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THE POSTPROCESSED MIXED FINITE-ELEMENT METHOD FOR THE NAVIER–STOKES EQUATIONS*

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Abstract. A postprocessing technique for mixed finite-element methods for the incompressible Navier–Stokes equations is studied. The technique was earlier developed for spectral and standard finite-element methods for dissipative partial differential equations. The postprocessing amounts to solving a Stokes problem on a finer grid (or higher-order space) once the time integration on the coarser mesh is completed. The analysis presented here shows that this technique increases the convergence rate of both the velocity and the pressure approximations. Numerical experiments are presented that confirm both this increase in the convergence rate and the corresponding improvement in computational efficiency.

Key words. Navier–Stokes equations, mixed finite-element methods

AMS subject classifications. 65M60, 65M20, 65M15, 65M12

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1. Introduction. This paper in a sense culminates the development of a postprocessing technique to increase the accuracy and computational efficiency of Galerkin methods for dissipative partial differential equations introduced in [18]. We turn to the equations which gave rise to this postprocessing technique, the incompressible Navier–Stokes equations, and we address those Galerkin methods for these equations which, when complex-shaped bodies are present, are acknowledged to be of wider applicability, mixed finite-element (MFE) methods.

The postprocessing technique we study here was originally developed for spectral methods [18], [19]. At that moment, either its analysis and understanding or its development seemed to depend heavily on the properties of the Fourier modes, although this was not a shortcoming to prove its usefulness in the study of nonlinear shell vibrations [27]. In later works [13], [14], the dependence on the Fourier modes was overcome. Of particular importance to the present work, besides [14], has been the development of the postprocessing technique for finite-element methods in [20], [15]. In [20], it was devised how to carry out the postprocessing without the help of an approximate inertial manifold [11], [12], a concept more suited to spectral methods and eigenfunction expansion. In [15], it is shown what gains can be expected when postprocessing low-order elements.

As is usually the case with MFE methods, it is the experience and understanding gained in previous works (see [14], [15], [16], [17], [18], [19], [20], and the references cited therein) with simpler equations and methods which has allowed the present one to be written. Furthermore, although for simplicity we focus on Hood–Taylor [26] elements, the postprocessing technique can be easily adapted to other kinds of mixed

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elements. In fact, in [3] (see also [5]) the so-called mini-element is shown to render similar gains as Hood–Taylor elements when postprocessed if the provisions in [15] are taken into account.

Let us describe what this postprocessing technique is. We consider the incompressible Navier–Stokes equations, which, in appropriate dimensionless variables, can be written as

$$\begin{aligned} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p &= f,\\ \operatorname{div}(u) &= 0 \end{aligned} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with smooth boundary subject to homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. In (1.1), u is the velocity field, p the pressure, and f a given force field. Suppose that for the solution u and p corresponding to a given initial condition

$$u(\cdot, 0) = u_0; \tag{1.2}$$

we are interested in its value at a certain time $T > 0$. We first compute MFE approximations u_h and p_h to the velocity and pressure, respectively, by integrating in time the corresponding discretization of (1.1)–(1.2) from $t = 0$ to $t = T$. Then, in the postprocessing step, we obtain an approximation to the solution $\tilde u$, $\tilde p$ of the Stokes problem

$$\left.\begin{aligned} -\nu\Delta\tilde u + \nabla\tilde p = f - \tfrac{d}{dt}u_h(T) - (u_h(T)\cdot\nabla)u_h(T)\\ \operatorname{div}(\tilde u) = 0\end{aligned}\right\}\quad \text{in } \Omega,\qquad \tilde u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

The MFE of this last step is either the same-order Hood–Taylor element over a finer grid or a higher-order Hood–Taylor element over the same grid. The rate of convergence of the discrete velocity and pressure in the resulting method is proved to be the same as the rate of convergence of the MFE used in the postprocessed step. The overcost of the postprocessed procedure is nearly negligible since the Stokes problem using the enhanced MFE is solved only once, when the time integration has been completed. In this respect, it radically differs from some other research [2], [32], with low-order MFEs for the Navier–Stokes equations that also developed from the ideas in [11] and [12], since in [2] and [32] computations with the enhanced element or on the finer grid are carried out all the way through the interval $(0, T]$.

Some superconvergence results are obtained in the paper and are used as a tool to get the rate of convergence of the postprocessed method. In particular, we derive a superconvergence result for the error between the MFE approximation to the velocity and the discrete Stokes projection introduced in [24]. For simplicity of analysis, we derive these results under the strong regularity hypotheses in (2.2), which, as pointed out in [24], are unrealistic in practical situations. In a more practical setting, assumptions (2.2) should be assumed from some positive time $t_0 > 0$ onwards, and, as we comment in section 2, computations (and their analysis) up to this time should take into account the lower regularity at $t = 0$.

Finally, we remark that recent research [16], [17] has shown the usefulness of the postprocessing technique in obtaining efficient a posteriori error estimators in partial differential equations of evolution, a field much less developed than in the case of steady problems. The application of the postprocessing technique to get a posteriori error estimates for Navier–Stokes equations using the results obtained in this paper will be the subject of future work.

The rest of the paper is as follows. In section 2 we recall some properties of MFE methods and collect some inequalities to be used later. In section 3 we first specify the postprocessing technique and then carry out the convergence analysis. Finally, in section 4 numerical experiments are presented to assess the capabilities of the new technique.

2. Preliminaries and notations. Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, not necessarily convex, but of class $\mathcal{C}^m$, $m \geq 3$, and let H and V be the Hilbert spaces $H = \{u \in \big(L^2(\Omega)\big)^d, \mid \operatorname{div}(u) = 0,\ u \cdot n_{|\partial\Omega} = 0\}$, $V = \{u \in \big(H_0^1(\Omega)\big)^d, \mid \operatorname{div}(u) = 0\}$, endowed with the inner product of $L^2(\Omega)^d$ and $H_0^1(\Omega)^d$, respectively. For $1 \leq q \leq \infty$ and $l \geq 0$, we consider the standard Sobolev spaces, $W^{l,q}(\Omega)^d$, of functions with derivatives up to order l in $L^q(\Omega)$, and $H^l(\Omega)^d = W^{l,2}(\Omega)^d$. The norm in $H^l(\Omega)^d$ will be denoted by $\|\cdot\|_l$ while $\|\cdot\|_{-l}$ will represent the norm of its dual space. We consider also the quotient spaces $H^l(\Omega)/\mathbb{R}$ with norm $\|p\|_{H^l/\mathbb{R}} = \inf\{\|p + c\|_l \mid c \in \mathbb{R}\}$.

We shall frequently use the following Sobolev's imbeddings [1]. There exists a constant $C = C(\Omega, q)$ such that for $q \in [1, \infty)$, $q' < \infty$, it holds that

$$\|v\|_{L^{q'}(\Omega)^d} \leq C\|v\|_{W^{s,q}(\Omega)^d}, \qquad \frac{1}{q} \geq \frac{1}{q'} \geq \frac{1}{q} - \frac{s}{d} > 0, \quad v \in W^{s,q}(\Omega)^d. \tag{2.1}$$

For $q' = \infty$, (2.1) holds with $\frac{1}{q} < \frac{s}{d}$.

Let $\Pi : L^2(\Omega)^d \longrightarrow H$ be the Leray projector that maps each function in $L^2(\Omega)^d$ onto its divergence-free part. We denote by $\mathcal{A}$ the Stokes operator in Ω:

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H \longrightarrow H, \quad \mathcal{A} = -\Pi\Delta, \quad \mathcal{D}(\mathcal{A}) = H^2(\Omega)^d \cap V.$$

Applying Leray's projector to (1.1), the equations can be written in the form

$$u_t + \nu\mathcal{A}u + B(u, u) = \Pi f \qquad \text{in } \Omega,$$

where $B(u, u) = \Pi((u \cdot \nabla)u)$.

In what follows we will assume that the solution (u, p) of (1.1)–(1.2) satisfies

$$\max_{0\leq t\leq T} \big(\|u(t)\|_r + \|p(t)\|_{H^{r-1}/\mathbb{R}}\big) < \infty, \quad \max_{0\leq t\leq T} \big(\|u_t(t)\|_r + \|p_t(t)\|_{H^{r-1}/\mathbb{R}}\big) < \infty. \tag{2.2}$$

We refer the reader to [30] for a study about the regularity of the solutions of the Navier–Stokes equations. Notice, however, that, as pointed out in [24], it is unrealistic to assume such a strong regularity up to time $t = 0$. The assumption in (2.2) is for simplicity in the analysis. In a more realistic setting, $t = 0$ should be replaced by some positive time t_0, and error bounds requiring less regularity such as those in [24] and [25] should be considered from $t = 0$ to $t = t_0$. In order to maintain the accuracy levels that a higher regularity would allow from t_0 onwards, computations up to $t = t_0$ should be carried out on an adequate finer grid. Notice also that among the conditions to ensure (2.2) (see, e.g., Theorem 4 in [23]) is that Ω is of class $\mathcal{C}^r$.

Let $\mathcal{T}_h = (\tau_i^h, \phi_i^h)_{i\in I_h}$, $h > 0$, be a family of partitions of suitable domains Ω_h, where the parameter h is the maximum diameter of the elements $\tau_i^h \in \mathcal{T}_h$ and ϕ_i^h are the mappings of the reference simplex τ_0 onto τ_i^h. We restrict ourselves to quasi-uniform and regular meshes $\mathcal{T}_h$.

Let $r \geq 2$, we consider the finite-element spaces

$$\widehat{S}_{h,r} = \big\{\chi_h \in \mathcal{C}^0(\overline{\Omega_h}) \,|\, \chi_{h|_{\tau_i^h}} \circ \phi_i^h \in P^{r-1}(\tau_0)\big\} \subset H^1(\Omega_h),$$

$$\overset{\circ}{S}_{h,r} = \big\{\chi_h \in \mathcal{C}^0(\overline{\Omega_h}) \,|\, \chi_{h|_{\tau_i^h}} \circ \phi_i^h \in P^{r-1}(\tau_0),\ \chi_h(x) = 0\ \forall x \in \partial\Omega_h\big\} \subset H_0^1(\Omega_h),$$

where $P^{r-1}(\tau_0)$ denotes the space of polynomials of degree at most $r-1$ on τ_0. As a consequence of restricting our study to quasi-uniform partitions, the following inverse inequality holds (see, e.g., [9, Theorem 3.2.6]) $\forall \tau = \tau_i^h \in \mathcal{T}_h$, with $\mathrm{diam}(\tau) = h_\tau \le h$, $v_h \in (\overset{\circ}{S}_{h,r})^d$:

(2.3)
$$\|v_h\|_{W^{m,q}(\tau)^d} \le C h^{l-m-d(\frac{1}{q'}-\frac{1}{q})} \|v_h\|_{W^{l,q'}(\tau)^d}, \quad 0 \le l \le m \le 2, \quad 1 \le q' \le q \le \infty.$$

In order to guarantee convergence of the MFE approximation, we choose a stable combination of two finite-element spaces (see [7]). We introduce the finite-element spaces in which our MFE approximation to (u,p) will be carried out. We shall denote by $(X_{h,r}, Q_{h,r-1})$ the so-called Hood–Taylor element, where

$$X_{h,r} = \left(\overset{\circ}{S}_{h,r}\right)^d, \quad Q_{h,r-1} = \widehat{S}_{h,r-1} \cap L^2(\Omega_h)/\mathbb{R}, \qquad r \ge 3.$$

For this mixed element a uniform inf-sup condition is satisfied (see [26], [6]), that is, there exists a constant $\beta > 0$ independent of the mesh grid size h such that

$$\inf_{q_h \in Q_{h,r-1}} \sup_{v_h \in X_{h,r}} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\|_1 \|q_h\|_{L^2/\mathbb{R}}} \ge \beta. \tag{2.4}$$

The approximate velocity solution belongs to the discretely divergence-free space

$$V_{h,r} = X_{h,r} \cap \left\{ \chi_h \in H_0^1(\Omega_h) \,:\, \int_{\Omega_h} q_h \,\mathrm{div}(\chi_h) = 0 \;\; \forall q_h \in Q_{h,r-1} \right\}.$$

We observe that for the Hood–Taylor element, $V_{h,r}$ is not a subspace of V.

For any $v \in C_0(\Omega)^d$, we consider the standard interpolant operator $I_h : C_0(\Omega)^d \longrightarrow X_{h,r}$. Let $v \in H^r(\Omega)^d \cap H_0^1(\Omega)^d$; it is well known that I_h satisfies

$$\|v - I_h(v)\|_{L^2(\Omega\cap\Omega_h)^d} + h\|v - I_h(v)\|_{H^1(\Omega\cap\Omega_h)^d} \le C h^r \|v\|_{H^r(\Omega)^d}. \tag{2.5}$$

We briefly discuss next under what circumstances (2.5) can be extended to a global estimate (i.e., to an estimate in Ω and not just in $\Omega\cap\Omega_h$). The interpolation operator $I_h(v)$ is extended by zero in $\Omega \setminus \Omega_h$, and defining $\delta(h) = \max_{x\in\partial\Omega_h} \mathrm{dist}(x, \partial\Omega)$, one obtains

$$\|v - I_h(v)\|_{L^2(\Omega)^d} + h\|v - I_h(v)\|_{H^1(\Omega\cap\Omega_h)^d} \le C(h^r + \delta(h))\|v\|_{H^r(\Omega)^d}. \tag{2.6}$$

For $x \in \Omega \cap \Omega_h$, (2.5) (and so (2.6)) follows from standard theory of interpolation and the Bramble–Hilbert lemma (see, e.g., [9, p. 192]). For $x \in \Omega \setminus \Omega_h$, $v(x)$ can be bounded by means of the mean-value theorem,

$$\|v - I_h(v)\|_{L^2(\Omega\setminus\Omega_h)^d} = \|v\|_{L^2(\Omega\setminus\Omega_h)^d} \le \delta(h)\|\nabla v\|_{L^2(\Omega)^d}.$$

We observe that using isoparametric elements $\delta(h) \le Ch^r$, and so in (2.6) the right-hand side is further bounded by $Ch^r\|v\|_{W^{r,q}(\Omega)^d}$ (see [9, section 4.4]). As regards the global estimate for the gradient, isoparametric modification is not enough to preserve the optimal approximability properties of the finite-element space. Following [3], we shall assume in what follows the use of superparametric elements at the boundary. By this type of approximation we mean that $\delta(h) \le Ch^{2r-2}$ so that the outside effects will not pollute the optimal estimate. Under these assumptions [3], [4], the interpolant I_h satisfies

$$\|v - I_h(v)\|_{L^2(\Omega)^d} + h\|v - I_h(v)\|_{H^1(\Omega)^d} \leq Ch^r\|v\|_{H^r(\Omega)^d}. \tag{2.7}$$

Notice then that the condition $\delta(h) \leq Ch^{2r-2}$ allows us to forget about the discrepancies between Ω and Ω_h in most of the arguments that follow. Observe, however, that one must then assume that Ω is piecewise of class $\mathcal{C}^{2r-2}$.

For each fixed time $t \in [0, T]$ the solution (u, p) of (1.1)–(1.2) is also the solution of a Stokes problem with right-hand side $f - u_t - (u \cdot \nabla)u$. We will denote by $(s_h, q_h) \in (X_{h,r}, Q_{h,r-1})$, its MFE approximation satisfying

$$\begin{aligned} \nu(\nabla s_h, \nabla \phi_h) - (q_h, \nabla \cdot \phi_h) &= \nu(\nabla u, \nabla \phi_h) - (p, \nabla \cdot \phi_h) \\ &= (f - u_t - (u \cdot \nabla u), \phi_h) \quad \forall \phi_h \in X_{h,r}, \\ (\nabla \cdot s_h, \psi_h) &= 0 \quad \forall \psi_h \in Q_{h,r-1}. \end{aligned} \tag{2.8}$$

We observe that $s_h = S_h(u) : V \longrightarrow V_{h,r}$ is the so-called discrete Stokes projection of the solution (u, p) of (1.1)–(1.2) (see [24]) and satisfies

$$(\nabla S_h(u), \nabla \chi_h) = (\nabla u, \nabla \chi_h) - (p, \nabla \cdot \chi_h) = (f - u_t - (u \cdot \nabla)u, \chi_h) \quad \forall\, \chi_h \in V_{h,r}.$$

The following bound holds for $2 \leq l \leq r$:

$$\|u - s_h\|_0 + h\|u - s_h\|_1 \leq Ch^l \left(\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}\right). \tag{2.9}$$

The proof of (2.9) for $\Omega = \Omega_h$ can be found in [25]. For the general case superparametric approximation at the boundary is assumed; see [3], [4]. Under the same conditions, the bound for the pressure is [21]

$$\|p - q_h\|_{L^2/\mathbb{R}} \leq C_\beta h^{l-1} \left(\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}\right), \tag{2.10}$$

where the constant C_β depends on the constant β in the inf-sup condition (2.4).

Since we are assuming that Ω is of class $\mathcal{C}^m$ with $m \geq 3$ (and that $\delta(h) \leq Ch^{2r-2}$) using standard duality arguments and (2.9), one obtains [3], [4]

$$\|u - s_h\|_{-s} \leq Ch^{r+s}(\|u\|_r + \|p\|_{H^{r-1}/\mathbb{R}}), \qquad 0 \leq s \leq \min(r-2, 1). \tag{2.11}$$

Let $\Pi_{h,r} : L^2(\Omega)^d \longrightarrow V_{h,r}$ be the discrete Leray's projection defined by demanding that $(\Pi_{h,r}(u), \chi_h) = (u, \chi_h)\ \forall \chi_h \in V_{h,r}$. By definition, the projection is stable in the L^2 norm. For divergence-free functions, by writing $\Pi_{h,r}u = (\Pi_{h,r}u - S_h(u)) + S_h(u)$ and using the quasi-uniformity of the meshes, one easily shows that

$$\|\Pi_{h,r}u\|_1 \leq C\|u\|_1 \qquad \forall\, u \in V. \tag{2.12}$$

We will denote by $\mathcal{A}_h$ the discrete Stokes operator defined by

$$(\nabla v_h, \nabla \phi_h) = (\mathcal{A}_h v_h, \phi_h) = \left(\mathcal{A}_h^{1/2} v_h, \mathcal{A}_h^{1/2} \phi_h\right) \qquad \forall v_h, \phi_h \in V_{h,r}.$$

Since $\mathcal{A}_h$ is a discrete self-adjoint operator, it is easy to show that, for each $0 \leq \alpha < 1$, there exists a positive constant C_α, which is independent of h, such that

$$\|\mathcal{A}_h^\alpha e^{-t\mathcal{A}_h}\|_0 \leq C_\alpha t^{-\alpha} \qquad \forall\ 0 \leq \alpha < 1. \tag{2.13}$$

In our analysis we shall frequently use the following relations for $f \in L^2(\Omega)^d$:

$$\|\mathcal{A}_h^{-s/2}\Pi_{h,r} f\|_0 \le Ch^s\|f\|_0 + \|\mathcal{A}^{-s/2}\Pi f\|_0, \quad s = 1, 2, \tag{2.14}$$

$$\|\mathcal{A}^{-s/2}\Pi f\|_0 \le Ch^s\|f\|_0 + \|\mathcal{A}_h^{-s/2}\Pi_{h,r} f\|_0, \quad s = 1, 2. \tag{2.15}$$

These inequalities are readily deduced from the estimates $\|\mathcal{A}^{-s/2} - \mathcal{A}_h^{-s/2}\Pi_{h,r}\|_0 \le Ch^s$ for $s = 1, 2$ [29]. Similarly, since $\forall v_h \in V_{h,r}$, $(\mathcal{A}_h^{-1/2}\Pi_{h,r} f, v_h) = (f, \mathcal{A}_h^{-1/2} v_h)$, it follows that

$$\|\mathcal{A}_h^{-1/2}\Pi_{h,r} f\|_0 \le C\|f\|_{-1}, \tag{2.16}$$

and since $\forall v \in V$, we have $(\mathcal{A}^{-1/2}\Pi(\Pi_{h,r} f), v) = (\Pi_{h,r} f, \mathcal{A}^{-1/2} v) = (f, \Pi_{h,r}\mathcal{A}^{-1/2} v)$, from (2.12) it follows that

$$\|\mathcal{A}^{-1/2}\Pi(\Pi_{h,r} f)\|_0 \le C\|f\|_{-1}, \qquad f \in L^2(\Omega)^2. \tag{2.17}$$

2.1. The suggested method. Let us suppose that we want to approximate the solution of (1.1)–(1.2) at time T. For $d = 3$, the final time T is assumed to satisfy $0 < T < T^*$, where T^* is the critical time until which the existence and uniqueness of a strong solution of (1.1)–(1.2) has been proven. The postprocessing technique can be seen as a two-level method. We first compute the MFE approximation to (1.1)–(1.2) at time T. Given $u^h(0)$ an initial approximation to $u(0)$, we find that $u_h : [0, T] \longrightarrow X_{h,r}$ and $p_h : [0, T] \longrightarrow Q_{h,r-1}$ satisfy

$$(\dot{u}_h, \phi_h) + \nu(\nabla u_h, \nabla \phi_h) + b_h(u_h, u_h, \phi_h) + (\nabla p_h, \phi_h) = (f, \phi_h) \quad \forall\, \phi_h \in X_{h,r}, \tag{2.18}$$

$$(\nabla \cdot u_h, \psi_h) = 0 \quad \forall\, \psi_h \in Q_{h,r-1}, \tag{2.19}$$

where $b_h(\cdot,\cdot,\cdot)$ is a suitable discrete approximation to its continuous counterpart. As an initial condition we will take $u_h(0) = S_h(u_0)$, although other choices are possible.

In the second step, the discrete velocity and pressure $(u_h(T), p_h(T))$ are postprocessed. Basically, we enhance this approximation by solving a single discrete Stokes problem, via MFE. The MFE in this step, denoted by $(\widetilde{X}, \widetilde{Q})$, is either

- the same-order Hood–Taylor element over a finer grid $(\widetilde{X}, \widetilde{Q}) = (X_{\tilde{h},r}, Q_{\tilde{h},r-1})$, $r \ge 3$, $\tilde{h} < h$, or
- a higher-order Hood–Taylor element over the same grid $(\widetilde{X}, \widetilde{Q}) = (X_{\tilde{h},r+1}, Q_{\tilde{h},r})$, $r \ge 3$, $\tilde{h} = h$.

That is, we shall search for $(\tilde{u}_h, \tilde{p}_h) \in (\widetilde{X}, \widetilde{Q})$ satisfying

$$\nu\big(\nabla \tilde{u}_{\tilde{h}}, \nabla \tilde{\phi}\big) + \big(\nabla \tilde{p}_{\tilde{h}}, \tilde{\phi}\big) = (f, \tilde{\phi}) - b_{\tilde{h}}(u_h(T), u_h(T), \tilde{\phi}) - (\dot{u}_h(T), \tilde{\phi}) \quad \forall\, \tilde{\phi} \in \widetilde{X}, \tag{2.20}$$

$$\big(\nabla \cdot \tilde{u}_{\tilde{h}}, \tilde{\psi}\big) = 0 \quad \forall\, \tilde{\psi} \in \widetilde{Q}. \tag{2.21}$$

We will denote by $\widetilde{V}$ the corresponding discretely divergence-free space that can be either $\widetilde{V} = V_{\tilde{h},r}$ or $\widetilde{V} = V_{h,r+1}$ depending on the selection of the postprocessed space. The discrete Leray's projection into $\widetilde{V}$ will be denoted by $\widetilde{\Pi}_{\tilde{h}}$, and we will represent by $\widetilde{\mathcal{A}}_{\tilde{h}}$ the discrete Stokes operator acting on functions in $\widetilde{V}$.

The postprocessed Hood–Taylor approximation to the velocity, $\tilde{u}_{\tilde{h}}$, is the solution of the pressure-free formulation

$$\nu\big(\nabla \tilde{u}_{\tilde{h}}, \nabla \tilde{\chi}_h\big) = \big(f, \tilde{\chi}_h\big) - b_{\tilde{h}}\big(u_h(T), u_h(T), \tilde{\chi}_h\big) - \big(\dot{u}_h(T), \tilde{\chi}_h\big) \quad \forall\, \tilde{\chi}_h \in \widetilde{V}. \tag{2.22}$$

In the next section, we show that the solution $(\tilde{u}_h, \tilde{p}_h)$ of (2.20)–(2.21) is a more accurate approximation to the solution of (1.1)–(1.2) than the Galerkin MFE approximation (u_h, p_h) that solves (2.18)–(2.19).

For the discrete approximation to the nonlinear term, following [24], we define b_h in the following way:

$$b_h(u_h, v_h, \phi_h) = ((u_h \cdot \nabla)v_h, \phi_h) + \frac{1}{2}(\operatorname{div}(u_h)v_h, \phi_h) \quad \forall\, u_h, v_h, \phi_h \in X_{h,r} \subset H_0^1(\Omega)^d.$$

For all $u, v \in H_0^1(\Omega)^d$, the corresponding continuous operator will be denoted by $F(u, v) = (u \cdot \nabla)v + (1/2)\operatorname{div}(u)v$. Extending the definition of b_h to functions in $H_0^1(\Omega)^d$ (not necessarily in $X_{h,r}$), we observe that $\forall\, u, v, w \in H_0^1(\Omega)^d$, $b_h(u, v, w) = (F(u, v), w)$. It is straightforward to verify that b_h enjoys the skew-symmetry property

$$b_h(u, v, w) = -b_h(u, w, v) \qquad \forall\, u, v, w \in H_0^1(\Omega)^d. \tag{2.23}$$

Let us observe that $B(u, v) = \Pi F(u, v)$ if $u \in V$. Finally, we shall denote by

$$B_h(u, v) = \Pi_{h,r} F(u, v) \qquad \forall\, u,\, v \,\in\, H_0^1(\Omega)^d.$$

3. Analysis of the postprocessed method. This section is devoted to the analysis of convergence of the postprocessed MFE method. Our first aim will be to show a superconvergence result for the error between the MFE approximation to the velocity u_h and the Stokes projection of the velocity field u, s_h. This superconvergence behavior occurs for both the L^2 and H^1 norms, as will be shown in Theorem 3.7 and Corollary 3.8, respectively. In the first part of the section, we shall concentrate our efforts in Theorem 3.7. It will be achieved by a stability plus consistency argument (Propositions 3.2 and 3.6, respectively). For the purpose of analysis, we shall mainly be concerned with the pressure-free formulation associated with (2.18)–(2.19). If (u_h, p_h) is the MFE approximation to the solution (u, p) of (1.1)–(1.2), then $u_h \in V_{h,r}$ is the solution of

$$(\dot{u}_h, \chi_h) + \nu(\nabla u_h, \nabla \chi_h) + b_h(u_h, u_h, \chi_h) = (f, \chi_h) \qquad \forall \chi_h \in V_{h,r}, \tag{3.1}$$

which can also be expressed in abstract operator form as

$$\dot{u}_h + \nu \mathcal{A}_h u_h + B_h(u_h, u_h) = \Pi_{h,r} f. \tag{3.2}$$

The Stokes projection s_h satisfies the abstract equation

$$\dot{s}_h + \nu \mathcal{A}_h s_h + B_h(s_h, s_h) = \Pi_{h,r} f + T_h, \tag{3.3}$$

where $T_h(t)$ is the truncation error, defined as

$$T_h(t) = \dot{s_h} - \Pi_{h,r}(u_t) + B_h(s_h, s_h) - B_h(u, u). \tag{3.4}$$

Let us now consider mappings $v_h : [0, T] \longrightarrow V_{h,r}$ satisfying the following threshold condition:

$$\|s_h(t) - v_h(t)\|_0 \le c_\tau h^2 \qquad \forall\, t \in [0, t_1], \quad 0 < t_1 \le T. \tag{3.5}$$

We define their truncation error as

$$\widehat{T}_h = \dot{v}_h + \nu \mathcal{A}_h v_h + B_h(v_h, v_h) - \Pi_{h,r} f. \tag{3.6}$$

Prior to establishing the stability restricted to the threshold (3.5) (Proposition 3.2), we prove a lemma which provides some estimates for the convective term.

LEMMA 3.1. *Let (u, p) be the solution of the Navier–Stokes problem* (1.1)–(1.2). *Let $s_h = S_h(u)$ be the discrete Stokes projection of the velocity field u and let v_h : $[0, T] \longrightarrow V_{h,r}$ satisfy the threshold condition* (3.5)*. Then, there exists a constant $K > 0$, independent of t_1 in* (3.5)*, such that $\forall\, t \in [0, t_1]$,*

$$\|F(s_h(t), s_h(t)) - F(v_h(t), v_h(t))\|_0 \leq K\|s_h(t) - v_h(t)\|_1, \tag{3.7}$$

$$\|F(s_h(t), s_h(t)) - F(v_h(t), v_h(t))]\|_{-1} \leq K\|s_h(t) - v_h(t)\|_0, \tag{3.8}$$

where the constant $K = K\big(c_\tau, \max_{0\leq t\leq T}(\|u(t)\|_2 + \|p(t)\|_{H^1/\mathbb{R}})\big)$.

Proof. In order to simplify the notation, we shall omit the dependence on t in the proof. Denote by $e_h = v_h - s_h$. We proceed by standard duality arguments, using the splitting

$$F(v_h, v_h) - F(s_h, s_h) = F(v_h, e_h) + F(e_h, s_h). \tag{3.9}$$

We start by showing (3.7). We first observe that

$$\begin{aligned}\|F(e_h, s_h)\|_0 &= \sup_{\|\phi\|_0=1} \left|(e_h \cdot \nabla s_h, \phi) + \frac{1}{2}((\nabla \cdot e_h)s_h, \phi)\right| \\ &\leq C\|e_h\|_{L^{2d/(d-1)}(\Omega)^d}\|\nabla s_h\|_{L^{2d}(\Omega)^d} + C\|e_h\|_1\|s_h\|_\infty.\end{aligned}$$

Let us show that both $\|s_h\|_\infty$, $\|\nabla s_h\|_{L^{2d}(\Omega)^d}$ are bounded. Since, by virtue of Sobolev's imbeddings (2.1), we have $\|s_h\|_\infty \leq C\|\nabla s_h\|_{L^{2d}(\Omega)^d}$, we only need to bound the second term. Application of the inverse inequality (2.3) and the error estimates (2.9) and (2.7) together with (2.1) give

$$\begin{aligned}\|\nabla s_h\|_{L^{2d}(\Omega)^d} &\leq Ch^{\frac{-(1+d)}{2}}(\|s_h - u\|_0 + \|u - I_h u\|_0) + \|\nabla I_h u\|_{L^{2d}(\Omega)^d} \\ &\leq Ch^{(3-d)/2}(\|u\|_2 + \|p\|_{H^1/\mathbb{R}}) + C\|u\|_{W^{1,2d}(\Omega)^d} \leq K.\end{aligned} \tag{3.10}$$

Using again (2.1) we obtain

$$\|e_h\|_{L^{2d/(d-1)}(\Omega)^d} \leq C\|e_h\|_{1/2} \leq C\|e_h\|_1,$$

and so $\|F(e_h, s_h)\|_0 \leq K\|e_h\|_1$. As regards the other term in (3.9), the same arguments lead to

$$\begin{aligned}\|F(v_h, e_h)\|_0 &= \sup_{\|\phi\|_0=1} \left|(v_h \cdot \nabla e_h, \phi) + \frac{1}{2}((\nabla \cdot v_h)e_h, \phi)\right| \\ &\leq C\|v_h\|_\infty\|e_h\|_1 + C\|\nabla v_h\|_{L^{2d}(\Omega)^d}\|e_h\|_{L^{2d/(d-1)}(\Omega)^d}.\end{aligned}$$

As before, to conclude we must show that the above norms of v_h are bounded. We only need to handle $\|\nabla v_h\|_{L^{2d}(\Omega)^d}$. Using the inverse inequality (2.3) and the threshold conditions (3.5) and (3.10), we find

$$\|\nabla v_h\|_{L^{2d}(\Omega)^d} \leq h^{\frac{-(1+d)}{2}}\|v_h - s_h\|_0 + \|\nabla s_h\|_{L^{2d}(\Omega)^d} \leq c_\tau h^{(3-d)/2} + K \leq K.$$

Therefore, (3.7) follows. We now show (3.8). Applying (3.9), we find

$$\|F(v_h, v_h) - F(s_h, s_h)\|_{-1} \leq \|F(v_h, e_h)\|_{-1} + \|F(e_h, s_h)\|_{-1}, \tag{3.11}$$

so that the proof is reduced to estimate each of the above negative norms on the right-hand side. Using the skew-symmetry property (2.23), one gets for the first term:

$$\begin{aligned}\|F(v_h, e_h)\|_{-1} &= \sup_{\|\phi\|_1=1} \left| -((v_h \cdot \nabla)\phi, e_h) - \frac{1}{2}((\nabla \cdot v_h)\phi, e_h) \right| \\ &\le \sup_{\|\phi\|_1=1} \left(\|e_h\|_0 \|v_h\|_\infty \|\phi\|_1 + \|e_h\|_0 \|\nabla \cdot v_h\|_{L^{2d/(d-1)}} \|\phi\|_{L^{2d}(\Omega)^d} \right) \le K\|e_h\|_0.\end{aligned}$$

Regarding the other term in (3.11), integrating by parts, we obtain

$$\begin{aligned}\|F(e_h, s_h)\|_{-1} &= \sup_{\|\phi\|_1=1} \left| \frac{1}{2}\big((e_h \cdot \nabla)s_h, \phi\big) - \frac{1}{2}\big((e_h \cdot \nabla)\phi, s_h\big) \right| \\ &\le \sup_{\|\phi\|_1=1} \left(\|e_h\|_0 \|\nabla s_h\|_{L^{2d/(d-1)}(\Omega)^d} \|\phi\|_{L^{2d}(\Omega)^d} + \|e_h\|_0 \|\phi\|_1 \|s_h\|_\infty \right) \le K\|e_h\|_0.\end{aligned}$$

This finishes the proof of (3.8). □

PROPOSITION 3.2 (stability). *Let $T > 0$ be fixed; let $s_h = S_h(u)$ be the discrete Stokes projection of the velocity field u solution of* (1.1)–(1.2) *and let $v_h : [0,T] \longrightarrow V_{h,r}$ satisfy the threshold condition* (3.5). *Then, there exists a positive constant $K_s > 0$ such that $\forall\, t_1 \le T$, the following estimate holds:*

$$\max_{0\le t\le t_1} \|s_h(t) - v_h(t)\|_0 \le e^{K_s t_1} \left(\|s_h(0) - v_h(0)\|_0 + \max_{0\le t\le t_1} \left\| \int_0^t e^{-\nu(t-s)\mathcal{A}_h} [T_h(s) - \widehat{T}_h(s)] ds \right\|_0 \right), \tag{3.12}$$

where $T_h(s)$ and $\widehat{T}_h(s)$ are the truncation errors given in (3.4) *and* (3.6), *respectively.*

Proof. We denote by $e_h = s_h - v_h$. Subtracting (3.6) from (3.3), it follows that e_h satisfies the error equation

$$\dot{e}_h(t) + \nu \mathcal{A}_h e_h(t) = B_h(v_h(t), v_h(t)) - B_h(s_h(t), s_h(t)) + T_h(t) - \widehat{T}_h(t).$$

Then, by integrating the above error equation from time 0 up to time t, we find that

$$\begin{aligned}e_h(t) = e^{-\nu t \mathcal{A}_h} \Pi_{h,r} e_h(0) &+ \int_0^t e^{-\nu(t-s)\mathcal{A}_h} \Pi_{h,r} [B_h(v_h, v_h) - B_h(s_h, s_h)] ds \\ &+ \int_0^t e^{-\nu(t-s)\mathcal{A}_h} \Pi_{h,r} [T_h(s) - \widehat{T}_h(s)] ds.\end{aligned}$$

Since $\{e^{-\nu t \mathcal{A}_h} \Pi_{h,r}\}_{t>0}$ is a contraction $\|e^{-\nu t\mathcal{A}_h} \Pi_{h,r} e_h(0)\|_0 \le \|e_h(0)\|_0$. As regards the second term, estimates (2.13), (2.16), and (3.8) from Lemma 3.1 lead to

$$\begin{aligned}&\left\| \int_0^t e^{-\nu(t-s)\mathcal{A}_h} [B_h(s_h, s_h) - B_h(v_h, v_h)] ds \right\|_0 \\ &\quad \le \frac{C_{1/2}}{\sqrt{\nu}} \int_0^t \frac{\left\| \mathcal{A}_h^{-1/2} \big(\Pi_{h,r} F(s_h, s_h) - \Pi_{h,r} F(v_h, v_h) \big) \right\|_0}{\sqrt{t-s}} ds \le \frac{K C_{1/2}}{\sqrt{\nu}} \int_0^t \frac{\|e_h(s)\|_0}{\sqrt{t-s}} ds.\end{aligned}$$

Then,

$$\|e_h(t)\|_0 \le \|e_h(0)\|_0 + \frac{K C_{1/2}}{\sqrt{\nu}} \int_0^t \frac{\|e_h(s)\|_0}{\sqrt{t-s}} ds + \int_0^t e^{-\nu(t-s)\mathcal{A}_h} [T_h(s) - \widehat{T}_h(s)] ds.$$

And now, a standard application of the generalized Gronwall lemma (see [22, pp. 188–189]) allows us to conclude the proof. ∎

Proposition 3.2 is an example of stability restricted to h-dependent thresholds. This kind of stability is an alternative to establishing the a priori bounds for the approximate solution u_h required in order to handle the nonlinear term [28].

The following lemmas will be required in the proof of Proposition 3.6.

LEMMA 3.3. *For any $f \in C([0,T];L^2(\Omega)^d)$, the following estimate holds $\forall\, t \in [0,T]$:*

$$\int_0^t \left\|\mathcal{A}_h e^{-\nu(t-s)\mathcal{A}_h}\Pi_{h,r} f(s)\right\|_0 ds \le \frac{C}{\nu}|\log(h)| \max_{0\le t\le T}\|f(t)\|_0.$$

Proof. The proof follows essentially the same steps as in [14] and [20]. ∎

LEMMA 3.4. *Let $v \in (H^2(\Omega))^d \cap V$. Then, there exists a constant $K = K(\|v\|_2)$ such that $\forall\, w \in H_0^1(\Omega)^d$, we have that*

$$\|\mathcal{A}^{-1}\Pi[F(v,v) - F(w,w)]\|_0 \le K\big(\|v-w\|_{-1} + \|v-w\|_1\|v-w\|_0\big). \tag{3.13}$$

Proof. Throughout the proof, we shall designate $e = v - w$. We rewrite the difference of the nonlinear terms as

$$\mathcal{A}^{-1}\Pi\,(F(v,v) - F(w,w)) = \mathcal{A}^{-1}\Pi F(v,e) + \mathcal{A}^{-1}\Pi F(e,v) - \mathcal{A}^{-1}\Pi F(e,e). \tag{3.14}$$

Let us first estimate the last term in (3.14). Using (2.23) and (2.1), we have

$$\begin{aligned}
\|\mathcal{A}^{-1}\Pi F(e,e)\|_0 &\le \sup_{\|\phi\|_0=1}\left| -\big((e\cdot\nabla)\mathcal{A}^{-1}\Pi\phi, e\big) - \frac{1}{2}\big((\nabla\cdot e)(\mathcal{A}^{-1}\Pi\phi), e\big)\right| \\
&\le \sup_{\|\phi\|_0=1}\Big(\|e\|_{L^{2d/(d-1)}(\Omega)^d}\|\nabla\mathcal{A}^{-1}\Pi\phi\|_{L^{2d}(\Omega)^d} + \|e\|_1\|\mathcal{A}^{-1}\Pi\phi\|_\infty\Big)\|e\|_0 \\
&\le \sup_{\|\phi\|_0=1}\Big(C\|e\|_{1/2}\|\mathcal{A}^{-1}\Pi\phi\|_2 + C\|e\|_1\|\mathcal{A}^{-1}\Pi\phi\|_2\Big)\|e\|_0 \le C\|e\|_1\|e\|_0.
\end{aligned}$$

For the first term in the splitting (3.14), taking into account that $\mathrm{div}(v) = 0$, we find

$$\|\mathcal{A}^{-1}\Pi F(v,e)\|_0 = \sup_{\|\phi\|_0=1}\left|\big((v\cdot\nabla)\mathcal{A}^{-1}\Pi\phi, e\big)\right| \le \|e\|_{-1}\sup_{\|\phi\|_0=1}\|\nabla((v\cdot\nabla)\mathcal{A}^{-1}\Pi\phi)\|_0.$$

Therefore, we must show that the last supremum above is bounded. Using again Sobolev's imbeddings (2.1) for $\phi \in L^2(\Omega)$ with $\|\phi\|_0 = 1$, we obtain

$$\begin{aligned}
\|\nabla\big((v\cdot\nabla)\mathcal{A}^{-1}\Pi\phi\big)\|_0^2 &\le \sum_{k=1}^d \|\big(\partial_k v\cdot\nabla\big)(\mathcal{A}^{-1}\Pi\phi) + (v\cdot\nabla)\big(\partial_k(\mathcal{A}^{-1}\Pi\phi)\big)\|_0^2 \\
&\le \sum_{k=1}^d \|\partial_k v\|^2_{L^{2d/(d-1)}(\Omega)^d}\|\nabla\mathcal{A}^{-1}\Pi\phi\|^2_{L^{2d}(\Omega)^d} + \|v\|_\infty^2\|\partial_k(\nabla\mathcal{A}^{-1}\Pi\phi)\|_0^2 \\
&\le C\big[\|\nabla v\|_{1/2}^2\|\mathcal{A}^{-1}\Pi\phi\|_2^2 + \|v\|_2^2\|\mathcal{A}^{-1}\Pi\phi\|_2^2\big] \le C\|v\|_2^2,
\end{aligned}$$

so that $\|\mathcal{A}^{-1}\Pi F(v,e)\|_0 \le K\|e\|_{-1}$. Finally, we deal with the second term in (3.14). Integrating by parts, we get

$$\|\mathcal{A}^{-1}\Pi F(e,v)\|_0 = \sup_{\|\phi\|_0=1}\left|\frac{1}{2}((e\cdot\nabla)v, \mathcal{A}^{-1}\Pi\phi) - \frac{1}{2}((e\cdot\nabla)\mathcal{A}^{-1}\Pi\phi, v)\right|.$$

We shall estimate each supremum in the above equation separately. For the first term, we have

$$\begin{aligned}\big|(e\cdot\nabla)v,\mathcal{A}^{-1}\Pi\phi)\big| &\le \|e\|_{-1}\big\|\big((\mathcal{A}^{-1}\Pi\phi)\cdot\nabla\big)v\|_1\\ &\le \|e\|_{-1}\Big(\|\nabla v\|_1\|\mathcal{A}^{-1}\Pi\phi\|_\infty+\|\nabla v\|_{L^{2d/(d-1)}(\Omega)^d}\|\nabla\mathcal{A}^{-1}\Pi\phi\|_{L^{2d}(\Omega)^d}\Big)\\ &\le C\|e\|_{-1}\Big(\|v\|_2\|\mathcal{A}^{-1}\Pi\phi\|_2+\|v\|_2\|\mathcal{A}^{-1}\Pi\phi\|_2\Big)\le C\|v\|_2\|e\|_{-1}\le K\|e\|_{-1}.\end{aligned}$$

As regards the other supremum, we note that

$$\begin{aligned}\big|(e\cdot\nabla)\mathcal{A}^{-1}\Pi\phi,v)\big| &\le \sum_{k=1}^d\|e^k\|_{-1}\Big(\|\nabla(\partial_k\mathcal{A}^{-1}\Pi\phi)\cdot v\|_0+\|\partial_k(\mathcal{A}^{-1}\Pi\phi)\cdot(\nabla v)\|_0\Big)\\ &\le \|e\|_{-1}\Big(\|\mathcal{A}^{-1}\Pi\phi\|_2\|v\|_\infty+\|\partial_k(\mathcal{A}^{-1}\Pi\phi)\|_{L^{2d}(\Omega)^d}\|\nabla v\|_{L^{2d/(d-1)}(\Omega)^d}\Big)\\ &\le \|e\|_{-1}\Big(\|\mathcal{A}^{-1}\Pi\phi\|_2\|v\|_2+\|\mathcal{A}^{-1}\Pi\phi)\|_2\|v\|_2\Big)\le C\|v\|_2\|e\|_{-1}\le K\|e\|_{-1},\end{aligned}$$

which concludes the proof. ◻

LEMMA 3.5. *Let* (u,p) *be the solution of* (1.1)–(1.2). *Then, there exists a positive constant* $K=K(u,p)$ *such that,* $\forall\, t\in[0,T]$, *the truncation error defined in* (3.4) *satisfies the following bound:*

$$\|\mathcal{A}_h^{-1}T_h(t)\|_0\le Kh^{r+1}. \tag{3.15}$$

Proof. In view of definition (3.4), we observe that

$$\|\mathcal{A}_h^{-1}T_h(t)\|_0\le\|\mathcal{A}_h^{-1}\Pi_{h,r}(\dot s_h-u_t)\|_0+\|\mathcal{A}_h^{-1}\Pi_{h,r}(F(s_h,s_h)-F(u,u))\|_0.$$

We will use (2.14) with $s=2$ to bound both terms on the right-hand side. For the first, we obtain

$$\begin{aligned}\|\mathcal{A}_h^{-1}\Pi_{h,r}(\dot s_h-u_t)\|_0 &\le Ch^2\|\dot s_h-u_t\|_0+\|\mathcal{A}^{-1}\Pi(\dot s_h-u_t)\|_0\\ &\le Ch^2\|\dot s_h-u_t\|_0+\|\dot s_h-u_t\|_{-2}\le Ch^{r+1}(\|u_t\|_r+\|p_t\|_{H^{r-1}/\mathbb{R}}),\end{aligned}$$

where in the last inequality we have used that $\|\cdot\|_{-2}\le\|\cdot\|_{-1}$ and applied (2.11). As regards the second term, applying (3.7) from Lemma 3.1 and (3.13) from Lemma 3.4, we get

$$\begin{aligned}&\|\mathcal{A}_h^{-1}\Pi_{h,r}\left(F(s_h,s_h)-F(u,u)\right)\|_0\\ &\quad\le Ch^2\|F(s_h,s_h)-F(u,u)\|_0+\|\mathcal{A}^{-1}\Pi(F(s_h,s_h)-F(u,u))\|_0\\ &\quad\le Kh^2\|s_h-u\|_1+K(\|s_h-u\|_{-1}+\|s_h-u\|_1\|s_h-u\|_0).\end{aligned}$$

We observe that although Lemma 3.1 has been stated for functions $v_h\in V_{h,r}$ satisfying (3.5) can equally be applied for $v_h=u$. To conclude, we apply estimates (2.9) and (2.11) to get

$$\|\mathcal{A}_h^{-1}\Pi_{h,r}\left(F(s_h,s_h)-F(u,u)\right)\|_0\le Kh^{r+1}\left(\|u\|_r+\|p\|_{H^{r-1}/\mathbb{R}}\right).$$ ◻

PROPOSITION 3.6 (consistency). *Let* (u, p) *be the solution of* (1.1)–(1.2). *Then, there exists a positive constant* $K = K(u, p, \nu)$ *such that*

$$\max_{0\le t\le T} \left\| \int_0^t e^{-\nu(t-s)\mathcal{A}_h} T_h(s)ds \right\|_0 \le K h^{r+1} |\log{(h)}|. \tag{3.16}$$

Proof. Let us start by noticing that

$$\left\| \int_0^t e^{-\nu(t-s)\mathcal{A}_h} T_h(s)ds \right\|_0 \le \int_0^t \left\| \mathcal{A}_h e^{-\nu(t-s)\mathcal{A}_h} \mathcal{A}_h^{-1} T_h(s) \right\|_0 ds.$$

By virtue of Lemma 3.3, the last integral reduces to

$$\int_0^t \left\| \mathcal{A}_h e^{-\nu(t-s)\mathcal{A}_h} \Pi_{h,r} \mathcal{A}_h^{-1} T_h(s) \right\|_0 ds \le \frac{C}{\nu} |\log(h)| \max_{0\le t\le T} \left\| \mathcal{A}_h^{-1} T_h(t) \right\|_0,$$

and then, since Lemma 3.5 provides the required estimate for the truncation error, we reach (3.16). ∎

THEOREM 3.7 (superconvergence for the velocity). *Let* (u, p) *be the solution of* (1.1)–(1.2), *let* s_h *be the Stokes projection of* u, *and let* u_h *be the Hood–Taylor element approximation to* u. *Then, there exist positive constants* $K(u, p, \nu)$ *and* h_0 *such that, for every* $h \in (0, h_0]$,

$$\max_{0\le t\le T} \|s_h(t) - u_h(t)\|_0 \le K(u, p, \nu) h^{r+1} |\log{(h)}|. \tag{3.17}$$

Proof. Since $u_h(0) = s_h(0)$, the proof follows from Proposition 3.2 (applied to $v_h = u_h$) and Proposition 3.6. The threshold condition (3.5) needed for Proposition 3.2 to be valid is easily proved by a standard bootstrap argument (see, e.g., [20] and [3]). ∎

Next, we derive the superconvergence result for the error between the MFE approximation u_h to the velocity and the Stokes projection s_h in the H^1 norm.

COROLLARY 3.8. *Let* (u, p) *be the solution of the Navier–Stokes problem* (1.1)–(1.2), *let* s_h *be the discrete Stokes projection of* u, *and let* u_h *be the Hood–Taylor element approximation to* u. *Then, there exist positive constants* $K(u, p, \nu)$ *and* h_0 *such that, for every* $h \in (0, h_0]$, *the following bound holds:*

$$\max_{0\le t\le T} \|s_h(t) - u_h(t)\|_1 \le K(u, p, \nu) h^r |\log(h)|. \tag{3.18}$$

Proof. The result follows from Theorem 3.7 and the inverse inequality (2.3). ∎

As a consequence of Theorem 3.7 and Corollary 3.8, the optimal rate of convergence for u_h is obtained.

COROLLARY 3.9. *Let* (u, p) *be the solution of the Navier–Stokes problem* (1.1)–(1.2), *and let the conditions of Theorem* 3.7 *be satisfied. Then, for* $s = 0, 1$,

$$\max_{0\le t\le T} \|u(t) - u_h(t)\|_s \le C h^{r-s} \Big(\max_{0\le t\le T} (\|u\|_r + \|p\|_{H^{r-1}/\mathbb{R}}) + K(u, p, \nu) h |\log(h)| \Big). \tag{3.19}$$

Proof. By rewriting $u - u_h = (u - s_h) + (s_h - u_h)$, and appealing to Theorem 3.7 and Corollary 3.8 together with estimate (2.9), we reach (3.19). ∎

The following lemma provides several estimations (in different norms) for the time derivative of the error in the MFE approximation to the velocity.

LEMMA 3.10. *Let (u,p) be the solution of* (1.1)–(1.2), *and let $u_h : [0,T] \to V_{h,r}$ be the Hood–Taylor approximation to the velocity. Then, the following estimates hold:*

$$\max_{0<t\le T} \left\| u_t(t) - \dot u_h(t) \right\|_0 \le K(u,p,\nu) h^{r-1} |\log(h)|, \tag{3.20}$$

$$\max_{0<t\le T} \left\| \mathcal{A}^{-1}\Pi(u_t(t) - \dot u_h(t)) \right\|_0 \le K(u,p,\nu) h^{r+1} |\log(h)|, \tag{3.21}$$

$$\max_{0<t\le T} \left\| u_t(t) - \dot u_h(t) \right\|_{-1} \le K(u,p,\nu) h^{r} |\log(h)|. \tag{3.22}$$

Proof. For simplicity, we shall drop the explicit dependence on the time t in the proof. We consider the splitting

$$u_t - \dot u_h = (u_t - \dot s_h) + (\dot s_h - \dot u_h).$$

Since the first term can be readily estimated in the different norms by means of (2.9) and (2.11), we will concentrate only on the second one in the rest of the proof. Let us denote $e_h = s_h - u_h$. The time derivative of e_h satisfies the equation

$$\dot e_h = -\nu \mathcal{A}_h e_h + B_h(u_h,u_h) - B_h(u,u) + \Pi_{h,r}(\dot s_h - u_t).$$

We shall start by proving (3.20). Applying the inverse inequality (2.3), the stability of $\Pi_{h,r}$ in the L^2 norm, (3.7) from Lemma 3.1 and (2.9), we get

$$\begin{aligned}
\|\dot e_h\|_0 &\le \nu \left\| \mathcal{A}_h^{1/2}\mathcal{A}_h^{1/2} e_h \right\|_0 + \|B_h(u_h,u_h) - B_h(u,u)\|_0 + \|\Pi_{h,r}(\dot s_h - u_t)\|_0 \\
&\le C\nu h^{-1} \left\| \mathcal{A}_h^{1/2} e_h \right\|_0 + K\|e_h\|_1 + Ch^r(\|u_t\|_r + \|p_t\|_{H^{r-1}/\mathbb{R}}) \\
&\le (C\nu h^{-1} + K)\|e_h\|_1 + O(h^r) \le (C\nu h^{-1} + K) K(u,p,\nu) h^r |\log(h)| + O(h^r)
\end{aligned}$$

after applying Corollary 3.8 in the last inequality, and so (3.20) is shown. Notice that Lemma 3.1 has been applied for $v_h = u_h$ and taking u instead of s_h. It is immediate to check that the proof of the lemma remains valid in this case.

We deal next with (3.21). We first observe that

$$\begin{aligned}
\|\mathcal{A}^{-1}\Pi \dot e_h\|_0 \le \nu \|\mathcal{A}^{-1}\Pi A_h e_h\|_0 + \|\mathcal{A}^{-1}\Pi(B_h(u_h,u_h) - B_h(u,u))\|_0 \\
+ \|\mathcal{A}^{-1}\Pi(\dot s_h - \Pi_{h,r} u_t)\|_0.
\end{aligned} \tag{3.23}$$

Let us now bound each term on the right-hand side of (3.23). For the first, taking into account the relation (2.15), and applying the inverse inequality (2.3) and Theorem 3.7, we obtain

$$\|\mathcal{A}^{-1}\Pi\mathcal{A}_h e_h\|_0 \le Ch^2 \|\mathcal{A}_h e_h\|_0 + \left\| \mathcal{A}_h^{-1}\mathcal{A}_h e_h \right\|_0 \le C\|e_h\|_0 \le K(u,p,\nu) h^{r+1} |\log(h)|.$$

As regards the second term, by writing $\mathcal{A}^{-1}\Pi\Pi_{h,r} = (\mathcal{A}^{-1}\Pi - \mathcal{A}_h^{-1}\Pi_{h,r})\Pi_{h,r} + (\mathcal{A}_h^{-1}\Pi_{h,r} - \mathcal{A}^{-1}\Pi) + \mathcal{A}^{-1}\Pi$, then (3.7) and Lemma 3.4 give

$$\begin{aligned}
\left\| \mathcal{A}^{-1}\Pi(B_h(u_h,u_h) - B_h(u,u)) \right\|_0 &= \left\| \mathcal{A}^{-1}\Pi[\Pi_{h,r}(F(u_h,u_h) - F(u,u))] \right\|_0 \\
&\le \mathcal{C}h^2 \left\| F(u_h,u_h) - F(u,u) \right\|_0 + K(\|u_h - u\|_{-1} + \|u_h - u\|_1 \|u_h - u\|_0) \\
&\le Kh^2 \|u_h - u\|_1 + K(\|e_h\|_0 + \|s_h - u\|_{-1} + \|u_h - u\|_1\|u_h - u\|_0),
\end{aligned}$$

so that applying Theorem 3.7, (2.11), and (3.19) the desired bound for this term is reached. Finally, for the last term on the right-hand side of (3.23), we use (2.17) and (2.11) to get

$$\begin{aligned}\|\mathcal{A}^{-1}\Pi(\dot{s}_h - \Pi_{h,r}u_t)\|_0 &\le \left\|\mathcal{A}^{-1/2}\Pi(\dot{s}_h - \Pi_{h,r}u_t)\right\|_0 \le C\|s_h - u_t\|_{-1} \\ &\le Ch^{r+1}(\|u_t\|_r + \|p_t\|_{H^{r-1}/\mathbb{R}}).\end{aligned}$$

To conclude, we now show (3.22). As we show in Lemma 3.11

$$\|\dot{e}_h\|_{-1} \le Ch\|\dot{e}_h\|_0 + C\left\|\mathcal{A}^{-1/2}\Pi\dot{e}_h\right\|_0.$$

We have already proved that $\|\dot{e}_h\|_0 \le Kh^{r-1}|\log(h)|$. Reasoning exactly as we did with $\|\mathcal{A}^{-1}\Pi\dot{e}_h\|_0$, we also get $\|A^{-1/2}\Pi\dot{e}_h\|_0 \le Kh^r|\log(h)|$, and then the proof is complete. ∎

Lemma 3.11. *There exists a positive constant independent of h such that*

$$\|f_h\|_{-1} \le Ch\|f_h\|_0 + C\left\|\mathcal{A}^{-1/2}\Pi f_h\right\|_0 \qquad \forall f \in V_{h,r}.$$

Proof. For $\phi \in H_0^1(\Omega)$ we have the (L^2-orthogonal) decomposition $\phi = \Pi\phi + (I-\Pi)\phi$, for which we have that $(I-\Pi)\phi = \nabla\chi$ for some $\chi \in H^2(\Omega)$ and, for some constant $C > 0$,

$$\|\Pi\phi\|_1 \le C\,\|\phi\|_1\,, \qquad \|\nabla\chi\|_1 \le C\,\|\phi\|_1 \tag{3.24}$$

(see, e.g., [10]). Thus, $(f_h, \phi) = (f_h, \Pi\phi) + (f_h, \nabla\chi)$. But, on the one hand, $(f_h, \Pi\phi) = (\Pi f_h, \Pi\phi) = (\mathcal{A}^{-1/2}\Pi f_h, \mathcal{A}^{1/2}\Pi\phi)$; on the other hand, since $f_h \in V_{h,r}$, we may write $(f_h, \nabla(\chi - I_h(\chi))$, where $I_h(\chi)$ is the standard interpolant of χ in $\hat{S}_{h,r-1}$. Now, standard interpolation bounds and (3.24) finish the proof. ∎

Theorem 3.12 (superconvergence for the pressure). *Let (u,p) be the solution of the Navier–Stokes equations* (1.1)–(1.2)*; let p_h be the Hood–Taylor approximation to the pressure p, and let q_h be the MFE approximation to p in the Stokes problem* (2.8)*. Then, there exist positive constants $K(u,p,\nu)$ and h_0 such that, for every $h \in (0, h_0]$,*

$$\max_{0\le t\le T} \|p_h(t) - q_h(t)\|_{L^2/\mathbb{R}} \le \frac{1}{\beta}K(u,p,\nu)h^r|\log(h)|, \tag{3.25}$$

where β is the constant in the inf-sup *condition* (2.4).

Proof. Subtracting (2.8) from (2.18), we obtain for the difference $p_h - q_h$

$$(p_h - q_h, \nabla\cdot\phi_h) = \nu(\nabla(u_h - s_h), \nabla\phi_h) + (F(u_h,u_h) - F(u,u), \phi_h) + (\dot{u}_h - u_t, \phi_h)$$

$\forall \phi_h \in X_{h,r}$. Using the inf-sup condition (2.4),

$$\beta\|p_h - q_h\|_{L^2/\mathbb{R}} \le \nu\|u_h - s_h\|_1 + \|F(u_h,u_h) - F(u,u)\|_{-1} + \|\dot{u}_h - u_t\|_{-1}.$$

Applying Corollary 3.8, (3.8) from Lemma 3.1, and (3.20) from Lemma 3.10, we get

$$\beta\|p_h - q_h\|_{L^2/\mathbb{R}} \le \nu K h^r|\log(h)| + \|u - u_h\|_0 + Kh^r|\log(h)|.$$

Finally, thanks to Corollary 3.9, (3.25) is reached. ∎

As a consequence of Theorem 3.12 and (2.10), and by writing $p - p_h = (p - q_h) + (q_h - p_h)$, we also obtain the optimal rate of convergence for of the pressure.

Corollary 3.13. *Let (u,p) be the solution of the Navier–Stokes equations* (1.1)–(1.2)*, and let the conditions of Theorem* 3.12 *be satisfied. Then,*

$$\max_{0\le t\le T} \|p(t) - p_h(t)\|_{L^2/\mathbb{R}} \le Ch^{r-1} \max_{0\le t\le T} \left(\|u\|_r + \|p\|_{H^{r-1}/\mathbb{R}}\right) + K(u,p,\nu)h^r. \tag{3.26}$$

Next, we state the rate of convergence of the postprocessed MFE approximation $(\tilde u_{\tilde h}, \tilde p_{\tilde h}) \in (\widetilde X, \widetilde Q)$ that solves (2.20)–(2.21).

THEOREM 3.14. *Let $T > 0$ be fixed. Let (u_h, p_h) be the MFE approximation to the solution (u, p) of (1.1)–(1.2), and let $(\tilde u_{\tilde h}, \tilde p_{\tilde h})$ be the postprocessed MFE approximation at time T. Then, there exist constants $K_1(u, p, \nu)$, $K_0(u, p, \nu)$ such that*

(i) *if the postprocessing element is $(\widetilde X, \widetilde Q) = (X_{\tilde h,r}, Q_{\tilde h,r-1})$, then*

$$\|u(T) - \tilde u_{\tilde h}\|_1 \le C(\tilde h)^{r-1}\big(\|u(T)\|_r + \|p(T)\|_{H^{r-1}/\mathbb{R}}\big) + K_1(u, p, \nu)h^r|\log(h)|, \tag{3.27}$$

$$\|u(T) - \tilde u_{\tilde h}\|_0 \le C(\tilde h)^{r}\big(\|u(T)\|_r + \|p(T)\|_{H^{r-1}/\mathbb{R}}\big) + K_0(u, p, \nu)h^{r+1}|\log(h)|; \tag{3.28}$$

(ii) *if at time T the solution $(u(T), p(T))$ belongs to $(H^{r+1}(\Omega)^d \cap V) \times H^r(\Omega)/\mathbb{R}$, and the postprocessing element is $(\widetilde X, \widetilde Q) = (X_{h,r+1}, Q_{h,r})$, then*

$$\|u(T) - \tilde u_{\tilde h}\|_1 \le Ch^{r}\big(\|u(T)\|_{r+1} + \|p(T)\|_{H^{r}/\mathbb{R}}\big) + K_1(u, p, \nu)h^r|\log(h)|, \tag{3.29}$$

$$\|u(T) - \tilde u_{\tilde h}\|_0 \le Ch^{r+1}\big(\|u(T)\|_{r+1} + \|p(T)\|_{H^{r}/\mathbb{R}}\big) + K_0(u, p, \nu)h^{r+1}|\log(h)|. \tag{3.30}$$

Proof. Let $\widetilde S_{\tilde h}(u) \in \widetilde V$ be the Stokes projection of the solution of (1.1)–(1.2) at time T that satisfies

$$\begin{aligned}\Big(\nabla \widetilde S_{\tilde h}(u), \nabla\tilde\chi_h\Big) &= \big(\nabla u(T), \nabla\tilde\chi_h\big) - (p(T), \nabla\cdot\tilde\chi_h)\\ &= \big(f(T) - u_t(T) - F(u(T), u(T)), \tilde\chi_h\big) \quad \forall\tilde\chi_h \in \widetilde V.\end{aligned} \tag{3.31}$$

Then, we consider the splitting $\|u(T) - \tilde u_{\tilde h}\|_l \le \|u(T) - \widetilde S_{\tilde h}(u)\|_l + \|\widetilde S_{\tilde h}(u) - \tilde u_{\tilde h}\|_l$, $l = 0, 1$. The first term can be readily estimated by using (2.9), so that, for $l = 0, 1$,

$$\|u(T) - \widetilde S_{\tilde h}(u)\|_l \le \begin{cases} C(\tilde h)^{r-l}(\|u(T)\|_r + \|p(T)\|_{H^{r-1}/\mathbb{R}}), & \widetilde V = \widetilde V_{\tilde h,r},\\ Ch^{r+1-l}(\|u(T)\|_{r+1} + \|p(T)\|_{H^{r}/\mathbb{R}}), & \widetilde V = \widetilde V_{h,r+1}.\end{cases}$$

We will concentrate now on the second term. Subtracting (3.31) from (2.22), one finds

$$\begin{aligned}\nu(\nabla(\tilde u_{\tilde h} - \widetilde S_{\tilde h}(u)), \nabla\tilde\chi_h) &= b_{\tilde h}(u(T), u(T), \tilde\chi_h) - b_{\tilde h}(u_h(T), u_h(T), \tilde\chi_h)\\ &\quad + (u_t(T) - \dot u_h(T), \tilde\chi_h) \quad \forall\tilde\chi_h \in \widetilde V.\end{aligned} \tag{3.32}$$

Then, by setting $\tilde\chi_h = \tilde u_{\tilde h} - \widetilde S_{\tilde h}(u) \in \widetilde V$, we find

$$\nu\|\nabla(\tilde u_{\tilde h} - \widetilde S_{\tilde h}(u))\|_0 \le \big\|F(u(T), u(T)) - F(u_h(T), u_h(T))\big\|_{-1} + \|u_t(T) - \dot u_h(T)\|_{-1}.$$

For the first term above, applying (3.8) from Lemma 3.1 and Corollary 3.9, we get

$$\begin{aligned}\big\|F(u(T), u(T)) - F(u_h(T), u_h(T))\big\|_{-1} &\le K\|u(T) - u_h(T)\|_0\\ &\le Kh^r(\|u(T)\|_r + \|p(T)\|_{H^{r-1}/\mathbb{R}}).\end{aligned}$$

For the second term, (3.22) from Lemma 3.10 gives $\|u_t - \dot u_h\|_{-1} \le Kh^r|\log(h)|$. Then

$$\|\tilde u_{\tilde h} - \widetilde S_{\tilde h}(u)\|_1 \le K_1(u, p, \nu)h^r|\log(h)|, \tag{3.33}$$

and the proof for the H^1 norm is complete. We next deal with the estimate in the L^2 norm. Writing (3.32) in abstract operator form, we find that

$$\nu\widetilde{\mathcal A}_{\tilde h}(\tilde u_{\tilde h} - \widetilde S_{\tilde h}(u)) = \widetilde\Pi_{\tilde h}[F(u(T), u(T)) - F(u_h(T), u_h(T))] + \widetilde\Pi_{\tilde h}[u_t(T) - \dot u_h(T)].$$

Then, applying $\widetilde{\mathcal{A}}_{\tilde h}^{-1}$ to both sides of the above equation, we obtain

$$\|\tilde u_{\tilde h} - \widetilde S_{\tilde h}(u)\|_0 \le \frac{1}{\nu}\Big(\big\|\widetilde{\mathcal{A}}_{\tilde h}^{-1}\widetilde\Pi_{\tilde h}[F(u(T),u(T)) - F(u_h(T),u_h(T))]\big\|_0 + \big\|\widetilde{\mathcal{A}}_{\tilde h}^{-1}\widetilde\Pi_{\tilde h}[u_t(T) - \dot u_h(T)]\big\|_0\Big).$$

Thus, our aim is reduced to estimate each of the above norms. As regards the nonlinear term, taking into account (2.14), with $s = 2$, we find

$$\big\|\widetilde{\mathcal{A}}_{\tilde h}^{-1}\widetilde\Pi_{\tilde h}[F(u,u) - F(u_h,u_h)]\big\|_0 \le C\tilde h^2\big\|F(u,u) - F(u_h,u_h)\big\|_0 + \big\|\mathcal{A}^{-1}\Pi[F(u,u) - F(u_h,u_h)]\big\|_0.$$

Now, using estimates (3.7) from Lemma 3.1 and (3.13) from Lemma 3.4, we get

$$\|\widetilde{\mathcal{A}}_{\tilde h}^{-1}\widetilde\Pi_{\tilde h}[F(u,u)-F(u_h,u_h)]\big\|_0 \le C\tilde h^2\|u-u_h\|_1 + C(\|u-u_h\|_{-1} + \|u-u_h\|_0\|u-u_h\|_1).$$

To conclude, we shall estimate each term in both sums. The required estimates in the L^2 and H^1 norms are granted by Corollary 3.9. As regards the estimate in the H^{-1} norm, note that by means of (2.11) and (3.17), one readily finds

$$\begin{aligned}\|u - u_h\|_{-1} &\le \|u - s_h\|_{-1} + \|s_h - u_h\|_{-1} \le \|u - s_h\|_{-1} + \|s_h - u_h\|_0\\ &\le Ch^{r+1}(\|u\|_r + \|p\|_{H^{r-1}/\mathbb{R}}) + Kh^{r+1}|\log(h)|.\end{aligned}$$

Then, we finally get $\big\|\widetilde{\mathcal{A}}_{\tilde h}^{-1}\widetilde\Pi_{\tilde h}[F(u,u) - F(u_h,u_h)]\big\|_0 \le Kh^{r+1}|\log(h)|$. We next deal with the estimate for the time derivative. Applying again (2.14) with $s = 2$ together with estimates (3.20) and (3.21) from Lemma 3.10, we reach

$$\begin{aligned}\big\|\widetilde{\mathcal{A}}_{\tilde h}^{-1}\widetilde\Pi_{\tilde h}\,[u_t(T) - \dot u_h(T)]\big\|_0 &\le \tilde h^2\,\|u_t(T) - \dot u_h(T)\|_0 + \|\mathcal{A}^{-1}\Pi[u_t(T) - \dot u_h(T)]\|_0\\ &\le K\tilde h^2 h^{r-1}|\log(h)| + Kh^{r+1}|\log(h)| \le Kh^{r+1}|\log(h)|.\end{aligned}$$

Hence the proof for the L^2 norm is also finished. □

THEOREM 3.15. *Let $T > 0$ be fixed. Let (u_h, p_h) be the MFE approximation to the solution (u, p) of* (1.1)–(1.2). *Let $(\tilde u_{\tilde h}, \tilde p_{\tilde h})$ be the postprocessed MFE approximation at time T. Then, there exists a constant $K(u, p, \nu)$ such that*

(i) *if the postprocessing element is $(\widetilde X, \widetilde Q) = (X_{\tilde h,r}, Q_{\tilde h,r-1})$, then*

$$\|p(T) - \tilde p_{\tilde h}\|_{L^2/\mathbb{R}} \le C_\beta(\tilde h)^{r-1}\big(\|u(T)\|_r + \|p(T)\|_{H^{r-1}/\mathbb{R}}\big) + K(u,p,\nu,\beta)h^r|\log(h)|; \tag{3.34}$$

(ii) *if at time T the solution $(u(T), p(T))$ belongs to $(H^{r+1}(\Omega)^d \cap V) \times H^r(\Omega)/\mathbb{R}$, and the postprocessing element is $(\widetilde X, \widetilde Q) = (X_{h,r+1}, Q_{h,r})$, then*

$$\|p(T) - \tilde p_{\tilde h}\|_{L^2/\mathbb{R}} \le C_\beta h^r\big(\|u(T)\|_{r+1} + \|p(T)\|_{H^r/\mathbb{R}}\big) + K(u,p,\nu,\beta)h^r|\log(h)|. \tag{3.35}$$

Proof. Let us denote by $\tilde q_{\tilde h}$ the MFE approximation to the pressure $p(T)$ obtained by solving the Stokes problem (2.8) at time T in the postprocessed space $(\widetilde X, \widetilde Q)$. Adding and subtracting $\tilde q_{\tilde h}$, we get

$$\|p(T) - \tilde p_{\tilde h}\|_{L^2/\mathbb{R}} \le \|p(T) - \tilde q_{\tilde h}\|_{L^2/\mathbb{R}} + \|\tilde q_{\tilde h} - \tilde p_{\tilde h}\|_{L^2/\mathbb{R}}.$$

The first term can easily be estimated applying (2.10):

$$\|p(T) - \tilde q_{\tilde h}\|_{L^2/\mathbb{R}} \le \begin{cases} C_\beta(\tilde h)^{r-1}\big(\|u(T)\|_r \\ + \|p(T)\|_{H^{r-1}/\mathbb{R}}\big), & \big(\widetilde X, \widetilde Q\big) = (X_{\tilde h,r}, Q_{\tilde h,r-1}), \\ C_\beta h^r\big(\|u(T)\|_{r+1} \\ + \|p(T)\|_{H^r/\mathbb{R}}\big), & \big(\widetilde X, \widetilde Q\big) = (X_{h,r+1}, Q_{\tilde h,r}). \end{cases}$$

Let us now bound the second term. Using the equations that satisfy $\tilde p_{\tilde h}$ and $\tilde q_{\tilde h}$ ((2.20), (2.8), respectively), we deduce

$$\Big(\tilde p_{\tilde h} - \tilde q_{\tilde h}, \nabla\cdot\tilde\phi\Big) = \nu\Big(\nabla\big(\tilde u_{\tilde h} - \widetilde S_{\tilde h}(u)\big), \nabla\tilde\phi\Big) + \big(F(u_h,u_h) \\ - F(u,u), \tilde\phi\big) + \big(\dot u_h - u_t, \tilde\phi\big) \qquad \forall\tilde\phi \in \widetilde X.$$

Using the inf-sup condition (2.4), we obtain

$$\beta\|\tilde p_{\tilde h} - \tilde q_{\tilde h}\|_{L^2/\mathbb{R}} \le \nu\|\tilde u_{\tilde h} - \widetilde S_{\tilde h}\|_1 + \|F(u_h,u_h) - F(u,u)\|_{-1} + \|u_h - u_t\|_{-1}.$$

Taking into account (3.33), (3.8) from Lemma 3.1, and (3.22) from Lemma 3.10, we reach

$$\|\tilde p_{\tilde h} - \tilde q_{\tilde h}\|_{L^2/\mathbb{R}} \le \frac{1}{\beta}\left(Kh^r|\log(h)| + \|u - u_h\|_0 + Kh^r|\log(h)|\right),$$

so that, applying Corollary 3.8, we have completed the proof. ◻

Remark 3.1. Observe that for the velocity we used piecewise polynomials of degree at least 2. In general, the postprocessed method does not increase the rate of convergence in the L^2 norm in the linear case although an improvement in the energy norm is obtained. The application of the postprocessing technique to the mini-element approximation to Navier–Stokes equations is studied in [3], [5].

4. Numerical experiments.

In this section, we present some numerical experiments in order to support the analysis developed in the paper and to assess the merit of the postprocessed method when compared with the standard MFE method. We consider the Navier–Stokes equations (1.1) over the domain $\Omega = [0,1]\times[0,1]$ subject to homogeneous Dirichlet boundary conditions. The value of the viscosity in the experiments is $\nu = 1$, and the final time is $T = 1.2$. We set to zero the initial velocity field u_0 (1.2) and choose the external force f so that the exact solution is

$$\begin{aligned} u^1(x,y,t) &= -6\cdot[1-\cos(\pi t)]\big(\sin^3(\pi x)\sin^2(\pi y)\cos(\pi y)\big), \quad (x,y,t)\in\Omega\times[0,T], \\ u^2(x,y,t) &= 6\cdot\big(1-\cos(\pi t)\big)\big(\sin^2(\pi x)\sin^3(\pi y)\cos(\pi x)\big), \quad (x,y,t)\in\Omega\times[0,T], \\ p(x,y,t) &= (\sin(2\pi t)/2)\big(\sin^4(\pi x)+\sin^3(\pi y)\big) - \mathfrak{p}_0, \quad (x,y,t)\in\Omega\times[0,T], \end{aligned}$$

where $\mathfrak{p}_0$ denotes the mean of the pressure. In spite of the simplicity of this solution and its lack of physical meaning, we remark that our main interest has been to check the improvement in the rate of convergence achieved with the postprocessing technique and whether this also increases the efficiency of the standard MFE approximation.

In our calculations we take the so-called regular pattern triangulations of Ω, which are induced by the set of nodes $(i/N, j/N)$, $0 \le i,\, j \le N$, where $N = |\Omega|/h$ is an integer. The MFE approximation to (1.1)–(1.2) is carried out using the Hood–Taylor element $(X_{h,3}, Q_{h,2})$ that we will denote by $P2P1$. That is, we use Lagrange quadratic

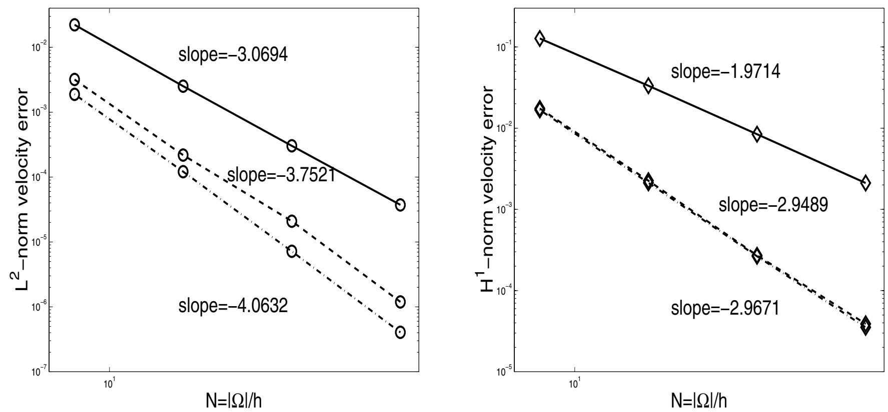

FIG. 4.1. *Convergence diagrams for the first component of the velocity with $P2P1$ (continuous line), $P3P2$ (dashed-dotted line), and the postprocessed method with $P3P2$ (dashed line). On the left the errors are measured in the L^2 norm (circles $\circ$) and on the right in the H^1 norm (diamonds $\diamond$).*

elements for the approximation to the velocity and linear elements to approximate the pressure. For the postprocessing step, due to the smoothness of the solution (u, p), we perform the experiments not only with the same MFE over a finer grid, $(X_{h',3}, Q_{h',2})$, $h' < h$, but also with the higher-order Hood–Taylor element over the same grid, $(X_{h,4}, Q_{h,3})$; i.e., Lagrange cubic for the velocity and Lagrange quadratic for the pressure. This element will be denoted by $P3P2$.

For the time integration we use the well-known semi-implicit method where linear terms are approximated by the implicit midpoint rule (i.e., the Crank–Nicolson method) and nonlinear terms by the two-step explicit Adams formula (see, e.g., [8, p. 105]). The modified Stokes problems that arise at each step are solved by means of a standard projection method [31, pp. 27–28] (see also [3, section 4.6]).

For each h used in the triangulations of Ω, every experiment was carried out with different values of the time step dt. There is always a point, depending on h, at which further reduction of the time step dt does not reduce the errors anymore. This means that the error arising from the time discretization is smaller than the error arising from the MFE discretization. To avoid wrong conclusions from our numerical experiments, we have been careful to ensure that the dominant error in all the computations presented here is the spatial discretization error. For the computational cost in the efficiency diagrams shown here, we use the largest time step among those in which the spatial discretization error is dominant.

In what follows, we use the same symbols in all the plots to represent the relative errors. For the velocity we plot the errors in the first component. Similar errors are obtained for the second. The different methods are distinguished by the line used to join the symbols. For the MFE-$P2P1$ approximation, we use continuous line, and for the MFE-$P3P2$ dashed-dotted line. The MFE-$P2P1$ has been postprocessed in two different ways: using $P3P2$ (dashed line) and refining the mesh (dotted line).

In Figure 4.1 we present two convergence diagrams showing the errors committed by the methods when used with $h = |\Omega|/N$, $N = 8, 16, 32, 64$, both in the L^2 norm (left) and the H^1 norm (right). We have plotted the errors of the MFE-$P2P1$ and $P3P2$ methods and the postprocessed errors with $P3P2$. One can observe that the postprocessing technique with $P3P2$ provides an approximate velocity with about the

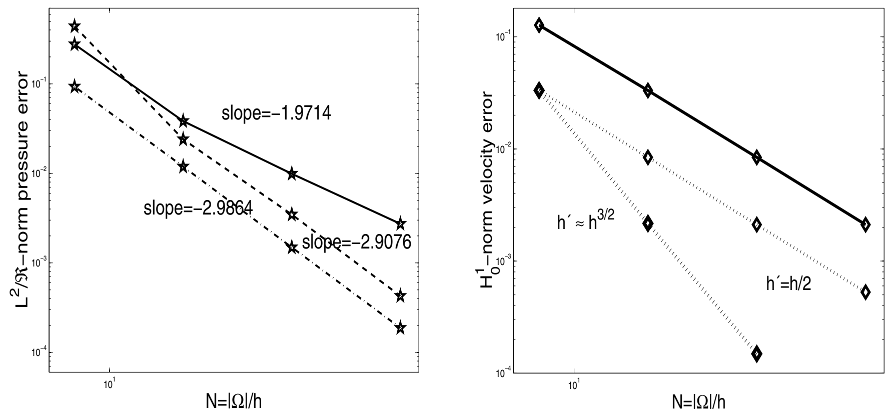

FIG. 4.2. *Left: convergence diagram for the pressure approximation with $P2P1$ (continuous line), $P3P2$ (dashed-dotted lines), and the postprocessed method with $P3P2$ (dashed lines). Right: convergence diagram for the first component of the velocity approximation with $P2P1$ (continuous line) and the postprocessed $P2P1$ over finer grids.*

same accuracy as that corresponding to the MFE-$P3P2$ method. This is especially true for the H^1 norm, in which the two methods produce virtually the same errors. Measures of the slopes of the plots confirm the rates predicted by the theory (i.e., the errors in the plots decrease like $N^{\text{slope}} = \text{const}.h^{-\text{slope}}$).

Similar conclusions can be reached from the errors of the approximations to the pressure in Figure 4.2 (left). Except for the first point, which correspond to $h = 1/8$, the postprocessed errors lies on a line (almost) parallel to the one joining the MFE-$P3P2$ errors. The rate of convergence of these two methods is one unit larger than that of the MFE-$P2P1$ in agreement with what the theory predicts.

In Figure 4.2 (right), we plot the errors obtained postprocessing the MFE-$P2P1$ refining the grid. We have represented the errors measured in the H^1 norm; similar results have been obtained for the L^2 norm. In view of Theorem 3.14, in order to get a gain of one order of convergence in the H^1 norm, we should use a mesh of size $h' \approx h^{3/2}$. The improvement in the rate of convergence of the postprocessed method can be observed in the figure. We can also observe in the plot that using a refined mesh of size $h' = h/2$ (only one regular refinement), the errors are considerably reduced. In fact, observe that the postprocessed error with $h' = h/2$ is almost the same as that of the standard MFE-$P2P1$ carried out using a mesh of size $h/2$ over the full interval $[0, T]$. This fact can be of interest when the cost of the postprocessing step with a refined mesh of size a power of h is not affordable for computational reasons.

The relevant question now is whether the improvement in the rate of convergence also implies improved efficiency. In Figure 4.3, we have represented the same errors as in Figure 4.1 (right) and Figure 4.2 (left) against the smallest amount of time needed to achieve them. We have also plotted the errors of the postprocessed method refining the mesh (Figure 4.2 (right)). In the plot we observe that the efficiency of the two postprocessing procedures is very similar. We can conclude that the postprocessed method really improves the efficiency of the standard MFE method for both approximations to the velocity and to the pressure. For any error that we may demand, the postprocessed method achieves that error in less computing time than the standard $P2P1$ and $P3P2$-MFE methods. The reason for this improvement is that the error of

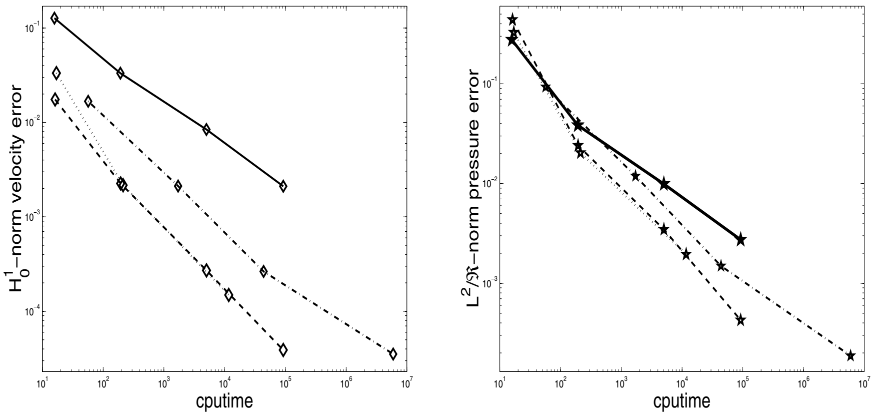

FIG. 4.3. *Efficiency diagrams for the first component of the velocity in the H^1 norm (left) and the pressure (right) with $P2P1$ (continuous line), $P3P2$ (dashed-dotted line) and the postprocessed method with $P3P2$ (dashed line) and refining the grid (dotted line).*

the MFE-$P2P1$ method is reduced when the postprocessing is done, but this is done at very little cost: that of solving a single discrete Stokes problem at the final time.

All numerical experiments were carried out on a Pentium IV, with 1 GB of Rimm memory, under the Solaris8 (Intel) operating system, with SUN Workshop 5 compilers. The programs were written in Fortran 77.

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