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Homogenization of the Poisson equation with Dirichlet conditions in random perforated domains

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Abstract

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1 Extended abstract

For a bounded open set $O \subset \mathbb{R}^N$, and a sequence of open sets $O_\varepsilon \subset O$, we consider the Poisson equation with Dirichlet conditions on ∂O_ε. The problem is to study the asymptotic behavior of the solutions, when ε tends to zero, by finding the equation satisfied by its limit. This allows us to describe the macroscopic behavior of the material corresponding to the mixture of the solid part O_ε and the holes $K_\varepsilon = O \setminus O_\varepsilon$. The obtention of this limit equation is a classical problem which has been considered since the early 80's. Usually, the set K_ε is a union of very small connected components distributed in O. It is well known that the homogenized equation is an elliptic problem in the whole of O which contains in general a new term of order zero depending on the distribution and size of the holes, the Cioranescu-Murat "strange term" ([6]). The most classical result corresponds to the homogenization of the Poisson equation with holes of size $\varepsilon^{\frac{N}{N-2}}$ if $N \geq 3$, or $\varepsilon^{-\frac{1}{\varepsilon^2}}$ if $N = 2$, which are periodically distributed with period ε. In this case the coefficient corresponding to the new term of zero order is a positive constant. The case of arbitrary holes and nonlinear equations has been considered in several papers such as [4], [5], [6], [7], [9], [10], [16].

Our purpose in the present work is to consider the case where the holes are randomly distributed. To simplify, we assume $N \geq 3$, and similarly to the classical periodic setting, we consider holes of size $\varepsilon^{\frac{N}{N-2}}$ such that the distance between two holes is of order ε. Namely:

We consider a probability space $(\Omega, \mathcal{F}, P)$, a subset $\tilde{\Omega} \subset \Omega$, and a function $T(x) : \Omega \to \Omega$, for every $x \in \mathbb{R}^N$, which defines an ergodic measure preserving dynamical system in $\mathbb{R}^N$. Then, for a compact set $K \subset \mathbb{R}^N$, we define the sequence of random holes as

$$K_\varepsilon(\omega) = \bigcup_{z\in\mathbb{R}^N,\, T(z)\omega\in\tilde{\Omega}} \left(\varepsilon z + \varepsilon^{\frac{N}{N-2}} K\right), \quad P\text{-a.e. } \omega \in \Omega.$$

Denoting by $O_\varepsilon(\omega)$ the open set obtained from O by removing the random holes, $K_\varepsilon(\omega)$, we want to study the asymptotic behavior of the solutions u_ε of

$$\begin{cases} -\Delta_x u_\varepsilon(\omega, x) = f_\varepsilon(\omega, x) \ \text{ in } O_\varepsilon(\omega) \\ u_\varepsilon(\omega, x) = 0 \ \text{ on } \partial O_\varepsilon(\omega) \end{cases} \qquad P\text{-a.e. } \omega \in \Omega, \tag{0.1}$$

where f_ε converges strongly in $L^2_P(\Omega; H^{-1}(O))$ to a function f.

The homogenization of random problems has been considered in several papers, see e.g. [2], [3], [8], [11], [12], [14], [15]. In particular, the G. Neguetseng and G. Allaire two-scale convergence method ([1], [13]), which is a very useful tool in periodic homogenization, has been extended in [2] to the setting of random homogenization problems. In general, the heterogeneities considered in those papers are given by functions of the form $F(T(\frac{x}{\varepsilon})\omega)$, with ω taking values on the probability space, T a dynamical system as above and F a random variable. However, in our problem, the homogenization process contains two different sizes ($\varepsilon^{\frac{N}{N-2}}$ and ε) to describe it, and then, it can not be analyzed by the results of [2]. To solve this difficulty, we introduce in the present paper a new extension of the two-scale method, which is based on some ideas used in [4] for the homogenization of Dirichlet elliptic problems in (deterministic) periodic domains. We show that the solution u_ε of (0.1) converges weakly in $L^2_P(\Omega; H^1_0(O))$ to the unique solution u of the problem

$$\begin{cases} -\Delta_x u(\omega, x)) + \gamma\kappa u(\omega, x) = f(\omega, x) \ \text{ in } O \\ u(\omega, x) = 0 \ \text{ on } \partial O \end{cases} \qquad P\text{-a.e. } \omega \in \Omega,$$

where the new term $\gamma\kappa u$ is the equivalent for our random problem of the Cioranescu-Murat strange term in the deterministic case ([6]). Similarly to the classical result, it is given by the capacity κ of the closed set K in $\mathbb{R}^N$, multiplied by γ, the mean density of holes in O. Thanks to the ergodic theory we prove that γ does not depend on $\omega \in \Omega$ or $x \in O$. From the physical point of view this means that the limit behavior of the material corresponding to the mixture of the solid part $O_\varepsilon(\omega)$ and the holes $K_\varepsilon(\omega)$ is deterministic. Similar results but assuming different assumptions about the random distribution of the holes and using different techniques have been obtained in [3] and [14].

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