UNIFORM APPROXIMATION OF CONTINUOUS FUNCTIONS BY SMOOTH FUNCTIONS WITH NO CRITICAL POINTS ON HILBERT MANIFOLDS

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Abstract. We prove that every continuous function on a separable infinite-dimensional Hilbert space X can be uniformly approximated by C^∞ smooth functions *with no critical points*. This kind of result can be regarded as a sort of very strong approximate version of the Morse-Sard theorem. Some consequences of the main theorem are as follows. Every two disjoint closed subsets of X can be separated by a one-codimensional smooth manifold which is a level set of a smooth function with no critical points; this fact may be viewed as a nonlinear analogue of the geometrical version of the Hahn-Banach theorem. In particular, every closed set in X can be uniformly approximated by open sets whose boundaries are C^∞ smooth one-codimensional submanifolds of X. Finally, since every Hilbert manifold is diffeomorphic to an open subset of the Hilbert space, all of these results still hold if one replaces the Hilbert space X with any smooth manifold M modelled on X.

1. Introduction and main results

A fundamental result in differential topology and analysis is the Morse-Sard theorem [17, 18], which states that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a C^r smooth function, with $r > \max\{n-m, 0\}$, and C_f stands for the set of critical points of f (that is, the points of X at which the differential of f is not surjective), then the set of critical values, $f(C_f)$, is of (Lebesgue) measure zero in $\mathbb{R}^m$. This result also holds true for smooth functions $f : X \longrightarrow Y$ between two smooth manifolds of dimensions n and m respectively.

Several authors have dealt with the question as to what extent one can obtain a similar result for infinite-dimensional spaces or manifolds modelled on such spaces. Let us recall some of their results.

Smale [20] proved that if X and Y are separable connected smooth manifolds modelled on Banach spaces and $f : X \longrightarrow Y$ is a C^r Fredholm map (that is, every differential $df(x)$ is a Fredholm operator between the corresponding tangent spaces) then $f(C_f)$ is meager, and in particular $f(C_f)$ has no interior points, provided that $r > \max\{\text{index}(df(x)), 0\}$ for all $x \in X$; here index$(df(x))$ stands for the index of the Fredholm operator $df(x)$, that is, the difference between the dimension of the kernel of $df(x)$ and the codimension of the image of $df(x)$, which are both finite. However, these assumptions are quite restrictive: for instance, if X is infinite-dimensional then

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there is no Fredholm map $f : X \longrightarrow \mathbb{R}$. In general, the existence of a Fredholm map f from a manifold X into another manifold Y implies that Y is infinite-dimensional whenever X is.

On the other hand, one cannot dream of extending the Morse-Sard theorem to infinite dimensions without imposing strong restrictions. Indeed, as shown by Kupka's counterexample [15], there are C^∞ smooth functions $f : X \longrightarrow \mathbb{R}$, where X is a Hilbert space, so that their sets of critical values $f(C_f)$ contain intervals and in particular have non-empty interior.

More recently, S. M. Bates has carried out a deep study concerning the sharpness of the hypothesis of the Morse-Sard theorem and the geometry of the sets of critical values of smooth functions. In particular he has shown that the above C^r smoothness hypothesis in the statement of the Morse-Sard theorem can be weakened to $C^{r-1,1}$. See [3, 4, 5, 6, 7].

Nevertheless, for many applications of the Morse-Sard theorem, it is often enough to know that any given function can be uniformly approximated by a map whose set of critical values has empty interior. In this direction, Eells and McAlpin established the following theorem [13]: if X is a separable Hilbert space, then every continuous function from X into $\mathbb{R}$ can be uniformly approximated by a smooth function f whose set of critical values $f(C_f)$ is of measure zero. This allowed them to deduce a version of this theorem for mappings between smooth manifolds M and N modelled on X and a Banach space F respectively, which they called an *approximate Morse-Sard theorem*: every continuous mapping from M into N can be uniformly approximated by a smooth function $f : X \longrightarrow Y$ so that $f(C_f)$ has empty interior. However, this seemingly much more general version of the result is a bit tricky: indeed, as they already observed ([13], Remark 3A), when F is infinite-dimensional, the function f they obtain satisfies that $C_f = X$, although $f(X)$ has empty interior in Y. Unfortunately, even though all the results of that paper seem to be true, some of the proofs are not correct.

In this paper we will prove a much stronger result: if M is a C^∞ smooth manifold modelled on a separable infinite-dimensional Hilbert space X (in the sequel such a manifold will be called a Hilbert manifold), then every continuous function on M can be uniformly approximated by C^∞ smooth functions *with no critical points*. This kind of result might be regarded as the strongest possible one of any class of approximate Morse-Sard theorems, when the target space is $\mathbb{R}$.

As a by-product we also obtain the following: for every open set U in a separable Hilbert space X there is a C^∞ smooth function f whose support is the closure of U and so that $f'(x) \neq 0$ for every $x \in U$. This result could be summed up by saying that for every open subset U of X there is a function f whose open support is U and which does not satisfy Rolle's theorem; one should compare this result with the main theorem from [2] (see also the references therein).

Either of these results has in turn interesting consequences related to smooth approximation and separation of closed sets. For instance, every closed set in a separable Hilbert manifold M can be uniformly approximated by open sets whose boundaries are smooth one-codimensional submanifolds of M. Moreover, every two disjoint closed subsets in M can be separated by a smooth one-codimensional submanifold of M which is a level set of a smooth function with no critical points. The latter may in turn be regarded as a nonlinear analogue of the geometrical version of the Hahn-Banach theorem.

Let us now formally state our main results.

Theorem 1.1. *Let U be an open subset of a separable infinite-dimensional Hilbert space X. Then, for all continuous functions $f : U \longrightarrow \mathbb{R}$ and $\varepsilon : U \longrightarrow (0, +\infty)$, there are C^∞ smooth functions ψ on U such that $|f(x) - \psi(x)| \leq \varepsilon(x)$ and $\psi'(x) \neq 0$ whenever $x \in X$.*

We will prove this result in the following section. Let us now establish the announced consequences of Theorem 1.1.

One could probably adapt the ideas in our proof to extend Theorem 1.1 to the setting of Hilbert manifolds but, for simplicity, we will instead use another approach. Indeed, bearing in mind a fundamental result on Hilbert manifolds due to Eells and Elworthy [12] that every separable Hilbert manifold can be C^∞ embedded as an open subset of the Hilbert space, it is a triviality to observe that Theorem 1.1 still holds if we replace U with a a separable Hilbert manifold.

Theorem 1.2. *Let M be a separable Hilbert manifold. Then, for all continuous functions $f : M \longrightarrow \mathbb{R}$ and $\varepsilon : M \longrightarrow (0, +\infty)$, there are C^∞ smooth functions $\psi : M \longrightarrow \mathbb{R}$ so that $|f(x) - \psi(x)| \leq \varepsilon(x)$, and $d\psi(x) \neq 0$, for all $x \in X$.*

Proof. According to the main theorem of [12], there is a C^∞ embedding of M onto an open subset of the Hilbert space X. Therefore M is C^∞ diffeomorphic to an open subset U of X; let $h : U \longrightarrow M$ be such a C^∞ diffeomorphism. Consider the continuous functions $g = f \circ h : U \longrightarrow \mathbb{R}$ and $\delta = \varepsilon \circ h : U \longrightarrow (0, +\infty)$. By Theorem 1.1 there is a C^∞ smooth function $\varphi : U \longrightarrow \mathbb{R}$ so that φ has no critical points, and

$$|g(y) - \varphi(y)| \leq \delta(y)$$

for all $y \in U$. Now define $\psi = \varphi \circ h^{-1} : M \longrightarrow \mathbb{R}$. Since h is a diffeomorphism it is clear that h takes the critical set of ψ onto the critical set of $\varphi = \psi \circ h$. But, as the latter is empty, so is the former; that is, ψ has no critical points either. On the other hand, it is clear that

$$|f(x) - \psi(x)| = |g(h^{-1}(x)) - \varphi(h^{-1}(x))| \leq \delta(h^{-1}(x)) = \varepsilon(x)$$

for all $x \in M$. □

As an easy corollary we can deduce our promised nonlinear version of the geometrical Hahn-Banach theorem.

We will say that an open subset U of a Hilbert manifold M is *smooth* provided that its boundary ∂U is a smooth one-codimensional submanifold of M.

Corollary 1.3. *Let M be a separable Hilbert manifold. Then, for every two disjoint closed subsets C_1, C_2 of M, there exists a C^∞ smooth function $\varphi : X \longrightarrow \mathbb{R}$ with no critical points, such that the level set $N = \varphi^{-1}(0)$ is a 1-codimensional C^∞*

smooth submanifold of M that separates C_1 and C_2, in the following sense. Define $U_1 = \{x \in M : \varphi(x) < 0\}$ and $U_2 = \{x \in M : \varphi(x) > 0\}$; then U_1 and U_2 are disjoint C^∞ smooth open sets of M so that $C_i \subset U_i$ for $i = 1, 2$, and $\partial U_1 = \partial U_2 = N$.

Proof. By Urysohn's lemma there exists a continuous function $f : M \longrightarrow [0,1]$ so that $C_1 \subset f^{-1}(0)$ and $C_2 \subset f^{-1}(1)$. Taking $\varepsilon = 1/3$ and applying Theorem 1.2 we get a C^∞ smooth function $\psi : M \longrightarrow \mathbb{R}$ which has no critical points and is so that

$$|f(x) - \psi(x)| \leq 1/3$$

for all $x \in M$; in particular

$$C_1 \subseteq f^{-1}(0) \subseteq \psi^{-1}(-\infty, 1/2) := U_1,$$

and

$$C_2 \subseteq f^{-1}(1) \subseteq \psi^{-1}(1/2, +\infty) := U_2.$$

The open sets U_1 and U_2 are smooth because their common boundary $N = \psi^{-1}(1/2)$ is a smooth one-codimensional submanifold of M (thanks to the implicit function theorem and the fact that $d\psi(x) \neq 0$ for all $x \in N$). In order to obtain the result in the above form it is enough to set $\varphi = \psi - 1/2$. □

A trivial consequence of this result is that every closed subset of X can be uniformly approximated by smooth open subsets of X. In fact,

Corollary 1.4. *Every closed subset of a separable Hilbert manifold M can be approximated by smooth open subsets of M, in the following sense: for every closed set $C \subset M$ and every open set W containing C there is a C^∞ smooth open set U so that $C \subset U \subseteq W$.*

Finally, the following result, which also implies the above corollary, tells us that for every open set U in X there always exists a function whose open support is U and which does not satisfy Rolle's theorem.

Theorem 1.5. *For every open subset U of a Hilbert manifold M there is a continuous function f on M whose support is the closure of U, so that f is C^∞ smooth on U and yet f has no critical point in U.*

Proof. For the same reasons as in the proof of Theorem 1.2 we may assume that U is an open subset of the Hilbert space $X = \ell_2$. Let $\varepsilon : X \longrightarrow [0, +\infty)$ be the distance function to $X \setminus U$, that is,

$$\varepsilon(x) = \text{dist}(x, X \setminus U) = \inf\{\|x - y\| : y \in X \setminus U\}.$$

The function ε is continuous on X and satisfies that $\varepsilon(x) > 0$ if and only if $x \in U$. According to Theorem 1.1, and setting $f(x) = 2\varepsilon(x)$, there exists a C^∞ smooth function $\psi : U \longrightarrow \mathbb{R}$ which has no critical points on U, and such that ε-approximates f on U, that is,

$$|2\varepsilon(x) - \psi(x)| \leq \varepsilon(x)$$

for all $x \in U$. This inequality implies that

$$\lim_{x \to z} \psi(x) = 0$$

for every $z \in \partial U$. Therefore, if we set $\psi = 0$ on $X \setminus U$, the extended function $\psi : X \longrightarrow [0, +\infty)$ is continuous on the whole of X, is C^∞ smooth on U and has no critical points on U. On the other hand, $\psi(x) \geq \varepsilon(x) > 0$ for all $x \in U$, hence the support of ψ is $\overline{U}$. □

2. Proof of the main result

The main ideas behind the proof of Theorem 1.1 are as follows. First we use a perturbed smooth partition of unity to approximate the given continuous function f. The summands of this perturbed partition of unity are functions supported on scalloped balls and carefully constructed in such a way that the critical points of the approximating sum φ are kept under control. More precisely, those critical points consist of a sequence of compact sets K_n that are suitably isolated in pairwise disjoint open sets U_n of small diameter so that the oscillation of both f and φ on U_n is small as well.

Then we have to eliminate all of those critical points without losing much of the approximation. To this end we compose the approximating function φ with a sequence of deleting diffeomorphisms $h_n : X \longrightarrow X \setminus K_n$ which extract each of the compact sets of critical points K_n and restrict to the identity outside each of the open sets U_n. The infinite composition of deleting diffeomorphisms with our function, $\psi = \varphi \circ \bigcirc_{n=1}^{\infty} h_n$, is locally finite, in the sense that only a finite number of diffeomorphisms are acting on some neighborhood of each point, while all the rest restrict to the identity on that neighborhood. In this way we obtain a smooth function ψ which has no critical points, and which happens to approximate the function φ (which in turn approximates the original f) because the perturbation brought on φ by that infinite composition is not very important: indeed, recall that each h_n restricts to the identity outside the set U_n (on which φ has a small oscillation), and the U_n are pairwise disjoint.

We will make the proof of Theorem 1.1 in the case of a constant $\varepsilon > 0$ so as to avoid bearing an unnecessary burden of notation. Later on we will briefly explain what additional technical precautions must be taken in order to deduce the general form of this result (see Remark 2.8).

The following proposition shows the existence of a function φ with the above properties. Recall that C_φ stands for the set of critical points of φ.

Proposition 2.1. *Let U be an open subset of the separable Hilbert space X. Let $f : U \longrightarrow \mathbb{R}$ be a continuous function on X, and $\varepsilon > 0$. Then there exist a C^∞ smooth function $\varphi : U \longrightarrow \mathbb{R}$, a sequence (K_n) of compact sets, a sequence (U_n) of open sets, and a sequence $(B(y_n, r_n))$ of open balls which are contained in U and whose union covers U, such that:*

(a) $C_\varphi \subseteq \bigcup_{n=1}^{\infty} K_n$;
(b) $K_n \subset U_n \subseteq B(y_n, r_n)$ *for all* $n \in \mathbb{N}$, *and* $U_n \cap U_m = \emptyset$ *whenever* $n \neq m$;
(c) $|\varphi(x) - f(y)| \leq 2\varepsilon$ *for all* $x, y \in B(y_n, r_n)$, $n \in \mathbb{N}$;
(d) *for every* $x \in U$ *there exist an open neighborhood* V_x *of* x *and some* $n_x \in \mathbb{N}$ *such that* $V_x \cap U_m = \emptyset$ *for all* $m > n_x$.

The following theorem ensures the existence of the diffeomorphisms h_n.

Theorem 2.2. *Let X be an infinite-dimensional Hilbert space. Then, for every compact set K and every open subset U of X with $K \subset U$, there exists a C^∞ smooth diffeomorphism $h : X \longrightarrow X \setminus K$ so that h restricts to the identity outside U.*

This result may be regarded, in the Hilbert case, as a (rather technical, but crucial to our purposes) improvement of some known results on smooth negligibility of compact sets (see [1, 14]; there h is known to be the identity only outside a ball containing K).

Assume for a while that Proposition 2.1 and Theorem 2.2 are already established, and let us see how we can deduce Theorem 1.1.

Proof of Theorem 1.1

For a given continuous function f and a number $\varepsilon > 0$, take a function φ and sequences (K_n) and (U_n) with the properties of Proposition 2.1. For each compact set K_n and each open set U_n, use Theorem 2.2 to find a C^∞ diffeomorphism $h_n : X \longrightarrow X \setminus K_n$ so that $h_n(x) = x$ if $x \notin U_n$. Note that, since the U_j contain the K_j and are pairwise disjoint,

$$h_n(x) \notin \bigcup_{j=1}^{\infty} K_j \supseteq C_\varphi \tag{1}$$

for all $x \in U$, $n \in \mathbb{N}$. Define then $\psi : U \longrightarrow \mathbb{R}$ by

$$\psi = \varphi \circ \bigcirc_{n=1}^{\infty} h_n.$$

This formula makes sense and the function ψ is C^∞ smooth because the infinite composition is in fact locally finite. Indeed, for a given $x \in U$, according to Proposition 2.1(d), we can find an open neighborhood V_x of x and some $n_x \in \mathbb{N}$ so that $V_x \cap U_m = \emptyset$ for all $m > n_x$; hence $h_m(y) = y$ for all $y \in V_x$ and $m > n_x$, and therefore

$$\psi(y) = \varphi \circ h_{n_x} \circ h_{n_x - 1} \circ ... \circ h_2 \circ h_1(y) \tag{2}$$

for all $y \in V_x$. The derivative $\psi'(y)$ is given by

$$\psi'(y) = \varphi'\big(\bigcirc_{j=1}^{n_x} h_j(y)\big) \circ Dh_{n_x}\big(\bigcirc_{j=1}^{n_x-1} h_j(y)\big) \circ ... \circ Dh_2(h_1(y)) \circ Dh_1(y) \tag{3}$$

for all $y \in V_x$. Since $U_n \subseteq X \setminus U_m$ for $n \neq m$, we have that h_m is the identity on U_n, and therefore $Dh_m(x) = I$ (the identity isomorphism of ℓ_2) for all $x \in U_n$. By the continuity of Dh_n it follows that $Dh_m(x) = I$ for all $x \in \overline{U}_n$, if $m \neq n$. This implies that, for $y \in \overline{U_n} \cap V_x$, all the differentials $Dh_j(z)$ in (3) are the identity, except perhaps for $j = n$. Hence we get that either

$$\psi'(y) = \varphi'(h_n(y)) \circ Dh_n(y), \text{ and } \psi(y) = \varphi(h_n(y)), \tag{4}$$

if y belongs to some $\overline{U}_n$, or else

$$\psi'(y) = \varphi'(y), \text{ and } \psi(y) = \varphi(y), \tag{5}$$

when $y \notin \bigcup_{n=1}^{\infty} \overline{U}_n$.

Now we can easily check that $C_\psi = \emptyset$. Take $x \in U$. If we are in the case that $x \in \overline{U}_n$ for some n then $\psi'(x) = \varphi'(h_n(x)) \circ Dh_n(x) \neq 0$, because $Dh_n(x)$ is a linear isomorphism and, according to (1) above, $\varphi'(h_n(x)) \neq 0$. Otherwise we have that $x \notin \bigcup_{n=1}^\infty \overline{U}_n \supseteq C_\varphi$, so $\psi'(x) = \varphi'(x) \neq 0$ trivially.

It only remains to check that ψ still approximates f. As before, for a given $x \in U$, either $\psi(x) = \varphi(x)$ or $\psi(x) = \varphi(h_n(x))$ for some n (with $x \in U_n$). In the first case, from Proposition 2.1(c) we get that $|\psi(x) - f(x)| \leq 2\varepsilon$. In the second case, bearing in mind that $h_n(x) \in U_n \subseteq B(y_n, r_n)$, and for the same reason, we have that

$$|\psi(x) - f(x)| = |\varphi(h_n(x)) - f(x)| \leq 2\varepsilon;$$

in either case we obtain that $|\psi(x) - f(x)| \leq 2\varepsilon$.

Proof of Proposition 2.1

We will assume that $U = X$, since the proof is completely analogous in the case of a general open set. One only has to take some (easy but rather rambling) technical precautions in order to make sure that the different balls considered in the argument are in U.

Let $B(x, r)$ and $\overline{B}(x, r)$ stand for the open ball and closed ball, respectively, of center x and radius r, with respect to the usual hilbertian norm $\|\cdot\|$ of X.

Let $f : X \longrightarrow \mathbb{R}$ be a continuous function, and let $\varepsilon > 0$. By continuity, for every $x \in X$ there exists $\delta_x > 0$ so that $|f(y) - f(x)| \leq \varepsilon/4$ whenever $y \in B(x, 2\delta_x)$. Since $X = \bigcup_{x \in X} B(x, \delta_x/2)$ is separable, there exists a countable subcovering,

$$X = \bigcup_{n=1}^{\infty} B(x_n, s_n/2),$$

where $s_n = \delta_{x_n}$, for some sequence of centers (x_n). By induction we can choose a sequence of *linearly independent* vectors (y_n), with $y_n \in B(x_n, s_n/2)$, so that

$$X = \bigcup_{n=1}^{\infty} B(y_n, s_n).$$

Moreover, we have that

$$|f(y) - f(y_n)| \leq \varepsilon/2$$

provided $\|y - y_n\| \leq \frac{3}{2}s_n$, as is immediately checked.

Notation 2.3. In the sequel $\mathcal{A}[z_1, ..., z_k]$ stands for the affine subspace spanned by a finite sequence of points $z_1, ..., z_k \in X$.

The following lemma shows that we can slightly move the radii s_n so that, for any finite selection of centers y_n, the spheres that are the boundaries of the balls $B(y_n, s_n)$ have empty intersection with the affine subspace spanned by those centers.

Lemma 2.4. *We can find a sequence of positive numbers (r_n) with $s_n \leq r_n \leq \frac{3}{2}s_n$ so that, if we denote $S_n = \partial B(y_n, r_n)$ then,*

(i) *for each finite sequence of positive integers* $k_1 < k_2 < ... < k_m$,

$$\mathcal{A}[y_{k_1}, ..., y_{k_m}] \cap S_{k_1} \cap ... \cap S_{k_m} = \emptyset.$$

(ii) *for any* $n, k \in \mathbb{N}$, $y_n \notin S_k$.

Proof. We will define the r_n inductively.

For $n = 1$ we may take $r_1 \in [s_1, \frac{3}{2}s_1]$ so that r_1 does not belong to the countable set $\{\|y_1 - y_k\| : k \in \mathbb{N}\}$; this means that $y_k \notin S_1$ for any $k \in \mathbb{N}$. On the other hand, it is obvious that $\{y_1\} \cap S_1 = \emptyset$.

Assume now that $r_1, ..., r_n$ have already been chosen in such a way that the spheres S_1, ..., S_n satisfy (i) and (ii), and let us see how we can find r_{n+1}. For any finite sequence of integers $0 < k_1 < ... < k_j \leq n + 1$, let us denote

$$\mathcal{A}_{k_1,...,k_j} = \mathcal{A}[y_{k_1}, ..., y_{k_j}].$$

For simplicity, and up to a suitable translation (which obviously does not affect our problem), we may assume that $y_{n+1} = 0$, so that $\mathcal{A}_{k_1,...,k_m,n+1}$ is the m-dimensional vector subspace of X spanned by y_{k_1}, ..., y_{k_m}. Now, for each finite sequence of integers $0 < k_1 < ... < k_m \leq n$, consider the map $F_{k_1,...,k_m} : \mathcal{A}_{k_1,...,k_m,n+1} \longrightarrow \mathbb{R}^m$ defined by

$$F_{k_1,...,k_m}(x) = \big(\|x - y_{k_1}\|^2 - {r_{k_1}}^2, ..., \|x - y_{k_m}\|^2 - {r_{k_m}}^2\big).$$

Note that

$$DF_{k_1,...,k_m}(x) = \big(2(x - y_{k_1}), ..., 2(x - y_{k_m})\big)$$

and therefore $\text{rank}\big(DF_{k_1,...,k_m}(x)\big) < m$ if and only if $x \in \mathcal{A}_{k_1,...,k_m}$. By the induction assumption we know that

$$S_{k_1} \cap ... \cap S_{k_m} \cap \mathcal{A}_{k_1,...,k_m} = \emptyset,$$

hence it is clear that $\text{rank}\big(DF_{k_1,...,k_m}(x)\big) = m$ for all $x \in S_{k_1} \cap ... \cap S_{k_m} \cap \mathcal{A}_{k_1,...,k_m,n+1}$. This implies that

$$M_{k_1,...,k_m} := S_{k_1} \cap ... \cap S_{k_m} \cap \mathcal{A}_{k_1,...,k_m,n+1}$$

is a compact $m - m = 0$-dimensional submanifold of $\mathcal{A}_{k_1,...,k_m,n+1}$, and in particular $M_{k_1,...,k_m}$ consists of a finite number of points (in fact two points, but we do not need to know this). Therefore

$$M = \bigcup M_{k_1,...,k_m}$$

(where the union is taken over all the finite sequences of integers $0 < k_1 < ... < k_n \leq n$) is a finite set as well. Now we have that

$$I := \big[s_{n+1}, \frac{3}{2}s_{n+1}\big] \setminus \Big(\{\|z\| : z \in M\} \cup \{\|y_j\| : j \in \mathbb{N}\}\Big)$$

is an uncountable subset of the real line, so we can find a number $r_{n+1} \in I$. With this choice it is clear that

$$S_{k_1} \cap ... \cap S_{k_m} \cap S_{n+1} \cap \mathcal{A}_{k_1,...,k_m,n+1} = M_{k_1,...,k_m} \cap S_{n+1} = \emptyset$$

for all finite sequences of integers $0 < k_1 < ... < k_m < n+1$, and also $y_j \notin S_{n+1} = \partial B(0, r_{n+1})$ for all $j \in \mathbb{N}$. Therefore the spheres S_1, ..., S_n, S_{n+1} satisfy (i) and (ii) as well. By induction the sequence (r_n) is thus well defined. □

Since $s_n \leq r_n \leq \frac{3}{2} s_n$ for all n, it is clear that the new balls $B(y_n, r_n)$ keep the two important properties of the old balls $B(y_n, s_n)$, namely,

$$X = \bigcup_{n=1}^{\infty} B(y_n, r_n), \tag{6}$$

and

$$|f(y) - f(y_n)| \leq \varepsilon/2 \text{ whenever } \|y - y_n\| \leq r_n. \tag{7}$$

Now we define the scalloped balls B_n that are the basis for our perturbed partition of unity: set $B_1 = B(y_1, r_1)$, and for $n \geq 2$ define

$$B_n = B(y_n, r_n) \setminus \Big(\bigcup_{j=1}^{n-1} \overline{B}(y_j, \lambda_n r_j) \Big);$$

where $1/2 < \lambda_2 < \lambda_3 < ... < \lambda_n < \lambda_{n+1} < ... < 1$, with $\lim_{n \to \infty} \lambda_n = 1$. The λ_n are to be fixed later on.

Taking into account that $\lim_{n \to \infty} \lambda_n = 1$, it is easily checked that the B_n form a locally finite open covering of X, with the nice property that

$$|f(y) - f(y_n)| \leq \varepsilon/2 \text{ whenever } y \in B_n.$$

Next, pick a C^∞ smooth function $g_1 : \mathbb{R} \longrightarrow [0, 1]$ so that:

(i) $g_1(t) = 1$ for $t \leq 0$,
(ii) $g_1(t) = 0$ for $t \geq {r_1}^2$,
(iii) $g_1'(t) < 0$ if $0 < t < {r_1}^2$;

and define then $\varphi_1 : X \longrightarrow \mathbb{R}$ by

$$\varphi_1(x) = g_1(\|x - y_1\|^2)$$

for all $x \in X$. Note that φ_1 is a C^∞ smooth function whose open support is B_1, and $B_1 \cap C_{\varphi_1} = \{y_1\}$, that is, y_1 is the only critical point of φ_1 that lies inside B_1.

Now, for $n \geq 2$, pick C^∞ smooth functions $\theta_{(n,j)} : \mathbb{R} \longrightarrow [0, 1]$, $j = 1, ..., n$, with the following properties. For $j = 1, ..., n-1$, $\theta_{(n,j)}$ satisfies that

(i) $\theta_{(n,j)}(t) = 0$ for $t \leq (\lambda_n r_j)^2$,
(ii) $\theta_{(n,j)}(t) = 1$ for $t \geq {r_j}^2$,
(iii) $\theta_{(n,j)}'(t) > 0$ if $(\lambda_n r_j)^2 < t < {r_j}^2$;

while for $j = n$ the function $\theta_{(n,n)}$ is such that

(i) $\theta_{(n,n)}(t) = 1$ for $t \leq 0$,
(ii) $\theta_{(n,n)}(t) = 0$ for $t \geq {r_n}^2$,
(iii) $\theta_{(n,n)}'(t) < 0$ if $0 < t < {r_n}^2$.

Then define the function $g_n : \mathbb{R}^n \longrightarrow [0,1]$ as

$$g_n(t_1, ..., t_n) = \prod_{i=1}^{n} \theta_{(n,i)}(t_i)$$

for all $t = (t_1, ..., t_n) \in \mathbb{R}^n$. This function is clearly C^∞ smooth on $\mathbb{R}^n$ and satisfies the following properties:

(i) $g_n(t_1, ..., t_n) > 0$ if and only if $t_j > (\lambda_n r_j)^2$ for all $j = 1, ..., n-1$, and $t_n < {r_n}^2$; and g_n vanishes elsewhere;
(ii) $g_n(t_1, ..., t_n) = \theta_{(n,n)}(t_n)$ whenever $t_j \geq {r_j}^2$ for all $j = 1, ..., n-1$;
(iii) $\nabla g_n(t_1, ..., t_n) \neq 0$ provided $(\lambda_n r_j)^2 < t_j$ for all $j = 1, ..., n-1$, and $0 < t_n < {r_n}^2$.

Moreover, under the same conditions as in (iii) just above we have that

$$\frac{\partial g_n}{\partial t_n}(t_1, ..., t_n) = \frac{\partial \theta_{(n,n)}}{\partial t_n}(t_n) \prod_{i=1}^{n-1} \theta_{(n,i)}(t_i) < 0, \tag{8}$$

since no function in this product vanishes on the specified set, while for $j < n$, according to the corresponding properties of the functions $\theta_{(n,j)}$ we have that

$$\frac{\partial g_n}{\partial t_j}(t_1, ..., t_n) = \frac{\partial \theta_{(n,j)}}{\partial t_j}(t_j) \prod_{i=1, i\neq j}^{n} \theta_{(n,i)}(t_i) > 0. \tag{9}$$

If we are not in the conditions of (iii) then the corresponding inequalities do still hold but are not strict.

Let us now define $\varphi_n : X \longrightarrow [0,1]$ by

$$\varphi_n(x) = g_n(\|x - y_1\|^2, ..., \|x - y_n\|^2).$$

It is clear that φ_n is a C^∞ smooth function whose open support is precisely the scalloped ball B_n.

As above, let us denote by C_{φ_n} the critical set of φ_n, that is,

$$C_{\varphi_n} = \{x \in X : \varphi_n'(x) = 0\}.$$

Since our norm $\|\cdot\|$ is hilbertian we have that, if $x \in C_{\varphi_n} \cap B_n$, then x belongs to the affine span of $y_1, ..., y_n$. Indeed, if $x \in B_n$,

$$\varphi_n'(x) = \sum_{j=1}^{n} \frac{\partial g_n}{\partial t_j}(\|x - y_1\|^2, ..., \|x - y_n\|^2)\, 2(x - y_j) = 0, \tag{10}$$

which (taking into account (8) and the fact that the y_j are all linearly independent) means that x is in the affine span of $y_1, ..., y_n$. Here, as is usual, we identify the Hilbert space X with its dual X^*, and we make use of the fact that the derivative of the function $x \mapsto \|x\|^2$ is the mapping $x \mapsto 2x$.

Similarly, by using (8) it can be shown that $x \in C_{\varphi_1 + \cdots + \varphi_m} \cap (B_1 \cup ... \cup B_m)$ implies that x belongs to the affine span of $y_1, ..., y_m$.

In order that our approximating function has a small critical set we cannot use the standard approximation provided by the partition of unity associated with the functions $(\varphi_j)_{i\in\mathbb{N}}$, namely

$$x \mapsto \frac{\sum_{n=1}^{\infty} \alpha_n \varphi_n(x)}{\sum_{n=1}^{\infty} \varphi_n(x)},$$

where $\alpha_n = f(y_n)$. Indeed, such a function would have a huge set of critical points since it would be constant (equal to α_n) on a lot of large places (at least on each B_n minus the union of the rest of the B_j). Instead, we will modify this standard approximation by letting the α_n be functions (and not mere numbers) of very small oscillation and with only one critical point (namely y_n). So, for every $n \in \mathbb{N}$ let us pick a C^∞ smooth real function $a_n : [0, +\infty) \longrightarrow \mathbb{R}$ with the following properties:

(i) $a_n(0) = f(y_n)$;
(ii) $a_n'(t) < 0$ whenever $t > 0$;
(iii) $|a_n(t) - a_n(0)| \leq \varepsilon/2$ for all $t \geq 0$;

and define $\alpha_n : X \longrightarrow \mathbb{R}$ by

$$\alpha_n(x) = a_n(\|x - y_n\|^2)$$

for every $x \in X$. It is clear that α_n is a C^∞ smooth function on X whose only critical point is y_n. Besides,

$$|\alpha_n(x) - f(y_n)| \leq \varepsilon/2 \text{ for all } x \in X.$$

Now we can define our approximating function $\varphi : X \longrightarrow \mathbb{R}$ by

$$\varphi(x) = \frac{\sum_{n=1}^{\infty} \alpha_n(x)\varphi_n(x)}{\sum_{n=1}^{\infty} \varphi_n(x)}$$

for every $x \in X$. Since the sums are locally finite, it is clear that φ is a well-defined C^∞ smooth function.

Fact 2.5. *The function φ approximates f nicely. Namely, we have that*

(i) $|\varphi(x) - f(x)| \leq \varepsilon$ *for all* $x \in X$, *and*
(ii) $|\varphi(y) - f(x)| \leq 2\varepsilon$ *for all* $x, y \in B(y_n, r_n)$ *and each* $n \in \mathbb{N}$.

Proof. Indeed, for every n we have that $|\alpha_n(x) - f(y_n)| \leq \varepsilon/2$ for all $x \in X$. On the other hand, by (7) above we know that $|f(x) - f(y_n)| \leq \varepsilon/2$ whenever $x \in B(y_n, r_n)$. Then, by the triangle inequality, it follows that

$$|\alpha_n(x) - f(x)| \leq \varepsilon \tag{11}$$

whenever $x \in B(y_n, r_n)$. In the same way we deduce that

$$|\alpha_m(x) - f(y_n)| \leq \varepsilon \tag{12}$$

whenever $x \in B(y_n, r_n) \cap B(y_m, r_m)$. Since $\varphi_m(y) = 0$ when $y \notin B(y_m, r_m)$, from (11) we get that

$$|\varphi(x) - f(x)| = \left| \frac{\sum_{m=1}^{\infty} (\alpha_m(x) - f(x))\varphi_m(x)}{\sum_{m=1}^{\infty} \varphi_m(x)} \right| \leq \frac{\sum_{m=1}^{\infty} \varepsilon\varphi_m(x)}{\sum_{m=1}^{\infty} \varphi_m(x)} = \varepsilon$$

for all $x \in X$, which shows (i). Similarly, we deduce from (12) that

$$|\varphi(y) - f(y_n)| = \left| \frac{\sum_{m=1}^{\infty} (\alpha_m(y) - f(y_n))\varphi_m(y)}{\sum_{m=1}^{\infty} \varphi_m(y)} \right| \leq \frac{\sum_{m=1}^{\infty} \varepsilon \varphi_m(y)}{\sum_{m=1}^{\infty} \varphi_m(y)} = \varepsilon$$

for every $y \in B(y_n, r_n)$, which, combined with the fact that $|f(x) - f(y_n)| \leq \varepsilon/2$ for $x \in B(y_n, r_n)$, yields that

$$|\varphi(y) - f(x)| \leq \varepsilon + \varepsilon/2,$$

for every $x, y \in B(y_n, r_n)$, so (ii) is satisfied as well. □

Now let us have a look at the derivative of φ. To this end let us introduce the auxiliary functions f_n defined by

$$f_n(x) = \frac{\sum_{k=1}^{n} \alpha_k(x)\varphi_k(x)}{\sum_{k=1}^{n} \varphi_k(x)}, \text{ for all } x \in \bigcup_{i=1}^{n} B_i.$$

Notice that φ can be expressed as

$$\varphi(x) = \lim_{n \to \infty} f_n(x),$$

that the domains of the f_n form an increasing tower of open sets whose union is X, and that each f_n restricts to f_{n-1} on $\bigcup_{i=1}^{n-1} B_i \setminus B_n$. Moreover, for each $x \in X$ there is some open neighborhood V_x of x and some $n_x \in \mathbb{N}$ so that $\varphi(y) = f_{n_x}(y)$ for all $y \in V_x$. In fact we have that

$$\varphi(x) = f_n(x) \quad \text{for all} \quad x \in V_n := \Big(\bigcup_{j=1}^{n} B_j\Big) \setminus \Big(\bigcup_{i=n+1}^{\infty} \overline{B}_i\Big),$$

for every n, the V_n are open, $V_n \subseteq V_{n+1}$, and $\bigcup_{i=1}^{\infty} V_i = X$, because the covering of X formed by the B_j is locally finite.

Hence, by looking at the derivatives of the functions f_n we will get enough information about the derivative of φ.

If $x \in \bigcup_{j=1}^{n} B_j$ then the expression for the derivative of f_n is given by

$$f_n'(x) = \frac{\sum_{j=1}^{n} [\alpha_j'(x)\varphi_j(x) + \alpha_j(x)\varphi_j'(x)] \sum_{i=1}^{n} \varphi_i(x) - \sum_{j=1}^{n} \varphi_j'(x) \sum_{i=1}^{n} \alpha_i(x)\varphi_i(x)}{(\sum_{j=1}^{n} \varphi_j(x))^2}.$$

Therefore, for $x \in \bigcup_{j=1}^{n} B_j$ we have that $f_n'(x) = 0$ if and only if

$$\sum_{j=1}^{n} \sum_{i=1}^{n} \varphi_i(x) \Big[\alpha_j'(x)\varphi_j(x) + \big(\alpha_j(x) - \alpha_i(x)\big)\varphi_j'(x)\Big] = 0. \tag{13}$$

By inserting the expressions for the derivatives of φ_j and α_j in equation (13), we can express the condition $f_n'(x) = 0$ as a nontrivial linear dependence link on the vectors $(x - y_j)$, which yields that x is in the affine span of the points $y_1, ..., y_n$. Indeed, we are going to prove the following.

Fact 2.6. *If $x \in C_{f_n} \cap B_n$, then $x \in \mathcal{A}_n := \mathcal{A}[y_1, ..., y_n]$. Moreover, for each $n \in \mathbb{N}$ and for every finite sequence of positive integers $k_1 < k_2 < ... < k_m < n$ we have that*

$$C_{f_n} \cap \big(B_n \setminus \bigcup_{j=1}^{m} B_{k_j}\big) \subseteq \mathcal{A}\big[\{y_1, ..., y_n\} \setminus \{y_{k_1}, ..., y_{k_m}\}\big].$$

Proof. As above, in all the subsequent calculations, we will identify the Hilbert space X with its dual X^*, and the derivative of $\|\cdot\|^2$ with the mapping $x \mapsto 2x$. To save notation, let us simply write

$$\frac{\partial g_n}{\partial t_j}(\|x-y_1\|^2, ..., \|x-y_n\|^2) = \mu_{(n,j)},$$

and

$$a_j'(\|x-y_j\|^2) = \eta_j.$$

Notice that, according to (8) and (9) above, $\mu_{(n,j)} \geq 0$ for $j = 1, ..., n-1$, while $\mu_{(n,n)} \leq 0$; and $\mu_{(n,n)} \neq 0$ provided $x \in B_n$ and $x \neq y_n$; on the other hand it is clear that $\eta_j < 0$ for all j unless $x = y_j$ (in which case $\eta_j = 0$).

Assuming $x \in C_{f_n} \cap B_n$, and taking into account the expression (10) for $\varphi_j'(x)$ and the fact that $\alpha_j'(x) = 2\eta_j(x-y_j)$, we can write condition (13) above in the form

$$2\sum_{j=1}^{n}\sum_{i=1}^{n}\varphi_i(x)\Big[\eta_j\varphi_j(x)\,(x-y_j) + \big(\alpha_j(x)-\alpha_i(x)\big)\sum_{\ell=1}^{j}\mu_{(j,\ell)}\,(x-y_\ell)\Big] = 0,$$

which in turn is equivalent (taking the common factors of each $(x-y_j)$ together) to the following one

$$\sum_{j=1}^{n}\left[\eta_j\varphi_j(x)\sum_{i=1}^{n}\varphi_i(x) + \sum_{k=j}^{n}\Big(\sum_{i=1}^{n}\big(\alpha_k(x)-\alpha_i(x)\big)\varphi_i(x)\Big)\mu_{(k,j)}\right](x-y_j) = 0. \quad (14)$$

Now notice that, if we can prove that at least one of the expressions multiplying the $(x-y_j)$ does not vanish then we are done; indeed, we will have that the vectors $x-y_1$, ..., $x-y_n$ are linearly dependent, which means that x belongs to the affine span of the points $y_1, ..., y_n$.

So let us check that not all of those expressions in (14) vanish. In fact we are going to see that at least one of the terms is strictly negative. We can obviously assume that x is not any of the points $y_1, ..., y_n$ (which are already in $\mathcal{A}_n$). In this case we have that $\mu_{(n,n)} < 0$ and $\eta_j < 0$ for all $j = 1, ..., n$. For simplicity, we will only make the argument in the case $n = 3$; giving a proof in a more general case would be as little instructive as tedious to read.

Let us first assume that $\varphi_j(x) \neq 0$ for $j = 1, 2, 3$. We begin by looking at the term that multiplies $(x-y_3)$ in (14), that is

$$\beta_3 := \eta_3\varphi_3(x)\sum_{i=1}^{3}\varphi_i(x) + \sum_{i=1}^{3}\big(\alpha_3(x)-\alpha_i(x)\big)\varphi_i(x)\mu_{(3,3)}.$$

If $\sum_{i=1}^{3}\big(\alpha_3(x)-\alpha_i(x)\big)\varphi_i(x)\geq 0$ we are done, since in this case we easily see that $\beta_3<0$ (remember that $\mu_{(3,3)}\leq 0$, $\eta_3<0$, and $\varphi_3(x)>0$). Otherwise we have that

$$\sum_{i=1}^{3}\Big(\alpha_3(x)-\alpha_i(x)\Big)\varphi_i(x)<0,$$

and then we look at the term β_2 multiplying $(x-y_2)$ in (14), namely,

$$\beta_2:=\eta_2\varphi_2(x)\sum_{i=1}^{3}\varphi_i(x)+\sum_{k=2}^{3}\left(\sum_{i=1}^{3}\Big(\alpha_k(x)-\alpha_i(x)\Big)\varphi_i(x)\right)\mu_{(k,2)}.$$

Now, since $\mu_{(3,2)}\geq 0$, we have $\sum_{i=1}^{3}\big(\alpha_3(x)-\alpha_i(x)\big)\varphi_i(x)\mu_{(3,2)}\leq 0$, and on the other hand $\eta_2\varphi_2(x)\sum_{i=1}^{3}\varphi_i(x)<0$ so that, if $\sum_{i=1}^{3}\big(\alpha_2(x)-\alpha_i(x)\big)\varphi_i(x)$ happens to be nonnegative, then we also have $\sum_{i=1}^{3}\big(\alpha_2(x)-\alpha_i(x)\big)\varphi_i(x)\mu_{(2,2)}\leq 0$, and then we are done since β_2, being a sum of negative terms (one of them strictly negative) must be negative as well. Otherwise,

$$\sum_{i=1}^{3}\Big(\alpha_2(x)-\alpha_i(x)\Big)\varphi_i(x)$$

is negative, and then we finally pass to the term β_1 multiplying $(x-y_1)$ in (14), that is,

$$\beta_1:=\eta_1\varphi_1(x)\sum_{i=1}^{3}\varphi_i(x)+\sum_{k=1}^{3}\left(\sum_{i=1}^{3}\Big(\alpha_k(x)-\alpha_i(x)\Big)\varphi_i(x)\right)\mu_{(k,1)}.$$

Here, by the assumptions we have made so far and taking into account the signs of $\mu_{(k,j)}$ and η_j, we see that $\sum_{i=1}^{3}\big(\alpha_k(x)-\alpha_i(x)\big)\varphi_i(x)\mu_{(k,1)}\leq 0$ for $k=2,3$. Having arrived at this point, it is sure that $\sum_{i=1}^{3}\big(\alpha_1(x)-\alpha_i(x)\big)\varphi_i(x)$ must be nonnegative (otherwise the numbers $\sum_{i=1}^{3}\big(\alpha_k(x)-\alpha_i(x)\big)\varphi_i(x)$ should be strictly negative for all $k=1,2,3$, which is impossible if one takes $\alpha_k(x)$ to be the maximum of the $\alpha_i(x)$), and now we can deduce as before that $\beta_1<0$.

Finally let us consider the case when some of the $\varphi_i(x)$ vanish, for $i=1,2$ (remember that $\varphi_3(x)\neq 0$ since $x\in B_3$, the open support of φ_3). From the definitions of $\mu_{(k,j)}$, g_n and φ_n, it is clear that $\mu_{(k,j)}=0$ whenever $\varphi_j(x)=0$ or $\varphi_k(x)=0$, and bearing this fact in mind we can simplify equality (14) to a great extent by dropping all the terms that now vanish.

If $\varphi_1(x)=\varphi_2(x)=0$ then (14) reads

$$\varphi_3(x)^2\eta_3\,(x-y_3)=0,$$

which cannot happen since we assumed $x\neq y_j$ (this means that the only critical point that f_n can have in $B_3\setminus(B_1\cup B_2)$ is y_3).

If $\varphi_1(x) = 0$ and $\varphi_2(x) \neq 0$ then the term β_1 accompanying $(x - y_1)$ in (14) vanishes, and hence (14) is reduced to

$$\sum_{j=2}^{3}\left[\eta_j\varphi_j(x)\sum_{i=2}^{3}\varphi_i(x) + \sum_{k=j}^{3}\Big(\sum_{i=2}^{3}(\alpha_k(x) - \alpha_i(x))\varphi_i(x)\Big)\mu_{(k,j)}\right](x - y_j) = 0.$$

Since at least one of the numbers $\sum_{i=2}^{3}(\alpha_k(x) - \alpha_i(x))\varphi_i(x)$, $k = 2, 3$, is nonnegative, the same reasoning as in the first case allows us to conclude that either β_3 or β_2 is strictly negative. Finally, in the case $\varphi_1(x) \neq 0$ and $\varphi_2(x) = 0$, it is β_2 that vanishes, and (14) reads $\beta_1\,(x - y_1) + \beta_3\,(x - y_3) = 0$, where

$$\beta_3 = \eta_3\varphi_3(x)\sum_{i=1, i\neq 2}^{3}\varphi_i(x) + \sum_{i=1, i\neq 2}^{3}\Big(\alpha_3(x) - \alpha_i(x)\Big)\varphi_i(x)\mu_{(3,3)},$$

and

$$\beta_1 = \eta_1\varphi_1(x)\sum_{i=1, i\neq 2}^{3}\varphi_i(x) + \sum_{k=1, i\neq 2}^{3}\sum_{i=1, i\neq 2}^{3}\Big(\alpha_k(x) - \alpha_i(x)\Big)\varphi_i(x)\mu_{(k,1)}.$$

Again, at least one of the numbers $\sum_{i=1, i\neq 2}^{3}\big(\alpha_k(x) - \alpha_i(x)\big)\varphi_i(x)$, $k = 1, 3$, is nonnegative, and the same argument as above applies.

To finish the proof of the proposition we will need even more accurate information about the location of the critical points of f_n. Bearing in mind the definition of the functions φ_j, whose open support are the B_j, it is clear that the above discussion shows, in fact, the following inclusions:

$C_{f_3} \cap B_3 \subseteq \mathcal{A}[y_1, y_2, y_3]$;
$C_{f_3} \cap (B_3 \setminus B_1) \subseteq \mathcal{A}[y_2, y_3]$, and $C_{f_3} \cap (B_3 \setminus B_2) \subseteq \mathcal{A}[y_1, y_3]$;
$C_{f_3} \cap (B_3 \setminus (B_1 \cup B_2)) \subseteq \mathcal{A}[y_3]$.

An analogous argument in the case $n \geq 4$ proves the second part of the statement of Fact 2.6. □

Remark 2.7. Note that from Fact 2.6 it follows that the set of critical points C_φ is locally compact, since it is closed and it is locally a bounded set of a finite-dimensional affine subspace.

So far, all the properties we have shown about our functions f_n are independent of the way we may choose the numbers λ_j in the definitions of B_j and φ_j. Now we are going to be more accurate and see how we can select those numbers λ_j so as to have more control over the set C_φ of critical points of φ. Indeed, we want C_φ not only to be locally compact, but to consist of a sequence of suitably isolated small compact sets K_n. That is, we want to write $C_\varphi \subseteq \bigcup_{n=1}^{\infty} K_n$, where the K_n are compact sets which are associated with open sets U_n so that $K_n \subset U_n \subset B(y_n, r_n)$, and $U_n \cap U_m = \emptyset$ whenever $n \neq m$.

We will choose the numbers λ_n and the open sets U_n inductively.

First step. Define φ_1 as above and put $f_1(x) = \alpha_1(x)$ for all $x \in B_1 = B(y_1, r_1)$. Set $\mu_2 = 1/2$, $K_1 = C_{f_1} \cap B_1 = \{y_1\}$, and $U_1 = B(y_1, \mu_2 r_1)$.

Second step. Fix $\lambda_2 \in (\mu_2, 1)$, and define B_2, φ_2, and f_2 as above. According to Fact 2.6, we have that

$C_{f_2} \cap B_2 \subset \mathcal{A}[y_1, y_2]$, and
$C_{f_2} \cap (B_2 \setminus B_1) \subseteq \mathcal{A}[y_2]$.

We claim that there must exist some $\mu_3 \in (\lambda_2, 1)$ so that $\overline{C_{f_2} \cap B_2 \cap B_1} \subset B(y_1, \mu_3 r_1)$. Otherwise there would exist a sequence (x_j) in $C_{f_2} \cap B_2 \cap B_1$ so that $\|x_j - y_1\|$ goes to r_1 as j goes to ∞. Since $C_{f_2} \cap B_2 \subset \mathcal{A}[y_1, y_2]$, we may assume, by compactness, that x_j converges to some point $x_0 \in \partial B(y_1, r_1) = S_1$. If $x_0 \in B(y_2, r_2)$ then $f_2'(x_0) = 0$ (by continuity of f_2'), and $x_0 \neq y_2$ (because $y_2 \notin S_1$ by ii) of Lemma 2.4), so

$$f_2'(x_0) = \alpha_2'(x_0) \neq 0,$$

a contradiction. Therefore it must be the case that $x_0 \in \partial B(y_2, r_2) = S_2$. But then

$$x_0 \in S_1 \cap S_2 \cap \mathcal{A}[y_1, y_2],$$

and this contradicts Lemma 2.4.

So let us take $\mu_3 \in (\lambda_2, 1)$ so that $\overline{C_{f_2} \cap B_2 \cap B_1} \subset B(y_1, \mu_3 r_1)$. In the case that $y_2 \in B_1$, let us simply set

$U_2 = B(y_2, r_2) \cap B(y_1, \mu_3 r_1) \setminus \overline{B}(y_1, \mu_2 r_1)$, and
$K_2 = \overline{C_{f_2} \cap B_2 \cap B_1} \subset U_2$.

In the case that $y_2 \notin B_1$, find $\delta_2 \in (0, \mu_3 r_2)$ so that $B(y_2, \delta_2) \subset B_2 \setminus \overline{B_1}$, and set

$U_2 = \left[B(y_2, r_2) \cap B(y_1, \mu_3 r_1) \setminus \overline{B}(y_1, \mu_2 r_1)\right] \cup B(y_2, \delta_2)$, and
$K_2 = \overline{C_{f_2} \cap B_2 \cap B_1} \cup \{y_2\} \subset U_2$.

Clearly, we have that $C_{f_2} \subseteq K_1 \cup K_2$, and $U_1 \cap U_2 = \emptyset$.

Third step. Now choose $\lambda_3 \in (\mu_3, 1)$ with $\lambda_3 > 1 - 1/3$, and define B_3, φ_3, and f_3 as above. We have that f_3 and f_2 coincide on $(B_1 \cup B_2) \setminus B_3$. On B_3, according to Fact 2.6, we know that

$$\begin{aligned} & C_{f_3} \cap B_3 \cap B_2 \cap B_1 \subseteq \mathcal{A}[y_1, y_2, y_3]; \\ & C_{f_3} \cap (B_3 \cap B_2 \setminus B_1) \subseteq \mathcal{A}[y_2, y_3], \text{ and } C_{f_3} \cap (B_3 \cap B_1 \setminus B_2) \subseteq \mathcal{A}[y_1, y_3]\ ; \\ & C_{f_3} \cap (B_3 \setminus (B_1 \cup B_2)) \subseteq \mathcal{A}[y_3]. \end{aligned} \tag{15}$$

Again, there must be some $\mu_4 \in (\lambda_3, 1)$ so that

$$\overline{C_{f_3} \cap B_3 \cap (B_1 \cup B_2)} \subset B(y_1, \mu_4 r_1) \cup B(y_2, \mu_4 r_2).$$

Otherwise (bearing in mind the local compactness of $\mathcal{A}[y_1, y_2, y_3]$), there would exist a sequence (x_j) in $C_{f_3} \cap B_3 \cap (B_1 \cup B_2)$ so that (x_j) converges to some point x_0 and (x_j) is not contained in $B(y_1, \mu_4 r_1) \cup B(y_2, \mu_4 r_2)$ for any $\mu_4 < 1$. Since a subsequence of (x_j) must be contained in one of the sets listed in (15), we deduce that the limit point x_0 must belong to one of the following sets:

$S_2 \cap S_1 \cap \mathcal{A}[y_1, y_2, y_3]$;
$S_2 \cap \mathcal{A}[y_2, y_3] \setminus B_1$;
$S_1 \cap \mathcal{A}[y_1, y_3] \setminus B_2$,

Now we have two cases: either $x_0 \in B_3$, or $x \in \partial B_3$. If $x_0 \in B_3$ then $f_3'(x_0) = 0$ (by continuity of f_3'), and $x_0 \neq y_3$ (because $y_3 \notin S_1 \cup S_2$ by (ii) of Lemma 2.4), so it follows that

$$f_3'(x_0) = \alpha_3'(x_0) \neq 0,$$

a contradiction. On the other hand, if $x_0 \in \partial B_3$ then $x_0 \in S_3$ as well, and now one of the following must hold:

$x_0 \in S_3 \cap S_2 \cap S_1 \cap \mathcal{A}[y_1, y_2, y_3]$;
$x_0 \in S_3 \cap S_2 \cap \mathcal{A}[y_2, y_3]$;
$x_0 \in S_3 \cap S_1 \cap \mathcal{A}[y_1, y_3]$,

but in any case this contradicts Lemma 2.4.

Hence we can take $\mu_4 \in (\lambda_3, 1)$ so that

$$\overline{C_{f_3} \cap B_3 \cap (B_1 \cup B_2)} \subset B(y_1, \mu_4 r_1) \cup B(y_2, \mu_4 r_2).$$

Now two possibilities arise. If $y_3 \in B_1 \cup B_2$, let us define

$$U_3 = \left[B(y_3, r_3) \setminus \bigcup_{j=1}^{2} \overline{B}(y_j, \mu_3 r_j) \right] \bigcap \left[\bigcup_{j=1}^{2} B(y_j, \mu_4 r_j) \right],$$

and

$$K_3 = \overline{C_{f_3} \cap B_3 \cap (B_1 \cup B_2)} \subset U_3.$$

If $y_3 \notin B_1 \cup B_2$, since $y_3 \notin S_1 \cup S_2$ we can find $\delta_3 \in (0, \mu_4 r_3)$ so that $B(y_3, \delta_3) \subseteq B_3 \setminus (B_1 \cup B_2)$, and then we can set

$$U_3 = \left[\Big(B(y_3, r_3) \setminus \bigcup_{j=1}^{2} \overline{B}(y_j, \mu_3 r_j) \Big) \bigcap \Big(\bigcup_{j=1}^{2} B(y_j, \mu_4 r_j) \Big) \right] \bigcup B(y_3, \delta_3),$$

and

$$K_3 = \overline{[C_{f_3} \cap B_3 \cap (B_1 \cup B_2)] \cup \{y_3\}} \subset U_3.$$

Notice that U_3 does not meet U_1 or U_2, and $C_{f_3} \subseteq K_1 \cup K_2 \cup K_3$.

N-th step. Suppose now that μ_j, λ_j, φ_j, B_j, f_j, K_j, U_j have already been fixed for $j = 1, ..., n$ (and also μ_{n+1} has been chosen) in such a manner that f_j agrees with f_{j-1} on $(B_1 \cup ... \cup B_{j-1}) \setminus B_j$, and K_j and U_j are of the form

$$K_j = \overline{C_{f_j} \cap B_j \cap (B_1 \cup ... \cup B_{j-1})} \tag{16}$$

and

$$U_j = \left[B(y_j, r_j) \setminus \Big(\bigcup_{i=1}^{j-1} \overline{B}(y_i, \mu_j r_i) \Big) \right] \bigcap \Big[\bigcup_{i=1}^{j-1} B(y_i, \mu_{j+1} r_i) \Big] \tag{17}$$

in the case that $y_j \in B_1 \cup ... \cup B_{j-1}$, and are of this form plus $\{y_j\}$ and $B(y_j, \delta_j)$ respectively when $y_j \notin B_1 \cup ... \cup B_{j-1}$; assume additionally that $U_j \cap U_k = \emptyset$ whenever $j \neq k$, that $C_{f_j} \subseteq \bigcup_{i=1}^{j} K_i$, and that $\lambda_j > 1 - 1/j$. Let us see how we can choose λ_{n+1}, μ_{n+2}, K_{n+1} and U_{n+1} so that the extended bunch keeps the required properties.

Pick any $\lambda_{n+1} \in (\mu_{n+1}, 1)$ so that $\lambda_{n+1} > 1 - 1/(n+1)$, and define φ_{n+1}, B_{n+1} and f_{n+1} as above. We know that f_{n+1} agrees with f_n on the set $(B_1 \cup ... \cup B_n) \setminus B_{n+1}$. On B_{n+1}, according to Fact 2.6, we have that

$$C_{f_{n+1}} \cap \big(B_{n+1} \setminus \bigcup_{j=1}^{m} B_{k_j}\big) \subseteq \mathcal{A}\big[\{y_1, ..., y_{n+1}\} \setminus \{y_{k_1}, ..., y_{k_m}\}\big]$$

for every finite sequence of integers $0 < k_1 < k_2 < ... < k_m < n+1$.

We claim that there exists some $\mu_{n+2} \in (\lambda_{n+1}, 1)$ so that

$$\overline{C_{f_{n+1}} \cap B_{n+1} \cap (B_1 \cup ... \cup B_n)} \subseteq \bigcup_{i=1}^{n} B(y_i, \mu_{n+2} r_i).$$

Otherwise there would exist a finite (possibly empty!) sequence of integers $0 < k_1 < k_2 < ... < k_m < n+1$, and a sequence $(x_j)_{j=1}^{\infty}$ contained in

$$\left[C_{f_{n+1}} \cap B_{n+1} \cap \Big(\bigcap_{j=1}^{\ell} B_{i_j}\Big)\right] \setminus \Big(\bigcup_{j=1}^{m} B_{k_j}\Big) \subseteq \mathcal{A}[y_{i_1}, ..., y_{i_\ell}, y_{n+1}]$$

(where $i_1, ..., i_\ell$ are the positive integers less than or equal to n that are left when we remove $k_1, ..., k_m$), such that (x_j) converges to some point $x_0 \in S_{i_1} \cap ... \cap S_{i_\ell}$ with $x_0 \notin \bigcup_{j=1}^{m} B_{k_j}$.

If $x_0 \in B_{n+1}$ then $f'_{n+1}(x_0) = 0$ (by continuity of f'_{n+1}), and $x_0 \neq y_{n+1}$, so we easily see that

$$f'_{n+1}(x_0) = \alpha'_{n+1}(x_0) \neq 0,$$

a contradiction.

If $x_0 \in \partial B_{n+1}$ then $x_0 \in S_{n+1}$ as well, and in this case we have

$$x_0 \in S_{i_1} \cap ... \cap S_{i_\ell} \cap S_{n+1} \cap \mathcal{A}[y_{i_1}, ..., y_{i_\ell}, y_{n+1}],$$

but this contradicts Lemma 2.4.

Therefore we may take $\mu_{n+2} \in (\lambda_{n+1}, 1)$ so that

$$\overline{C_{f_{n+1}} \cap B_{n+1} \cap (B_1 \cup ... \cup B_n)} \subseteq \bigcup_{i=1}^{n} B(y_i, \mu_{n+2} r_i).$$

As before, now we face two possibilities. If $y_{n+1} \in \bigcup_{i=1}^{n} B_i$, let us define

$$U_{n+1} = \left[B(y_{n+1}, r_{n+1}) \setminus \bigcup_{i=1}^{n} \overline{B}(y_i, \mu_{n+1} r_i)\right] \bigcap \left[\bigcup_{i=1}^{n} B(y_i, \mu_{n+2} r_i)\right],$$

and

$$K_{n+1} = \overline{C_{f_{n+1}} \cap B_{n+1} \cap (B_1 \cup ... \cup B_n)}.$$

If $y_{n+1} \notin \bigcup_{i=1}^{n} B_i$, since $y_{n+1} \notin S_i$ we may find $\delta_{n+1} \in (0, \mu_{n+2} r_{n+1})$ so that $B(y_{n+1}, \delta_{n+1}) \subseteq B_{n+1} \setminus \bigcup_{i=1}^{n} B_i$, and then we can add this ball to the above U_{n+1}, and the point $\{y_{n+1}\}$ to that K_{n+1}, in order to obtain sets U_{n+1}, K_{n+1} with the required properties.

By induction, the sequences (φ_n), (f_n), (U_n), (K_n) are well defined and satisfy the above properties.

From the construction it is clear that $U_n \cap U_m = \emptyset$ whenever $n \neq m$, and

$$C_{f_n} \subseteq \bigcup_{j=1}^{n} K_j$$

for all n. Note also that $U_n \subseteq B(y_n, r_n)$ for all n, and $\lim_{n\to\infty} \lambda_n = 1$.

As observed before,

$$\varphi(x) = \frac{\sum_{n=1}^{\infty} \alpha_n(x)\varphi_n(x)}{\sum_{n=1}^{\infty} \varphi_n(x)} = \lim_{n\to\infty} f_n(x)$$

and, moreover, for each $x \in X$ there exists an open neighborhood V_x of x and some $n_x \in \mathbb{N}$ so that $\varphi(y) = f_{n_x}(y)$ for all $y \in V_x$. Bearing these facts in mind, it is immediately checked that $C_\varphi \subseteq \bigcup_{n=1}^{\infty} K_n$. Now it is clear that φ satisfies (a), (b) and (d) in the statement of Proposition 2.1. On the other hand, remember that (c) is a consequence of fact 2.5.

Remark 2.8. Let us say a few words as to the way one has to modify the above proofs in order to establish Theorem 1.1 when ε is a positive continuous function. At the beginning of the proof of Proposition 2.1, before choosing the δ_x, we have to take some number $\alpha_x > 0$ so that $|\varepsilon(y) - \varepsilon(x)| \leq \varepsilon(x)/2$ whenever $\|y - x\| \leq 2\alpha_x$ and then we can find some $\delta_x \leq \alpha_x$ so that $|f(y) - f(x)| \leq \varepsilon(x)/4$ whenever $y \in B(x, 2\delta_x)$. Equation (7) above reads now

$$|f(y) - f(y_n)| \leq \varepsilon(y_n)/2$$

for all $y \in B(y_n, r_n)$. Some obvious changes must be made in the definition of the functions a_n and α_n. Fact 2.5 and Proposition 2.1(c) can be reduced to saying that

$$|\varphi(y) - f(x)| \leq 2\varepsilon(y_n)$$

for all $x, y \in B(y_n, r_n)$ and each $n \in \mathbb{N}$. Finally, at the end of the proof of Theorem 1.1 we get that

$$|\psi(x) - f(x)| \leq 2\varepsilon(y_n)$$

whenever $x, y \in B(y_n, r_n)$; now, taking into account that $r_n \leq \alpha_{y_n}$, we have that $\varepsilon(y_n) \leq 2\varepsilon(x)$ for all $x \in B(y_n, r_n)$. Hence, by combining these inequalities, we obtain that $|\psi(x) - f(x)| \leq 4\varepsilon(x)$ for all $x \in X$.

Proof of Theorem 2.2

The proof of Theorem 2.2 is done in two steps. The first one uses the noncomplete norm technique of deleting compact sets introduced in [1, 14]. We only sketch the guidelines of this part, referring to the proof of Theorem 2.1 in [1] for the details. We will show that a mapping of the form $G(x) = x + p(f(x))$, $x \in X \setminus K$, for a certain function $f : X \to [0, +\infty)$ with $f^{-1}(K) = 0$ and a path $p : (0, +\infty) \to X$, establishes a C^∞ diffeomorphism between $X \setminus K$ and X. The map G can be viewed as a *small* perturbation of the identity. In order that the perturbation $p \circ f$ be small, p and f must satisfy some Lipschitzian-type conditions with respect to a certain distance induced by a smooth *noncomplete norm* ω. Lemma 2.10 provides us with a required function $f(x)$ which can be viewed as a smooth substitute for the ω-distance function from x to the set K. Lemma 2.11 gives us a required path $p(t)$ which avoids compact sets and gets lost in the infinitely many dimensions of X as t goes to 0; by pushing away ω-neighborhoods of K along the path p, the mapping G^{-1} will make K disappear. By combining all these tools, the C^∞ diffeomorphism G can be constructed in such a way that G restricts to the identity outside a given ω-neighborhood of K.

So far this is the same negligibility scheme as in [1, 14]. The second step of the proof is to construct a self-diffeomorphism F of X that fixes the compact set K and takes the open set U (which in general is *not* a ω-neighborhood of K) onto a ω-neighborhood of K, and then to adjust the definition of G so that it restricts to the identity outside $F(U)$. If we succeed in doing so then the composition $h = F^{-1} \circ G^{-1} \circ F$ will define a diffeomorphism from X onto $X \setminus K$ with the property that h is the identity outside U.

Definition 2.9. Let $(X, \|\cdot\|)$ be a Banach space. We say that a norm $\omega : X \longrightarrow [0, +\infty)$ is a C^p smooth noncomplete norm on X provided ω is C^p smooth (with respect to $\|\cdot\|$) away from the origin, but the norm ω is not equivalent to $\|\cdot\|$. Geometrically speaking, this means that the unit ball of ω is a symmetric C^p smooth convex body that contains no rays and yet is unbounded. We define the (open) ω-ball of center x and radius r as

$$B_\omega(x, r) = \{y \in X : \omega(y - x) < r\},$$

and the ω-distance from x to A as

$$d_\omega(x, A) = \inf\{\omega(x - z) : z \in A\}.$$

We say that a set V is an ω-neighborhood of a subset A of X provided that for every $x \in A$ there exists some $r > 0$ so that $B_\omega(x, r) \subseteq V$.

We next state the two facts we need for the first part of the proof. All the omitted proofs can be found in [1, 14].

Lemma 2.10. *Let $\omega : X \longrightarrow [0, +\infty)$ be a C^∞ smooth noncomplete norm in the Hilbert space X, and let K be a compact subset of X. Then, for each $\varepsilon > 0$ there exists a continuous function $f = f_\varepsilon : X \longrightarrow [0, +\infty)$ such that*

1. *f is C^∞ smooth on $X \setminus K$;*
2. *$f(x) - f(y) \leq \omega(x - y)$ for every $x, y \in X$;*
3. *$f^{-1}(0) = K$;*
4. *$\inf\{f(x) \mid d_\omega(x, K) \geq \eta\} > 0$ for every $\eta > 0$;*
5. *f is constant on the set $\{x \in X \mid d_\omega(x, K) \geq \varepsilon\}$.*

Lemma 2.11. *Let ω be a continuous noncomplete norm in the Hilbert space X. Then, for every $\delta > 0$, there exists a C^∞ path $p = p_\delta : (0, +\infty) \longrightarrow X$ such that*

1. *$\omega(p(\alpha) - p(\beta)) \leq \frac{1}{2}(\beta - \alpha)$ if $\beta \geq \alpha > 0$;*

2. *For every compact set* $A \subset X$ *there exists* $t_0 > 0$ *such that*
$$\inf\{\omega(z - p(t)) \mid 0 < t \leq t_0, z \in A\} > 0;$$
3. $p(t) = 0$ *if and only if* $t \geq \delta$.

The following lemma is the key to the second step of the proof, allowing us to improve, at least for the Hilbert case, the negligibility scheme introduced in [1, 14]. Note also that the norm ω that we will use in the first step is in fact the one provided by this lemma.

Lemma 2.12. *Let* $(X, \|\cdot\|)$ *be an infinite-dimensional Hilbert space (with its usual hilbertian norm). Then, for every compact set* K *and every open set* U *so that* $K \subset U$*, there exist a* C^∞ *diffeomorphism* $F : X \longrightarrow X$ *and a* C^∞ *smooth noncomplete norm* ω *on* X *such that* $F(K) = K$ *and* $F(U)$ *is an* ω*-neighborhood of* K*.*

Proof. Since K is compact and U is an open neighborhood of K we can find points $x_1, ..., x_n \in K$ and positive numbers $r_1, ..., r_n$ so that
$$K \subset \bigcup_{i=1}^{n} B_{\|.\|}(x_i, r_i) \subset \bigcup_{i=1}^{n} B_{\|.\|}(x_i, 2r_i) \subseteq U. \tag{18}$$
We may assume that $0 \in K$. Let $Y = \operatorname{span}\{x_1, ..., x_n\}$, and write $X = Y \oplus Z$, where Z is an infinite-dimensional space of finite codimension that is orthogonal to Y. Since the norm $\|\cdot\|$ is hilbertian we have that
$$\|x\| = \|(y, z)\| = \left(\|y\|^2 + \|z\|^2\right)^{1/2}$$
for every $x = (y, z) \in X = Y \oplus Z$.

Take a normalized basic sequence (z_i) in Z so that the vectors z_i are pairwise orthogonal, and let W be the closed linear subspace spanned by (z_i). Let us write $Z = W \oplus V$, where V is the orthogonal complement of W in Z. Define $\omega_Z : Z \longrightarrow [0, +\infty)$ by
$$\omega_Z(w, v) = \left[\sum_{j=1}^{\infty} \Big(\frac{< w, z_j >}{2^j}\Big)^2 + \|v\|^2\right]^{1/2},$$
where $<,>$ denotes the inner product on X. Then ω_Z is a C^∞ smooth noncomplete norm on Z, as it is easily checked. We also have that $\omega_Z(z) \leq \|z\|$ for every $z \in Z$. If we define now $\omega : X = Y \oplus Z \longrightarrow [0, +\infty)$ by
$$\omega(x) = \omega(y, z) = \left(\|y\|^2 + \omega_Z(z)^2\right)^{1/2},$$
it is clear that ω is a C^∞ smooth noncomplete norm on X (note that in fact both ω_Z and ω are real-analytic, as they are prehilbertian).

For each $i = 1, ..., n$, let us now pick C^∞ smooth functions $\theta_i : \mathbb{R} \longrightarrow [0, 1]$ so that θ_i is nondecreasing and $\theta_i^{-1}(0) = (-\infty, r_i]$, while $\theta_i^{-1}(1) = [2r_i, +\infty)$. Define then $g : X = Y \oplus Z \longrightarrow [0, 1]$ by
$$g(y, z) = g(x) = \prod_{i=1}^{n} \theta_i(\|x - x_i\|)$$

for all $x \in X$. Note that the function g is C^∞ smooth on X and has the following properties:

(i) the function $t \mapsto g(y,tz)$, $t \geq 0$, is nondecreasing, for all $(y,z) \in X = Y \oplus Z$;
(ii) $g(x) = 0$ if $x \in \bigcup_{i=1}^n B_{\|.\|}(x_i, r_i)$;
(iii) $g(x) = 1$ whenever $x \notin \bigcup_{i=1}^n B_{\|.\|}(x_i, 2r_i)$.

The first property is merely a consequence of the definition of g and the fact that the function $t \mapsto \|((y - x_i, tz)\|$, $t \geq 0$, is increasing for every $(y,z) \in Y \oplus Z$ and every $i = 1, ..., n$. Note that here we are using that $\| \cdot \|$ is a hilbertian norm; this property is not necessarily true for other norms.

Let us define our mapping $F : X = Y \oplus Z \longrightarrow X$ by

$$F(x) = F(y,z) = \left(y, \ \left(g(x) \frac{\|z\|}{\omega_Z(z)} + 1 - g(x) \right) z \right).$$

Clearly, F is C^∞ smooth. By using the facts that the functions $t \mapsto g(y,tz)$, $t \geq 0$, are nondecreasing, and that $\omega_Z(z) \leq \|z\|$ for every $z \in Z$, $y \in Y$, it is not difficult to see that F is a bijection from every ray $\{(y,tz) : t \geq 0\}$ onto itself, and therefore F is one-to-one from X onto X. Moreover, a standard application of the implicit function theorem allows to show that F^{-1} is C^∞ smooth as well, and hence F is a diffeomorphism.

Finally, by the definitions of g and F, it is clear that $F(K) = K$. In fact, F restricts to the identity on the set $\bigcup_{i=1}^n B_{\|.\|}(x_i, r_i)$, which contains K, because g takes the value 0 on this set.

On the other hand, if $x \notin \bigcup_{i=1}^n B_{\|.\|}(x_i, 2r_i)$ we have $g(x) = 1$, so $F(y,z) = (y, \frac{\|z\|}{\omega(z)} z)$, and therefore

$$\begin{aligned} \omega(F(x) - x_j) &= \omega\Big(y - x_j, \frac{\|z\|}{\omega(z)} z \Big) = \Big(\|y - x_j\|^2 + \omega\Big(\frac{\|z\|}{\omega(z)} z \Big)^2 \Big)^{1/2} \\ &= \big(\|y - x_j\|^2 + \|z\|^2 \big)^{1/2} = \|x - x_j\| \geq 2r_j \end{aligned}$$

for each $j = 1, ..., n$, which means that $F(x) \notin \cup_{i=1}^n B_\omega(x_i, 2r_i)$. Therefore, considering (18), and bearing in mind that, since $\omega(x) \leq \|x\|$, the ω-balls are larger than the $\| \cdot \|$-balls, we deduce that

$$K \subset \bigcup_{i=1}^n B_\omega(x_i, r_i) \subset \bigcup_{i=1}^n B_\omega(x_i, 2r_i) \subseteq F\big(\bigcup_{i=1}^n B_{\|.\|}(x_i, 2r_i) \big) \subseteq F(U);$$

in particular we see that $F(U)$ includes a finite union of ω-balls which in turn includes K, and this shows that $F(U)$ is a ω-neighborhood of K. □

Let us now see how we can finish the proof of Theorem 2.2. First, for the given sets $K \subset U$, take a non-complete norm ω and a diffeomorphism $F : X \longrightarrow X$ with the properties of Lemma 2.12. Since $F(U)$ is a ω-neighborhood of K and K is also compact in (X,ω), we can write

$$K \subset \bigcup_{i=1}^{n} B_\omega(x_i, r_i) \subseteq \bigcup_{i=1}^{n} B_\omega(x_i, 2r_i) \subseteq F(U)$$

for some points $x_1, ..., x_n \in K$ and positive numbers $r_1, ..., r_n$ (in fact such an expression appears in the proof of 2.12). This in turn implies that $d_\omega(x,K) \geq \min\{r_1, ..., r_n\} > 0$ whenever $x \in X \setminus F(U)$, as it is easily seen.

Now, for $\varepsilon = \min\{r_1, ..., r_n\}$, we can choose a function $f = f_\varepsilon$ satisfying the properties of Lemma 2.10 (for the already selected ω). Assuming $f(x) = \delta > 0$ whenever $d_\omega(x,K) \geq \varepsilon$, select a path $p = p_\delta$ from Lemma 2.11. With these choices, for every $x \in X \setminus K$, define

$$G(x) = x + p(f(x)).$$

Exactly as in the proof of Theorem 2.1 in [1], it can be checked that G is a C^∞ diffeomorphism from $X \setminus K$ onto X, with the property that $G(x) = x$ whenever $d_\omega(x,K) \geq \varepsilon$. In particular, since $d_\omega(x,K) \geq \varepsilon = \min\{r_1, ..., r_n\}$ whenever $x \in X \setminus F(U)$, we have that G restricts to the identity outside $F(U)$.

Finally, let us define $h = F^{-1} \circ G^{-1} \circ F$. Taking into account the properties of the diffeomorphisms $F : X \longrightarrow X$ and $G : X \setminus K \longrightarrow X$, it is clear that h is a C^∞ diffeomorphism from X onto $X \setminus K$ so that h is the identity outside U.

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