Maximum Weight Triangulation of A Special Convex Polygon [1]

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Abstract

In this paper, we investigate the maximum weight triangulation of a special convex polygon, called 'semi-circled convex polygon'. We prove that the maximum weight triangulation of such a polygon can be found in $O(n^2)$ time.

1 Introduction

Triangulation of a set of points is a fundamental structure in computational geometry. Among different triangulations, the *minimum weight triangulation* (MWT for short) of a set of points in the plane attracts special attention [1,2,3]. The construction of the MWT of a point set is still an outstanding open problem. When the given point set is the set of vertices of a convex polygon (so-called *convex point set*), then the corresponding MWT can be found in $O(n^3)$ time by dynamic programming [1,2].

On the contrast, there is not much research done on *maximum weight triangulation* (MAT for short). From the theoretical viewpoint, the maximum weight triangulation problem and the minimum weight triangulation problem attracts equally interest, and one seems not to be easier than the other. The study of maximum weight triangulation will help us to understand the nature of optimal triangulations.

The first work in the MAT [4] showed that if an n-sided polygon P inscribed on a circle, then $MAT(P)$ can be found in $O(n^2)$ time.

In this paper, we study the MAT of the following special convex polygon: let P be a convex $n-$sided polygon such that the entire polygon P is contained inside the circle with an edge of P as diameter. We call such a polygon as *semi-circled*. We propose an $O(n^2)$ algorithm for computing the $MAT(P)$ of a semi-circled convex polygon P. Recall that a straightforward dynamic programming method will take $O(n^3)$ to find $MAT(P)$.

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2 Preliminaries

Let S be a set of points in the plane. A *triangulation* of S, denoted by $T(S)$, is a maximal set of non-crossing line segments with their endpoints in S. It follows that the interior of the convex hull of S is partitioned into non-overlapping triangles. The weight of a triangulation $T(S)$ is given by

$$\omega(T(S)) = \sum_{\overline{s_i s_j} \epsilon T(S)} \omega(\overline{i,j}),$$

where $\omega(\overline{i,j})$ is the Euclidean length of line segment $\overline{s_i s_j}$.

A *maximum weight triangulation* of S ($MAT(S)$) is defined as for all possible $T(S)$, $\omega(MAT(S)) = max\{\omega(T(S))\}$.

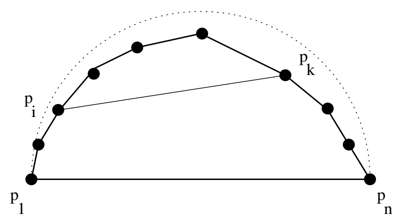

Figure 1: An illustration of semi-circled convex polygon.

Let P be a convex polygon (whose vertices form a *convex point set*) and $T(P)$ be its triangulation. A *semi-circled convex polygon* P is a special convex polygon such that its longest edge is the diameter and all the edges of P lie inside the semi-circle with this longest edge as diameter. (Refer to Figure 1).

The following two properties are easy to verify for a semi-circled convex polygon $P = (p_1, p_2, ..., p_k, ..., p_{n-1}, p_n)$.

Property 1: Let $\overline{p_i p_k}$ for $1 \leq i < k \leq n$ be an edge in P. Then the area bounded by $\overline{p_i p_k}$ and chain $(p_i, ..., p_k)$ is a semi-circled convex polygon.

Property 2: In P, edge $\overline{p_i p_j}$ for $1 \leq i < j < n$ or $1 < i < j \leq n$ is shorter than $\overline{p_1 p_n}$.

3 Finding an MAT of a semi-circled convex polygon

Lemma 1 *Let $P = (p_1, p_2, ..., p_k, ..., p_{n-1}, p_n)$ be a semi-circled convex polygon. Then, one of the edges: $\overline{p_1 p_{n-1}}$ and $\overline{p_n p_2}$ must belong to its maximum weight triangulation $MAT(P)$.*

proof: Suppose for contradiction that none of the two extreme edges: $\overline{p_1 p_{n-1}}$ and $\overline{p_n p_2}$ belongs to $MAT(P)$. Then, there must exist a vertex p_k for $2 < k < n-1$ such that both $\overline{p_1 p_k}$ and $\overline{p_n p_k}$ belong to $MAT(P)$. Without loss of generality, let p_k lie on one side

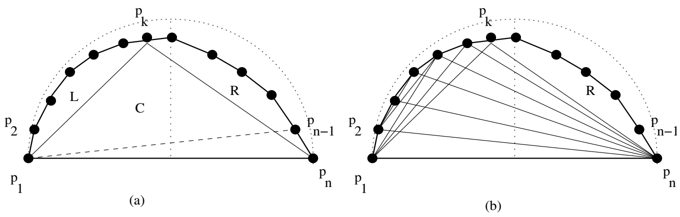

Figure 2: For the proof of Lemma 1.

of the perpendicular bisector of edge $\overline{p_1p_n}$, say the lefthand side (if p_k is on the bisector, the following argument is still applicable), The two edges $\overline{p_1p_k}$ and $\overline{p_np_k}$ partition P into three areas, denoted by R, L, and C. Both L and R are semi-circled convex polygons by Property 1. (Refer to part (a) of Figure 2.)

Let the edges of $MAT(P)$ lying inside area L be $(e_1, e_2, ..., e_{k-3})$ and let $E_L = (e_1, e_2, ..., e_{k-3}, \overline{p_1p_k})$. Let E_L^* denote the edges: $(\overline{p_np_2}, \overline{p_np_3}, ..., \overline{p_np_{k-1}})$. Then, there is a perfect matching between E_L and E_L^*. This is because E_L and E_L^* respectively triangulate the same area $L \cup C$, hence the number of internal edges of the two triangulations must be equal. Let us consider a pair in the matching $(e_i, \overline{p_np_j})$ for $1 < i, j < k$. It is not hard to see that $\omega(e_i) < \omega(\overline{n,j})$ because any edge in E_L is shorter than $\overline{p_1p_k}$ by Property 2, any edge in E_L^* is longer than $\overline{p_np_k}$, and $\overline{p_np_k}$ is longer than $\overline{p_1p_k}$ (due to p_k lying on the lefthand side of the perpendicular bisector of $\overline{p_1p_n}$). Therefore, $\omega(E_L)$ is less than $\omega(E_L^*)$. Now, we shall construct a new triangulation, say $T(P)$, which consists of all the edges in $MAT(P)$ except replacing the edges of E_L by E_L^*. We have that $\omega(T(P)) > \omega(MAT(P))$, which contradicts the $MAT(P)$ assumption. Then, such a p_k cannot exist and one of $\overline{p_1p_{n-1}}$ and $\overline{p_np_2}$ must belong to $MAT(P)$. □

By Lemma 1 and by Property 1, we have a recurence for the weight of a $MAT(P)$. Let $\omega(i,j)$ denote the weight of the $MAT(P_{i,j})$ of a semi-circled convex polygon $P_{i,j} = (p_i, p_{i+1}, ..., p_j)$. Let $\omega(\overline{i,j})$ denote the length of edge $\overline{p_ip_j}$.

$$\omega(i,j) = \begin{cases} \omega(\overline{i,i+1}) & j = (i+1) \\ max\{\omega(i,j-1) + \omega(\overline{j-1,j}), \omega(i+1,j) + \omega(\overline{i,i+1})\} + \omega(\overline{i,j}) & \text{otherwise} \end{cases}$$

It is a straight-forward matter to design a dynamic programming algorithm for finding the $MAT(P)$.

ALGORITHM $MAT - FIND(P)$

Input: a semicircled convex $n-$sided polygon: $(p_1, ..., p_n)$.

Output: $MAT(P)$.

Method:

1. **for** $i = 1$ **to** $n - 1$ **do**

   $\omega(i, i + 1) = \omega(\overline{i, i + 1})$

2. **for** $l = 2$ **to** $n - 1$ **do**

3. **for** $i = 1$ **to** $n - l$ **do**

$$\omega(i, i+l) = max\{\omega(i, i+l-1) + \omega(\overline{i+l-1, i+l}), \omega(i+1, i+l) + \omega(\overline{i, i+1})\}$$
$$+\omega(\overline{i, i+l})$$

4. Identify the edges of $MAT(P)$ by checking the ω's.

5. **end.**

Since the loop indices i and l range roughly from 1 to n and each evaluation of $\omega(i, j)$ takes constant time, all $\omega(i, j)$ for $1 \leq i, j \leq n$ can be evaluated in $O(n^2)$ time. If we record these ω's, we can find the edges in $MAT(P)$ by an extra $O(n)$ time to examine the record.

Therefore, we have the following theorem.

Theorem 1 *The maximum weight triangulation of a semi-circled convex $n-$gon P, $MAT(P)$, can be found in $O(n^2)$ time.*

Proof: The correctness is due to Property 1. It is clear that the number of executions of Step 3 dominates the time complexity. The number of executions is $(n - 3) + (n - 4) + ... + 2 + 1)\epsilon O(n^2)$. □

4 Conclusion

In this paper, we proposed an $O(n^2)$ dynamic programming algorithm for constructing the $MAT(P)$ of a semi-circled convex $n-$sided polygon.

It is still an open problem whether one can design an $o(n^3)$ algorithm for finding the $MAT(P)$ for a general convex $n-$sided polygon P.

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