Two Optimization Problems with Floodlights

Andreas Spillner

Institut für Informatik, Friedrich-Schiller-Universität Jena, 07743 Jena, Germany

Key words: illumination, floodlight

1. Introduction

The area of visibility and illumination problems is a very attractive field inside computational geometry. We refer the interested reader to [8] for an overview of recent results. In this paper we study two illumination problems involving floodlights. Loosely spoken a floodlight is a light source that illuminates a region in the plane which is bounded by two rays emanating from a common point, the apex of the floodlight. The size of a floodlight is the angle between its bounding rays.

In the first problem we are given a segment S of a straight line L in the plane and a finite set P of n points. P is contained in one of the open halfplanes bounded by L. We want to find a set $\mathfrak{F}$ of floodlights such that each point of S is contained in at least one floodlight, (ie the floodlights in $\mathfrak{F}$ illuminate S), such that the apex of each floodlight in $\mathfrak{F}$ is a point from P, and such that the sum of the sizes of the floodlights in $\mathfrak{F}$ is minimum. Furthermore we require that no two floodlights in $\mathfrak{F}$ share a common apex.

Variants of this problem have been studied before. In [1] the authors formulate a related decision problem and outline the difficulties in finding an efficient algorithm for it: We are given a segment S, called the *stage*, a set of points P and $|P|$ angles. Can floodlights whose sizes equal the given angles be placed at the given points to illuminate S? Please note that here it is not forbidden to place two or more floodlights at the same point.

A partial answer to the question about the computational complexity of the last mentioned variant was given in [6]. The variant considered there is stated as follows: We are given a segment S, a set of points P and a set of angles A. Every angle $\alpha \in A$ is mapped to a point $p(\alpha) \in P$. Can we find floodlights F_α, $\alpha \in A$, that together illuminate S, such that the apex of F_α is placed at $p(\alpha)$ and the size of F_α is α? It is shown that this problem is NP-complete.

The problem of illuminating a given segment S from a given set P of points such that the sum of the sizes of the floodlights used is minimized was studied in [4] and [3]. But there it was not required that the apices of the floodlights have to be pairwise distinct. The authors give an $O(n \log n)$ time and $O(n)$ space algorithm to solve this problem. Since it may happen that more than one floodlight is placed at a point from P, it could be interesting to study the problem with such placement explicitly ruled out. This was also posed as an open question in [8] and [2].

In the second problem that we consider in this paper we are given a convex polygon P with n vertices and a positive integer k. We want to illuminate P with k floodlights such that the sum of their sizes is minimum. Such a set of k floodlights is called optimal. The apices of the floodlights must be located in P.

For $k = 1$ the problem is easy: We have to find a vertex v of the convex polygon P such that the interior angle at v is minimum. This can trivially be done in $O(n)$ time. For $k = 2$ in [5] an algorithm is given that solves this problem in $O(n^2)$ time using $O(n)$ space. The algorithm exploits the fact that in an optimal set with two floodlights the apices of these two floodlights must be located at vertices of P. For $k \geq 3$ the problem was open [7]. We consider the case $k = 3$.

This paper is structured as follows. After this introduction we give some formal definitions. In the third section we present an $O(n \log n)$ time and $O(n)$ space algorithm for the first problem (illu-

Email address: spillner@minet.uni-jena.de (Andreas Spillner).

mination of a stage). In the fourth section we give an $O(n^2)$ time and $O(n)$ space algorithm for the second problem (illumination of a convex polygon with three floodlights). We end with some concluding remarks.

2. Preliminaries

Our setting will be the plane $\mathbb{R}^2$. For $M \subseteq \mathbb{R}^2$ we denote by $int(M)$ the interior of M, by $relint(M)$ the relative interior of M, by $bd(M)$ the boundary of M, and by $cl(M)$ the closure of M.

Definition 1 *Let p and q be points in the plane. By $\overline{pq}$ we denote the straight line segment with endpoints p and q.*

Definition 2 *Let $M \subseteq \mathbb{R}^2$. M is called convex iff for every $p, q \in M$ the straight line segment $\overline{pq}$ is contained in M.*

Definition 3 *Let R_l and R_r be two distinct rays with common starting point p. Let R be a ray rotating around p in clockwise direction starting at the position where $R = R_l$ and stopping at the position where $R = R_r$. Every point touched by R during its rotation belongs to the floodlight F with bounding rays $l(F) = R_l$ and $r(F) = R_r$. The point p is called the apex of F and denoted by $a(F)$. The angle from R_l to R_r in clockwise direction is called the size of F and denoted by $s(F)$.*

Definition 4 *Let $v_1, \ldots, v_n$ be pairwise distinct points in the plane and $n \in \mathbb{N}$, $n \geq 3$. The set of points $S = \overline{v_n v_1} \cup \bigcup_{i=1}^{n-1} \overline{v_i v_{i+1}}$ is called a simple closed polygonal curve iff it is a homeomorph of the boundary of an open disk and no two consecutive line segments on S are collinear. The points in $v(S) = \{v_1, \ldots, v_n\}$ are called the vertices of S. The line segments in $e(S) = \{\overline{v_n v_1}, \overline{v_1 v_2}, \ldots, \overline{v_{n-1} v_n}\}$ are called the edges of S.*

Definition 5 *A set P of points is called a simple polygon iff P is closed, bounded, and connected, $int(P) \neq \emptyset$ and $bd(P)$ is a simple closed polygonal curve. We set $v(P) = v(bd(P))$ and $e(P) = e(bd(P))$. A simple polygon P is called a convex polygon iff P is convex.*

Definition 6 *Let P be a simple polygon, p and q points, F a floodlight. We say that p and q can see each other and write this $p \leftrightarrow q$ iff $\overline{pq} \subseteq P$. The region visible from point p is $vis(p, P) = \{x \in \mathbb{R}^2 : p \leftrightarrow x\}$. The region illuminated by F is $ill(F, P) = F \cap vis(a(F), P)$.*

3. Illumination of a stage

Now we can state our *stage illumination problem* or *SIP* for short more formally: An instance (S, P) consists of a segment S of a straight line L and a finite set of points P such that P is contained in one of the open halfplanes bounded by L. We want to find a set $\mathfrak{F}$ of floodlights with the following properties:

(i) $S \subseteq \cup_{F \in \mathfrak{F}} F$
(ii) $\forall F \in \mathfrak{F}\ (a(F) \in P)$
(iii) $\forall F_1, F_2 \in \mathfrak{F}\ (F_1 \neq F_2 \Rightarrow a(F_1) \neq a(F_2))$
(iv) $s(\mathfrak{F}) = \sum_{F \in \mathfrak{F}} s(F)$ is minimum

Now it is clear that without loss of generality we can assume that L is the x-axis and all points in P have a positive y-coordinate, ie the points in P lie above the x-axis.

Definition 7 *Let (S, P) be an instance, $p \in P$ and z be a point on L. By $C_p(z)$ we denote the circle through the point p such that L is tangent to $C_p(z)$ at the point z.*

If we drop property (iii) in SIP we will obtain the problem considered in [4] and denote it by SIP*. From [4] we know that in a solution $\mathfrak{F}^*$ to an instance (S, P) for SIP* a point $z \in S$ is contained in a floodlight $F \in \mathfrak{F}^*$ if and only if no point of P lies outside $C_{a(F)}(z)$. Hence for SIP* to every instance there exists a unique solution. Please note that SIP* is meaningful even in the limiting case, ie if $S = L$.

Observation 8 *Consider the solution to an instance (S, P) for SIP*.*

(i) There are at most two floodlights placed at each point of P.
(ii) There is at most one point $p \in P$ such that there are two floodlights placed at p.
(iii) If there are two floodlights placed at $p \in P$ then p has a smaller y-coordinate than every other point in P.

If we have an instance (S, P) and solutions $\mathfrak{F}$ and $\mathfrak{F}^*$ to this instance for SIP and SIP* respectively then it is clear that $s(\mathfrak{F}) \geq s(\mathfrak{F}^*)$.

Observation 9 *Consider a solution $\mathfrak{F}$ to an instance (S, P) for SIP.*

(i) $\forall F \in \mathfrak{F}\ (F \cap L = F \cap S)$
(ii) $\forall F_1, F_2 \in \mathfrak{F}\ (F_1 \neq F_2 \Rightarrow relint(F_1 \cap S) \cap relint(F_2 \cap S) = \emptyset)$.

Definition 10 *Let p and q be two points on the x-axis. Then $[p, q] = \overline{pq}$ and the x-coordinate of q is not smaller than the x-coordinate of p.*

Definition 11 *Let $\mathfrak{F}$ be a solution to an instance (S, P) for SIP and $k = |\mathfrak{F}|$. According to observation 9 the segment S is partitioned into k internally disjoined subsegments, each of which is the intersection of a floodlight from $\mathfrak{F}$ with S.*

(i) Let $S_1, S_2, \ldots, S_k$ denote these subsegments in order they appear on S from left to right.

(ii) Let z_{i-1} and z_i denote the endpoints of S_i, that is $S_i = [z_{i-1}, z_i]$, $i \in \{1, \ldots, k\}$.

(iii) Let F_i denote the floodlight from $\mathfrak{F}$ that contains S_i, $i \in \{1, \ldots, k\}$.

The next three lemmas we give without proof.

Lemma 12 *Let $\mathfrak{F}$ be a solution to an instance (S, P) for SIP, $k = |\mathfrak{F}|$ and $i \in \{1, \ldots, k-1\}$. Then the point $a_{i+1} = a(F_{i+1})$ lies in clockwise direction between z_i and $a_i = a(F_i)$ on the circle $C = C_{a_i}(z_i)$.*

Lemma 13 *Let $I = (S, P)$ be an instance such that in the solution $\mathfrak{F}^*$ to I for SIP* there are two floodlights placed at the point $p \in P$. Then in every solution $\mathfrak{F}$ to I for SIP there exists a floodlight $F \in \mathfrak{F}$ such that $a(F) = p$.*

Lemma 14 *Let $I = (S, P)$ be an instance such that in the solution $\mathfrak{F}^*$ to I for SIP* there are two floodlights placed at the point $p \in P$. Let $\mathfrak{F}$ be a solution to I for SIP and $k = |\mathfrak{F}|$. Then it is $a(F_1) = p$ or $a(F_k) = p$ again using the notation from definition 11.*

Now it is not hard to see that the problem SIP can be solved in polynomial time employing the *dynamic programming* technique. We found an algorithm running in $O(n\log n)$ time and $O(n)$ space which first sorts the points in P according to their y-coordinates and then processes these points in the order obtained from sorting. We omit the details and summarize our results.

Theorem 15 *Let $I = ([a, b], P)$ be an instance and $n = |P|$. A solution to instance I for SIP can be computed in $O(nlogn)$ time using $O(n)$ space in the real RAM model of computation. In the worst case we need $\Omega(nlogn)$ time to find a solution.*

Remark 16 *The lower bound follows from the reduction of SORTING to SIP* presented in [4] and [2].*

4. Illumination of a convex polygon

First we state our illumination problem more formally. Fix a positive integer k. Given a convex polygon P we want to find a set of floodlights $\mathfrak{F}$ such that:

(i) $P = \bigcup_{F \in \mathfrak{F}} ill(F, P)$

(ii) $|\mathfrak{F}| = k$

(iii) $s(\mathfrak{F}) = \sum_{F \in \mathfrak{F}} s(F)$ is minimum

(iv) $\forall k^* \in \{1, \ldots, k-1\} \forall \mathfrak{F}^* \, (\mathfrak{F}^*$ is a solution to P of $\Pi_{k^*} \;\Rightarrow\; s(\mathfrak{F}^*) > s(\mathfrak{F}))$

We will refer to this problem as Π_k and call a set $\mathfrak{F}$ satisfying (i)-(iv) a solution to instance P of Π_k.

Remark 17 *An instance P of Π_k need not have a solution (consider for example an instance P of Π_2 such that P is a triangle). But for such an instance P there exists $k^* < k$ such that P has a solution for problem Π_{k^*} and using more than k^* (but at most k) floodlights does not help to decrease the total size.*

Observation 18 *Let P be an instance of Π_k and $\mathfrak{F}$ a solution to P. Then for every $F \in \mathfrak{F}$ holds $a(F) \in bd(P)$.*

But are there polygons where three floodlights are better than two? The answer is yes. Consider for example the convex polygon with vertices (0,0), (0,2), (10,2), (15,1), (10,0).

Definition 19 *Let P be a convex polygon and $p, q \in bd(P)$ such that $p \neq q$. By $[p, q]$ we denote the closed segment of the boundary of P that consists of those points in $bd(P)$ which we meet by traversing the boundary of P from p to q in clockwise direction. Furthermore we introduce $]p, q[= [p, q] \setminus \{p, q\}$.*

Definition 20 *Let P be a convex polygon and $\mathfrak{F}$ a finite set of floodlights such that $\forall F \in \mathfrak{F} \, (a(F) \in bd(P))$. Then $cl(P \setminus (\cup_{F \in \mathfrak{F}} F)) = \cup_{Q \in \mathfrak{Q}} Q$ where $\mathfrak{Q}$ is a finite set of convex polygons such that $\forall Q_1, Q_2 \in \mathfrak{Q} \, (Q_1 \neq Q_2 \;\Rightarrow\; Q_1 \cap Q_2 \subseteq bd(P))$. We set $v(cl(P \setminus (\cup_{F \in \mathfrak{F}} F))) = \cup_{Q \in \mathfrak{Q}} v(Q)$.*

The next lemmas tell us something about the structure of solutions. We give them without proof.

Lemma 21 *Let P be a convex polygon, $\mathfrak{F} = \{F_1, F_2, F_3\}$ a solution to P of Π_3, R_i, $i \in \{1, 2, 3\}$, a bounding ray of F_i such that R_1, R_2, R_3 are pairwise nonparallel. Then there is no point $s \in int(P)$ sucht that $s \in R_1 \cap R_2 \cap R_3$.*

Lemma 22 *Let P be a convex polygon and $\mathfrak{F}$ a solution to P of Π_3. Then $v(cl(P \setminus (\cup_{F \in \mathfrak{F}} F))) \cap int(P) = \emptyset$.*

Lemma 23 *Let P be a convex polygon and $\mathfrak{F}$ a solution to P of Π_3. Then $\forall F_1, F_2 \in \mathfrak{F} \, (F_1 \neq F_2 \Rightarrow int(F_1) \cap int(F_2) = \emptyset)$.*

Lemma 24 *Let P be a convex polygon and $\mathfrak{F}$ a solution to P of Π_3. Then we can number the elements of $\mathfrak{F}$ with elements of $\{1,2,3\}$ such that $a(F_1) \in l(F_2)$, $a(F_2) \in l(F_1)$, $a(F_2) \in r(F_3)$ and $a(F_3) \in r(F_2)$.*

Lemma 25 *Let P be a convex polygon and $\mathfrak{F}$ a solution to P of Π_3. Then $\{a(F) : F \in \mathfrak{F}\} \subseteq v(P)$.*

Lemma 26 *Let P be a convex polygon and $\mathfrak{F} = \{F_1, F_2, F_3\}$ a solution to P of Π_3. Let the elements of $\mathfrak{F}$ be numbered as in lemma 24 and let D denote the disk bounded by the circle through the points $a(F_1)$, $a(F_2)$ and $a(F_3)$. Then $[a(F_3), a(F_1)] \subseteq D$.*

Now we are ready to sketch an algorithm for problem Π_3. Given a convex polygon P if there exists a solution $\{F_1, F_2, F_3\}$ to P of Π_3 then from lemma 24 we know the way the floodlights in this solution are arranged. In connection with lemma 26 this suggests the following strategy: For every pair of distinct vertices u and w of P try to find a vertex $v \in]u, w[$ such that $[u, w]$ is contained in the disk bounded by the circle through the points u, v and w. That is we fix a position for $a(F_3)$ (vertex u) and a position for $a(F_1)$ (vertex w) and try to find a position for $a(F_2)$ (vertex v). It should be clear that by pursueing this strategy if there is a solution to P of Π_3 then we will find it. Since we may suppose that solutions to P of Π_1 and Π_2 are available (or we know that there is no solution to P of Π_2), it is also easy to detect those polygons that admit no solution of Π_3.

The question that remains is about an efficient implementation of the above sketched algorithm. We can achieve a running time in $O(n^2)$. But we omit the details and summarize our results.

Theorem 27 *Given a convex polygon P with n vertices we can compute in $O(n^2)$ time and $O(n)$ space a solution to P of Π_3 or find out that there is no such solution using the real RAM model of computation.*

5. Concluding remarks

In our view there are at least two interesting open questions regarding the illumination problem of convex polygons: Are there algorithms for problems Π_2 and Π_3 with running time in $o(n^2)$? Can our results be extended to problems Π_k with $k \geq 4$?

References

[1] P. Bose, L. J. Guibas, A. Lubiw, M. H. Overmars, D. L. Souvaine, and J. Urrutia. The floodlight problem. *International Journal of Computational Geometry & Applications*, 7:153–163, 1997.

[2] F. Contreras. Cutting polygons and a problem on illumination of stages. Master's thesis, University of Ottawa, 1998.

[3] F. Contreras, J. Czyzowicz, E. Rivera-Campo, and J. Urrutia. Optimal floodlight illumination of stages. In *Proc. 14th Annual Symposium on Computational Geometry*, pages 409–410, 1998.

[4] J. Czyzowicz, E. Rivera-Campo, and J. Urrutia. Optimal floodlight illumination of stages. In *Proc. 5th Canadian Conference on Computational Geometry*, pages 393–398, 1993.

[5] V. Estivill-Castro and J. Urrutia. Two-floodlight illumination of convex polygons. In *Proceedings of the Fourth Workshop on Algorithms and Data Structures*, volume 955 of *LNCS*, pages 62–73, 1995.

[6] H. Ito, H. Uehara, and M. Yokoyama. NP-completeness of stage illumination problems. In *Proc. Japan Conference Discrete Computational Geometry*, pages 88–92, 1998.

[7] J. O'Rourke. Open problems in the combinatorics of visibility and illumination. In *Advances in Discrete and Computational Geometry*, volume 223 of *Contemporary Mathematics*, pages 237–243, South Hadley, MA, 1996.

[8] J. Urrutia. Art gallery and illumination problems. In J. R. Sack and J. Urrutia, editors, *Handbook of Computational Geometry*, pages 973–1027. North Holland, 2000.