A family of quasi-birth-and-death processes coming from the theory of orthogonal matrix polynomials[1]

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[1] joint work with F. A. Grünbaum

Outline

1. Introduction
   - Random walks
   - Orthogonal matrix polynomials
   - Quasi-birth-and-death processes
2. The family of processes
3. Probabilistic aspects
   - Karlin-McGregor formula
   - Recurrence
   - The invariant measure

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Random walks

Transition probability matrix

$$P = \begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad b_n \geqslant 0, a_n, c_n > 0, \quad a_n + b_n + c_n = 1$$

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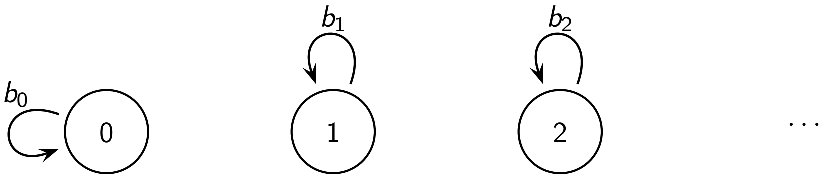

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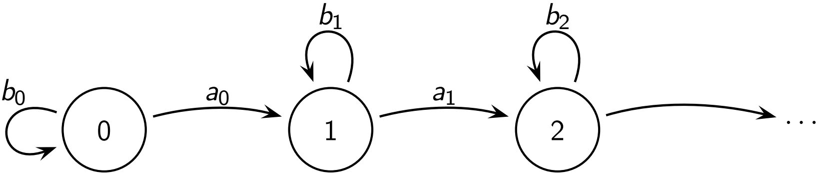

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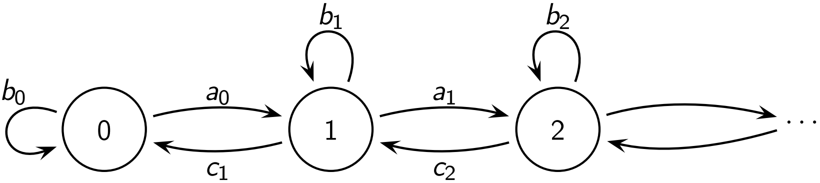

Properties

n-step transition probability matrix:

$$\text{Prob}\{E_i \to E_j \text{ in } n \text{ steps}\} = P_{ij}^n = \sum_{k_1, k_2, \ldots, k_{n-1}} P_{ik_1} P_{k_1 k_2} \cdots P_{k_{n-1} j}$$

Recurrence

- Transient
- Recurrent
  - Positive or ergodic
  - Null

Invariant distribution or measure

A non-null vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \ldots)$ with non-negative components

$$\boldsymbol{\pi} P = \boldsymbol{\pi}$$

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Introducing the polynomials $(q_n)_n$ by the conditions $q_{-1}(t) = 0$, $q_0(t) = 1$ and the recursion relation

$$t \begin{pmatrix} q_0(t) \\ q_1(t) \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}}_{P} \begin{pmatrix} q_0(t) \\ q_1(t) \\ \vdots \end{pmatrix}$$

i.e.

$$tq_n(t) = a_n q_{n+1}(t) + b_n q_n(t) + c_n q_{n-1}(t), \quad n = 0, 1, \dots$$

there exists a unique measure $d\omega(t)$ supported in $[-1, 1]$ such that

$$\int_{-1}^{1} q_i(t) q_j(t) d\omega(t) \Big/ \int_{-1}^{1} q_j(t)^2 d\omega(t) = \delta_{ij}$$

Karlin and McGregor (1959): integral representation of P^n

Karlin-McGregor formula

$$P_{ij}^n = \int_{-1}^{1} t^n q_i(t) q_j(t) \mathrm{d}\omega(t) \Big/ \int_{-1}^{1} q_j(t)^2 \mathrm{d}\omega(t)$$

Invariant measure or distribution

$\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \dots)$ with $\boldsymbol{\pi} P = \boldsymbol{\pi}$

$$\Rightarrow \pi_i = \frac{a_0 a_1 \cdots a_{i-1}}{c_1 c_2 \cdots c_i} = \frac{1}{\int_{-1}^{1} q_i^2(t) \mathrm{d}\omega(t)} = \frac{1}{\|q_i\|^2}$$

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Orthogonal matrix polynomials

Krein (1949): orthogonal matrix polynomials (OMP)

Orthogonality: weight matrix W. Matrix valued inner product:

$$\langle P, Q\rangle_W = \int_{\mathbb{R}} P(t) dW(t) Q^*(t) \in \mathbb{C}^{N\times N}, \quad P, Q \in \mathbb{C}^{N\times N}[t]$$

Using Gram-Schmidt we get a family of OMP $(Q_n)_n$.

One gets a three term recurrence relation

$$tQ_n(t) = A_n Q_{n+1}(t) + B_n Q_n(t) + C_n Q_{n-1}(t), \quad n \geqslant 0, \quad \det(A_n) \neq 0.$$

Jacobi operator (block tridiagonal)

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Quasi-birth-and-death processes

Transition probability matrix

$$P = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \begin{array}{l} (A_n)_{ij}, (B_n)_{ij}, (C_n)_{ij} \geqslant 0, \ \det(A_n), \det(C_n) \neq 0 \\ \displaystyle\sum_j (A_n)_{ij} + (B_n)_{ij} + (C_n)_{ij} = 1, \ i = 1, \ldots, N \end{array}$$

Particular case: pentadiagonal matrix

$$P = \left(\begin{array}{cc|cc|cc|cc|c} b_0 & a_0 & d_0 & 0 & & 0 & & & \\ c_1 & b_1 & a_1 & d_1 & & & & & \\ \hline e_2 & c_2 & b_2 & a_2 & d_2 & 0 & & 0 & \\ 0 & e_3 & c_3 & b_3 & a_3 & d_3 & & & \\ \hline & 0 & e_4 & c_4 & b_4 & a_4 & d_4 & 0 & \ddots \\ & & 0 & e_5 & c_5 & b_5 & a_5 & d_5 & \\ \hline & & & \ddots & & \ddots & & \ddots & \ddots \end{array}\right)$$

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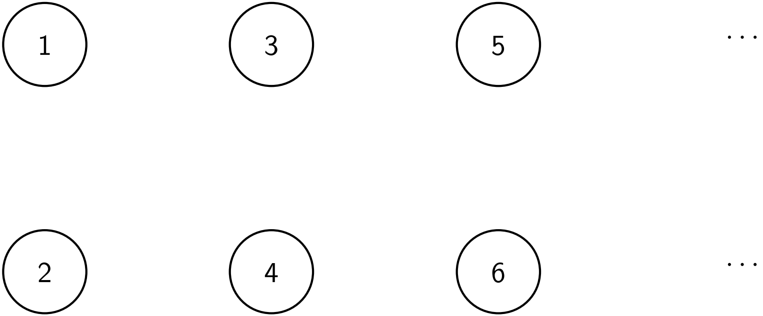
Network



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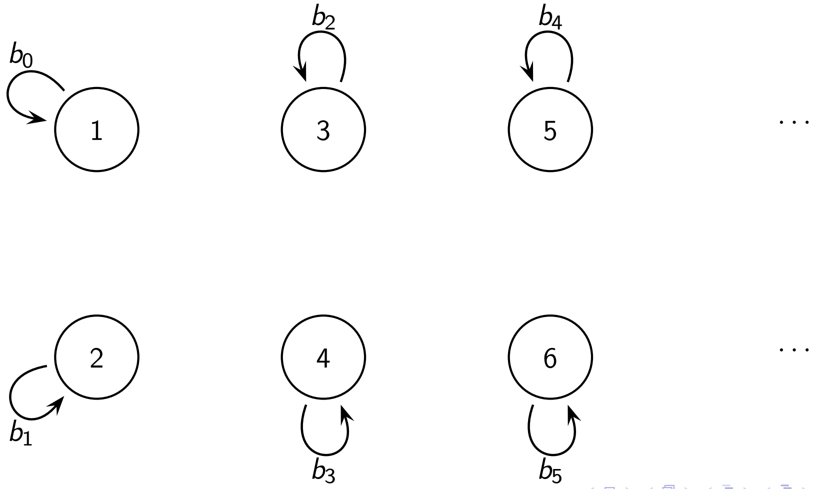

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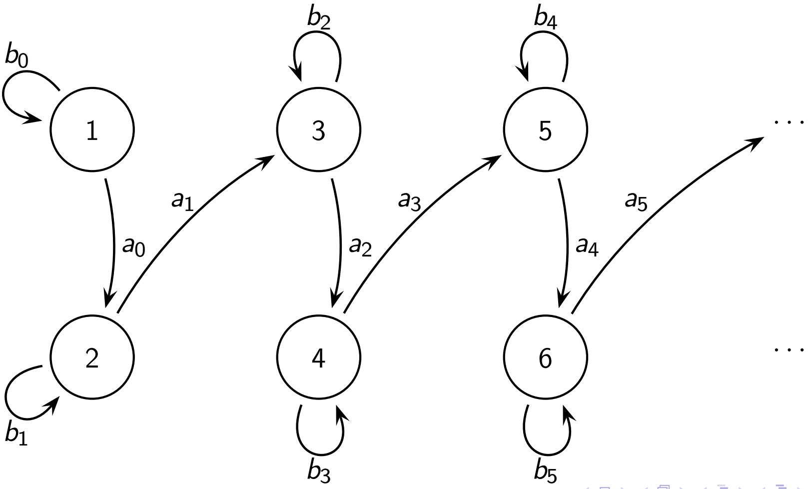

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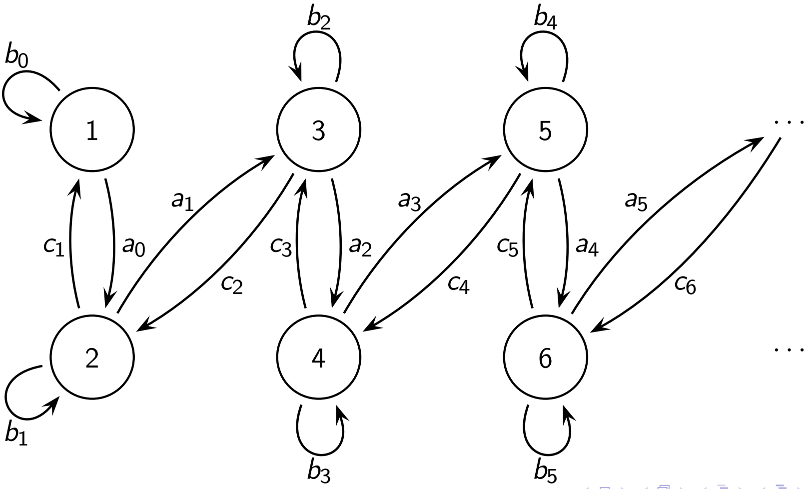

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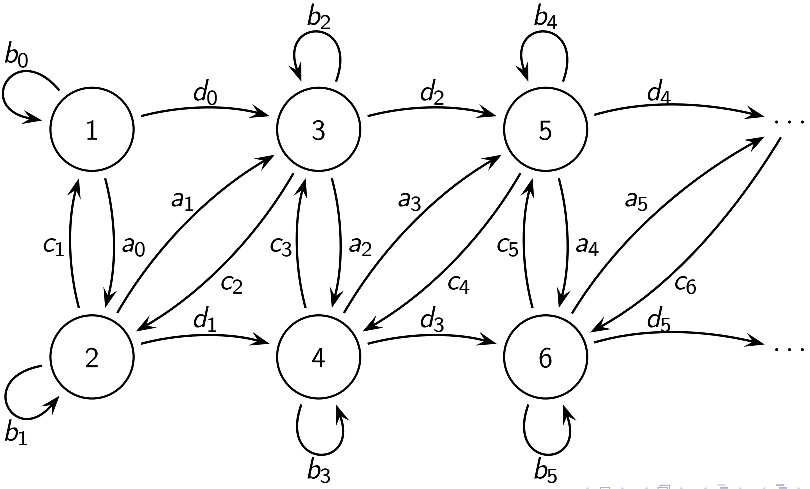
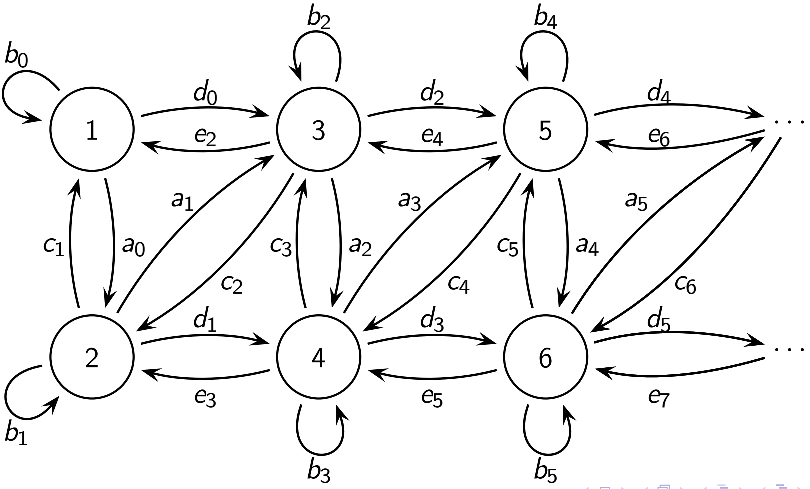

b_0
b_2
b_4
d_0
d_2
d_4
1
3
5
...
a_1
a_3
a_5
c_1
a_0
c_3
a_2
c_5
a_4
c_2
c_4
c_6
d_1
d_3
d_5
2
4
6
...
b_1
b_3
b_5

Network


b_0
b_2
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1
3
5
e_2
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$\cdots$
a_1
a_3
a_5
c_1
a_0
c_3
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c_6
d_1
d_3
d_5
2
4
6
$\cdots$
e_3
e_5
e_7
b_1
b_3
b_5

OMP: Grünbaum (2007) and Dette-Reuther-Studden-Zygmunt (2007): Introducing the matrix polynomials $(Q_n)_n$ by the conditions $Q_{-1}(t)=0$, $Q_0(t)=I$ and the recursion relation

$$t\begin{pmatrix} Q_0(t) \\ Q_1(t) \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}}_{P} \begin{pmatrix} Q_0(t) \\ Q_1(t) \\ \vdots \end{pmatrix}$$

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and under certain technical conditions over A_n, B_n, C_n, there exists an unique weight matrix $dW(t)$ supported in $[-1,1]$ such that

$$\left(\int_{-1}^{1} Q_i(t)dW(t)Q_j^*(t)\right)\left(\int_{-1}^{1} Q_j(t)dW(t)Q_j^*(t)\right)^{-1} = \delta_{ij} I$$

Karlin-McGregor formula

$$P_{ij}^n = \left(\int_{-1}^{1} t^n Q_i(t) \mathrm{d}W(t) Q_j^*(t) \right) \left(\int_{-1}^{1} Q_j(t) \mathrm{d}W(t) Q_j^*(t) \right)^{-1}$$

Invariant measure or distribution

Non-null vector with non-negative components

$$\boldsymbol{\pi} = (\boldsymbol{\pi}^0; \boldsymbol{\pi}^1; \cdots) = (\pi_1^0, \pi_2^0, \ldots, \pi_N^0; \pi_1^1, \pi_2^1, \ldots, \pi_N^1; \cdots)$$

such that

$$\boldsymbol{\pi} P = \boldsymbol{\pi}$$

$$\Rightarrow \pi_i^j = ?$$

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The family of processes ($N = 2$)

Conjugation

$$W(t) = T^* \widetilde{W}(t) T$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & -\dfrac{\alpha+\beta-k+2}{\beta-k+1} \end{pmatrix}$$

Grünbaum-MdI (2008)

$$\widetilde{W}(t) = t^{\alpha}(1-t)^{\beta} \begin{pmatrix} kt+\beta-k+1 & (1-t)(\beta-k+1) \\ (1-t)(\beta-k+1) & (1-t)^2(\beta-k+1) \end{pmatrix}$$

$t \in (0,1)$, $\alpha, \beta > -1$, $0 < k < \beta + 1$

Pacharoni-Tirao (2006)

We consider the family of OMP $(Q_n(t))_n$ such that

- Three term recurrence relation

$$tQ_n(t) = A_nQ_{n+1}(t) + B_nQ_n(t) + C_nQ_{n-1}(t), \quad n = 0, 1, \ldots$$

where the Jacobi matrix is stochastic

- Choosing $Q_0(t) = I$ the leading coefficient of Q_n is

$$\frac{\Gamma(\beta+2)\Gamma(\alpha+\beta+2n+2)}{\Gamma(\alpha+\beta+n+2)\Gamma(\beta+n+2)}\begin{pmatrix} \frac{k+n}{k} & -\frac{n(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \\ 0 & \frac{(n+\alpha+\beta-k+2)(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \end{pmatrix}$$

- Moreover, the corresponding norms are diagonal matrices:

$$\|Q_n\|_W^2 = \frac{\Gamma(n+\alpha+1)\Gamma(n+1)\Gamma(\beta+2)^2(n+\alpha+\beta-k+2)}{\Gamma(n+\alpha+\beta+2)\Gamma(n+\beta+2)}\times \begin{pmatrix} \frac{n+k}{k(2n+\alpha+\beta+2)} & 0 \\ 0 & \frac{(n+\alpha+1)(n+k+1)}{(\beta-k+1)(2n+\alpha+\beta+3)(n+\alpha+\beta+2)} \end{pmatrix}$$

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$$\|Q_n\|_W^2 = \frac{\Gamma(n+\alpha+1)\Gamma(n+1)\Gamma(\beta+2)^2(n+\alpha+\beta-k+2)}{\Gamma(n+\alpha+\beta+2)\Gamma(n+\beta+2)} \times \begin{pmatrix} \frac{n+k}{k(2n+\alpha+\beta+2)} & 0 \\ 0 & \frac{(n+\alpha+1)(n+k+1)}{(\beta-k+1)(2n+\alpha+\beta+3)(n+\alpha+\beta+2)} \end{pmatrix}$$

We consider the family of OMP $(Q_n(t))_n$ such that

- Three term recurrence relation
$$tQ_n(t) = A_nQ_{n+1}(t) + B_nQ_n(t) + C_nQ_{n-1}(t), \quad n = 0, 1, \ldots$$
where the Jacobi matrix is stochastic
- Choosing $Q_0(t) = I$ the leading coefficient of Q_n is

$$\frac{\Gamma(\beta+2)\Gamma(\alpha+\beta+2n+2)}{\Gamma(\alpha+\beta+n+2)\Gamma(\beta+n+2)}\begin{pmatrix} \frac{k+n}{k} & -\frac{n(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \\ 0 & \frac{(n+\alpha+\beta-k+2)(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \end{pmatrix}$$

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- The choice of the leading coefficient is motivated by the fact that

$$Q_n(1)\mathbf{e}_N = \mathbf{e}_N$$

where $\mathbf{e}_N = (1, 1, \cdots, 1)^T$.

- Consequently, the Jacobi matrix is stochastic:

$$\begin{array}{ccc} 1 \cdot Q_n(1)\mathbf{e}_N & = & A_n Q_{n+1}(1)\mathbf{e}_N + B_n Q_n(1)\mathbf{e}_N + C_n Q_{n-1}(1)\mathbf{e}_N \\ \parallel & & \parallel \\ \mathbf{e}_N & = & (A_n + B_n + C_n)\mathbf{e}_N \end{array}$$

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Particular case $\alpha = \beta = 0$, $k = 1/2$

$$A_n = \begin{pmatrix} \dfrac{(2n+1)(n+2)^2}{2(2n+3)^2(n+1)} & 0 \\ \dfrac{2(n+2)}{(2n+5)(2n+3)^2} & \dfrac{n+3}{2(2n+5)} \end{pmatrix}$$

$$B_n = \begin{pmatrix} \dfrac{1}{2} - \dfrac{4n^2+8n-1}{2(2n+1)^2(2n+3)^2} & \dfrac{n+2}{(2n+3)^2(n+1)} \\ \dfrac{2(n+1)}{(2n+1)(2n+3)^2} & \dfrac{1}{2} - \dfrac{1}{(2n+3)^2} \end{pmatrix}$$

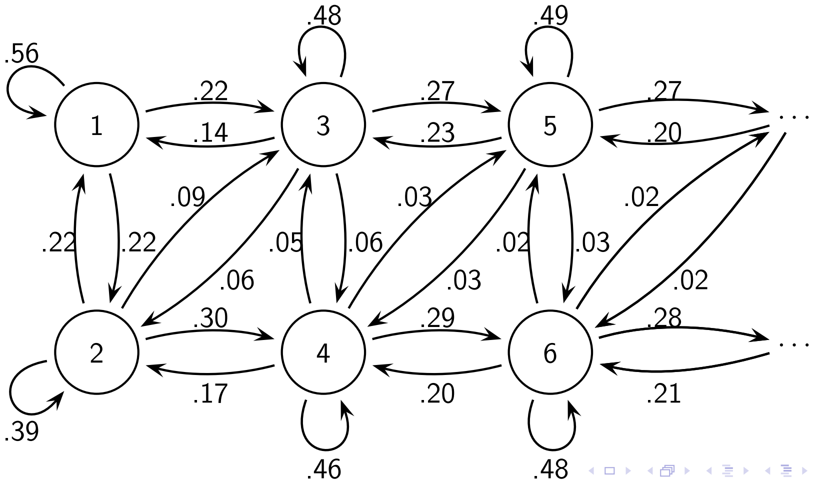
$$C_n = \begin{pmatrix} \dfrac{n^2(2n+3)}{2(2n+1)^2(n+1)} & \dfrac{n}{(n+1)(2n+1)^2} \\ 0 & \dfrac{n}{2(2n+1)} \end{pmatrix}$$

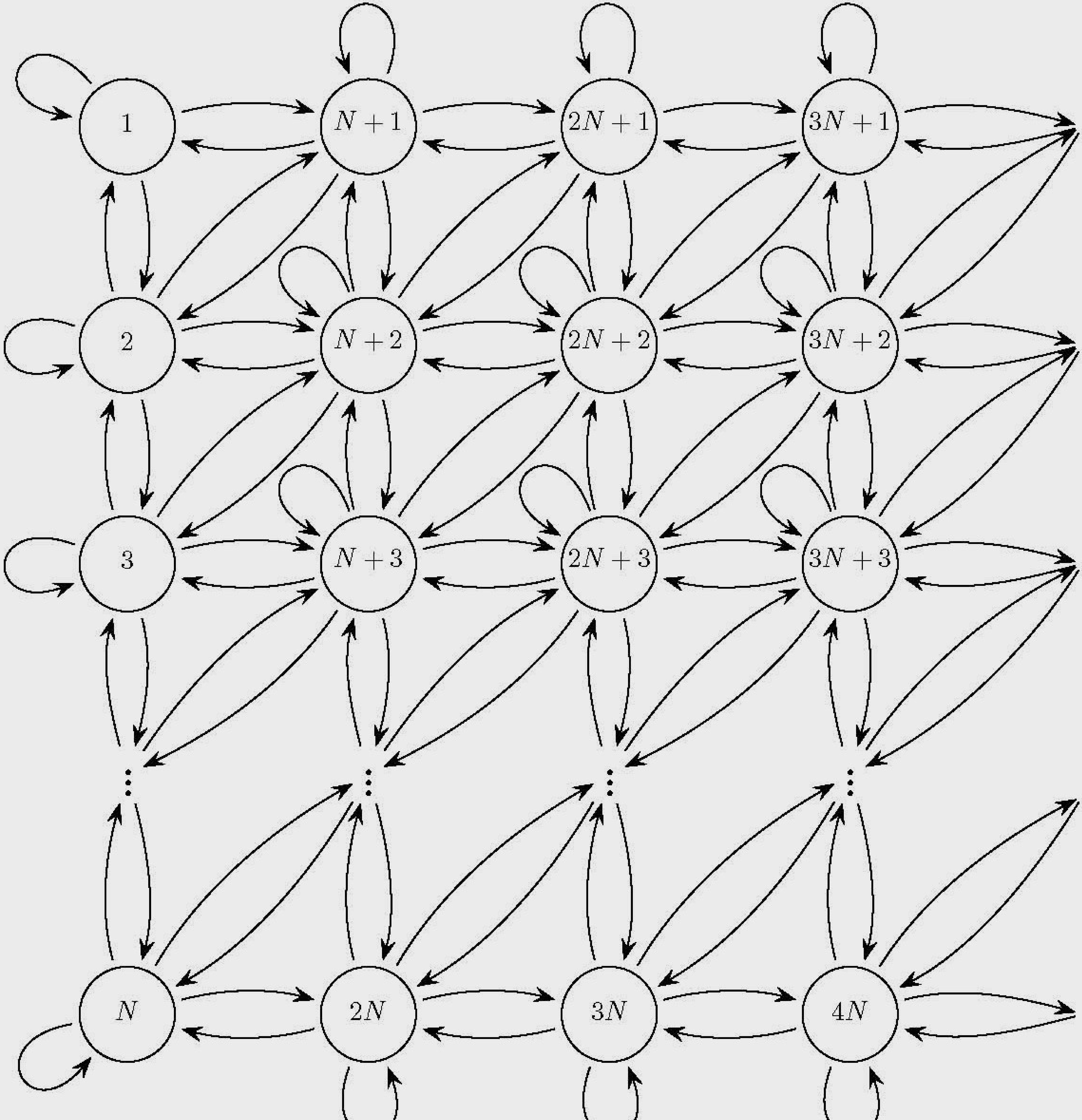
Particular case $\alpha = \beta = 0$, $k = 1/2$

Pentadiagonal Jacobi matrix:

$$P = \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{2}{9} & & & & & & & \\ \frac{2}{9} & \frac{7}{18} & \frac{4}{45} & \frac{3}{10} & & & & & & \\ \frac{5}{36} & \frac{1}{18} & \frac{107}{225} & \frac{3}{50} & \frac{27}{100} & & & & & \\ & \frac{1}{6} & \frac{4}{75} & \frac{23}{50} & \frac{6}{175} & \frac{2}{7} & & & & \\ & & \frac{14}{75} & \frac{2}{75} & \frac{597}{1225} & \frac{4}{147} & \frac{40}{147} & & & \\ & & & \frac{1}{5} & \frac{6}{245} & \frac{47}{98} & \frac{8}{441} & \frac{5}{18} & & \\ & & & & \frac{81}{392} & \frac{3}{196} & \frac{1955}{3969} & \frac{5}{324} & \frac{175}{648} & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Associated network




1
N + 1
2N + 1
3N + 1
2
N + 2
2N + 2
3N + 2
3
N + 3
2N + 3
3N + 3
N
2N
3N
4N

Outline

1. Introduction
   - Random walks
   - Orthogonal matrix polynomials
   - Quasi-birth-and-death processes
2. The family of processes
3. Probabilistic aspects
   - Karlin-McGregor formula
   - Recurrence
   - The invariant measure

n-step transition probability matrix

Let

$$P = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

the transition probability matrix. Then

Karlin-McGregor formula

$$P_{ij}^n = \left(\int_{-1}^{1} t^n Q_i(t) \mathrm{d}W(t) Q_j^*(t) \right) \left(\int_{-1}^{1} Q_j(t) \mathrm{d}W(t) Q_j^*(t) \right)^{-1}$$

Recurrence

Theorem (Grünbaum-MdI, 2008)

Let

$$P = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

be the transition probability matrix.

If $\beta > 0$ *then the process is transient.*

If $-1 < \beta \leqslant 0$ *then the process is null recurrent.*

Hence, the Markov process is never positive recurrent.

The invariant measure

Invariant measure

The row vector

$$\boldsymbol{\pi} = (\boldsymbol{\pi}^0; \boldsymbol{\pi}^1; \cdots)$$

$$\boldsymbol{\pi}^n = \left(\frac{1}{\left(\|Q_n\|_W^2\right)_{1,1}}, \frac{1}{\left(\|Q_n\|_W^2\right)_{2,2}}, \cdots, \frac{1}{\left(\|Q_n\|_W^2\right)_{N,N}} \right), \quad n \geqslant 0$$

is an invariant measure of P

Particular case $N = 2$, $\alpha = \beta = 0$, $k = 1/2$:

$$\boldsymbol{\pi}^n = \left(\frac{2(n+1)^3}{(2n+3)(2n+1)}, \frac{(n+1)(n+2)}{2n+3} \right), \quad n \geqslant 0$$

$$\boldsymbol{\pi} = \left(\frac{2}{3}, \frac{2}{3}; \frac{16}{15}, \frac{6}{5}; \frac{54}{35}, \frac{12}{7}; \frac{128}{63}, \frac{20}{9}; \frac{250}{99}, \frac{30}{11}; \frac{432}{143}, \frac{42}{13}; \frac{686}{195}, \frac{56}{15}; \cdots \right)$$

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The shape of the invariant measure ($N = 2$)

The invariant measure $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} P = \boldsymbol{\pi}$ is given by

$$\boldsymbol{\pi} = (\boldsymbol{\pi}^0; \boldsymbol{\pi}^1; \cdots)$$

where $\boldsymbol{\pi}^n$, $n \geqslant 0$, is a 2-dimensional vector.

We have

$$\lim_{n\to\infty} \pi^n = \begin{cases} (\infty, \infty), & \text{if} \quad \beta > -\frac{1}{2}, \\ \frac{4}{\pi}(2k, 1-2k), & \text{if} \quad \beta = -\frac{1}{2}, \\ (0,0), & \text{if} \quad -1 < \beta < -\frac{1}{2}. \end{cases}$$

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For $\beta > -1/2$

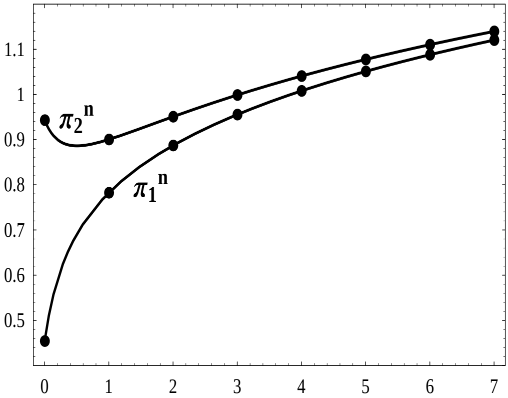

Figure: $\alpha = -0.8$, $\beta = -0.4$, $k = 0.3$

For $\beta = -1/2$

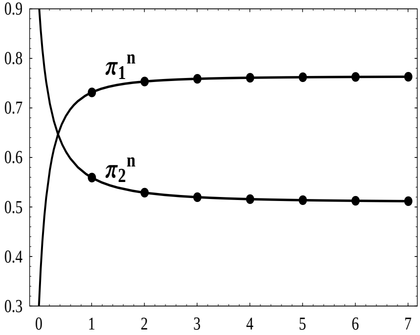

Figure:
$\alpha = -0.92, \beta = -0.5, k = 0.3$

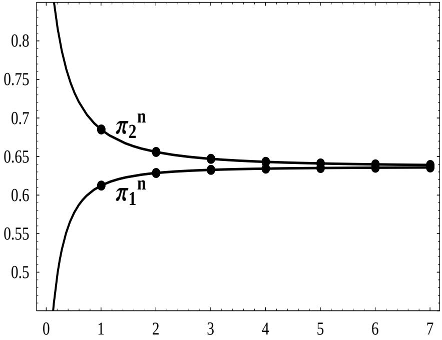

Figure:
$\alpha = -0.9, \beta = -0.5, k = 0.25$

For $-1 < \beta < -1/2$

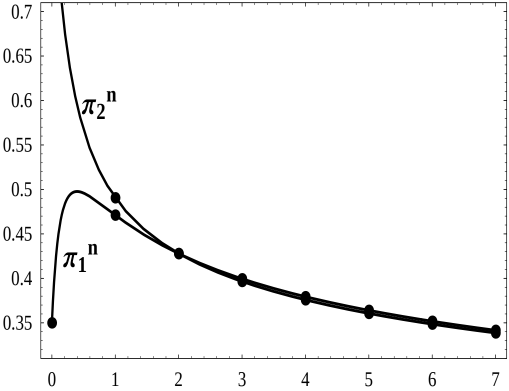

Figure: $\alpha = -0.9$, $\beta = -0.6$, $k = 0.2$

Summary

- A stochastic block tridiagonal matrix P gives rise to a quasi-birth-and-death process.
- The probabilistic aspects of these processes can be greatly simplified if we have the explicit expression of the weight matrix $W(t)$.
- We start from a rich group theoretical situation that yields $W(t)$ as well as an stochastic Jacobi matrix P.
- Therefore, we have a nonhomogeneous quasi-birth-and-death process depending on 4 parameters, α, β, k, N, where we can study recurrence.
- Also we have an explicit expression of the invariant measure π.

F. A. Grünbaum and M. D. de la Iglesia, *Matrix valued orthogonal polynomials arising from group representation theory and a family of quasi-birth-and-death processes*, SIMAX **30** (2008), 741–761.

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