identical to the ones of states within a given irrep. In summary, the states can be expressed as a coherent superposition of irreps that behave as a single one. Note that to show that a QDS exists, one has to fix a subset of states and the degree of accuracy for the comparison with the observables of the dynamical symmetry.

In our comparison between the IBM and the E(5) models we observe a phenomenon that resembles the QDS, i.e., part of the IBM spectrum behaves as having E(5) symmetry, although, indeed they do not have such a symmetry. We should emphasize that this is not a real QDS for two reasons: (i) E(5) cases are not dynamical symmetry limits of the IBM and (ii) the BM and the IBM have different Hilbert spaces. Indeed, it is not possible to define irreps in E(5) models and therefore embedded representations. We will call this situation quasicritical point symmetry (QCPS) [47].

To study in detail the QCPS one has to fix the degree of accuracy to be demanded to the observables. In our study, for the energies an accuracy of 1% for all the states belonging to a given ξ is set, whereas for the $B(E2)$ values an accuracy of 10% for all the studied intraband transitions in a given ξ is selected.

Tables IV and V, which correspond to $N = 60$ are analyzed below,

(i) E(5): only the states in the $\xi = 1$ band present E(5) QCPS.
(ii) E(5)-β^8: only the $\xi = 1$ states present E(5)-β^8 QCPS.
(iii) E(5)-β^6: only the $\xi = 1$ states present E(5)-β^6 QCPS.
(iv) E(5)-β^4: all the studied states, $\xi = 1$, $\xi = 2$, and $\xi = 3$, present E(5)-β^4 QCPS.

These results, regarding the energies, can be extended to larger values of N as well (see Fig. 1), i.e., the values of the energies remain stable when N increases, whereas for the $B(E2)$ values the observed differences become larger, especially in the E(5) case.

V. SUMMARY AND CONCLUSIONS

In this article, we studied the connection between the E(5) models and the IBM on the basis of a numerical mapping between models. To establish the mapping we have performed a best fit of the general U(5)-O(6) transitional IBM Hamiltonian to a selected set of energy levels produced by several E(5) models. Later on, a check to the wave functions, obtained with the best fit parameters, has been done by calculating relevant $B(E2)$ transition rates. All calculations have been done as a function of the number of bosons. Once the best fit IBM Hamiltonians to the different E(5) models are obtained, their energy surfaces are constructed and analyzed with the help of the catastrophe theory so as to know how close they are to a critical point. Finally, the concept of quasicritical point symmetry is introduced, as similar to the idea of quasidynamical symmetry.

We have shown that it is possible, in all cases, to establish a one-to-one mapping between the E(5) models and the IBM with a remarkable agreement for both the energies and the $B(E2)$ transition rates. In general, the goodness of the fit to the energies is independent on the number of bosons, but the corresponding $B(E2)$ transition rates are indeed sensitive to N. This is so specially in the E(5), for which the χ^2 value reaches a minimum for N small ($N \approx 7$) and from there on increases notably as a function of N. Globally, the best agreement is obtained for the E(5)-β^4 Hamiltonian and the worst for the E(5) case. For the case of very large number of bosons and Hamiltonians with O(5) symmetry we have confirmed the results of [29,30], i.e., the only E(5) model that can be reproduced exactly by the IBM is E(5)-β^4, corresponding such a Hamiltonian with the critical point of the model ($r_1 = 0$). A consequence of this excellent agreement is that it is impossible, from a experimental point of view, to discriminate between a E(5) model and its corresponding IBM Hamiltonian when only few low-lying states are considered (usually the four lowest states in the ground-state band, plus 0_2^+ and 2_3^+ in the $\xi = 2$ band).

We have also proved that all the E(5) models correspond to IBM Hamiltonians very close to the critical area, $|r_1| < 0.05$. Therefore, one can say that the E(5) models are appropriate to describe transitional γ-unstable regions close to the critical point.

We have found that the results presented in this article are consistent with the existence of something similar to a quasi-dynamical symmetry; we call this phenomenon quasicritical point symmetry.

Finally, it should be noted that the use of a more general U(5)-O(6) Hamiltonian, e.g., using κ_4 as free parameter, do not change the main conclusions of this work.

ACKNOWLEDGMENTS

We are grateful to D. J. Rowe for a careful reading of the manuscript and for his valuable comments. This work has been partially supported by the Spanish Ministerio de Educación y Ciencia and by the European regional development fund (FEDER) under project numbers FIS2005-01105, FPA2006-13807-C02-02, and FPA2007-63074 and by the Junta de Analucía under project numbers FQM160, FQM318, P05-FQM437, and P07-FQM-02962.

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