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POSTPROCESSING FINITE-ELEMENT METHODS FOR THE NAVIER–STOKES EQUATIONS: THE FULLY DISCRETE CASE*

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Abstract. An accuracy-enhancing postprocessing technique for finite-element discretizations of the Navier–Stokes equations is analyzed. The technique had been previously analyzed only for semidiscretizations, and fully discrete methods are addressed in the present paper. We show that the increased spatial accuracy of the postprocessing procedure is not affected by the errors arising from any convergent time-stepping procedure. Further refined bounds are obtained when the time-stepping procedure is either the backward Euler method or the two-step backward differentiation formula.

Key words. Navier–Stokes equations, mixed finite-element methods, time-stepping methods, optimal regularity, error estimates, backward Euler method, two-step BDF

AMS subject classifications. 65M60, 65M20, 65M15, 65M12

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1. Introduction.

The purpose of the present paper is to study a postprocessing technique for fully discrete mixed finite-element (MFE) methods for the incompressible Navier–Stokes equations

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f, \tag{1.1}$$

$$\mathrm{div}(u) = 0, \tag{1.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with smooth boundary subject to homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. In (1.1), u is the velocity field, p the pressure, and f a given force field. We assume that the fluid density and viscosity have been normalized by an adequate change of scale in space and time.

For semidiscrete MFE methods the postprocessing technique has been studied in [2, 3, 18] and is as follows. In order to approximate the solution u and p corresponding to a given initial condition

$$u(\cdot, 0) = u_0, \tag{1.3}$$

at a time $t \in (0, T]$, $T > 0$, consider first standard MFE approximations u_h and p_h to the velocity and pressure, respectively, solutions at time $t \in (0, T]$ of the corresponding discretization of (1.1)–(1.3). Then compute MFE approximations $\tilde{u}_h$ and $\tilde{p}_h$ to the

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solution $\tilde{u}$ and $\tilde{p}$ of the following Stokes problem,

$$-\Delta\tilde{u} + \nabla\tilde{p} = f - \frac{d}{dt}u_h - (u_h\cdot\nabla)u_h \quad \text{in } \Omega, \tag{1.4}$$

$$\text{div }(\tilde{u}){=}0 \quad \text{in } \Omega, \tag{1.5}$$

$$\tilde{u} = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

The MFE on this postprocessing step can be either the same MFE over a finer grid or a higher-order MFE over the same grid. In [2, 18] it is shown that if the errors in the velocity (in the H^1 norm) and the pressure of the standard MFE approximations u_h and p_h are $O(t^{-(r-2)/2}h^{r-1})$, $r = 2,3,4$, for $t \in (0,T]$, then those of the postprocessed approximations $\tilde{u}_h$ and $\tilde{p}_h$ are $O(t^{-(r-1)/2}h^r\,|\log(h)|)$, that is, an $O(h\,|\log(h)|)$ improvement with respect the standard MFE error bound) (see precise statement on Theorem 2.2 below), and if $r \geq 3$ (finite elements of degree at least two), the $O(h\,|\log(h)|)$ improvement is also obtained in the L^2 norm of the velocity.

In practice, however, the finite-element approximations u_h and p_h can rarely be computed exactly, and one has to compute approximations $U_h^{(n)} \approx u_h(t_n)$ and $P_h^{(n)} \approx p_h(t_n)$ at some time levels $0 = t_0 < t_1 \cdots < t_N = T$, by means of a time integrator. Consequently, instead of the postprocessed approximations $\tilde{u}_h(t_n)$ and $\tilde{p}_h(t_n)$, one obtains $\tilde{U}_h^{(n)}$ and $\tilde{P}_h^{(n)}$ as solutions of a system similar to (1.4)–(1.6) but with u_h on the right-hand side of (1.4) replaced by $U_h^{(n)}$ and $\dot{u}_h$ replaced by an appropriate approximation $d_t^*U_h^{(n)}$.

In the present paper we analyze the errors $u(t_n) - \tilde{U}_h^{(n)}$ and $p(t_n) - \tilde{P}_h^{(n)}$. We show that, if any convergent time stepping procedure is used to integrate the standard MFE approximation, then the error of the fully discrete postprocessed approximation, $u(t_n) - \tilde{U}_h^{(n)}$, is that of the semidiscrete postprocessed approximation $u(t_n) - \tilde{u}_h$ plus a term $\tilde{e}_n$ whose norm is proportional to that of the time-discretization error $e_n = u_h(t_n) - U_h^{(n)}$ of the MFE method, and, furthermore, we show $\tilde{e}_n = e_n$ plus higher-order terms for two particular time integration methods, the backward Euler method and the two-step backward differentiation formula (BDF) [9] (see also [25, section III.1]). We remark that the fact that $\tilde{e}_n$ is asymptotically equivalent to e_n has proved its relevance when developing a posteriori error estimators for dissipative problems [17] (see also [15, 16]). To prove $\tilde{e}_n \approx e_n$ we perform first a careful error analysis of the backward Euler method and the two-step BDF. This allows us to obtain error estimates for the pressure that improve by a factor of the time step k those in the literature [10, 34].

It must be noticed that the backward Euler method and the two-step BDF are the only G-stable methods (see, e.g., [26, section V.6]) in the BDF family of methods. G-stability makes it easier the use of energy methods in the analysis, and this has proved crucial in obtaining our error bounds. At present we ignore if error bounds similar to that obtained in the present paper can be obtained without resource to energy methods, so that error bounds for higher-order methods in the BDF family of methods can be proved.

The analysis of fully discrete postprocessed methods may be less trivial than it may seem at first sight, since although many results for postprocessed semidiscrete methods can be found in the literature (see next paragraph) as well as numerical experiments (carried out with fully discrete methods) showing an increase in accuracy similar to that theoretically predicted in semidiscrete methods [3, 11, 14, 12, 20, 22], the only analysis of postprocessed fully discrete methods is that by Yan [44]. There, for a semilinear parabolic equation of the type $u_t - \Delta u = F(u)$, Δ being the Laplacian operator and F a smooth and bounded function, the postprocess of a finite-element (FE) approximation when integrated in time with the backward Euler method with fixed stepsize k is analyzed (higher-order time-stepping methods are also considered, but only for linear homogeneous parabolic equations). Error estimates are obtained where an $O(k(1+h^2))$ term is added to the bounds previously obtained for the postprocessed semidiscrete approximation. It must be remarked, though, that in [44] no attempt is made to analyze methods for equations with convective terms. In fact, in [44], it is stated that "It is not quite clear how it is possible to generalize our method to deal with a nonlinear convection term". This is precisely what we do in the present paper.

The postprocess technique considered here was first developed for spectral methods in [20, 21]. Later it was extended to methods based on Chebyshev and Legendre polynomials [11], spectral element methods [12, 13], and finite element methods [22, 14]. In these works, numerical experiments show that, if the postprocessed approximation is computed at the final time T, the postprocessed method is computationally more efficient than the method to which it is applied. Similar results are obtained in the numerical experiments in [2, 3] for MFE methods. Due to this better practical performance, the postprocessing technique has been applied to the study of nonlinear shell vibrations [37], as well as to stochastic differential parabolic equations [38]. Also, it has been effectively applied to reduce the order of practical engineering problems modeled by nonlinear differential systems [42, 43].

The postprocess technique can be seen as a two-level method, where the postprocessed (or fine-mesh) approximations $\tilde u_h$ and $\tilde p_h$ are an improvement of the previously computed (coarse mesh) approximations u_h and p_h. Recent research on two-level finite-element methods for the transient Navier–Stokes equations can be found in [23, 27, 28, 40] (see also [30, 29, 36, 39] for spectral discretizations), where the fully nonlinear problem is dealt with on the coarse mesh, and a linear problem is solved on the fine mesh.

The rest of the paper is as follows. In section 2 we introduce some standard material and the methods to be studied. In section 3 we analyze the fully discrete postprocessed method. In section 4 we prove some technical results and, finally, section 5 is devoted to analyze the time discretization errors of the MFE approximation when integrated with the backward Euler method or the two-step BDF.

2. Preliminaries and notations.

2.1. The continuous solution. We will assume that Ω is a bounded domain in $\mathbb{R}^d$, $d=2,3$, of class $\mathcal{C}^m$, for $m\geq 2$, and we consider the Hilbert spaces

$$H=\left\{u\in L^2(\Omega)^d \mid \operatorname{div}(u)=0,\, u\cdot n_{|\partial\Omega}=0\right\},$$
$$V=\left\{u\in H_0^1(\Omega)^d \mid \operatorname{div}(u)=0\right\},$$

endowed with the inner product of $L^2(\Omega)^d$ and $H_0^1(\Omega)^d$, respectively. For $l\geq 0$ integer and $1\leq q\leq\infty$, we consider the standard Sobolev spaces, $W^{l,q}(\Omega)^d$, of functions with derivatives up to order l in $L^q(\Omega)$, and $H^l(\Omega)^d=W^{l,2}(\Omega)^d$. We will denote by $\|\cdot\|_l$ the norm in $H^l(\Omega)^d$, and $\|\cdot\|_{-l}$ will represent the norm of its dual space. We consider also the quotient spaces $H^l(\Omega)/\mathbb{R}$ with norm $\|p\|_{H^l/\mathbb{R}}=\inf\{\|p+c\|_l \mid c\in\mathbb{R}\}$.

Let $\Pi: L^2(\Omega)^d \longrightarrow H$ be the $L^2(\Omega)^d$ projection onto H. We denote by A the Stokes operator on Ω:

$$A:\mathcal{D}(A)\subset H\longrightarrow H,\quad A=-\Pi\Delta,\quad \mathcal{D}(A)=H^2(\Omega)^d\cap V.$$

Applying Leray's projector to (1.1), the equations can be written in the form

$$u_t + Au + B(u,u) = \Pi f \quad \text{in } \Omega,$$

where $B(u,v) = \Pi(u \cdot \nabla)v$ for u, v in $H_0^1(\Omega)^d$.

We shall use the trilinear form $b(\cdot,\cdot,\cdot)$ defined by

$$b(u,v,w) = (F(u,v), w) \quad \forall u,v,w \in H_0^1(\Omega)^d,$$

where

$$F(u,v) = (u \cdot \nabla)v + \frac{1}{2}(\nabla \cdot u)v \quad \forall u,v \in H_0^1(\Omega)^d.$$

It is straightforward to verify that b enjoys the skew-symmetry property

$$b(u,v,w) = -b(u,w,v) \quad \forall u,v,w \in H_0^1(\Omega)^d. \tag{2.1}$$

Let us observe that $B(u,v) = \Pi F(u,v)$ for $u \in V$, $v \in H_0^1(\Omega)^d$.

We shall assume that u is a strong solution up to time $t = T$, so that

$$\|u(t)\|_1 \le M_1, \quad \|u(t)\|_2 \le M_2, \quad 0 \le t \le T, \tag{2.2}$$

for some constants M_1 and M_2. We shall also assume that there exists another constant $\tilde{M}_2$ such that

$$\|f\|_1 + \|f_t\|_1 + \|f_{tt}\|_1 \le \tilde{M}_2, \quad 0 \le t \le T. \tag{2.3}$$

Let us observe, however, that if for $k \ge 2$

$$\sup_{0\le t\le T} \big\|\partial_t^{\lfloor k/2\rfloor} f\big\|_{k-1-2\lfloor k/2\rfloor} + \sum_{j=0}^{\lfloor (k-2)/2\rfloor} \sup_{0\le t\le T} \big\|\partial_t^j f\big\|_{k-2j-2} < +\infty,$$

and if Ω is of class $\mathcal{C}^k$, then, according to Theorems 2.4 and 2.5 in [32], there exist positive constants M_k and K_k such that the following bounds hold:

$$\|u(t)\|_k + \|u_t(t)\|_{k-2} + \|p(t)\|_{H^{k-1}/\mathbb{R}} \le M_k \tau(t)^{1-k/2}, \tag{2.4}$$

$$\int_0^t \sigma_{k-3}(s)\big(\|u(s)\|_k^2 + \|u_s(s)\|_{k-2}^2 + \|p(s)\|_{H^{k-1}/\mathbb{R}}^2 + \|p_s(s)\|_{H^{k-3}/\mathbb{R}}^2\big)\,\mathrm{d}s \le K_k^2, \tag{2.5}$$

where $\tau(t) = \min(t,1)$ and $\sigma_n = e^{-\alpha(t-s)}\tau^n(s)$ for some $\alpha > 0$. Observe that for $t \le T < \infty$, we can take $\tau(t) = t$ and $\sigma_n(s) = s^n$. For simplicity, we will take these values of τ and σ_n.

We note that although the results in the present paper require only (2.2) and (2.3) to hold, those in [18] that we summarize in section 2.3 require that for $r = 3, 4$, (2.4)–(2.5) hold for $k = r + 2$.

2.2. The spatial discretization. Let $\mathcal{T}_h = (\tau_i^h, \phi_i^h)_{i\in I_h}$, $h > 0$ be a family of partitions of suitable domains Ω_h, where h is the maximum diameter of the elements $\tau_i^h \in \mathcal{T}_h$, and ϕ_i^h are the mappings of the reference simplex τ_0 onto τ_i^h. We restrict ourselves to quasi-uniform and regular meshes $\mathcal{T}_h$.

Let $r \ge 3$, we consider the finite-element spaces

$$S_{h,r} = \Big\{\chi_h \in \mathcal{C}\left(\overline{\Omega}_h\right) \mid \chi_h|_{\tau_i^h} \circ \phi_i^h \in P^{r-1}(\tau_0)\Big\} \subset H^1(\Omega_h), \quad S_{h,r}^0 = S_{h,r} \cap H_0^1(\Omega_h),$$

where $P^{r-1}(\tau_0)$ denotes the space of polynomials of degree at most $r-1$ on τ_0. Since we are assuming that the meshes are quasi-uniform, the following inverse inequality holds for each $v_h \in (S^0_{h,r})^d$ (see, e.g., [7, Theorem 3.2.6])

$$\|v_h\|_{W^{m,q}(\tau)^d} \le Ch^{l-m-d\left(\frac{1}{q'}-\frac{1}{q}\right)} \|v_h\|_{W^{l,q'}(\tau)^d}, \tag{2.6}$$

where $0 \le l \le m \le 1$, $1 \le q' \le q \le \infty$, and τ is an element in the partition $\mathcal{T}_h$.

We shall denote by $(X_{h,r}, Q_{h,r-1})$ the so-called Hood–Taylor element [5, 35], when $r \ge 3$, where

$$X_{h,r} = \left(S^0_{h,r}\right)^d, \quad Q_{h,r-1} = S_{h,r-1} \cap L^2(\Omega_h)/\mathbb{R}, \quad r \ge 3,$$

and the so-called mini-element [6] when $r = 2$, where $Q_{h,1} = S_{h,2} \cap L^2(\Omega_h)/\mathbb{R}$, and $X_{h,2} = (S^0_{h,2})^d \oplus \mathbb{B}_h$. Here, $\mathbb{B}_h$ is spanned by the bubble functions b_τ, $\tau \in \mathcal{T}_h$, defined by $b_\tau(x) = (d+1)^{d+1}\lambda_1(x)\cdots\lambda_{d+1}(x)$, if $x \in \tau$ and 0 elsewhere, where $\lambda_1(x), \dots, \lambda_{d+1}(x)$ denote the barycentric coordinates of x. For these elements a uniform inf-sup condition is satisfied (see [5]), that is, there exists a constant $\beta > 0$ independent of the mesh grid size h such that

$$\inf_{q_h \in Q_{h,r-1}} \sup_{v_h \in X_{h,r}} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\|_1 \|q_h\|_{L^2/\mathbb{R}}} \ge \beta. \tag{2.7}$$

The approximate velocity belongs to the discretely divergence-free space

$$V_{h,r} = X_{h,r} \cap \left\{\chi_h \in H^1_0(\Omega_h)^d \mid (q_h, \nabla \cdot \chi_h) = 0 \quad \forall q_h \in Q_{h,r-1}\right\},$$

which is not a subspace of V. We shall frequently write V_h instead of $V_{h,r}$ whenever the value of r plays no particular role.

Let $\Pi_h : L^2(\Omega)^d \longrightarrow V_{h,r}$ be the discrete Leray's projection defined by

$$(\Pi_h u, \chi_h) = (u, \chi_h) \quad \forall \chi_h \in V_{h,r}.$$

We will use the following well-known bounds

$$\|(I - \Pi_h)u\|_j \le Ch^{l-j}\|u\|_l, \quad 1 \le l \le 2, \quad j = 0, 1. \tag{2.8}$$

We will denote by $A_h : V_h \to V_h$ the discrete Stokes operator defined by

$$(\nabla v_h, \nabla \phi_h) = (A_h v_h, \phi_h) = \left(A_h^{1/2} v_h, A_h^{1/2}\phi_h\right) \quad \forall v_h, \phi_h \in V_h.$$

Let $(u, p) \in (H^2(\Omega)^d \cap V) \times (H^1(\Omega)/\mathbb{R})$ be the solution of a Stokes problem with right-hand side g, we will denote by $s_h = S_h(u) \in V_h$ the so-called Stokes projection (see [33]) defined as the velocity component of solution of the following Stokes problem: find $(s_h, q_h) \in (X_{h,r}, Q_{h,r-1})$ such that

$$(\nabla s_h, \nabla \phi_h) + (\nabla q_h, \phi_h) = (g, \phi_h) \qquad \forall \phi_h \in X_{h,r}, \tag{2.9}$$

$$(\nabla \cdot s_h, \psi_h) = 0 \qquad \forall \psi_h \in Q_{h,r-1}. \tag{2.10}$$

Obviously, $s_h = S_h(u)$. The following bound holds for $2 \le l \le r$:

$$\|u - s_h\|_0 + h\|u - s_h\|_1 \le Ch^l\big(\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}\big). \tag{2.11}$$

The proof of (2.11) for $\Omega = \Omega_h$ can be found in [33]. For the general case, Ω_h must be such that the value of $\delta(h) = \max_{x\in\partial\Omega_h} \operatorname{dist}(x, \partial\Omega)$ satisfies

$$\delta(h) = O\left(h^{2(r-1)}\right). \tag{2.12}$$

This can be achieved if, for example, $\partial\Omega$ is piecewise of class $\mathcal{C}^{2(r-1)}$, and superparametric approximation at the boundary is used [1]. Under the same conditions, the bound for the pressure is [24]

$$\|p - q_h\|_{L^2/\mathbb{R}} \le C_\beta h^{l-1}\big(\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}\big), \tag{2.13}$$

where the constant C_β depends on the constant β in the inf-sup condition (2.7).

In the sequel we will apply the above estimates to the particular case in which (u, p) is the solution of the Navier–Stokes problem (1.1)–(1.3). In that case s_h is the discrete velocity in problem (2.9)–(2.10) with $g = f - u_t - (u \cdot \nabla u)$. Note that the temporal variable t appears here merely as a parameter and then, taking the time derivative, the error bounds (2.11) and (2.13) can also be applied to the time derivative of s_h changing u, p by u_t, p_t, respectively.

Since we are assuming that Ω is of class $\mathcal{C}^m$ and $m \ge 2$, from (2.11) and standard bounds for the Stokes problem [1, 19], we deduce that

$$\left\|\left(A^{-1}\Pi - A_h^{-1}\Pi_h\right) f\right\|_j \le Ch^{2-j}\|f\|_0 \quad \forall f \in L^2(\Omega)^d, \quad j = 0, 1. \tag{2.14}$$

In our analysis we shall frequently use the following relation, which is a consequence of (2.14) and the fact that any $f_h \in V_h$ vanishes on $\partial\Omega$. For some $c \ge 1$,

$$\frac{1}{c}\left\|A_h^{s/2} f_h\right\|_0 \le \|f_h\|_s \le c\left\|A_h^{s/2} f_h\right\|_0 \quad \forall f_h \in V_h, \quad s = 1, -1. \tag{2.15}$$

Finally, we will use the following inequalities whose proof can be obtained applying [32, Lemma 4.4]

$$\|v_h\|_\infty \le C\|A_h v_h\|_0 \qquad \forall v_h \in V_h, \tag{2.16}$$

$$\|\nabla v_h\|_{L^3} \le C\|\nabla v_h\|_0^{1/2}\|A_h v_h\|_0^{1/2} \qquad \forall v_h \in V_h. \tag{2.17}$$

We consider the finite-element approximation (u_h, p_h) to (u, p), solution of (1.1)–(1.3). That is, given $u_h(0) = \Pi_h u_0$, we compute $u_h(t) \in X_{h,r}$ and $p_h(t) \in Q_{h,r-1}$, $t \in (0, T]$, satisfying

$$(\dot u_h, \phi_h) + (\nabla u_h, \nabla\phi_h) + b(u_h, u_h, \phi_h) + (\nabla p_h, \phi_h) = (f, \phi_h) \quad \forall\, \phi_h \in X_{h,r}, \tag{2.18}$$

$$(\nabla \cdot u_h, \psi_h) = 0 \qquad \forall\, \psi_h \in Q_{h,r-1}. \tag{2.19}$$

For convenience, we rewrite this problem in the following way,

$$\dot u_h + A_h u_h + B_h(u_h, u_h) = \Pi_h f, \quad u_h(0) = \Pi_h u_0, \tag{2.20}$$

where $B_h(u, v) = \Pi_h F(u, v)$.

For $r = 2, 3, 4, 5$, provided that (2.11)–(2.13) hold for $l \le r$, and (2.4)–(2.5) hold for $k = r$, then we have

$$\|u(t) - u_h(t)\|_0 + h\|u(t) - u_h(t)\|_1 \le C\frac{h^r}{t^{(r-2)/2}}, \quad 0 \le t \le T, \tag{2.21}$$

(see, e.g., [18, 32, 33]), and also,

$$\|p(t)-p_h(t)\|_{L^2/\mathbb{R}} \le C\frac{h^{r-1}}{t^{(r'-2)/2}}, \quad 0\le t\le T, \tag{2.22}$$

where $r'=r$ if $r\le 4$ and $r'=r+1$ if $r=5$. Results in [18] hold for h sufficiently small. In the rest of the paper we assume h to be small enough for (2.21)–(2.22) to hold.

Observe that from (2.21) and (2.2) it follows that $\|u_h(t)\|_1$ is bounded for $0\le t\le T$. However, further bounds for $u_h(t)$ will be needed in the present paper, so we recall the following result, which, since we are considering finite times $0<T<+\infty$, it is a rewriting of [34, Proposition 3.2].

PROPOSITION 2.1. *Let the forcing term f in* (1.1) *satisfy* (2.3). *Then, there exists a constant $\tilde M_3>0$, depending only on $\tilde M_2$, $\|A_hu_h(0)\|_0$ and $\sup_{0\le t\le T}\|u_h(t)\|_1$, such that the following bounds hold for $0\le t\le T$:*

$$F_{0,2}(t)\equiv\|A_hu_h(t)\|_0^2\le\tilde M_3^2, \tag{2.23}$$

$$F_{1,r}(t)\equiv t^r\|A_h^{r/2}\dot u_h(t)\|_0^2\le\tilde M_3^2,\quad r=0,1,2, \tag{2.24}$$

$$F_{2,r}(t)\equiv t^{r+2}\|A_h^{r/2}\ddot u_h(t)\|_0^2\le\tilde M_3^2,\quad r=-1,0,1, \tag{2.25}$$

$$I_{1,r}(t)\equiv\int_0^t s^{r-1}\|A_h^{r/2}\dot u_h(s)\|_0^2\,\mathrm{d}s\le\tilde M_3^2,\quad r=1,2, \tag{2.26}$$

$$I_{2,r}(t)\equiv\int_0^t s^{r+1}\|A_h^{r/2}\ddot u_h(s)\|_0^2\,\mathrm{d}s\le\tilde M_3^2,\quad r=-1,0,1. \tag{2.27}$$

2.3. The postprocessed method. This method obtains for any $t\in(0,T]$ an improved approximation by solving the following discrete Stokes problem: find $(\tilde u_h(t),\tilde p_h(t))\in(\widetilde X,\widetilde Q)$ satisfying

$$\begin{aligned}\left(\nabla\tilde u_h(t),\nabla\tilde\phi\right)+\left(\nabla\tilde p_h(t),\tilde\phi\right)=\left(f,\tilde\phi\right)-b\left(u_h(t),u_h(t),\tilde\phi\right)&\\ -\left(\dot u_h(t),\tilde\phi\right)\quad\forall\,\tilde\phi\in\widetilde X&,\end{aligned} \tag{2.28}$$

$$\left(\nabla\cdot\tilde u_h(t),\tilde\psi\right)=0\quad\forall\,\tilde\psi\in\widetilde Q, \tag{2.29}$$

where $(\widetilde X,\widetilde Q)$ is either:

(a) The same-order MFE over a finer grid. That is, for $h'<h$, we choose $(\widetilde X,\widetilde Q)=(X_{h',r},Q_{h',r-1})$.

(b) A higher-order MFE over the same grid. In this case we choose $(\widetilde X,\ \widetilde Q)=(X_{h,r+1},Q_{h,r})$.

In both cases, we will denote by $\widetilde V$ the corresponding discretely divergence-free space that can be either $\widetilde V=V_{h',r}$ or $\widetilde V=V_{h,r+1}$ depending on the selection of the postprocessing space. The discrete orthogonal projection into $\widetilde V$ will be denoted by $\tilde\Pi_h$, and we will represent by $\widetilde A_h$ the discrete Stokes operator acting on functions in $\widetilde V$. Notice then that from (2.28) it follows that $\tilde u_h(t)\in\widetilde V$ and it satisfies

$$\tilde A_h\tilde u_h(t)=\tilde\Pi_h\big(f-F(u_h(t),u_h(t))-\dot u_h(t)\big). \tag{2.30}$$

In [18] the following result is proved.

THEOREM 2.2. *Let (u,p) be the solution of* (1.1)–(1.3) *and for $r=3,4$, let* (2.4)–(2.5) *hold with $k=r+2$ and let* (2.11) *hold for $2\le l\le r$. Then, there exists a* *positive constant C such that the postprocessed MFE approximation to u, $\tilde u_h$ satisfies the following bounds for $r = 3, 4$ and $t \in (0, T]$:*

(i) *if the postprocessing element is $(\widetilde X, \widetilde Q) = (X_{h',r}, Q_{h',r-1})$, then*

$$\|u(t) - \tilde u_h(t)\|_j \le \frac{C}{t^{(r-2)/2}}(h')^{r-j} + \frac{C}{t^{(r-1)/2}} h^{r+1-j} |\log(h)|, \quad j = 0, 1, \tag{2.31}$$

$$\|p(t) - \tilde p_h(t)\|_{L^2/\mathbb{R}} \le \frac{C}{t^{(r-2)/2}}(h')^{r-1} + \frac{C}{t^{(r-1)/2}} h^{r} |\log(h)|, \tag{2.32}$$

(ii) *if the postprocessing element is $(\widetilde X, \widetilde Q) = (X_{h,r+1}, Q_{h,r})$, then*

$$\|u(t) - \tilde u_h(t)\|_j \le \frac{C}{t^{(r-1)/2}} h^{r+1-j} |\log(h)|, \qquad j = 0, 1, \tag{2.33}$$

$$\|p(t) - \tilde p_h(t)\|_{L^2/\mathbb{R}} \le \frac{C}{t^{(r-1)/2}} h^{r} |\log(h)|. \tag{2.34}$$

Since the constant C depends on the type of element used, the result is stated for a particular kind of MFE methods, but it applies to any kind of MFE method satisfying the LBB condition (2.7), the approximation properties (2.11)–(2.13), as well as negative norm estimates, that is,

$$\|u - s_h\|_{-m} \le C h^{l + \min(m, r-2)} (\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}})$$

for $m = 1, 2$ and $1 \le l \le r$. For these negative norm estimates to hold, it is necessary on the one hand that Ω is of class $\mathcal{C}^{2+m}$, and, on the other hand, that $X_{h,r} \subset H_0^1(\Omega)^d$, so that $X_{h,r}$ consists of continuous functions vanishing on $\partial\Omega$ (i.e., discontinuous elements are excluded).

As pointed out in [18, Remark 4.2], with a much simpler analysis than that needed to prove Theorem 2.2, together with results in [2], the previous result applies to the so-called mini element ($r = 2$) but excluding the case $j = 0$ (L^2 errors) in (2.31) and (2.33).

3. Analysis of fully discrete postprocessed methods.

3.1. The general case. As mentioned in the Introduction, in practice, it is hardly ever possible to compute the MFE approximation exactly, and, instead, some time-stepping procedure must be used to approximate the solution of (2.18)–(2.19). Hence, for some time levels $0 = t_0 < t_1 < \cdots < t_N = T$, approximations $U_h^{(n)} \approx u_h(t_n)$ and $P_h^{(n)} \approx p_h(t_n)$ are obtained. Then, given an approximation $d_t^* U_h^{(n)}$ to $\dot u_h(t_n)$, the fully discrete postprocessed approximation $(\tilde U_h^{(n)}, \tilde P_h^{(n)})$ is obtained as the solution of the following Stokes problem:

$$\left(\nabla \tilde U_h^{(n)}, \nabla\tilde\phi\right) + \left(\nabla \tilde P_h^{(n)}, \tilde\phi\right) = \left(f, \tilde\phi\right) - b\left(U_h^{(n)}, U_h^{(n)}, \tilde\phi\right) - \left(d_t^* U_h^{(n)}, \tilde\phi\right) \quad \forall\, \tilde\phi \in \widetilde X, \tag{3.1}$$

$$\left(\nabla\cdot \tilde U_h^{(n)}, \tilde\psi\right) = 0 \quad \forall\, \tilde\psi \in \widetilde Q, \tag{3.2}$$

where $(\widetilde X, \widetilde Q)$ is as in (2.28)–(2.29). Notice then that $\tilde U_h^{(n)} \in \tilde V$ and it satisfies

$$\tilde A_h \tilde U_h^{(n)} = \tilde\Pi_h \left(f - F\left(U_h^{(n)}, U_h^{(n)}\right) - d_t^* U_h^{(n)}\right). \tag{3.3}$$

For reasons already analyzed in [17] and confirmed in the arguments that follow, we propose

$$d_t^* U_h^{(n)} = \Pi_h f - A_h U_h^{(n)} - B_h\left(U_h^{(n)}, U_h^{(n)}\right) \tag{3.4}$$

as an adequate approximation to the time derivative $\dot u_h(t_n)$, which is very convenient from the practical point of view.

We decompose the errors $u(t) - \tilde U_h^{(n)}$ and $p(t) - \tilde P_h^{(n)}$ as follows,

$$u(t) - \tilde U_h^{(n)} = (u(t) - \tilde u_h(t_n)) + \tilde e_n, \tag{3.5}$$

$$p(t_n) - \tilde P_h^{(n)} = (p(t_n) - \tilde p_h(t_n)) + \tilde \pi_n, \tag{3.6}$$

where $\tilde e_n = \tilde u_h(t_n) - \tilde U_h^{(n)}$ and $\tilde\pi_n = \tilde p_h(t_n) - \tilde P_n$ are the temporal errors of the fully discrete postprocessed approximation $(\tilde U_h^{(n)}, \tilde P_h^{(n)})$. The first terms on the right-hand sides of (3.5)–(3.6) are the errors of the (semidiscrete) postprocessed approximation whose size is estimated in Theorem 2.2. In the present section we analyze the time discretization errors $\tilde e_n$ and $\tilde \pi_n$.

To estimate the size of $\tilde e_n$ and $\tilde\pi_n$, we bound them in terms of

$$e_n = u_h(t_n) - U_h^{(n)},$$

the temporal error of the MFE approximation. We do this for any time-stepping procedure satisfying the following assumption

$$\lim_{k\to 0} \max_{0\le n\le N} \|e_n\|_0 = 0, \quad \text{and} \quad \limsup_{k\to 0} \max_{0\le n\le N} \|e_n\|_1 = O(1), \tag{3.7}$$

where $k = \max\{t_n - t_{n-1} \mid 1 \le n \le N\}$. Bounds for $\|e_n\|_0$ and $\|e_n\|_1$ of size $O(k^2/t_n)$ and $O(k/t_n^{1/2})$, respectively, have been proven for the Crank–Nicolson method in [34] (see also [41]). The arguments in [10] can be adapted to show that, for the two-step BDF, $\|e_n\|_j \le Ck^{2-j/2}/t_n$, for $2 \le n \le N$, $j = 0, 1$ (although in section 5 we shall obtain sharper bounds of $\|e_n\|_1$). For problems in two spatial dimensions, bounds for a variety of methods can be found in the literature (see a summary in [31]).

In the arguments in the present section we use the following inequalities [34, (3.7)] which hold for all $v_h, w_h \in V_h$ and $\phi \in H_0^1(\Omega)^d$:

$$|b(v_h, v_h, \phi)| \le c\|v_h\|_1^{3/2} \|A_h v_h\|_0^{1/2} \|\phi\|_0, \tag{3.8}$$

$$|b(v_h, w_h, \phi)| + |b(w_h, v_h, \phi)| \le c\|v_h\|_1 \|A_h w_h\|_0 \|\phi\|_0, \tag{3.9}$$

$$|b(v_h, w_h, \phi)| + |b(w_h, v_h, \phi)| \le \|v_h\|_1 \|w_h\|_1 \|\phi\|_1. \tag{3.10}$$

PROPOSITION 3.1. *Let* (2.11) *hold for* $l = 2$. *Then, there exists a positive constant* $C = C(\max_{0\le t\le T} \|A_h u_h(t)\|_0)$ *such that*

$$\|\tilde e_n - e_n\|_j \le Ch^{2-j}\big(\|e_n\|_1 + \|e_n\|_1^3 + \|A_h e_n\|_0\big), \quad j = 0, 1, \quad 1 \le n \le N, \tag{3.11}$$

$$\|\tilde \pi_n\|_{L^2(\Omega)/\mathbb{R}} \le C\big(\|\tilde e_n\|_1 + \|e_n\|_1 + \|e_n\|_1^2\big), \qquad 1 \le n \le N. \tag{3.12}$$

Proof. From (3.4) and (2.20) it follows that

$$\dot u_h(t_n) - d_t^* U_h^{(n)} = -A_h e_n + \Pi_h\left(F\left(U_h^{(n)}, U_h^{(n)}\right) - F(u_h(t_n), u_h(t_n))\right), \tag{3.13}$$

so that subtracting (3.3) from (2.30) and multiplying by $\widetilde{A}_h^{-1}$ we get

$$\tilde{e}_n = -\widetilde{A}_h^{-1}\widetilde{\Pi}_h(I-\Pi_h)g + \widetilde{A}_h^{-1}\widetilde{\Pi}_h A_h e_n, \tag{3.14}$$

where $g = F(u_h(t_n), u_h(t_n)) - F(U_h^{(n)}, U_h^{(n)})$. By writing

$$\widetilde{A}_h^{-1}\widetilde{\Pi}_h A_h e_n = e_n + \left(\widetilde{A}_h^{-1}\widetilde{\Pi}_h - A_h^{-1}\right) A_h e_n,$$

and applying (2.14) we get

$$\|\tilde{e}_n - e_n\|_j \le \left\|\widetilde{A}_h^{-1}\widetilde{\Pi}_h(I-\Pi_h)g\right\|_j + Ch^{2-j}\|A_h e_n\|_0, \quad j=0,1. \tag{3.15}$$

Similarly, for g we write

$$\widetilde{A}_h^{-1}\widetilde{\Pi}_h(I-\Pi_h)g = \left(\widetilde{A}_h^{-1}\widetilde{\Pi}_h - A^{-1}\Pi\right)(I-\Pi_h)g + A^{-1}\Pi(I-\Pi_h)g. \tag{3.16}$$

In order to bound the first term on the right-hand side above we first apply (2.14), and then we observe that $\|(I-\Pi_h)g\|_0 \le \|g\|_0$. For the second term on the right-hand side of (3.16), we may use a simple duality argument and (2.8), so that we have $\|\widetilde{A}_h^{-1}\widetilde{\Pi}_h(I-\Pi_h)g\|_j \le Ch^{2-j}\|g\|_0$. Now, by writing g as

$$g = F(e_n, u_h(t_n)) + F(u_h(t_n), e_n) - F(e_n, e_n), \tag{3.17}$$

a duality argument and (3.8)–(3.9) show that

$$\left\|\widetilde{A}_h^{-1}\widetilde{\Pi}_h(I-\Pi_h)g\right\|_j \le Ch^{2-j}\left(\|A_h u_h(t_n)\|_0\|e_n\|_1 + \|e_n\|_1^{3/2}\|A_h e_n\|_0^{1/2}\right).$$

Applying Hölder's inequality to the last term on the right-hand side above, the bound (3.11) follows from (3.14) and (3.15).

For the pressure, subtracting (3.1) from (2.28) and recalling (3.13) we have

$$\left(\tilde{\pi}_n, \nabla\cdot\tilde{\phi}\right) = \left(\nabla\tilde{e}_n, \nabla\tilde{\phi}\right) + \left(g, \tilde{\phi}\right) + \left(\dot{u}_h - d_t^* U_h^{(n)}, \tilde{\phi}\right),$$

for all $\tilde{\phi} \in \tilde{X}$, where g is as in (3.17). Then, thanks to the inf-sup condition (2.7), we have

$$\|\tilde{\pi}_n\|_{L^2(\Omega)/\mathbb{R}} \le C\left(\|\tilde{e}_n\|_1 + \sup_{\tilde{\phi}\in\tilde{X}}\frac{|(g,\tilde{\phi})|}{\|\tilde{\phi}\|_1} + \left\|\dot{u}_h(t_n) - d_t^* U_h^{(n)}\right\|_{-1}\right).$$

Taking into account the expression of g in (3.17) and applying (3.10) it follows that

$$\|\tilde{\pi}_n\|_{L^2(\Omega)/\mathbb{R}} \le C\left(\|\tilde{e}_n\|_1 + \|e_n\|_1(\|u_h(t_n)\|_1 + \|e_n\|_1) + \|\dot{u}_h - d_t^* U_h^{(n)}\|_{-1}\right),$$

so that (3.12) follows by applying Lemma 3.2 below, (2.15), and using the fact that $\|u_h(t_n)\|_1 \le C\|A_h u_h(t_n)\|_0$. □

Lemma 3.2. *Under the hypotheses of Proposition* 3.1, *there exists a constant* $C = C(\max_{0\le t\le T}\|u_h(t)\|_1) > 0$ *such that the following bound holds for* $1 \le n \le N$*:*

$$\left\|\dot{u}_h - d_t^* U_h^{(n)}\right\|_{-1} \le C\left\|A_h^{1/2}e_n\right\|_0\left(1 + \|A_h^{1/2}e_n\|_0\right). \tag{3.18}$$

Proof. Since $\dot u_h - d_t^* U_h^{(n)} \in V_h$, we have

$$\left\| \dot u_h - d_t^* U_h^{(n)} \right\|_{-1} \le C \left\| A_h^{-1/2} \left(\dot u_h - d_t^* U_h^{(n)} \right) \right\|_0,$$

due to (2.15). Thus, in view of (3.13), the lemma is proved if for

$$g = F(u_h(t_n), u_h(t_n)) - F\left(U_h^{(n)}, U_h^{(n)}\right),$$

we show that $\|A_h^{-1/2}\Pi_h g\|_0$ can be bounded by the right-hand side of (3.18). If we write g as in (3.17), a simple duality argument, (3.10), and the equivalence (2.15) show that, indeed,

$$\left\| A_h^{-1/2}\Pi_h g \right\|_0 \le C\|e_n\|_1 \big(\|u_h(t_n)\|_1 + \|e_n\|_1 \big).$$

Since according to (2.15), $\|e_n\|_1$ and $\|A_h^{1/2} e_n\|_0$ are equivalent, then the result follows. ∎

Since we are assuming that the meshes are quasiuniform and, hence, both (2.6) and (2.15) hold, we have $Ch^{2-j}\|A_h e_n\|_0 \le C\|e_n\|_j$ and $Ch\|e_n\|_1 \le C\|e_n\|_0$. Thus, from (3.11)–(3.12) and (3.7) the following result follows.

THEOREM 3.3. *Let* (2.11) *hold for* $l = 2$ *and let* (2.3) *hold. Then there exists a positive constant* C *depending on* $\max\{F_{2,0}(t) \mid 0 \le t \le T\}$, *such that if the errors* $e_n = u_h(t_n) - U_h^{(n)}$, $1 \le n \le N$ *of any approximation* $U_h^{(n)} \approx u_h(t_n)$ *for* $0 = t_0 < \cdots < t_N$ *satisfy* (3.7), *then the (fully discrete) postprocessed approximations* $(\tilde U_h^{(n)}, \tilde P_h^{(n)})$ *solution of* (3.1)–(3.2) *satisfy*

$$\left\| \tilde u_h(t_n) - \tilde U_h^{(n)} \right\|_j \le C \left\| u_h(t_n) - U_h^{(n)} \right\|_j, \quad 1 \le n \le N, \quad j = 0, 1, \tag{3.19}$$

$$\left\| \tilde p_h(t_n) - \tilde P_h^{(n)} \right\|_{L^2(\Omega)/\mathbb{R}} \le C \left\| u_h(t_n) - U_h^{(n)} \right\|_1, \quad 1 \le n \le N, \tag{3.20}$$

for k *sufficiently small, where* $(\tilde u_h(t_n), \tilde p_h(t_n))$ *is the (semidiscrete) postprocessed approximation defined in* (2.28)–(2.29).

3.2. The case of the BDF. Better estimates than (3.19) can be obtained when $\|A_h e_n\|_0$ can be shown to decay with k at the same rate as $\|e_n\|_0$. As mentioned in the introduction this will be shown to be the case of two (fixed time-step) time integration procedures in section 5: the backward Euler method and the two-step BDF [9] (see also [25, section III.1]). We describe them now.

For $N \ge 2$ integer, we fix $k = \Delta t = T/N$, and we denote $t_n = nk$, $n = 0, 1, \dots, N$. For a sequence $(y_n)_{n=0}^N$ we denote

$$Dy_n = y_n - y_{n-1}, \quad n = 1, 2 \dots, N.$$

Given $U_h^{(0)} = u_h(0)$, a sequence $(U_h^{(n)}, P_h^{(n)})$ of approximations to $(u_h(t_n), p_h(t_n))$, $n = 1, \dots N$, is obtained by means of the following recurrence relation:

$$\left(d_t U_h^{(n)}, \phi_h\right) + \left(\nabla U_h^{(n)}, \nabla\phi_h\right) + b\left(U_h^{(n)}, U_h^{(n)}, \phi_h\right) - \left(P_h^{(n)}, \nabla\cdot\phi_h\right) = (f, \phi_h) \quad \forall \phi_h \in X_{h,r}, \tag{3.21}$$

$$\left(\nabla\cdot U_h^{(n)}, \psi_h\right) = 0, \quad \forall\, \psi_h \in Q_{h,r-1}, \tag{3.22}$$

where $d_t = k^{-1}D$ in the case of the backward Euler method and $d_t = k^{-1}(D + \frac{1}{2}D^2)$ for the two-step BDF. In this last case, a second starting value $U_h^{(1)}$ is needed. In the present paper, we will always assume that $U_h^{(1)}$ is obtained by one step of the backward Euler method. Also, for both the backward Euler and the two-step BDF, we assume that $U_h^{(0)} = u_h(0)$, which is usually the case in practical situations.

In order to cope for the minor differences between the two methods, we set

$$l_0 = \begin{cases} 1, & \text{for the backward Euler method,} \\ 2, & \text{for the two-step BDF.} \end{cases} \tag{3.23}$$

Under these conditions, we show in Lemma 5.2 and Theorems 5.4 and 5.7 in section 5 that the errors e_n of these two time integration procedures satisfy that

$$\|e_n\|_0 + t_n\|A_h e_n\|_0 \le \mathcal{C}_{l_0} \frac{k^{l_0}}{{t_n}^{l_0 - 1}}, \quad 1 \le n \le N, \tag{3.24}$$

for a certain constants $\mathcal{C}_1$ and C_2. These are, respectively, the terms between parentheses in (5.23) and (5.33) below, which as Proposition 2.1 above and Lemma 4.3 below show, can be bounded for $T > 0$ fixed. Thus, from Proposition 3.1 and (3.24) the following result follows readily.

THEOREM 3.4. *Under the hypotheses of Proposition* 3.1, *let the approximations* $U_h^{(n)}$, $n = 1, \dots, N$ *be obtained by either the backward Euler method or the two-step BDF under the conditions stated above. Then, there exist positive constants* $C_l' = C(\mathcal{C}_l)$, *for* $l = 1, 2$, *and* k', *such that for* $k < k'$ *the temporal errors* $\tilde{e}_n$ *of the fully discrete postprocessed approximation satisfy that* $\tilde{e}_n = e_n + r_n$, *and*

$$\|r_n\|_j \le C_{l_0}' h^{2-j} \frac{k^{l_0}}{t_n^{l_0}}, \quad j = 0, 1, \quad 1 \le n \le N.$$

We remark that a consequence of the above result is that for these two methods the temporal errors of the postprocessed and MFE approximations are asymptotically the same as $h \to 0$. This allows to use the difference $\gamma_h^{(n)} = \tilde{U}_h^{(n)} - U_h^{(n)}$ as an a posteriori error estimator of the spatial error of the MFE approximation, since, as shown in [15, 16, 17], its size is that of $u(t) - u_h(t)$ so long as the spatial and temporal errors are not too unbalanced.

We also remark that at a price of lengthening the already long and elaborate analysis in the present paper, variable stepsizes could have been considered following ideas in [4], but, for the sake of simplicity we consider only fixed stepsize in the analysis that follows.

4. Technical results.

4.1. Inequalities for the nonlinear term. We now obtain several estimates for the quadratic form $B_h(v, w) = \Pi_h F(u, v)$ that will be frequently used in our analysis. We start by proving an auxiliary result.

LEMMA 4.1. *Let* (2.11) *hold for* $l = 2$, *Then, the following bound holds for any* f_h, g_h, *and* ψ_h *in* V_h*:*

$$|b(f_h, g_h, \psi_h)| + |b(g_h, f_h, \psi_h)| \le C\|A_h f_h\|_0 \|A_h^{-1/2} g_h\|_0 \|A_h \psi_h\|_0. \tag{4.1}$$

Proof. To prove this bound we will use the following identity,

$$I = A^{-1}\Pi A_h + \left(A_h^{-1} - A^{-1}\Pi\right) A_h.$$

It will be applied to either f_h or ψ_h whenever any of their derivatives appears in the expressions of $b(f_h, g_h, \psi_h)$ and $b(g_h, f_h, \psi_h)$. We deal first with the second term on the left-hand side of (4.1). Integrating by parts we may write

$$\begin{aligned} b(g_h, f_h, \psi_h) &= \frac{1}{2}((g_h \cdot \nabla) f_h, \psi_h) - \frac{1}{2}((g_h \cdot \nabla)\psi_h, f_h) \\ &= \frac{1}{2}((g_h \cdot \nabla) A^{-1}\Pi A_h f_h, \psi_h) - \frac{1}{2}((g_h \cdot \nabla) A^{-1}\Pi A_h \psi_h, f_h) \\ &\quad + \frac{1}{2}((g_h \cdot \nabla)\left(A_h^{-1} - A^{-1}\Pi\right) A_h f_h, \psi_h) \\ &\quad - \frac{1}{2}\left((g_h \cdot \nabla)\left(A_h^{-1} - A^{-1}\Pi\right) A_h \psi_h, f_h\right). \end{aligned}$$

Using (2.14) with $j = 1$ and (2.16), the last two terms on the right-hand side above can be bounded by

$$Ch\big(\|A_h f_h\|_0 \|\psi_h\|_\infty + \|A_h \psi_h\|_0 \|f_h\|_\infty\big)\|g_h\|_0 \le Ch\|A_h f_h\|_0 \|A_h \psi_h\|_0 \|g_h\|_0.$$

By writing $\|g_h\|_0 \le \|A_h^{1/2}\|_0 \|A_h^{-1/2} g_h\|_0 \le Ch^{-1}\|A_h^{-1/2} g_h\|_0$, we thus have

$$\begin{aligned} |b_h(g_h, f_h, \psi_h)| &\le \frac{1}{2}\left|\left((g_h \cdot \nabla) A^{-1}\Pi A_h f_h, \psi_h\right)\right| + \frac{1}{2}\left|\left((g_h \cdot \nabla) A^{-1}\Pi A_h \psi_h, f_h\right)\right| \\ &\quad + C\|A_h f_h\|_0 \|A_h \psi_h\|_0 \left\|A_h^{-1/2} g_h\right\|_0. \end{aligned} \tag{4.2}$$

Now, applying Hölder's inequality it easily follows that

$$\begin{aligned} &\left|\left((g_h \cdot \nabla) A^{-1}\Pi f_h, \psi_h\right)\right| \le \\ &\qquad C\|g_h\|_{-1}\left(\left\|A^{-1}\Pi A_h f_h\right\|_2 \|\psi_h\|_\infty + \left\|\nabla A^{-1}\Pi A_h f_h\right\|_{L^6} \|\nabla \psi_h\|_{L^3}\right), \end{aligned}$$

so that, applying (2.16)–(2.17) and regularity estimates for the Stokes problem, and standard Sobolev's inequalities we have

$$\left|\left((g_h \cdot \nabla) A^{-1}\Pi f_h, \psi_h\right)\right| \le C\|g_h\|_{-1} \|A_h f_h\|_0 \|A_h \psi_h\|_0. \tag{4.3}$$

Also, arguing similarly, $|((g_h \cdot \nabla) A^{-1}\Pi A_h \psi_h, f_h)| \le C\|g_h\|_{-1}\|A_h f_h\|_0 \|A_h \psi_h\|_0$, so that from (4.2) and (4.3) it follows that

$$|b(g_h, f_h, \psi_h)| \le C\left(\|g_h\|_{-1} + \left\|A_h^{-1/2} g_h\right\|_0\right) \|A_h f_h\|_0 \|A_h \psi_h\|_0.$$

Now, recalling the equivalence (2.15) we have

$$|b(g_h, f_h, \psi_h)| \le C\left\|A_h^{-1/2} g_h\right\|_0 \|A_h f_h\|_0 \|A_h \psi_h\|_0. \tag{4.4}$$

For the second term on the left-hand side of (4.1), thanks to (2.1) we may write

$$\begin{aligned} |b(f_h, g_h, \psi_h)| &\le \left|\left((f_h \cdot \nabla) A^{-1}\Pi A_h \psi_h, g_h\right)\right| + \left|\left(\left(\nabla \cdot A^{-1}\Pi A_h f_h\right) \psi_h, g_h\right)\right| \\ &\quad + \left|\left((f_h \cdot \nabla)\left(A_h^{-1} - A^{-1}\Pi\right) A_h \psi_h, g_h\right)\right| \\ &\quad + \left|\left(\left(\nabla \cdot \left(A_h^{-1} - A^{-1}\Pi\right) A_h f_h\right) \psi_h, g_h\right)\right|, \end{aligned}$$

so that

$$\begin{aligned} |b(f_h, g_h, \psi_h)| &\le \left(\left\|(f_h \cdot \nabla) A^{-1}\Pi A_h \psi_h\right\|_1 + \left\|\left(\nabla \cdot A^{-1}\Pi A_h f_h\right) \psi_h\right\|_1\right) \|g_h\|_{-1} \\ &\quad + Ch\|A_h f_h\|_0 \|A_h \psi_h\|_0 \|g_h\|_0. \end{aligned}$$

Then, recalling that $\|g_h\|_0 \le Ch^{-1}\|A_h^{-1/2}g_h\|_0$ and (2.15), arguments like those used from (4.2) to (4.4) also show that

$$|b_h(f_h, g_h, \psi_h)| \le C\|A_h^{-1/2}g_h\|_0\|A_h f_h\|_0\|A_h\psi_h\|_0,$$

so that, in view of (4.4), the proof of (4.1) is finished. ∎

LEMMA 4.2. *Under the conditions of Lemma* 4.1, *there exists a constant* $C > 0$ *such that the following bounds hold for* $v_h, w_h \in V_h$

$$\left\|A_h^{j/2}B_h(v_h, w_h)\right\|_0 + \left\|A_h^{j/2}B_h(w_h, v_h)\right\|_0 \le C\left\|A_h^{(j+1)/2}v_h\right\|_0\|A_h w_h\|_0, \tag{4.5}$$

for $j = -2, -1, 0, 1$, *and*

$$\|B_h(v_h, v_h)\|_0 \le C\|v_h\|_1^{3/2}\|A_h v_h\|_0^{1/2}, \tag{4.6}$$

$$\|A_h^{-1}B_h(v_h, v_h)\|_0 \le C\|v_h\|_0\|v_h\|_1. \tag{4.7}$$

Proof. The cases $j = -1, 0$ in (4.5) as well as (4.6) are easily deduced from the fact that for every $v_h \in V_h$, $\|A_h^{1/2}v_h\|_0 = \|\nabla v_h\|_0$, (2.16), and from standard bounds (e.g., (3.8), [34, (3.7)]) .

If we denote $f_h = w_h$, $g_h = v_h$, and, for $\phi_h \in V_h$ $\psi_h = A_h^{-1}\phi_h$, case $j = -2$ in (4.5) is a direct consequence of standard duality arguments and (4.1). Also, arguing by duality the bound (4.7) is a straightforward consequence of well-known bounds for the trilinear form b (e.g., [34, (3.7)]).

Finally, for the case $j = 1$ in (4.5), we argue by duality. For $\phi_h \in V_h$, thanks to (2.1), we have

$$b\left(v_h, w_h, A_h^{1/2}\phi_h\right) + b\left(w_h, v_h, A_h^{1/2}\phi_h\right) = -b\left(v_h, A_h^{1/2}\phi_h, w_h\right) - b\left(w_h, A_h^{1/2}\phi_h, v_h\right),$$

so that, by denoting $g_h = A_h^{1/2}\phi_h$, the case $j = 1$ in (4.5) is a direct consequence of (4.1). ∎

4.2. Further a priori estimates for the finite-element solution. We maintain the notation tacitly introduced in Proposition 2.1,

$$F_{l,r} = t^{r+2(l-1)}\left\|A_h^{r/2}\frac{d^l}{ds^l}u_h(t)\right\|_0^2, \quad \text{and} \quad I_{l,r} = \int_0^t s^{r+2l-3}\left\|A_h^{r/2}\frac{d^l}{ds^l}u_h(t)\right\|_0^2 \mathrm{d}s.$$

LEMMA 4.3. *Under the conditions of Proposition* 2.1, *there exists a positive constant* $\tilde M_4$ *such that for* $0 \le t \le T$ *the following bounds hold:*

$$F_{2,-2}(t) = \left\|A_h^{-1}\ddot u_h(t)\right\|_0 \le \tilde M_4, \tag{4.8}$$

$$I_{3,-3}(t) = \int_0^t \left\|A_h^{-3/2}\dddot u_h(s)\right\|_0^2 \mathrm{d}s \le \tilde M_4, \tag{4.9}$$

$$I_{2,2}(t) = \int_0^t s^3\|A_h\ddot u_h(s)\|_0^2 \mathrm{d}s \le \tilde M_4, \tag{4.10}$$

$$I_{3,-1}(t) = \int_0^t s^2\left\|A_h^{-1/2}\dddot u_h(s)\right\|_0^2 \mathrm{d}s \le \tilde M_4, \tag{4.11}$$

$$I_{3,1}(t) + F_{2,2}(t) = \int_0^t s^4\left\|A_h^{1/2}\dddot u_h(s)\right\|_0^2 \mathrm{d}s + t^4\|A_h\ddot u_h(t)\|_0^2 \le \tilde M_4. \tag{4.12}$$

Proof. Taking derivatives with respect to t in (2.20) and multiplying by A_h^{-1} we have

$$A_h^{-1}\ddot u_h = A_h^{-1}\Pi_h f_t - \dot u_h - A_h^{-1}\big(B_h(\dot u_h, u_h) + B_h(u_h, \dot u_h)\big).$$

Applying Lemma 4.2 we have

$$\left\|A_h^{-1}\left(B_h(\dot u_h, u_h) + B_h(u_h, \dot u_h)\right)\right\|_0 \le C\|A_h u_h\|_0 \left\|A_h^{-1/2}\dot u_h\right\|_0 \le C\|A_h u_h\|_0\|\dot u_h\|_0,$$

so that (4.8) follows from (2.3), (2.23), and (2.24) with $r = 0$.

We now prove (4.9). Taking derivatives twice with respect to t in (2.20) and multiplying by $A_h^{-3/2}$ we have

$$A_h^{-3/2}\dddot u_h = A_h^{-3/2}\Pi_h f_{tt} - A_h^{-1/2}\ddot u_h - A_h^{-3/2}\frac{d^2}{dt^2}B_h(u_h, u_h). \tag{4.13}$$

Taking into account that for $v_h \in V_h$, we have $\|A_h^{-3/2}v_h\|_0 \le C\|A_h^{-1}v_h\|_0$, and that

$$\frac{d^2}{dt^2}B_h(u_h, u_h) = B_h(\ddot u_h, u_h) + 2B_h(\dot u_h, \dot u_h) + B_h(u_h, \ddot u_h), \tag{4.14}$$

then, applying the bound (4.5) for $j = -2$ with $v_h = \ddot u_h$, and $w_h = u_h$, on the one hand, and, on the other, (4.7) with $v_h = \dot u_h$, it follows

$$\left\|A_h^{-3/2}\frac{d^2}{dt^2}B_h(u_h, u_h)\right\|_0 \le C\|A_h u_h\|_0\left\|A_h^{-1/2}\ddot u_h\right\|_0 + \|\dot u_h\|_0\|\dot u_h\|_1,$$

so that the bound (4.9) follows from (4.13), (2.3), and the fact that $A_h^{-3/2}$ is bounded independently of h, together with (2.23), (2.24) with $r = 0$, (2.26) with $r = 1$, and (2.27) with $r = -1$.

We now prove (4.10). Taking derivatives twice with respect to t in (2.18) and then setting $\phi = t^3 A_h\ddot u_h$, we have

$$\frac{1}{2}t^3\frac{d}{dt}\left\|A_h^{1/2}\ddot u_h\right\|_0^2 + t^3\|A_h\ddot u_h\|_0^2 = t^3\left(f_{tt} - \frac{d^2}{dt^2}B_h(u_h, u_h), A_h\ddot u_h\right).$$

Since $|(f_{tt} - \frac{d^2}{dt^2}B_h(u_h, u_h), A_h\ddot u_h)| \le \|f_{tt}\|_0^2 + \|\frac{d^2}{dt^2}B_h(u_h, u_h)\|_0^2 + \frac{1}{2}\|A_h\ddot u_h\|_0^2$, it follows that

$$\begin{aligned}\frac{d}{dt}\left(t^3\|A_h^{1/2}\ddot u_h\|_0^2\right) + t^3\|A_h\ddot u_h\|_0^2 \le 2t^3\|f_{tt}\|_0^2 \left\|\frac{d^2}{dt^2}B_h(u_h, u_h)\right\|_0^2 \\ +\ 3t^2\left\|A_h^{1/2}\ddot u_h\right\|_0^2.\end{aligned} \tag{4.15}$$

Now recall (4.14) and apply (4.5) with $v_h = \ddot u_h$, and $w_h = u_h$, on the one hand, and, on the other, (4.6) with $v_h = \dot u_h$ to get

$$\left\|\frac{d^2}{dt^2}B_h(u_h, u_h)\right\|_0 \le C\left(\|A_h u\|_0\left\|A_h^{1/2}\ddot u\right\|_0 + \left\|A_h^{1/2}\dot u\right\|_0^{3/2}\|A_h\dot u\|_0^{1/2}\right),$$

so that, in view of (2.23)–(2.25) it follows that

$$t^3\left\|\frac{d^2}{dt^2}B_h(u_h, u_h)\right\|_0^2 \le C\left(1 + t^{1/2}\right)\tilde M_3^4. \tag{4.16}$$

Integrating with respect to t in (4.15) and taking into account (2.3), (2.25) with $r = 1$, and (4.16), the bound (4.10) follows.

To prove (4.12), we take derivatives twice with respect to t in (2.18) and then we set $\phi = t^4 A_h \dddot{u}_h$, so that

$$t^4 \left\| A_h^{1/2} \dddot{u}_h \right\|_0^2 + \frac{1}{2} t^4 \frac{d}{dt} \|A_h \ddot{u}_h\|_0^2 = t^4 \left(f_{tt} - \frac{d^2}{dt^2} B_h(u_h, u_h), A_h \dddot{u}_h \right),$$

and, since $\|A_h^{1/2} \Pi_h f_{tt}\|_0 \le C \|f_{tt}\|_1$,

$$\begin{aligned} t^4 \left\| A_h^{1/2} \dddot{u}_h \right\|_0^2 + \frac{d}{dt} \left(t^4 \|A_h \ddot{u}_h\|_0^2 \right) \le & 4t^3 \|A_h \ddot{u}_h\|_0^2 \\ & + 2t^4 \left(C^2 \|f_{tt}\|_1^2 + \left\| A_h^{1/2} \frac{d^2}{dt^2} B_h(u_h, u_h) \right\|_0^2 \right). \end{aligned} \tag{4.17}$$

Applying (4.5) to bound the third term on the right-hand side above we have

$$\begin{aligned} t^4 \left\| A_h^{1/2} \frac{d^2}{dt^2} B_h(u_h, u_h) \right\|_0^2 & \le C t^4 \left(\|A_h u_h\|_0^2 \|A_h \ddot{u}_h\|_0^2 + \|A_h \dot{u}_h\|_0^4 \right) \\ & \le C t \tilde{K} \left(1 + \tilde{M}_4^2 \right) \left(t^3 \|A_h \ddot{u}_h\|_0^2 + t \|A_h \dot{u}_h\|_0^2 \right), \end{aligned} \tag{4.18}$$

the last inequality being a consequence of (2.23) and (2.24) with $r = 2$. Thus, integrating with respect to t in (4.17) and applying (2.3), (2.26) with $r = 2$, (4.18), and (4.10), the bound (4.12) follows.

Finally, since standard spectral theory of positive self-adjoint operators shows that $\|A_h^{-1/2} \dddot{u}_h\|_0^2 \le \|A_h^{-3/2} \dddot{u}_h\|_0 \|A_h^{1/2} \dddot{u}_h\|_0$, by applying Hölder's inequality the bound (4.11) follows from (4.9) and (4.12). ▯

5. Error estimates. In this section we obtain error estimates for the temporal errors e_n of the two BDF described in section 3.2, the backward Euler method and the two-step formula (3.21)–(3.22), for which, an equivalent formulation is

$$d_t U_h^{(n)} = -A_h U_h^{(n)} - B_h \left(U_h^{(n)}, U_h^{(n)} \right) + \Pi_h f(t_n), \quad l_0 \le n \le N. \tag{5.1}$$

We remark that although higher regularity was required in section 2.3, in what follows it is only required that Ω is of class $\mathcal{C}^2$ and that (2.11)–(2.13) hold for $l = 2$.

A simple calculation shows that for a sequence $(y_n)_{n=0}^N$ in V_h,

$$\sum_{j=l}^n (y_j, D y_j) = \frac{1}{2} \|y_n\|_0^2 - \frac{1}{2} \|y_{l-1}\|_0^2 + \frac{1}{2} \sum_{j=l}^n \|D y_j\|_0^2, \quad 1 \le l \le n \le N, \tag{5.2}$$

and, for $2 \le l \le n \le N$, (see, e.g., [10, (2.4b)])

$$\begin{aligned} \sum_{j=l}^n \left(y_j, \left(D + \frac{1}{2} D^2 \right) y_j \right) = & \frac{1}{4} \|y_n\|_0^2 + \frac{1}{4} \|y_n + D y_n\|_0^2 + \frac{1}{4} \sum_{j=l}^n \|D^2 y_j\|^2 \\ & - \frac{1}{4} \|y_{l-1}\|_0^2 - \frac{1}{4} \|y_{l-1} + D y_{l-1}\|_0^2. \end{aligned} \tag{5.3}$$

As mentioned in section 3.2, we shall assume that $e_0 = 0$ although in some of the previous lemmas this condition will not be required. It must be noticed that $e_0 = 0$ is not a serious restriction, since, on the one hand, it is usually satisfied in practice, and, on the other hand, were it not satisfied, there are standard ways to show that the effect of $e_0 \neq 0$ decays exponentially with time.

The finite-element approximation u_h to the velocity satisfies

$$d_t u_h(t_n) + A_h u_h(t_n) + B_h(u_h(t_n), u_h(t_n)) - \Pi_h f(t_n) = \tau_n, \tag{5.4}$$

where

$$\tau_n = d_t u_h(t_n) - \dot{u}_h(t_n) = \frac{1}{k}\int_{t_{n-1}}^{t_n} (t - t_{n-1})\ddot{u}_h(t)\,\mathrm{d}t, \quad n = 1, 2, \ldots, N, \tag{5.5}$$

for the backward Euler method, and, for the two-step BDF,

$$\tau_n = \frac{1}{k}\int_{t_{n-2}}^{t_n} \left(2(t - t_{n-1})_+ - \frac{1}{2}(t - t_{n-2})\right)\ddot{u}_h(t)\,\mathrm{d}t, \tag{5.6}$$

where for $x \in \mathbb{R}$, $x_+ = \max(x, 0)$, and, also,

$$\tau_n = \frac{1}{2k}\int_{t_{n-2}}^{t_n} \left(2{(t - t_{n-1})_+}^2 - \frac{1}{2}(t - t_{n-2})^2\right)\frac{d^3}{dt^3}u_h(t)\,\mathrm{d}t. \tag{5.7}$$

Subtracting (5.1) from (5.4), we obtain that the temporal error e_n satisfies

$$d_t e_n + A_h e_n + B_h(e_n, u_h(t_n)) + B\left(U_h^{(n)}, e_n\right) = \tau_n, \qquad n = 2, 3, \ldots, N. \tag{5.8}$$

We shall now prove a result valid for both the backward Euler method and the two-step BDF.

LEMMA 5.1. *Fix $T > 0$ and $M > 0$. Then, there exist positive constants k_0 and C, such that for any for $k \le k_0$ with $Nk = T$, and any four sequences $(Y_n)_{n=0}^N$, $(V_n)_{n=0}^N$, $(W_n)_{n=0}^N$, and $(g_n)_{n=0}^N$ in V_h satisfying*

$$\max\left(\|A_h V_n\|, \|A_h W_n\|\right) \le M, \quad n = 0, 1, \ldots, N, \tag{5.9}$$

and

$$d_t Y_i + A_h Y_i + B_h(Y_i, V_i) + B_h(W_i, Y_i) = g_i, \qquad i = l_0, \ldots, N, \tag{5.10}$$

where l_0 is the value defined in (3.23), the following bound holds for $n = l_0, \ldots, N$, and $j = -2, -1, 0, 1, 2$.

$$\begin{aligned}\left\|A_h^{j/2} Y_n\right\|_0^2 &+ k\sum_{i=l_0}^{n}\left\|A_h^{(j+1)/2} Y_i\right\|_0^2 \\ &\le C^2\left(\left\|A_h^{j/2} Y_0\right\|_0^2 + (l_0 - 1)\left\|A_h^{j/2} Y_1\right\|_0^2 + k\sum_{i=l_0}^{n}\left\|A_h^{(j-1)/2} g_i\right\|_0^2\right).\end{aligned} \tag{5.11}$$

When $j = 0$, condition (5.9) can be relaxed to $\|A_h V_n\| \le M$, for $n = 0, 1 \ldots, N$.

Proof. Take inner product with $A^j Y_i$ in (5.10) so that we have

$$\left(d_t A_h^{j/2} Y_i, A_h^{j/2} Y_i\right) + \left\| A_h^{(j+1)/2} Y_i \right\|_0^2 \le \left| \left(Z_i, A_h^j Y_i\right) \right| + \left| \left(A_h^{(j-1)/2} g_i, A_h^{(j+1)/2} Y_i\right) \right|, \tag{5.12}$$

where

$$Z_i = B(Y_i, V_i) + B(W_i, Y_i). \tag{5.13}$$

Applying Hölder's inequality to the last term on the right-hand side of (5.12) and rearranging terms we have

$$\left(d_t A_h^{j/2} Y_i, A_h^{j/2} Y_i\right) + \frac{1}{2} \left\| A_h^{(j+1)/2} Y_i \right\|_0^2 \le \left| \left(Z_i, A_h^j Y_i\right) \right| + \frac{1}{2} \left\| A_h^{(j-1)/2} g_i \right\|_0^2. \tag{5.14}$$

For $j > -2$, we write $(Z_i, A_h^j Y_i) = (A_h^{(j-1)/2} Z_i, A_h^{(j+1)/2} Y_i)$, so that applying Hölder's inequality and Lemma 4.2 with V_i and W_i taking the role of v_h and w_h in (4.5), and recalling (5.9) we have

$$\begin{aligned} \left| \left(Z_i, A_h^j Y_i\right) \right| &\le \left\| A_h^{(j-1)/2} Z_i \right\|_0^2 + \frac{1}{4} \left\| A_h^{(j+1)/2} Y_i \right\|_0^2 \\ &\le C^2 M^2 \left\| A_h^{j/2} Y_i \right\|_0^2 + \frac{1}{4} \left\| A_h^{(j+1)/2} Y_i \right\|_0^2. \end{aligned} \tag{5.15}$$

Notice that when $j = 0$, due to the skew-symmetry property (2.1) of the trilinear form b we have $|(Z_i, Y_i)| = |b(Y_i, V_i, Y_i)|$, so that only $\|A_h V_i\|_0 \le M$ is necessary for (5.15) to hold; that is, no condition on $A_h W_i$ is required. For $j = 2$, on the other hand, we write $(Z_i, A_h^j Y_i) = (A_h^{j/2} Z_i, A_h^{j/2} Y_i)$, so that arguing similarly we have

$$\begin{aligned} \left| \left(Z_i, A_h^{-2} Y_i\right) \right| &\le \left\| A_h^{-1} Z_i \right\|_0 \left\| A_h^{-1} Y_i \right\|_0 \le CM \left\| A_h^{-1/2} Y_i \right\|_0 \left\| A_h^{-1} Y_i \right\|_0 \\ &\le C^2 M^2 \left\| A_h^{-1} Y_i \right\|_0^2 + \frac{1}{4} \left\| A_h^{-1/2} Y_i \right\|_0^2. \end{aligned} \tag{5.16}$$

In all cases, then, from (5.14) it follows that for an appropriate constant $C_0 > 0$

$$\left(d_t A_h^{j/2} Y_i, A_h^{j/2} Y_i\right) + \frac{1}{4} \left\| A_h^{(j+1)/2} Y_i \right\|_0^2 \le C_0^2 M^2 \left\| A_h^{j/2} Y_i \right\|_0^2 + \frac{1}{2} \left\| A_h^{(j-1)/2} g_i \right\|_0^2,$$

so that multiplying this inequality by k and summing from l_0 to n, and recalling (5.2–5.3), after some rearrangements we can write

$$\begin{aligned} &\frac{1}{2l_0} \left\| A_h^{j/2} Y_n \right\|_0^2 + \frac{k}{4} \sum_{i=l_0}^{n} \left\| A_h^{(j+1)/2} Y_i \right\|_0^2 \\ &\qquad \le \frac{1}{2l_0} \left(\left\| A_h^{j/2} Y_{l_0-1} \right\|_0^2 + (l_0 - 1) \left\| A_h^{j/2} (Y_{l_0-1} + DY_{l_0-1}) \right\|_0^2 \right) \\ &\qquad\quad + C_0^2 M^2 k \sum_{i=l_0}^{n} \left\| A_h^{j/2} Y_i \right\|_0^2 + \frac{k}{2} \sum_{i=l_0}^{n} \left\| A_h^{(j-1)/2} g_i \right\|_0^2. \end{aligned} \tag{5.17}$$

A simple calculation shows that

$$\left\| A_h^{j/2} Y_1 \right\|_0^2 + \left\| A_h^{j/2} (Y_1 + DY_1) \right\|_0^2 \le 7 \left\| A_h^{j/2} Y_1 \right\|_0^2 + 3 \left\| A_h^{j/2} Y_0 \right\|_0^2,$$

so that multiplying by $2l_0$ in (5.17) and taking into account that $2l_0/4 \le 1$, for an appropriate constant C' we may write

$$\left\|A_h^{j/2}Y_n\right\|_0^2 + k\sum_{i=l_0}^n \left\|A_h^{(j+1)/2}Y_i\right\|_0^2 \le 2l_0C_0^2M^2k\sum_{i=l_0}^n \left\|A_h^{j/2}Y_i\right\|_0^2$$
$$+ C'\left(\left\|A_h^{j/2}Y_0\right\|_0^2 + (l_0-1)\left\|A_h^{j/2}Y_1\right\|_0^2 + k\sum_{i=l_0}^n \left\|A_h^{(j-1)/2}g_i\right\|_0^2\right).$$

Now, for k sufficiently small so that $2l_0C_0^2M^2k < 1/2$, applying a standard discrete Gronwall lemma (e.g., [34, Lemma 5.1]) we have that (5.11) holds, with C^2 being $C'\exp(4l_0C_0^2M^2T)$. □

LEMMA 5.2. *Let* (2.11) *hold for* $l = 2$. *Then, there exist positive constants* k_0 *and* c_1 *such that the errors* e_n *satisfy the following bound for* $1 \le n \le N$, $k \le k_0$*:*

$$E_n \equiv \|e_n\|_0^2 + k\sum_{i=l_0}^n \left\|A_h^{1/2}e_i\right\|_0^2 \le c_1^2\left(\|e_0\|_0^2 + (l_0-1)\|e_1\|_0^2 + k^2I_{2,-1}(t_n)\right). \tag{5.18}$$

Proof. We apply Lemma 5.1 to (5.8) in the case where $j = 0$ and $Y_i = e_i$, $V_i = u_h(t_i)$, and $W_i = U_h^{(i)}$. Observe that since we are in the case $j = 0$, only one of the two sequences $(A_hu_h(t_i))_{i=0}^N$, $(A_hU_h^{(i)})_{i=0}^N$ has to be bounded, and, in the present case, the first one is bounded according to (2.23). Thus, we have

$$\|e_n\|_0^2 + k\sum_{i=l_0}^n \left\|A_h^{1/2}e_i\right\|_0^2 \le C^2\left(\|e_0\|_0^2 + (l_0-1)\|e_1\|_0^2 + k\sum_{i=1}^n \left\|A_h^{-1/2}\tau_j\right\|_0^2\right). \tag{5.19}$$

Now, applying Hölder's inequality to the right-hand side of (5.5) we have

$$k\left\|A_h^{-1/2}\tau_j\right\|_0^2 \le \frac{1}{k}\int_{t_{j-1}}^{t_j}(t-t_{j-1})^2\mathrm{d}t\int_{t_{j-1}}^{t_j}\left\|A_h^{-1/2}\ddot{u}_h(t)\right\|_0^2\mathrm{d}t$$
$$= \frac{k^2}{3}\int_{t_{j-1}}^{t_j}\left\|A_h^{-1/2}\ddot{u}_h(t)\right\|_0^2\mathrm{d}t.$$

Similarly, for the two-step BDF, applying Hölder's inequality to the right-hand side of (5.6) we have

$$k\left\|A_h^{-1/2}\tau_n\right\|_0^2 \le \frac{1}{4k}\int_{t_{n-2}}^{t_n}(4(t-t_{n-1})_+ - (t-t_{n-2}))^2\,\mathrm{d}t\int_{t_{n-2}}^{t_n}\left\|A_h^{-1/2}\ddot{u}_h(t)\right\|_0^2\mathrm{d}t$$
$$\le \frac{5}{3}k^2\int_{t_{n-2}}^{t_n}\left\|A_h^{-1/2}\ddot{u}_h(t)\right\|_0^2\mathrm{d}t.$$

Thus, the statement of the lemma follows from (5.19). □

The previous result allows us to deduce a bound for $\|A_he_h\|_0$ in the following lemma. The values of $I_{2,-1}$, $I_{2,0}$ are those of Proposition 2.1.

LEMMA 5.3. *Under the conditions of Lemma* 5.2, *there exist positive constants* $\tilde{k}_0$ *and* $\tilde{c}_0$ *such that if* $e_0 = 0$ *and, in the case of the two-step BDF, also* $U_h^{(1)}$ *is given by the backward Euler method, the following bound holds for* $k \le \tilde{k}_0$ *and* $n = 1, 2, \ldots, N$*:*

$$\|A_he_n\|_0 \le \tilde{c}_0J_n, \tag{5.20}$$

where $J_n = (I_{2,-1}(t_n) + I_{2,0}(t_n))^{1/2}$.

Proof. If $e_0 = 0$, from (5.18) for the Euler method (for which $l_0 = 1$) it follows that

$$\|e_n\|_0 \le c' k (I_{2,-1}(t_n))^{1/2} \le c' k J_n, \quad n = 1, 2 \ldots, N \tag{5.21}$$

for $c' = c_1$. In the case of the two-step BDF, if $U_h^{(1)}$ is obtained by the backward Euler method, if we allow for a larger value of c', it is clear that (5.21) also holds.

Furthermore, it is immediate to check that $\|d_t e_n\|_0 \le 2 l_0 k^{-1} \max_{0\le i\le l_0} \|e_n\|_0$, so that, in view of (5.21), from (5.8) it follows that

$$\|A_h e_n\|_0 \le c'' J_n + \left\| B_h(e_n, u_h(t_n)) + B\left(U_h^{(n)}, e_n\right)\right\|_0 + \|\tau_n\|_0,$$

for some constant $c'' > 0$. For the Euler method, recalling the expression of τ_n in (5.5) we can write

$$\|\tau_n\|_0^2 = \left\| \frac{1}{k}\int_{t_{n-1}}^{t_n} (t - t_{n-1}) \ddot{u}_h \, \mathrm{d}t \right\|_0^2 \le \frac{1}{k^2} \int_{t_{n-1}}^{t_n} \frac{(t - t_{n-1})^2}{t} \, \mathrm{d}t \int_{t_{n-1}}^{t_n} t \|\ddot{u}_h\|_0^2 \, \mathrm{d}t.$$

A simple calculation shows that the first factor on the right-hand side above can be bounded by $k/t_n \le 1$ for $n = 1, 2, \ldots, N$. Furthermore, a similar bound can be also obtained in the case of the two-step BDF. Thus, we have

$$\|A_h e_n\|_0 \le c''' J_n + \left\| B_h(e_n, u_h(t_n)) + B\left(U_h^{(n)}, e_n\right)\right\|_0,$$

for an appropriate constant $c''' > 0$. Finally,

$$\left\| B_h(e_n, u_h(t_n)) + B\left(U_h^{(n)}, e_n\right)\right\|_0 = \|B_h(e_n, u_h(t_n)) + B(u_h(t_n), e_n) - B_h(e_n, e_n)\|_0.$$

Applying (4.5) and (4.6) we get

$$\|A_h e_n\|_0 \le c''' J_n + C \left\| A_h^{1/2} e_n \right\|_0 \|A_h u_h\|_0 + C \left\| A_h^{1/2} e_n \right\|_0^{3/2} \|A_h e_n\|_0^{1/2},$$

and, thus,

$$\frac{1}{2}\|A_h e_n\|_0 \le c''' J_n + C \left\| A_h^{1/2} e_n \right\|_0 \|A_h u_h\|_0 + \frac{1}{2} C^2 \left\| A_h^{1/2} e_n \right\|_0^3.$$

Since $\|A_h u_h\|_0$ is bounded (recall Proposition 2.1) and, arguing as in (5.21), we have $\|A_h^{1/2} e_n\|_0 \le c_1 (k I_{2,-1}(t_n))^{1/2}$, the bound (5.20) follows for k sufficiently small. ∎

Remark 5.1. Observe that from the previous lemma and Proposition 2.1 it follows that $\|A_h U_h^{(n)}\|_0 \le c\tilde{M}_3$ where $c = 1 + \tilde{c}_0\sqrt{2}$. Thus, as long as $e_0 = 0$ and, in case of the two-step BDF, also $U_h^{(1)}$ is given by the Euler method, we may apply Lemma 5.1 for $j \ne 0$, with V_n and W_n replaced by $u_h(t_n)$ and $U_h^{(n)}$, respectively.

We now study the errors $A_h t_n e_n$. We deal first with the backward Euler method. Observe that $D(t_n e_n) = t_n D e_n + k e_{n-1}$, so that multiplying by t_n in (5.8), after some rearrangements we get

$$d_t(t_n e_n) + A_h(t_n e_n) + B_h(t_n e_n, u_h) + B_h\left(U_h^{(n)}, t_n e_n\right) = e_{n-1} + t_n \tau_n. \tag{5.22}$$

We have the following result.

THEOREM 5.4. *Let* (2.11) *hold for* $l = 2$. *Then, there exist positive constants* k_2 *and* c_2 *such that for* $k \le k_2$, *if* $e_0 = 0$ *the errors* $\epsilon_n = t_n e_n$ *satisfy*

$$\left(\|A_h \epsilon_n\|_0^2 + k\sum_{i=1}^{n}\left\|A_h^{3/2}\epsilon_i\right\|_0^2\right)^{1/2} \le c_2 k (I_{2,-1}(t_n) + I_{2,1}(t_n))^{1/2}, \quad 1 \le n \le N. \tag{5.23}$$

Proof. Applying Lemma 5.1 with $j = 2$ to (5.22) we have

$$\|A_h \epsilon_n\|_0^2 + k\sum_{i=1}^{n}\left\|A_h^{3/2}\epsilon_i\right\|_0^2 \le c\left(k\sum_{i=1}^{n-1}\left\|A_h^{1/2}e_i\right\|_0^2 + k\sum_{i=1}^{n}\left\|t_i A_h^{1/2}\tau_i\right\|_0^2\right).$$

The first term on the right-hand side above is bounded Lemma 5.2 by $c_1^2 k^2 I_{2,-1}(t_n)$. For the second one we notice that

$$k\left\|t_i A_h^{1/2}\tau_i\right\|_0^2 = k\left\|\frac{t_i}{k}\int_{t_{i-1}}^{t_i}\frac{t - t_{i-1}}{t} t A_h^{1/2}\ddot{u}_h(t)\,\mathrm{d}t\right\|_0^2,$$

so that, for $i \ge 2$ since $t_i/t \le t_i/t_{i-1} \le 2$, if $t \in (t_{i-1}, t_i)$, we may bound $k\|t_j A_h^{1/2}\tau_j\|_0^2$ by

$$\frac{4}{k}\int_{t_{i-1}}^{t_i}(t - t_{i-1})^2\,\mathrm{d}t\int_{t_{i-1}}^{t_i} t^2\left\|A_h^{1/2}\ddot{u}_h(t)\right\|_0^2\mathrm{d}t \le \frac{4k^2}{3}\int_{t_{i-1}}^{t_i} t^2\left\|A_h^{1/2}\ddot{u}_h(t)\right\|_0^2\mathrm{d}t,$$

and, for $i = 1$, since $t_1/k = 1$, and $(t - t_0)/t = 1$, we may bound $k\|t_i A_h^{1/2}\tau_i\|_0^2$ by

$$k\int_{t_{i-1}}^{t_i}\mathrm{d}t\int_{t_{i-1}}^{t_i} t^2\left\|A_h^{1/2}\ddot{u}_h(t)\right\|_0^2\mathrm{d}t \le k^2\int_{t_{i-1}}^{t_i} t^2\left\|A_h^{1/2}\ddot{u}_h(t)\right\|_0^2\mathrm{d}t.$$

Thus, (5.23) follows. ∎

LEMMA 5.5. *Under the conditions of Lemma* 5.2, *there exist positive constants* k_0' *and* c_1' *such that for* $k \le k_0'$, *if* $e_0 = 0$, *in the backward Euler method* e_1 *satisfies*

$$\left\|A_h^{-1}e_1\right\|_0^2 + k\left\|A_h^{-1/2}e_1\right\|_0^2 \le c_1' k^4 G_1, \tag{5.24}$$

where $G_1 = \max_{0\le s\le k} F_{2,-2}(s)$.

Proof. We take inner product with $2kA_h^{-2}e_1$ in (5.8) for $n = 1$, and recalling (5.2) and taking into account that $e_0 = 0$, after some rearrangements we have

$$\left\|A_h^{-1}e_1\right\|_0^2 + 2k\left\|A_h^{-1/2}e_1\right\|_0^2 \le 2k\left|\left(Z_1, A_h^{-2}e_1\right)\right| + 2k\left|\left(A_h^{-1}\tau_1, A_h^{-1}e_1\right)\right|, \tag{5.25}$$

where Z_1 is as in (5.13) but with Y_1, V_1, and W_1 replaced by e_1, $u_h(t_1)$, and $U_h^{(1)}$, respectively. Thus, arguing as in (5.16)

$$\left(1 - 2kC^2\tilde{M}_3^2\right)\left\|A_h^{-1}e_1\right\|_0^2 + \frac{3}{2}k\left\|A_h^{-1/2}e_1\right\|_0^2 \le 2k\left|\left(A_h^{-1}\tau_1, A_h^{-1}e_1\right)\right|,$$

so that taking into account that

$$2k\left|\left(A_h^{-1}e_1, A_h^{-1}\tau_1\right)\right| \le \left\|A_h^{-1}e_1\right\|_0^2/2 + k^2\left\|A_h^{-1}\tau_1\right\|_0^2,$$

from (5.25) it follows that

$$\left(\frac{1}{2} - 2kC^2\tilde{M}_3^2\right)\left\|A_h^{-1}e_1\right\|^2 + \frac{3}{2}k\left\|A_h^{-1/2}e_1\right\|_0^2 \leq k^2\left\|A_h^{-1}\tau_1\right\|_0^2. \tag{5.26}$$

Recalling the expression of τ_n in (5.5) we can write

$$\left\|A_h^{-1}\tau_1\right\|_0^2 \leq \max_{0\leq t\leq t_1}\left\|A_h^{-1}\ddot{u}_h(t)\right\|_0^2\left(\frac{1}{k}\int_0^k t\,\mathrm{d}t\right)^2 = \frac{k^2}{4}\max_{0\leq t\leq t_1}\left\|A_h^{-1}\ddot{u}_h(t)\right\|_0^2,$$

we then have that (5.24) follows from (5.26) provided that k is sufficiently small. ◻

LEMMA 5.6. *Under the conditions of Lemma* 5.2, *let* $e_0 = 0$ *and let* $U_h^{(1)}$ *be given by the backward Euler method. Then, there exist positive constants* k_0 *and* c_1 *such that the errors* e_n *of the two-step BDF satisfy*

$$E_n' \equiv \left\|A_h^{-1}e_n\right\|_0^2 + k\sum_{j=2}^n\left\|A_h^{-1/2}e_j\right\|_0^2 \leq c_1k^4(G_1 + I_{3,-3}(t_n)), \quad 2\leq n\leq N, \tag{5.27}$$

where G_1 *is given after* (5.24).

Proof. In view of the comments in Remark 5.1, we can apply Lemma 5.1 with $j = -2$ to (5.8), so that, recalling that $e_0 = 0$, we have

$$\left\|A_h^{-1}e_n\right\|^2 + k\sum_{i=2}^n\left\|A_h^{-1/2}e_i\right\|_0^2 \leq c\left(\left\|A_h^{-1}e_1\right\|_0^2 + k\sum_{j=2}^n\left\|A_h^{-3/2}\tau_j\right\|_0^2\right). \tag{5.28}$$

Notice that, as we showed in Lemma 5.5, the first term on the right-hand side above is bounded by $c_1'k^4G_1$. For the second term on the right-hand side of (5.28), in view of (5.7) a simple calculation shows that

$$k\left\|A_h^{-3/2}\tau_j\right\|_0^2 \leq Ck^4\int_{t_{n-2}}^{t_n}\left\|A_h^{-3/2}\frac{d^3u_h(s)}{ds^3}\right\|_0^2\mathrm{d}s.$$

Thus, (5.27) follows. ◻

For any two sequences $(y_n)_{n=0}^\infty$ and $(z_n)_{n=0}^\infty$, it is easy to check that $D(y_nz_n) = y_nDz_n + z_{n-1}Dy_n$, for $n = 1, 2, \ldots$, and, also,

$$D^2(y_nz_n) = y_nD^2z_n + 2Dy_nDz_{n-1} + z_{n-2}D^2y_n.$$

Thus, for the two-step BDF, multiplying (5.8) by t_n and t_n^2 and rearranging terms, for $j = 2, 3, \ldots, N$, we have

$$\begin{aligned} &d_t(t_ne_n) + A_ht_ne_n + B_h(u_h(t_n), t_ne_n) \\ &\qquad - t_nB\left(U_h^{(n)}, t_ne_n\right) = t_n\tau_n + (e_{n-1} + De_{n-1}), \end{aligned} \tag{5.29}$$

and

$$d_t\left(t_n^2e_n\right) + A_ht_n^2e_n + B_h\left(t_n^2e_n, u_h(t_n)\right) - B\left(U_h^{(n)}, t_n^2e_n\right) = t_n^2\tau_j + \sigma_{n-1}, \tag{5.30}$$

where

$$\sigma_{n-1} = (t_n + t_{n-1})(e_{n-1} + De_{n-1}) + ke_{n-2}. \tag{5.31}$$

THEOREM 5.7. *Under the conditions of Theorem* 5.4, *there exist positive constants k_1 and c_2 such that for $k \le k_1$ if $e_0 = 0$ and $U_h^{(1)}$ be given by the backward Euler method, the errors $\epsilon_n = t_n e_n$ and $\epsilon_n' = t_n^2 e_n$ of the two-step BDF satisfy the following bounds for $2 \le n \le N$:*

$$\mathcal{E}_n \equiv \left(\|\epsilon_n\|_0^2 + k \sum_{i=2}^{n} \left\| A_h^{1/2} \epsilon_i \right\|_0^2 \right)^{\frac{1}{2}} \le c_2 k^2 \left(H_1 + I(t_n) \right)^{\frac{1}{2}}, \tag{5.32}$$

$$\mathcal{E}_n' \equiv \left(\|A_h \epsilon_n'\|_0^2 + k \sum_{i=2}^{n} \left\| A_h^{3/2} \epsilon_i' \right\|_0^2 \right)^{\frac{1}{2}} \le c_2 k^2 \left(H_1 + I(t_n) + J_1^2 + I_{3,1}(t_n) \right)^{1/2}, \tag{5.33}$$

where J_1 is given after (5.20), $H_1 = I_{2,-1}(t_1) + G_1$, *where G_1 is given after* (5.24), *and* $I(t) = I_{3,-1}(t) + I_{3,-3}(t)$.

Proof. To prove (5.32) we apply Lemma 5.1 with $j = 0$ to (5.29), so that taking into account that $e_0 = 0$ we have

(5.34)
$$\|\epsilon_n\|_0^2 + k \sum_{j=2}^{n} \left\| A_h^{1/2} \epsilon_j \right\|_0^2 \le c \left(\|\epsilon_1\|_0^2 + k \sum_{i=1}^{n-1} \left\| A_h^{-1/2} (e_i + De_i) \right\|_0^2 + k \sum_{i=2}^{n} \left\| t_i A_h^{-1/2} \tau_i \right\|_0^2 \right).$$

Notice that since $e_i + De_i = 2e_i - e_{i-1}$, the second term on the right-hand side above can be bounded by $7k \sum_{i=1}^{n-1} \|A_h^{-1/2} e_i\|_0^2$, a quantity that has already been bounded in Lemma 5.6. Also, by writing

$$t_i \tau_i = \frac{t_i}{2k} \int_{t_{i-2}}^{t_i} \frac{1}{t} \left(2(t - t_{i-1})_+{}^2 - \frac{1}{2} (t - t_{i-2})^2 \right) t \frac{d^3}{dt^3} u_h(t)\, \mathrm{d}t,$$

and noticing that $t_i / t_{i-2} \le 3$ for $i \ge 3$, and $t_2 / k \le 2$, a straightforward calculation shows that

$$k \left\| A_h^{-1/2} t_i \tau_j \right\|_0^2 \le C k^4 \int_{t_{i-2}}^{t_i} t^2 \left\| A_h^{-1/2} \dddot{u}_h(t) \right\|_0^2 \mathrm{d}t, \quad j = 2, \dots, N. \tag{5.35}$$

Finally, noticing that $\|\epsilon_1\|_0^2 = k^2 \|e_1\|_0^2$ and recalling Lemma 5.2, we have that (5.32) follows from (5.34) and (5.35).

To prove (5.33) we apply Lemma 5.1 with $j = 2$ to (5.30) to get

(5.36)
$$\|A_h \epsilon_n'\|_0^2 + k \sum_{j=2}^{n} \left\| A_h^{3/2} \epsilon_j' \right\|_0^2 \le c \left(\|A_h \epsilon_1'\|_0^2 + k \sum_{i=1}^{n-1} \left\| A_h^{1/2} \sigma_i \right\|_0^2 + k \sum_{i=2}^{n} \left\| t_i^2 A_h^{1/2} \tau_i \right\|_0^2 \right).$$

For the first term on the right-hand side above, in view of Lemma 5.3 we can write

$$\|A_h \epsilon_1'\|_0 = k^2 \|A_h e_1\| \le k^2 \tilde{c}_0 J_1. \tag{5.37}$$

For the second term on the right-hand side of (5.36), we first recall the expression of σ_i in (5.31) and then we notice that for $i \ge 1$ we have that $k \le t_i$, $t_{i+2}/t_i \le 3$ and $t_{i+1}/t_i \le 2$, so that, recalling that $e_0 = 0$, for an appropriate constant $C > 0$ we may write

(5.38)
$$k \sum_{i=1}^{n-1} \left\| A_h^{1/2} \sigma_i \right\|_0^2 \le C \left(k \left\| A_h^{1/2} \epsilon_1 \right\|_0^2 + k \sum_{i=2}^{n-1} \left\| A_h^{1/2} \epsilon_i \right\|_0^2 \right) \le C \left(k^3 \left\| A_h^{1/2} e_1 \right\|_0^2 + \mathcal{E}_n^2 \right),$$

$\mathcal{E}_n$ being the quantity in (5.32). Also, by writing

$$t_i^2\tau_i = \frac{t_i^2}{2k}\int_{t_{i-2}}^{t_i} \frac{1}{t^2}\left(2(t-t_{i-1})_+{}^2 - \frac{1}{2}(t-t_{i-2})^2\right) t^2 \frac{d^3}{dt^3}u_h(t)\,\mathrm{d}t,$$

and noticing that $t_i^2/t_{i-2}^2 \le 9$ for $i \ge 3$, and $t_2^2/k \le 4k$, we get

$$k\left\|A_h^{1/2}t_i^2\tau_i\right\|_0^2 \le Ck^4\int_{t_{i-2}}^{t_i} t^4\left\|A_h^{1/2}\dddot{u}_h(t)\right\|_0^2\mathrm{d}t, \quad j=2,\ldots,N. \tag{5.39}$$

Thus, (5.33) follows from (5.36), (5.37), (5.38), (5.32), (5.18), and (5.39). ◻

Although not strictly necessary for the analysis of the postprocessed approximation, for the sake of completeness we include an error bound for the pressure. We first notice that as a consequence of the LBB condition (2.7) we have that the error $\pi_n = p_h(t_n) - P_h^{(n)}$ satisfies

$$\|\pi_n\|_{L^2(\Omega)/\mathbb{R}} \le \frac{1}{\beta}\sup_{\phi_h\in X_{h,r}}\frac{|(\pi_n,\nabla\cdot\phi_h)|}{\|\phi_h\|_1}.$$

Furthermore subtracting (3.21) from (2.18), we have

$$(\pi_n,\nabla\cdot\phi_h) = \left(\dot{u}_h - d_tU_h^{(n)},\phi_h\right) + (\nabla e_n,\nabla\phi_h) + b(e_n,u_h,\phi_h) + b\left(U_h^{(n)},e_n,\phi_h\right)$$

for all $\phi_h \in X_{h,r}$. Using standard bounds for the trilinear form b (e.g., [34, (3.7)]) we can write

$$\|\pi_n\|_{L^2(\Omega)/\mathbb{R}} \le C\left(\|e_n\|_1 + \left\|\dot{u}_h - d_tU_h^{(n)}\right\|_{-1}\right). \tag{5.40}$$

Recalling the expression of $d_t^*U_h^{(n)}$ in (3.4), we see that $\dot{u}_h - d_tU_h^{(n)} = \dot{u}_h - d_t^*U_h^{(n)}$, so that applying Lemma 3.2 and taking into account the equivalence (2.15) between $\|e_n\|_1$ and $\|A_h^{1/2}e_n\|_0$, we have $\|\pi_n\|_{L^2(\Omega)/\mathbb{R}} \le C\|A_h^{1/2}e_n\|_0$. Since using standard spectral theory of positive self-adjoint operators it is straightforward to show that $\|A_h^{1/2}e_n\|_0 \le C\|e_n\|_0^{1/2}\|A_he_n\|_0^{1/2}$, applying Lemma 5.2 and Theorem 5.4 in the case of the backward Euler method, and Theorem 5.7 in the case of the two-step BDF, we conclude the following result.

THEOREM 5.8. *Under the conditions of Theorem* 5.4, *there exist positive constants k_3 and c_4 such that if $e_0 = 0$ and $U_h^{(1)}$ is obtained by the backward Euler method, the following bound holds for $k < k_3$ and for $n = l_0,\ldots,N$: For the backward Euler method,*

$$\left\|p_h(t_n) - P_h^{(n)}\right\|_{L^2(\Omega)/\mathbb{R}} \le c_4C_1^{1/2}\frac{k}{t_n^{1/2}},$$

where $C_1 = (I_{2,-1}(I_{2,-1}+I_{2,1}))^{1/2}$, and, for the two-step BDF,

$$\left\|p_h(t_n) - P_h^{(n)}\right\|_{L^2(\Omega)/\mathbb{R}} \le c_4C_2^{1/2}\frac{k^2}{t_n^{3/2}},$$

where C_2 is the product of the quantities between parentheses on the right-hand sides of (5.32) *and* (5.33).

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