Constructing inferential contexts

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Representation of inferential contexts may need logical systems that could be different from classical ones, particularly when the pretension is to model how inferences are in concrete scientific practices. Such contexts will be represented by means of different consequence relations, the so called "consequence modulo X", for X as a set of formulas, a set of models or a set of additional rules, which are defined by following (in part) Makinson's method. Structural rules of the classical relation leads us to consider a variety of contexts that share a closure relation, though compactness may fail and uniform substitution is not accomplished.

Given sets of sentences Γ, Θ —Θ countable— and a sentence φ, we say that φ is **logical consequence of Γ modulo the set Θ** —in symbols $\Gamma \models_\Theta \varphi$— iff for all interpretative structures M,

$$\text{if } M \models \Gamma \text{ and } M \models \Theta, \text{ then } M \models \varphi$$

or, which is the same

$$\text{if } M \in MOD(\Gamma) \text{ and } M \in MOD(\Theta), \text{ then } M \in MOD(\varphi).$$

It is equivalent to

1. $\Gamma \models_\Theta \varphi$ iff when $M \in MOD(\Theta \cup \Gamma)$, then $M \in MOD(\varphi)$, and
2. $\Gamma \models_\Theta \varphi$ iff $\Gamma, \Theta \models \varphi$.

Consequence relation modulo a set of sentences is a closure relation, since it verifies reflexivity, monotonicity and transitivity. On the other hand, compactnes is accomplished but uniform substitution is not verified, because of which is not reducible at a mere classical consequence relation. Nevertheless when a consequence relation is a closure one, it is considered as belongins to classical style of inference.

Two consistent sets of formulas Θ and Θ' are *deductively equivalent* —two inconsistent sets of formulas are equivalent *a fortiori*—, iff for every φ,

$$\Theta \models \varphi \text{ iff } \Theta' \models \varphi$$

Then $\models_\Theta$ and $\models_{\Theta'}$ are *equivalent consequence relations*: for all set of sentences Γ and a sentence φ,

$$\Gamma \models_\Theta \varphi \text{ iff } \Gamma \models_{\Theta'} \varphi$$

For a class of interpretative structures $\mathcal{M}$, a set of sentences Γ and a sentence φ, we can define new consequence relations: φ is **logical consequence of** Γ **modulo** $\mathcal{M}$ —in symbols $\Gamma \models_{\mathcal{M}} \varphi$— iff for all $M \in \mathcal{M}$,

$$\text{if } M \models \Gamma, \text{ ten } M \models \varphi.$$

This is also a closure relation since it verifies reflexivity, monotonicity and transitivity. A specific class of models is $\mathcal{M}_n$, whose universes of discourse have cardinality $n \geq 1$, then $\Gamma \models_n \varphi$ represents the consequence relation modulo n, a special case of consequence relation modulo a class of models. If $\mathcal{M}$ represents the class of finite models

$$\Gamma \models_{\mathcal{M}} \varphi \text{ iff } \Gamma \models_i \varphi$$

for all $i \in \mathbb{N} - \{0\}$. In general that relations do not verify compactness nor uniform sustitution. If $\mathcal{M} = MOD(\Theta)$, then $\models_{\mathcal{M}}$ and $\models_{\Theta}$ are equivalent.

Taking sets of sentences Θ_i as modulos, contexts could be represented by consequence relations $\models_{\Theta_i}$, though in general if $\models_{\Theta_i}$ and $\models_{\Theta_j}$ are equivalent, then both of them represent the same inferential context. In a similar way taking sete of models as modulos.

Abductive problems and solutions for that can be defined with respect to such contexts: given a finite set of sentences Θ and a sentence φ, (Θ, φ) is an *abductive problem with respect to* $\models_{\Theta_i}$, named as $\mathbf{AbdProb}_{\models_{\Theta_i}(\Theta,\varphi)}$, iff

1. $\Theta \not\models_{\Theta_i} \varphi$ and
2. $\Theta \not\models_{\Theta_i} \neg\varphi$.

On the other hand, α is an *abductive solution* for such abductive problem iff:

1. $\Theta, \alpha \models_{\Theta_i} \varphi$
2. $\Theta \cup \alpha$ is Θ_i-satisfiable —taking into account the character of such modulo–
3. $\alpha \not\models_{\Theta_i} \varphi$.

The set of abductive solutions is

$$\mathcal{A}bd_{\models_{\Theta_i}}(\Theta, \varphi) = \{\psi \in L \mid \Theta, \psi \models_{\Theta_i} \varphi\}$$

Now inferential contexts can be analyzed in terms of a process from an initial context until a final one. Given an initial (consistent) context Θ_0, and formulas φ_1, ..., φ_n that are consistent with Θ_0, $n <\mid \mathbb{N} \mid$, $\models_{\Theta_n}$ represents the context that results of obtaining the modulo Θ_n according to the following rule, for every $i < n$

1. If $\mathbf{not\text{-}AbdProb}_{\models_{\Theta_i}}(\Theta_i, \varphi_{i+1})$, then $\Theta_{i+1} = \Theta_i$
2. In another case, $\Theta_{i+1} = \Theta_i \cup \{\varphi_{i+1}\} \cup \{f(\Theta_i, \varphi_{i+1})\}$, where

$$f : (\Theta_i \times \varphi_{i+1}) \longmapsto \mathcal{A}bd_{\models_{\Theta_i}}(\Theta_i, \varphi_{i+1})$$

—f chooses a "good" explanation α, according to a preferential relation defined in $\mathcal{A}bd_{\models_{\Theta_i}}(\Theta_i, \varphi_{i+1})$—

To define calculi that could be used to search abductive solutions, a criterion of compactness/completeness could be given. Two formulations of sound-completeness are equivalent, namely

1. There exists an axiomatic system $\vdash_\Sigma$ such that for every set of formulas Γ and the formula φ, $\Gamma \vdash_\Sigma \varphi$ iff $\Gamma \models \varphi$
2. There exists an axiomatic system $\vdash_\Sigma$ such that for every set of formulas Γ and the formula φ, $\Gamma \models \varphi$ iff
   (a) $\vdash_\Sigma$ has *modus ponens*
   (b) for formulas α, β, $\vdash_\Sigma \alpha \to (\neg\alpha \to \beta)$ (contradiction implies triviality in $\vdash_\Sigma$)
   (c) for all formulas α, $\vdash_\Sigma (\neg\alpha \to \alpha) \to \alpha$ (it verifies *consequentia mirabilis*)
   (d) the meta-theorem of deduction is verified in $\vdash_\Sigma$
   (e) Γ is consistent in $\vdash_\Sigma$ —that is to say $\Gamma \nvdash_\Sigma \bot$— iff Γ is satisfiable [it has a model]

However problems arise when the semantics is considered w.r.t. the class of finite models. Suppose sets of sentences defined as

$$\Gamma_n = \{\exists x_1, ..., x_n (\bigwedge_{(i \neq j)=1}^{n} x_i \neq x_j) \in L \mid n < |\mathbb{N}|\},$$

and let that set be:

$$\Gamma = \bigcup \Gamma_n.$$

By definition, $\Gamma \nvdash \bot$, but there is no finite model M that satisfies all sentences of Γ, this implies the failure of compactness, then it is not possible to state completeness, that is to say, for such set Γ and a sentence φ, it could be the case that $\Gamma \models_{\mathcal{M}} \varphi$ but $\Gamma \nvdash \varphi$. On the other hand, similar problems arise when the language is a second order one with standard semantics, since compactness, completeness and Löwenheim-Skolem theorem are not accomplished.

So, conditions should be weaked in the following sense. In classical style, a calculus $\vdash$ is *suitable* for a consequence relation $\models$ iff $\vdash$ accomplishes

1. $\vdash$ is sound
2. Deduction theorem and *modus ponens* hold in $\vdash$
3. *Consequentia mirabilis* holds
4. Contradiction implies triviality

An abductive calculus $\hookrightarrow$, to obtain solutions for abductive problems with respect to a consequence relation $\models_{\Theta_i}$, is definable if there exists a suitable calculus $\vdash_{\Theta_i}$.

On the other hand, $\mathbf{REL}(\Theta, \alpha, \varphi)$ expresses that it is verified

1. $\Theta, \alpha \nvdash_{\Theta_i} \bot$

2. $\alpha \nvdash_{\Theta_i} \varphi$

3. One of the following conditions is verified

   (a) $\alpha \vdash_{\Theta_i} \theta_1 \wedge ... \wedge \theta_n \rightarrow \varphi$, where $\Theta = \{\theta_1, ..., \theta_n\}$

   (b) $\Theta, \neg\varphi \vdash_{\Theta_i} \neg\alpha$

A framework rule is defined to apply deductive calculi to obtain solutions for abductive problems (with respect to a consequence relation):

$$\frac{\mathbf{AbdProb}_{\models_{\Theta_i}}(\Theta, \varphi); \mathbf{REL}(\Theta, \alpha, \varphi)}{\alpha}$$

Then, we can say that any abductive calculus $\hookrightarrow$ defined according to such framework rule will be sound, that is to say

$$\text{if } \Theta, \varphi \hookrightarrow \alpha, \text{ then } \alpha \in \mathcal{A}bd_{\models_{\Theta_i}}(\Theta, \varphi)$$

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