# New Lower Bounds for the Number of Straight-Edge Triangulations of a Planar Point Set

Extended Abstract

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**Abstract**

We present new lower bounds on the number of straight-edge triangulations that every set of $n$ points in plane must have. These bounds are better than previous bounds in case of sets with either many or few extreme points.

## 1. Introduction

For a finite set $S$ of points in the plane let $T(S)$ denote the number of straight-edge triangulations of $S$. Let $t(h,m)$ and $T(h,m)$ denote the minimum and the maximum of $T(S)$ with $S$ ranging over all sets with $h$ extreme and $m$ non-extreme ("interior") points. It is known that [4]

$$T(h,m) \leq \frac{59^m \cdot 7^h}{\binom{m+h+6}{6}}$$

and [1] that for $h+m \geq 1212$

$$t(h,m) \geq 0.092 \cdot 2.33^{h+m} = \Omega(2.33^{h+m}) .$$

We prove the following new bounds for $t(h,m)$:

**Theorem 1** *For constant $c = 4829/116640 > 0.0414$ we have for all $h+m \geq 11$*

$$t(h,m) \geq c\Big(\frac{30}{11}\Big)^h \cdot \Big(\frac{11}{5}\Big)^m = c \cdot 2.\overline{7272}^h \cdot 2.2^m .$$

**Theorem 2** *For every fixed $h$ we have*

$$t(h,m) \geq \Omega(2.63^m) .$$

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[1] Part of this research was done while the first author was with the International-Max-Planck-Research-School in Saarbrücken.

## 2. Proof Outline for Theorem 1

Let $S$ be a finite planar set (in non-degenerate position), let $p$ be an extreme point of $S$, and let $S_p = S \setminus \{p\}$. We call $p$ of type $(k,c)$ if the following holds: 1) The chain $C_p$ of convex hull edges of $S_p$ that are visible from $p$ consists of exactly $k+1$ edges. 2) The chain $C_p$ admits at most $c$ simultaneous non-intersecting chords.

What is a chord? This is an edge $e$ connecting vertices of $C_p$ so that the polygon bounded by $e$ and the relevant portion of $C_p$ contains no point of $S_p$ in its interior.

Note that if $p$ is of type $(k,c)$ then the difference of the number of extreme points of $S-p$ and the number of extreme points of $S$ is $k-1$.

Theorem 1 is a consequence of the following two Lemmas:

**Lemma 3** *Assume $p$ is of type $(k,c)$ and let $I_p$ be the $k$ interior vertices of the chain $C_p$. The following can be done $c$ times, starting with $U = S_p$:*

*Find a vertex $u \in I_p \cap U$ that is of type $(0,0)$ with respect to $U$ and remove it from $U$.*

**Lemma 4** *Let $p$ be an extreme point of $S$ of type $(k,c)$ so that $S_p \setminus C_p$ contains at least 4 points:*

$$T(S) \geq \Phi(k,c) \cdot T(S_p) ,$$

*where $\Phi(0,0) = 3$, $\Phi(1,0) = 11/5$, and for all other pairs $c \leq k$*

$$\Phi(k,c) = \Big(\frac{2^c}{2^{c+1}-1}\Big)\beta(k,c) + 1$$

$$\beta(k,c) = \begin{cases} \dfrac{2C_{k+1}-1}{2C_{k+1}-2} & \textit{if}\; c = k \\ \dfrac{2C_{k+2}-1}{2C_{k+2}-2} & \textit{if}\; c < k, \end{cases}$$

*and* $C_j = \binom{2j}{j}/(j+1)$ *is the* $j$*-the Catalan number.*

Chasing through the definitions in Lemma 4 one finds that extreme points of type $(k,0)$ yield a nice increase in triangulation count from $S_p$ to $S$, with the worst being type $(1,0)$. If on the other hand $p$ has type $(k,c)$ with $c > 0$, which does not lead to such a large triangulation count increase, then Lemma 3 guarantees the existence of $c$ other nice vertices of type $(0,0)$ that provide sufficient increase if $S$ is built up by adding extreme points.

## 3. Proof Outline for Theorem 2

This proofs expands an initial idea of Francisco Santos [3]. It suffices to consider the case of $t(3,m)$, in other words the case of $m$ points inside a triangle $\Delta$.

Consider each of the $m$ interior points in turn. Connect it to the three corners of $\Delta$. This partitions $\Delta$ into three triangles, each of which can be triangulated recursively. This leads to the following recursive relation:

$$t(3,m) \geq m \cdot \min_{m_1+m_2+m_3=m-1} \{t(3,m_1) \cdot t(3,m_2) \cdot t(3,m_2)\}$$

Now we would like to set $t(3,m) \geq 2^{cm-a}$ for appropriate constants $a$ and $c$ and use induction based on the above recursive relation. The problem now is to make sure that this induction has a good base. For this purpose we construct using various computational methods explicit lower bounds $b(m)$ for $t(3,m)$ for $m$ from 0 to some $N$ (we used $N = 300$). Then we choose $a$ and $c$ so that $2^{cm-a} \leq b(m)$ for all $m \leq N$ and so that with $t(3,m) \geq 2^{cm-a}$ the above recursive relation is satisfied for all $m > N$. This amounts to the constraint that $2^{c+2a} \leq N+1$.

Values for $c$ and $a$ can then be found using linear programming, since the constraints for $c$ and $a$ after taking logarithms are linear. By optimizing $c$ we then got that

$$t(3,m) \geq 0.093 \cdot 2.63^m \,.$$

## 4. Acknowledgments

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## References

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