New Bound for Incremental Constructing Arrangements of Curves

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Abstract

Let $A(\Gamma)$ be the arrangement induced by a set Γ of n unbounded Jordan curves in the plane that intersect each other in at most two points. The upper bound for constructing those arrangements by an incremental method is, up to now, $O(n\lambda_4(n))$. In this paper we improve this bound to $O(n\lambda_3(n))$.

Key words: arrangements of curves, incremental algorithms, zones

1. Introduction

The arrangement of a set Γ of n unbounded Jordan curves, denoted by $A(\Gamma)$, is the plane subdivision induced by the curves in Γ. Let γ_0 be a curve not belonging to Γ. The *zone* of γ_0 in $A(\Gamma)$, denoted by $z(\gamma_0)$, is the set of faces of $A(\Gamma)$ intersected by γ_0.

Up to now, the best known upper bound for the zone of γ_0 in $A(\Gamma)$ is $O(\lambda_4(n)) = O(n\,2^{\alpha(n)})$, obtained from its relation with the complexity of a face in an arrangement of Jordan arcs: by deleting small pieces in the curves of Γ just containing the intersection points with γ_0, all faces of $z(\gamma_0)$ become a part of the same face of an arrangement of Jordan arcs. The total amount of new elements in the arrangement is a constant factor of the size of the arrangement ([3], pg. 23). By using an incremental algorithm, in which curves are inserted in the arrangement one by one, the construction of $A(\Gamma)$ takes $O(n\,\lambda_4(n))$ time. In this paper we improve this bound to $O(n\,\lambda_3(n))$ if the curves intersect each other in at most two points (remember that $\lambda_4(n) = O(n2^{\alpha(n)})$ and $\lambda_3(n) = O(n\alpha(n))$).

The algorithm that computes $A(\Gamma)$ in $O(n\lambda_3(n))$ time is based in three lemmas.

The first lemma considers the frontier of a face in the external zone composed by the edges of the face not belonging to γ_0 and says that if p and q are the extreme points of the frontier of a face in the external zone, then the segment $\overline{pq}$ in γ_0 is an edge of the arrangement. This property allows to travel through the frontiers of the successive faces of the external zone thus obtaining the intersection points of γ_0 with the curves in the arrangement just in the order they appear in γ_0. Therefore, the time complexity for the insertion of γ_0 depends on the complexity of the external zone which is $O(\lambda_3(n))$.

The second lemma considers the case in which the infinite extremes of all the curves that intersect γ_0 in two points are in the same side of γ_0, and proves that the complexity of the zone in this side of γ_0 is $O(\lambda_3(n))$. We call *external zone* to this part of the zone.

Last lemma shows that it is always possible to list the curves of Γ in order $\gamma_1, \gamma_2, \ldots, \gamma_n$, such that each γ_i with $2 \le i \le n$ verifies that the infinite extremes of every curve in the set $\{\gamma_1, \gamma_2, \ldots, \gamma_{i-1}\}$ that intersects γ_i in zero or two points are in the same side of γ_i.

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1 Partially supported by *MCYT* TIC2003-08933-C02-01
2 Partially supported by *MCYT* TIC2002-01541

2. Notation and definitions

Every curve in Γ intersecting γ_0 is decomposed by γ_0 in two or three connected pieces. The unbounded pieces will be called *branches*. Branches will be oriented, being the starting point the intersection point with γ_0. This point will be called the *base* of the branch. Base of a branch r will be noted by $\overline{r}$. Every curve in Γ intersecting γ_0 in two points give rise to two branches.

If all branches of the curves intersecting γ_0 in two points are in the same side of γ_0 we will say that γ_0 is a pseudo-convex curve in the arrangement and the *external zone* of γ_0 will be the half-zone containing all these branches. From now on γ_0 will be pseudo-convex in the arrangement $A(\Gamma)$ and we will only consider the external zone of γ_0, noted by $z^+(\gamma_0)$.

We will call *external frontier* every connected component obtained by deleting edges contained in γ_0 in the frontier of a face of the external zone. As a consequence, extreme points of the external frontiers belong to γ_0.

Without lost of generality, we will suppose that every face of the external zone, but the first and the last, are bounded. If this where not the case, one can add a curve to Γ that intersects only the unbounded faces.

3. The external frontier

Lemma 1 *The extreme points of an external frontier are the extreme points of an edge contained in γ_0 in the arrangement* $\mathrm{A}(\Gamma \cup \{\gamma_0\})$.

PROOF. Let F^* be the external frontier of a face in the external zone. Let p and q be the extreme points of F^*. If $\overline{pq}$, the arc of γ_0 with extreme points p and q, were not an edge in the mentioned arrangement, $\overline{pq}$ would contain at least one intersection point c in its interior. As $\overline{pq}$ is contained in γ_0, then necessarily c is the base of a branch. Because branches are not bounded, c and F^* intersect each other. This is a contradiction with the fact that F^* is an external frontier of a face.

4. External zone complexity

In this section one proves the $O(\lambda_3(n))$ complexity of the external zone of γ_0 in $A(\Gamma)$. It is proved by bounding the number of its edges.

The idea is to visit the external frontiers in the order they appear along γ_0 and write down, for every edge encountered, the branch or supporting curve. Let $U = \langle u_1, u_2, \ldots, u_m\rangle$ be the obtained list of symbols. One realizes that U is a Davenport-Schinzel sequence.

(n, s) Davenport-Schinzel sequences have to satisfy the following properties (see [4], pg. 1):

(i) To contain at most n different symbols.
(ii) $u_i \neq u_{i+1}$ for every $1 \leq i < m$.
(iii) there cannot be $s + 2$ indices $1 \leq i_1 < i_2 < \ldots < i_{s+2} \leq m$ such that $u_{i_1} = u_{i_3} = u_{i_5} \ldots = a$, $u_{i_2} = u_{i_4} = u_{i_6} \ldots = b$ and $a \neq b$.

In order to have condition (ii) satisfied one uses different symbols for each side of the same branch. In this way, n curves give rise to $4n$ symbols at most (two for every of the $2n$ possible branches).

Proposition 2 *U is a $(4n, 3)$ Davenport-Schinzel sequence.*

PROOF.

(i) $4n$ symbols corresponds to the, at most, $2n$ branches.
(ii) In the same face there are not two consecutive edges from the same branch (nor curve, if it does not intersect γ_0). This is due to the fact that curves intersect in a transversal way. When passing from one face to the consecutive one, there are neither two equal consecutive symbols because they corresponds to the two sides of the same branch, marked with different symbols.
(iii) Third condition is due to the fact that two branches intersect in at most two points. One can prove that, under this condition, the maximum length of an alternating sub-chain of two symbols is 4.

Lemma 3 *The external zone of γ_0 has $O(\lambda_3(n))$ complexity.*

PROOF. By proposition 2, it suffices to note that $\lambda_3(4n) = O(\lambda_3(n))$.

5. Insertion order

In this section it's seen how to sort the set of curves Γ in such a way that, if they are inserted one by one in this order in the arrangement, curve γ_i is pseudo-convex with respect to the previous γ_j inserted curves, $2 \leq j \leq i-1$.

Without lost of generality one can suppose that the external zone of every curve is always to its left.

The idea for sorting is the following: a curve γ is fixed while the others are classified with respect to γ into two sets that will be recursively sorted: the set of curves that must be inserted before γ in the arrangement and the set of curves that have to be inserted after γ. We insert before γ all the curves that give rise to unbounded branches to the left of γ (γ must be oriented before doing this classification). Let $|\delta \cap \gamma|$ denote the number of intersection points between δ and γ.

Procedure for sorting the set Γ;

(1) If $\Gamma = \emptyset$ nothing to do. Return $\emptyset$.
(2) Select at random a curve $\gamma \in \Gamma$.
(3) If it is still not oriented, select an orientation for γ.
(4) Classify the rest of curves into two sets $C_1(\gamma)$ and $C_2(\gamma)$ containing the curves of Γ - $\{\gamma\}$ that have to be inserted in the arrangement before and after γ respectively:
- Curve δ is included in $C_1(\gamma)$ whenever one of the three following conditions holds:
  $|\delta \cap \gamma| = 0$ and δ is to the left of γ.
  $|\delta \cap \gamma| = 1$.
  $|\delta \cap \gamma| = 2$ and the branches of δ are to the left of γ.
- Curve δ is included in $C_2(\gamma)$ if it has not been included in $C_1(\gamma)$, that is:
  $|\delta \cap \gamma| = 0$ and δ is to the right of γ or
  $|\delta \cap \gamma| = 2$ and the branches of δ are to the right of γ.

(5) Orient the curves in $C_2(\gamma)$ depending on the orientation of γ in such a way that the infinite extremes of γ lay to the left of each curve.
(6) Recursively repeat the procedure for $C_1(\gamma)$ and $C_2(\gamma)$, thus obtaining the sorted lists $L_1(\gamma)$ and $L_2(\gamma)$ respectively.
(7) The final sorting in Γ is $L_1(\gamma) + [\gamma] + L_2(\gamma)$, being "+" the concatenation of lists operation.

Definition 4 *Curve γ is called reference curve of the process. Sets $C_1(\gamma)$ and $C_2(\gamma)$ determined by γ are called associated sets.*

One can proof that the orientation that a reference curve γ determine in its associated set $C_2(\gamma)$ do not prevent a future sorting of this set. No care is necessary with curves in $C_1(\gamma)$ because they appear in the arrangement before γ.

Observation: Every curve is, at some point during the execution of the procedure, a reference curve having the corresponding associated sets (may be empty some of them).

In order to facilitate the proof of the pseudo-convexity of every curve with respect to the preceding ones, one consider a surrounding pseudo-circumference containing in its interior all intersection points of the arrangement. This extra curve represents the infinity and intersects every curve in two points.

Definition 5 *The intersection points between a curve $\gamma \in \Gamma$ and the pseudo-circumference are called the extreme points of γ. They are considered points at infinity.*

Definition 6 *Every curve γ divides the pseudo-circumference in two pieces called vaults. Once the curve γ is oriented, the corresponding vaults becomes left vault and right vault of γ. The left vault is denoted by $\widehat{\gamma}$.*

Vault concept allow us to define pseudo-convexity in the following way:

Definition 7 *γ is pseudo-convex with respect to a set of curves if all of them have at least one intersection point with $\widehat{\gamma}$.*

Proposition 8 *A curve γ and its associated sets verify the following properties:*

(i) *γ is pseudo-convex with respect to curves in $C_1(\gamma)$.*
(ii) *Curves in $C_2(\gamma)$ are pseudo-convex with respect to γ.*
(iii) *Curves in $C_2(\gamma)$ are pseudo-convex with respect to those in $C_1(\gamma)$.*

PROOF. Items (i) and (ii) are verified by construction. For item (iii), let be $\gamma_1 \in C_1(\gamma)$. By construction, γ_1 have at least an intersection point with $\widehat{\gamma}$. Let be $\gamma_2 \in C_2(\gamma)$. By construction, $\widehat{\gamma} \subset \widehat{\gamma_2}$. That means that γ_1 have at least an intersection point with $\widehat{\gamma_2}$. Therefore, by definition 7, γ_2 is pseudo-convex with respect to γ_1.

Finally, we can proof the third lemma:

Lemma 9 *If $\gamma_1, \gamma_2, \ldots, \gamma_n$ is the sorted set of curves given by the sorting procedure, then γ_i is pseudo-convex with respect to $\gamma_1, \gamma_2, \ldots, \gamma_{i-1}$ for every $2 \leq i \leq n$.*

PROOF.
Let be $2 \leq i \leq n$ and let γ_j be a curve with $j < i$. By construction there is a reference curve γ_k, $j \leq k \leq i$, separating γ_i and γ_j, thus having one of the three following situations:

(i) $\gamma_k = \gamma_i$ and $\gamma_j \in C_1(\gamma_k)$.

(ii) $\gamma_k = \gamma_j$ and $\gamma_i \in C_2(\gamma_k)$.

(iii) $\gamma_j \in C_1(\gamma_k)$ and $\gamma_i \in C_2(\gamma_k)$.

Taking into account proposition 8 one verifies in all three cases that γ_i is pseudo-convex with respect to γ_j.

As a consequence of the previous lemmas, one verifies the main theorem of the paper:

Theorem 10 *The arrangement $\mathrm{A}(\Gamma)$ of unbounded Jordan curves that intersect each other in at most two points can be computed incrementally in $O(n\lambda_3(n))$ time.*

6. Applications

This result can be applied in geometric location problems. In particular in covering problems with geometric figures as wedges, circular annulus or strips.

In these problems one wants to locate the best position for a geometric figure in the plane in order to cover as much points of a given set of points as possible. In the solution of these problems in the dual space, arrangements as the considered in this paper have to be managed. (Details can be found in [1]).

7. Open problems

An open problem is to determine the complexity of the internal zone of this kind of arrangements. Internal zone is the half-zone which is not the external zone. It is an open problem for $s \geq 1$ for specific families of curves. For arbitrary curves the known complexity is $O(\lambda_{s+2}(n))$ ([4], pág. 125).

Note that for $s = 1$ one have a known problem that is open from more than ten years (as mentioned M. Sharir in the Dagstuhl Workshop [2]):

Problem 11 *Given a circle and an arrangement of pseudo-lines determine the complexity of the half-zone of the circle contained in its interior.*

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