Defining discrete Morse functions on infinite surfaces

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Abstract

We present an algorithm which defines a discrete Morse function in Forman's sense on an infinite surface including a study of the minimality of this function.

Key words: discrete Morse function, infinite surface, critical point, Morse inequalities

1. Introduction

Under a classical or smooth point of view, Morse theory looks for links between global properties of a smooth manifold and critical points of a function defined on it. In [2], Forman introduced the notion of discrete Morse function defined on a finite *cw*-complex and, in this combinatorial context, he developed a discrete Morse theory as a tool for studying the homotopy type and homology groups of these complexes.

Once a Morse function has been defined on a complex, then its topological information can be deduced from the critical simplices of this function. Since we studied the problem concerning the definition of discrete Morse functions on infinite 1-complexes in other works [1,4], the goal of this paper is to continue with the following natural step: to develop a way of constructing discrete Morse functions on infinite 2-complexes, in particular on connected and non compact surfaces. Our methods are based on algorithms developed by T. Lewiner [3] for the finite case.

Given a simplicial complex M, R. Forman [2] introduces the notion of *discrete Morse function* as a function $f : M \longrightarrow R$ such that, for any p-simplex $\sigma \in M$:

(M1) $card\{\tau^{(p+1)} > \sigma / f(\tau) \leq f(\sigma)\} \leq 1$.

(M2) $card\{\upsilon^{(p-1)} < \sigma / f(\upsilon) \geq f(\sigma)\} \leq 1$.

A p-simplex $\sigma \in M$ is said to be *critical* with respect to f if:

(C1) $card\{\tau^{(p+1)} > \sigma / f(\tau) \leq f(\sigma)\} = 0$.

(C2) $card\{\upsilon^{(p-1)} < \sigma / f(\upsilon) \geq f(\sigma)\} = 0$.

2. Constructing discrete Morse functions

In order to define a discrete Morse function on an infinite surface S we recall that it can be expressed as a countable union of finite subcomplexes $S = \cup_{n \in N} K_n$, with $K_n \subseteq K_{n+1}$ for any $n \in N$. Indeed, let v_0 be any vertex of a triangulation of S and let K_1 be the closed star of v_0, that is, the smallest closed subcomplex of S which contains all edges and triangles including v_0. The following figure shows the closed star of v_0 in continous lines.

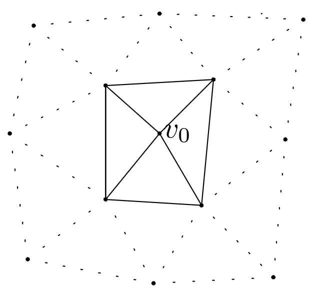

Fig. 1. Star of v_0.

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To define the rest of subcomplexes K_n, we can do a successive thickening of K_1: K_2 will be the closed star of K_1 and, in the general case, K_n will be the closed star of K_{n-1}, where the closed star of a subcomplex means the smallest closed subcomplex that contains all edges and triangles which contain some simplex of the given subcomplex.

Next, we shall use a special graph which contains information of part of any K_n, in which all triangles and some edges of K are represented. If T_1 is a spanning tree in K_1, then the *complementary graph* of T_1 in K_1, denoted by D_1, is the graph constructed as follows: each vertex of D_1 corresponds to a triangle of K_1 or to a bounding edge of K_1 (that is, an edge which is in a unique triangle of K_1 and not in T_1) and there is an edge between two vertices of D_1 if these ones correspond to two triangles sharing an edge or to a triangle and a bounding edge which is in this triangle but not in T_1. Considering the preceding figure, D_1 is the graph drawed with discontinous lines in the following figure:

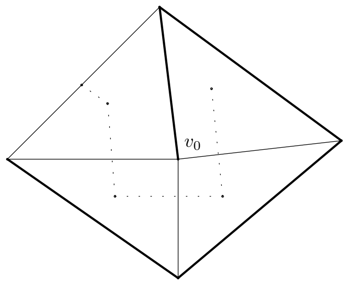

Fig. 2. D_1.

Now we enlarge the spanning tree T_1 until we get a spanning tree T_2 in K_2. Then, we obtain D_2 as an enlargement of D_1 and it is the complementary graph of T_2 in K_2. We can continue this process in successive steps. It is interesting to point out that this construction is only possible for 2-dimensional complexes because the union of T_n and D_n covers whole K_n, for any $n \in N$.

The definition of a discrete Morse function f on S starts with the definition of f in K_1. This process is divided in two steps: first, we define f on a spanning tree T_1 in K_1 starting at v_0. This assignment is made by an increasing way and such that we do not introduce any critical vertex or edge but the vertex v_0, which is a global minimum and hence it is a critical vertex [4].

On the other hand, in order to complete the definition of f on the whole K_1, we shall need to define f on D_1. To this end, we define the **degree of a vertex** corresponding to a triangle as the number of edges which are in this triangle such that we have not assigned them any value. Since S is a surface, the degree of any vertex is a number between 0 and 3. Now we start the definition of f on vertices of degree 1 assigning them the greatest value of f on T_1 plus one unit. This means that we have assigned that value to the triangles of K_1 corresponding to such vertices of degree one. Next, we assign the same value to the free (no values assigned) edges of such triangles. Then, we re-write the degrees of the vertices of D_1 because they could have changed and continue assigning values to vertices of degree one by increasing in one unit until finishing with all vertices of D_1. See the following figure as an example.

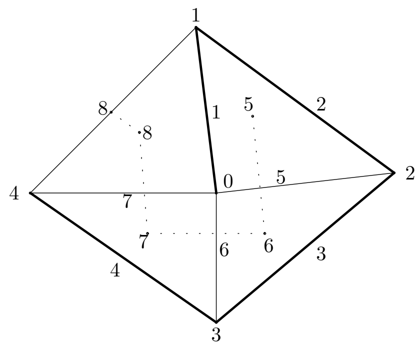

Fig. 3. f on K_1.

Now, to extend f to the subcomplex K_2, we repeat the increasing procedure to assign values of f on $T_2 - T_1$ and on $D_2 - D_1$. Since this procedure is well defined for any $n \in N$, it gives us a function f defined on whole surface S. It's important to point out that in this process may appear situations in which there are no degree one vertices in a D_n. It implies that the corresponding edges are in a cycle of non assigned edges.

3. Study of f

The following result states that the above defined function f is a discrete Morse function in

Forman's sense and give us information about its critical elements.

Proposition. *The function f given by the preceding procedure is a discrete Morse function defined on the infinite surface S and has a unique critical vertex, the initial vertex v_0, as many critical edges as many independent cycles of non assigned edges are found and do not have any critical triangle.*

References

[1] R. Ayala, L.M. Fernández and J.A. Vilches, Desigualdades de Morse generalizadas sobre grafos, *Actas de las III jornadas de Matemática Discreta y Algorítmica* (Universidad de Sevilla, Spain, 2002) 159–164.

[2] R. Forman, Morse Theory for cell complexes, *Adv. in Math.* **134** (1998) 90-145.

[3] T. Lewiner, G. Tavares and H. Lopes. Optimal Morse-Forman functions for combinatorial 2-manifolds, *to appear in Computational Geometry: Theory and Applications* (2003).

[4] J.A. Vilches, Funciones de Morse discretas sobre complejos infinitos, *book* (Edición Digital @tres, Sevilla, 2003).