Planar embeddability of the vertices of a graph using a fixed point set is NP-hard [1]

Sergio Cabello

Institute of Information and Computing Sciences
Utrecht University
The Netherlands

Abstract

Let $G = (V, E)$ be a graph with n vertices and let P be a set of n points in the plane. We show that deciding whether there is a planar straight-line embedding of G such that the vertices V are embedded onto the points P is NP-complete, even when G is 2-connected and 2-outerplanar. This settles an open problem posed in [2,4,14].

1. Introduction

A *geometric graph* H is a graph $G(H)$ together with an injective mapping of its vertices into the plane. An edge of the graph is drawn as a straight-line segment joining its vertices. We use $V(H)$ for the set of points where the vertices of $G(H)$ are mapped to, and we do not make a distinction between the edges of $G(H)$ and H. A *planar* geometric graph is a geometric graph such that its edges intersect only at common vertices. In this case, we say that H is a *geometric planar embedding* of $G(H)$. See [15] for a survey on geometric graphs.

Let P be a set of n points in the plane, and let G be a graph with n vertices. What is the complexity of deciding if there is a straight-line planar embedding of G such that the vertices of G are mapped onto P? This question has been posed as open problem in [2,4,14], and here we show that this decision problem is NP-complete. Let us rephrase the result in terms of geometric graphs.

Theorem 1 *Let P be a set of n points, and let G be a graph on n vertices. Deciding if there exists a geometric planar embedding H of G such that $V(H) = P$ is an NP-complete problem.*

The reduction is from 3-partition, a strongly NP-hard problem to be described below, and it constructs a 2-connected graph G. The main ideas of the proof are given in Section 2, and we use that the maximal 3-connected blocks of a 2-connected planar graph can be embedded in different faces. In a 3-connected planar graph, all planar embeddings are topologically equivalent due to Whitney's theorem [10, Chapter 6]. Therefore, it does not seem possible to extend our technique to show the hardness for 3-connected planar graphs.

Related work A few variations of the problem of embedding a planar graph into a fixed point set have been considered. The problem of characterizing what class of graphs can be embedded into any point set in general position (no three points being collinear) was posed in [12]. They showed that the answer is the class of *outerplanar* graphs, that is, graphs that admit a straight-line planar embedding with all vertices in the outerface. This result was rediscovered in [6], and efficient algorithms for constructing such an embedding for a given graph and a given point set are described in [2]. The currently best algorithm runs in $O(n \log^3 n)$ time, although the best known lower bound is $\Omega(n \log n)$.

A tree is a special case of outerplanar graph. In this case, we also can choose to which point the root should be mapped. See [3,13,16] for the evolution on this problem, also from the algorithmical point of view. For this setting, there are algorithms running in $O(n \log n)$ time, which is worst case optimal. Bipartite embeddings of trees were considered in [1].

Email address: sergio@cs.uu.nl (Sergio Cabello).

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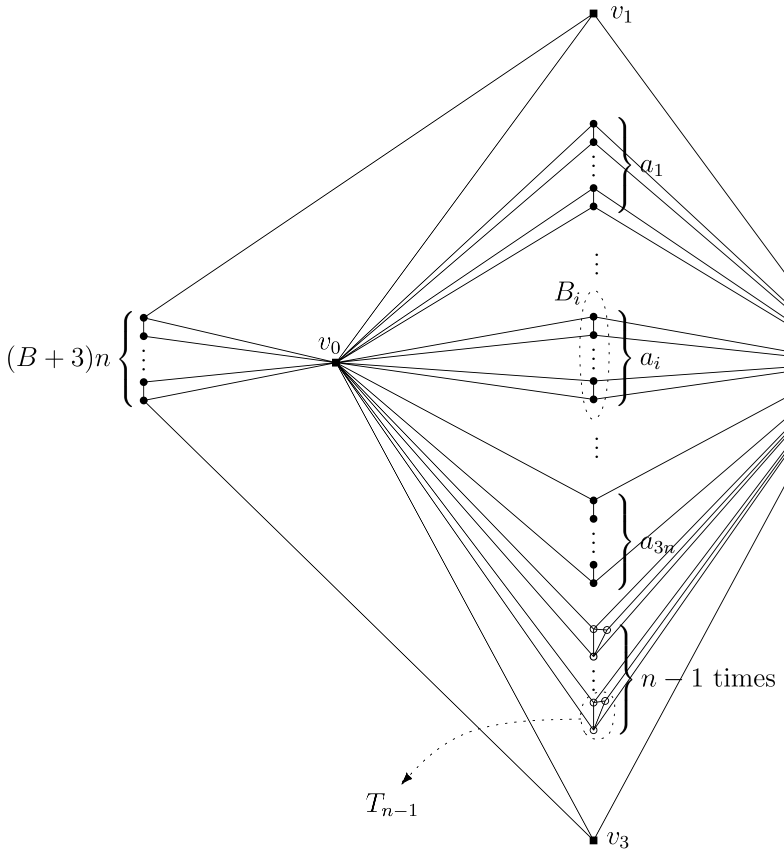

Fig. 1. Graph G for the NP-hardness reduction.

If we allow each edge to be represented by a polygonal path with at most two bends, then it is always possible to get a planar embedding of a planar graph that maps the vertices to a fixed point set [14]. If a bijection between the vertices and the point set is fixed, then we need $O(n^2)$ bends in total to get a planar embedding of the graph, which is also asymptotically tight in the worst case [17].

We finish by mentioning a related problem, which was the initial motivation for this research. A universal set for graphs with n vertices is a set of points S_n such that any planar graph with n vertices has a straight-line planar embedding whose vertices are a subset of S_n. Asymptotically, the smallest universal set is known to have size at least $1.098n$ [7], and it is bounded by $O(n^2)$ [8,18]. Characterizing the asymptotic size of the smallest universal set is an interesting open problem [9, Problem 45].

2. Planar embeddability is NP-complete

It is clear that the problem belongs to NP: a geometric graph H with $V(H) = P$ and $G(H) \equiv G$ can be described by the bijection between $V(G)$ and P, and for a given bijection we can test in polynomial time whether it actually is a planar geometric graph; therefore, we can take as certificate the bijection between $V(G)$ and P.

For showing the NP-hardness, the reduction is from 3-partition.

Problem: 3-partition

Input: A natural number B, and $3n$ natural numbers $a_1, \ldots, a_{3n}$ with $\frac{B}{4} < a_i < \frac{B}{2}$.

Output: n disjoint sets $S_1, \ldots, S_n$ such that for all S_j we have $S_j \subset \{a_1, \ldots, a_{3n}\}$, $|S_j| = 3$, and $\sum_{a \in S_j} a = B$.

We will use that 3-partition is a strongly NP-hard problem, that is, it is NP-hard even if B is bounded by a polynomial in n [11]. Observe that because $\frac{B}{4} < a_i < \frac{B}{2}$, it does not make sense to have sets

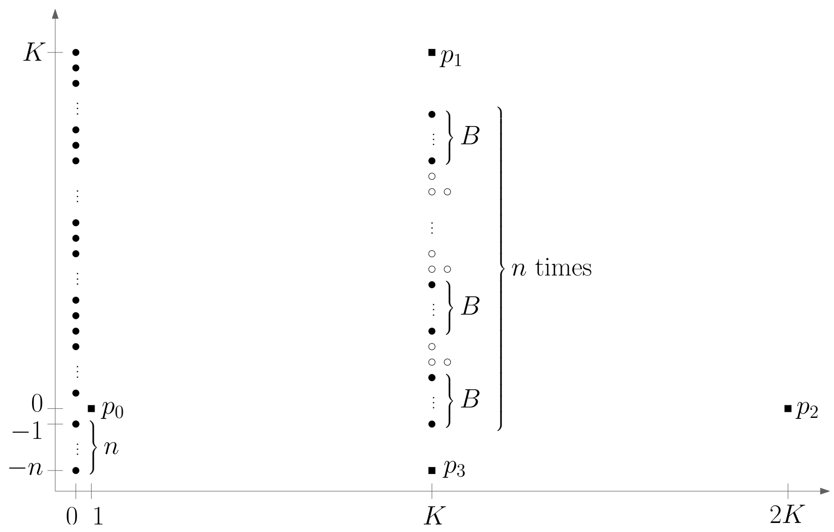

Fig. 2. Point set P for the NP-hardness reduction. $K = (B + 2)n$.

S_j with fewer or more than 3 elements. That is, it is equivalent to ask for subdividing all the numbers into disjoint sets that sum to B. Of course, we can assume that $\sum_{i=1}^{3n} a_i = Bn$, as otherwise it is impossible that a solution exists.

In the following, we only give the reduction, and do not discuss how the solution to the constructed problem relates to the original problem. Details can be found in the full version [5]. Given a 3-partition instance, we construct the following graph G (see Figure 1):

- Start with a 4-cycle with vertices $v_0, \ldots, v_3$, and edges $(v_{i-1}, v_{i \bmod 4})$. The vertices v_0 and v_2 will play a special role.
- For each a_i in the input, make a path B_i consisting of a_i vertices, and put an edge between each of those vertices and the vertices v_0, v_2.
- Construct $n-1$ triangles $T_1, \ldots, T_{n-1}$. For each triangle T_i, put edges between each of its vertices and v_2, and edges between two of its vertices and v_0. We call each of these structures a separator (the reason for this will become clear later).
- Make a path C of $(B+3)n$ vertices, and put edges between each of the vertices in C and v_0. Furthermore, put an edge between one end of the path and v_1, and another edge between the other end and v_3.

It is easy to see that G is planar; in fact, we are giving a planar embedding of it in Figure 1. The idea is to design a point set P such that $G \setminus (B_1 \cup \cdots \cup B_{3n})$ can be embedded onto P in essentially one way. Furthermore, the embedding of $G \setminus (B_1 \cup \cdots \cup B_{3n})$ will decompose the rest of the points into n groups, each of B vertices and lying in a different face. The embedding of the remaining vertices $B_1 \cup \cdots \cup B_{3n}$ will be possible in a planar way if and only if the paths B_i can be decomposed into groups of exactly B vertices, which is equivalent to the original 3-partition instance. The following point set P does the work (see Figure 2):

- Let $K := (B+2)n$.
- Place $(B+3)n$ points at the coordinates $(0,-n), (0,-(n-1)), \ldots, (0,-1)$ and at the coordinates $(0,1), (0,2), \ldots, (0,K)$.
- Place points $p_0 := (1,0), p_1 := (K,K), p_2 := (2K,0), p_3 := (K,-n)$. In the figure, these points are shown as boxes and are labeled.
- For each $i \in \{0, \ldots, n-1\}$, place the group of B points $(K, -2+(B+2)i+1), (K, -2+(B+2)i+2), \ldots, (K, -2+(B+2)i+B)$.
- For each $i \in \{1, \ldots, n-1\}$, place the group of three points $(K, (B+2)i-3), (K, (B+2)i-2), \ldots, (K+1, (B+2)i-3)$. In the figure, these points are shown as empty circles.

3. Concluding remarks

The point set P that we have constructed has many collinear points. However, in the proof we do not use this fact, and so it is easy to modify the reduction in such a way that no three points of P are collinear. Probably, the easiest way for keeping integer coordinates is replacing each of the points lying in a vertical line by points lying in a parabola, and adjusting the value K accordingly. Therefore, the result remains valid even if P is in general position, meaning that no 3 points are collinear.

In the proof, the graph G that we constructed is 2-outerplanar, as shown in Figure 1. k-outerplanarity is a generalization of outerplanarity that is defined inductively. A planar embedding of a graph is *k-outerplanar* if removing the vertices of the outer face produces a $(k-1)$-outerplanar embedding, where 1-outerplanar stands for an outerplanar embedding. A graph is k-outerplanar if it admits a k-outerplanar embedding. For outerplanar graphs, the embedding problem is polynomially solvable [2], but for 2-outerplanar we showed that it is NP-complete. Therefore, regarding outerplanarity, our result is tight.

The graph G that we constructed in the proof is 2-connected; removing the vertices v_1, v_3 disconnects the graph. As mentioned in the introduction, we use this fact in the proof because in a 2-connected graph the maximal 3-connected blocks can flip from one face to another one. Therefore, it would be interesting to find out the complexity of the problem when the graph G is 3-connected, and more generally, the complexity when the topology of the embedding is specified beforehand.

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