Computational Method for Obtaining Filiform Lie Algebras of Arbitrary Dimension

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Abstract: - This paper shows a new computational method to obtain filiform Lie algebras, which is based on the relation between some known invariants of these algebras and the maximal dimension of their abelian ideals. Using this relation, the law of each of these algebras can be completely determined and characterized by means of the triple consisting of its dimension and the invariants z_1 and z_2. As examples of application, we have included a table showing all valid triples determining filiform Lie algebras for dimension 13.

Key–Words: Filiform Lie algebra, abelian ideal, invariant z_1, invariant z_2, algorithm, computational methods.

1 Introduction

Nowadays, several questions about Lie algebras remain unanswered. One of these still open problem is to obtain the classification of nilpotent Lie algebras. Due to the difficulty of this problem, it seems convenient not to consider the whole classification but over a subclass of nilpotent Lie algebras. In the case of this paper, we study the classification of filiform Lie algebras — introduced by Vergne [9] in the late 1960s.

Note that classifying nilpotent Lie algebras can be worked from a geometric approach. In this sense, the set of n-dimensional Lie-algebra laws is an affine algebraic variety containing a Zariski closed subset made up of all nilpotent laws. Moreover, filiform laws correspond with a Zariski open subset in the variety of nilpotent laws.

The classification of filiform Lie algebras of dimensions 11 and 12 and the confirmation of the classifications of minors dimensions was obtained by Boza et al. [1] and Fedriani [7]. Additionally, Bratzlavsky [2] completely determined the classification of filiform Lie algebras $\mathfrak{g}$ such that their derived algebra $\mathcal{C}^2(\mathfrak{g})$ is abelian. More recently, Echarte et al. [6] characterized n-dimensional complex filiform Lie algebras such that $\mathcal{C}^3(\mathfrak{g})$ is abelian by means of the invariants z_1 and z_2 previously introduced in [3]. The present paper carries on the study started in [6] and also introduce a new computational method to obtain the law of all filiform Lie algebras only giving as input the maximum among the dimension of its abelian ideals (this maximum determines the value of z_2 and gives the interval in which the value of z_1 is running). In our opinion, the results obtained here can be considered a new step forward in the classification problem of filiform Lie algebras.

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2 Preliminaries

For an overall review about Lie algebras, the reader can consult [8]. In this paper, we only consider finite-dimensional Lie algebras over the complex number field $\mathbb{C}$.

Given a n-dimensional Lie algebra $\mathfrak{g}$, its *lower central series* is defined as

$$\mathcal{C}^1(\mathfrak{g}) = \mathfrak{g},\ \mathcal{C}^2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}],\ \ldots,\ \mathcal{C}^k(\mathfrak{g}) = [\mathcal{C}^{k-1}(\mathfrak{g}), \mathfrak{g}],\ \ldots$$

and $\mathfrak{g}$ is *nilpotent* if there exists $m \in \mathbb{N}$ such that $\mathcal{C}^m(\mathfrak{g}) \equiv 0$. Moreover, if the lower central series satisfies $\dim(\mathcal{C}^1(\mathfrak{g}))=n$, $\dim(\mathcal{C}^2(\mathfrak{g}))=n-2$, $\dim(\mathcal{C}^3(\mathfrak{g}))=n-3, \ldots$ and $\dim(\mathcal{C}^n(\mathfrak{g}))=0$, then $\mathfrak{g}$ is said to be *filiform*.

For these latter, a basis $\{e_i\}_{i=1}^n$ is called *adapted* if the brackets appearing in the law of this algebra are: $[e_1, e_h] = e_{h-1}$, for $3 \leq h \leq n$; $[e_2, e_h] = 0$, $\forall h$; $[e_3, e_h] = 0$, for $2 \leq h \leq n$. In this way, $\mathfrak{g}$ is *model* if the unique nonzero brackets in its law are $[e_1, e_h] = e_{h-1}$, for $3 \leq h \leq n$.

A subset $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is an *abelian ideal* if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$ and $\mathcal{C}^2(\mathfrak{h}) = 0$. The maximal dimension among all the abelian ideals of $\mathfrak{g}$ will be denoted by $\beta(\mathfrak{g})$. This value is an invariant of the Lie algebra $\mathfrak{g}$.

Next, we recall two invariants of filiform Lie algebras introduced in [3, 5]. If $\mathfrak{g}$ is a n-dimensional filiform Lie algebra, the invariant z_1 is defined as follows

$$z_1 = \max\{k \in \mathbb{N} \,|\, C_{\mathfrak{g}}(\mathcal{C}^{n-k+2}(\mathfrak{g})) \supset \mathcal{C}^2(\mathfrak{g})\} = \min\{k \geq 4 \,|\, [e_k, e_n] \neq 0\},$$

where $C_{\mathfrak{g}}(\mathfrak{h})$ is the centralizer of a subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and $\{e_i\}_{i=1}^n$ is an adapted basis of $\mathfrak{g}$. Note that this set is empty for model filiform Lie algebras. Additionally, the definition of z_1 involves that the ideal $\mathcal{C}^{n-i+2}(\mathfrak{g})$ is the largest one with centralizer containing $\mathcal{C}^2(\mathfrak{g})$. Moreover, the following assertion was proved in [3]

$$[e_h, e_k] = 0, \text{ for } 1 < h < z_1 \text{ and } k > 1. \tag{1}$$

The invariant z_2 is defined as $z_2 = \max\{k \in \mathbb{N} \,|\, \mathcal{C}^{n-k+1}(\mathfrak{g}) \text{ is abelian}\}$. Immediately, this definition implies that $\mathcal{C}^{n-z_2+1}(\mathfrak{g}) \equiv \langle e_2, \ldots, e_{z_2}\rangle$ is the largest abelian ideal in the lower central series of $\mathfrak{g}$.

In [3], the following relation was proved for these two invariants and the dimension of $\mathfrak{g}$

$$4 \leq z_1 \leq z_2 < n \leq 2z_2 - 2. \tag{2}$$

Finally, the following results were achieved for invariants z_1 and z_2 in [5]

Lemma 1 *Under the previous conditions, it is deduced that* $[e_{z_1+k-1}, e_{z_2+1}] = \alpha_1 e_{k+1} + \alpha_2 e_k + \ldots + \alpha_{k-1} e_3 + \alpha_k e_2$, *with* $1 \leq k \leq z_2 - z_1 + 1$. *Moreover,* $\alpha_p \neq 0$, *for some* p *such that* $1 \leq p \leq z_2 - z_1 + 1$.

Lemma 2 *Under the previous conditions, it is deduced that* $[e_{z_1}, e_{z_2+k}] = \alpha_1 e_{k+1} + \alpha_2^1 e_k + \ldots + \alpha_k^{k-1} e_2$, *with* $1 \leq k \leq n - z_2$. *Moreover,* $\alpha_q^{q-1} \neq 0$, *for some* q *such that* $1 \leq q \leq n - z_2$.

3 Theoretical results

Proposition 3 *Let* $\mathfrak{g}$ *be an* n*-dimensional non-model filiform Lie algebra. Then,* $\mathcal{C}^{n-z_2+1}(\mathfrak{g})$ *is the unique abelian ideal of maximal dimension. Consequently,* $\beta(\mathfrak{g}) = z_2 - 1$.

Proof: Let $\{e_h\}_{h=1}^n$ be an adapted basis of $\mathfrak{g}$. Since $\mathfrak{g}$ is filiform, its lower central series is given by the ideals $\mathcal{C}^k(\mathfrak{g}) = \langle \{e_i\}_{i=2}^{n-k+1}\rangle$, $\forall k \in \mathbb{N}$. In virtue of the definition of z_2, the ideal $\mathcal{C}^{n-z_2+1}(\mathfrak{g})$ is abelian but $\mathcal{C}^{n-z_2}(\mathfrak{g})$ is not.

If $\mathfrak{I}$ is an abelian ideal of $\mathfrak{g}$, we will prove that $\mathfrak{I} \subset \mathcal{C}^{n-z_2+1}(\mathfrak{g})$. In effect, assume $\mathfrak{I} \nsubseteq \mathcal{C}^{n-z_2+1}(\mathfrak{g})$. If $x \in \mathfrak{I} \setminus \mathcal{C}^{n-z_2+1}(\mathfrak{g})$, then $x = \sum_{h=1}^n \alpha_h e_h$ such that $\exists h \in \mathbb{N} \cap [z_2+1, n]$ with $\alpha_h \neq 0$.

Additionally, if $\alpha_p \neq 0$ for some $p \in \mathbb{N} \cap [z_2+1, n]$, then $\mathcal{C}^{n-p+1}(\mathfrak{g}) = \langle e_2, \ldots, e_p\rangle \subseteq \mathfrak{I}$. This is because $\mathfrak{I}$ is an ideal and hence $ad^q(e_1)(x) = \alpha_{q+2} e_2 + \ldots + \alpha_p e_{p-q} + \ldots + \alpha_n e_{n-q} \in \mathfrak{I}$, for each $q \in \mathbb{N} \cap [0, n-2]$. In consequence, since $p \geq z_2 + 1$, we have the following chain of inclusions: $\mathcal{C}^{n-z_2}(\mathfrak{g}) \subset \mathcal{C}^{n-p+1}(\mathfrak{g}) \subset \mathfrak{I}$. Hence, $\mathfrak{I}$ is non-abelian, which comes into contradiction with our initial assumption. $\square$

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Proposition 4 *Let $\mathfrak{g}$ be an n-dimensional non-model filiform Lie algebra. Then, the law of $\mathfrak{g}$ is given by the following brackets*
$[e_1, e_h] = e_{h-1}$, *for* $3 \leq h \leq n$;
$[e_{z_1}, e_{z_2+1}] = \alpha_1 e_2$;
$[e_{z_1+1}, e_{z_2+1}] = \alpha_1 e_3 + \alpha_2 e_2; \ldots$
$[e_{z_2}, e_{z_2+1}] = \alpha_1 e_{z_2-z_1+2} + \alpha_2 e_{z_2-z_1+1} + \ldots + \alpha_{z_2-z_1} e_3 + \alpha_{z_2-z_1+1} e_2$;
$[e_{z_1}, e_{z_2+k}] = \alpha_1 e_{k+1} + \alpha_2^1 e_k + \ldots + \alpha_k^{k-1} e_2$;
$[e_{z_1+p}, e_{z_2+k}] = \sum_{h=2}^{k+p} P_h([e_{z_1+p-1}, e_{z_2+k}] + [e_{z_1+p}, e_{z_2+k-1}]) e_{h+1} + \alpha_{k+p}^{k-1} e_2$,
where $2 \leq k \leq n - z_2$, $p < z_2 - z_1 + k$ *and* P_l *(for* $1 \leq l \leq n$*) is the following function*

$$P_l : \mathfrak{g} \to \mathbb{C} : u \mapsto P_l(u),$$

where $P_l(u)$ denotes the coordinate of $u \in \mathfrak{g}$ with respect to the basis vector e_l.

Proof: The brackets $[e_{z_1+k}, e_{z_2+1}] = \alpha_1 e_{k+2} + \alpha_2 e_{k+1} + \ldots + \alpha_k e_3 + \alpha_{k+1} e_2$ and $[e_{z_1}, e_{z_2+k}] = \alpha_1 e_{k+1} + \alpha_2^1 e_k + \ldots + \alpha_k^{k-1} e_2$ are obtained from Lemmas 1 and 2, respectively. We only need to prove that for $2 \leq k \leq n - z_2$ and $p < z_2 - z_1 + k$, $[e_{z_1+p}, e_{z_2+k}] = \sum_{h=2}^{k+p} P_h([e_{z_1+p-1}, e_{z_2+k}] + [e_{z_1+p}, e_{z_2+k-1}]) e_{h+1} + \alpha_{k+p}^{k-1} e_2$. We use the Jacobi identity $J(e_1, e_{z_1+p}, e_{z_2+k}) = 0$, which involves $[e_1, [e_{z_1+p}, e_{z_2+k}]] = [e_{z_1+p-1}, e_{z_2+k}] + [e_{z_1+p}, e_{z_2+k-1}]$, therefore $[e_{z_1+p}, e_{z_2+k}] = \sum_{h=2}^{k+p} P_h([e_{z_1+p-1}, e_{z_2+k}] + [e_{z_1+p}, e_{z_2+k-1}]) e_{h+1} + \alpha_{k+p}^{k-1} e_2$. □

Now, by using the invariant z_2 and Expression (2), we obtain the following two results

Corollary 5 *Let $\mathfrak{g}$ be an n-dimensional non-model filiform Lie algebra with $\beta(\mathfrak{g}) = k \in \mathbb{N}$. Then, $z_2 = k + 1$ and the following relation holds: $3 \leq z_1 - 1 \leq k < n - 1 \leq 2k - 1$.* □

Corollary 6 *Let $\mathfrak{g}$ be an n-dimensional non-model filiform Lie algebra with adapted basis $\{e_h\}_{h=1}^n$. If $\mathfrak{h}$ is the subalgebra $\langle \{e_i\}_{i=2}^n \rangle$, then the derived subalgebra $\mathcal{C}^2(\mathfrak{h})$ satisfies that $\mathcal{C}^2(\mathfrak{h}) \subset \left\langle \{e_i\}_{i=2}^{2n-(z_1+z_2)} \right\rangle$.* □

Let us note that Corollary 5 improves the well-known bound for $\beta(\mathfrak{g})$ in a complex non-abelian nilpotent Lie algebra $\mathfrak{g}$ given in [4]: $\frac{\sqrt{8n+1}-1}{2} \leq \beta(\mathfrak{g}) \leq n - 1$.

4 Algorithmic Method

This section introduces a computational method which returns the law of all n-dimensional non-model filiform Lie algebra $\mathfrak{g}$ such that $\beta(\mathfrak{g})$ equals to the value $k \in \mathbb{N}$ used as input. Next, we give a step-by-step explanation of this method.

Input

1. Dimension n of a non-model filiform Lie algebra $\mathfrak{g}$.
2. Value k of the invariant $\beta(\mathfrak{g})$.

Output

1. List with all triples (z_1, z_2, n) such that there exist non-model filiform Lie algebras associated with them.
2. Law of all filiform Lie algebras associated with each triple.

Method

1. Computing the value of invariant z_2 by using Proposition 3. According to this value and Expression (2), several possibilities appear for the invariant z_1.

2. For each value of z_1, computing all possible non-zero brackets of $\mathfrak{g}$ given by Proposition 4.
3. Ruling out those values of z_1 not satisfying the Jacobi identities $J(e_h, e_k, e_l) = 0$, for $z_1 \le h < k < l \le n$.
4. Obtaining the list with all the triples (z_1, z_2, n) such that there exist non-model filiform Lie algebras having such invariants.
5. By using Proposition 4 again, computing the law of all Lie algebras associated with each triple given in the previous step.

Regarding Step 3, let us note that Expression (1) implies that $J(e_a, e_b, e_c) = 0$, for $a < b < c < z_1$. Next, we give a result related to the trivial case for triples (z_1, z_2, n).

Proposition 7 *If $z_1 \ge n - \frac{z_2}{2}$, the triple (z_1, z_2, n) is associated with an n-dimensional non-model filiform Lie algebra.*

Proof: If $z_1 \ge n - \frac{z_2}{2}$, then $\mathcal{C}^2(\mathfrak{h}) \subset \langle e_2, \ldots, e_{z_1} \rangle$ in virtue of Corollary 6. We have to prove that $J(e_i, e_j, e_k) = 0$, for $z_1 \le i < j < k \le n$. From Expression (1), we obtain $[[e_a, e_b], e_c] = 0$, for $z_1 \le a < b < n$ and $c \ge 1$. Consequently, $J(e_i, e_j, e_k) = 0$, for $z_1 \le i < j < k \le n$. □

5 Implementation and application

Now, we show the implementation of the two main routines needed to apply the previous algorithm. To do so, we have used the symbolic computation package MAPLE 12. We start loading the library `DifferentialGeometry, LieAlgebras` to activate commands related to Lie algebras such as `BracketOfSubspaces`.

First, we show the implementation of a routine which computes the law of a filiform Lie algebra from the triple (z_1, z_2, n) by using Proposition 4. This routine, named `law`, receives the triple (z_1, z_2, n) as input and returns the filiform Lie algebra associated with this triple. For the implementation, we define a list as a local variable `L`. This list saves the indexes and the value of the structure constants corresponding with the non-zero brackets of the filiform Lie algebra. First, `L` saves the brackets given by filiformity (i.e. $[e_1, e_h] = e_{h-1}, \forall 3 \le h \le n$) and the bracket $[e_{z_1}, e_{z_2+1}] = \alpha_1 e_2$. Then, a loop is programmed for including the indexes of the rest of non-zero brackets in `L`.

```
> law:=proc(z_1,z_2,n)
> local L; L:=[seq([[1,h,h-1],1],h=3..n),
> [[z_1,z_2+1,2],a[1]]];
> for i from 1 to n do for j from 1 to n do c[i,i][j]:=0; od; od;
> for i from 1 to z_2-z_1 do for l from 1 to i+1 do
>    L:=[op(L),[[z_1+i,z_2+1,i+3-l],a[l]]];
>    c[z_1+i,z_2+1][i+3-l]:=a[l];
> od; od;
> for i from 2 to n-z_2 do
>    L:=[op(L),[[z_1,z_2+i,i+1],a[1]]];
>    c[z_1,z_2+i][i+1]:=a[1];
>    for j from 2 to i do
>       L:=[op(L),[[z_1,z_2+i,i+2-j],a[j][j-1]]];
>       c[z_1,z_2+i][i+2-j]:=a[j][j-1];
> od; od;
> for i from 2 to n-z_2 do for j from 1 to z_2-z_1+i-1 do
>    L:=[op(L),[[z_1+j,z_2+i,2],
>    a[i+j][i+j-1]]];
>    c[z_1+j,z_2+i][2]:=a[i+j][i+j-1];
>    for h from 2 to i+j do
>       L:=[op(L),[[z_1+j,z_2+i,h+1],
>       c[z_1+j-1,z_2+i][h]+c[z_1+j,z_2+i-1][h]]];
>       c[z_1+j,z_2+i][h+1]:=c[z_1+j-1,z_2+i][h]+
>       c[z_1+j,z_2+i-1][h];
```

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```
> od;od;od;
> return _DG([["LieAlgebra",Alg1,[n]],L]);
> end proc:
```

Next, we implement the routine `coefficients`, which receives as input the triple (z_1, z_2, n) and returns a list with all the structure constants α_i and α_i^j in the law of the associated filiform Lie algebra.

```
> coefficients:=proc(z_1,z_2,n)
> local M; M:={};
> for i from 1 to z_2-z_1+1 do M:={op(M),a[i]}; od;
> for j from 2 to 2*n-z_1-z_2-1 do M:={op(M),a[j][j-1]}; od;
> return M;
> end proc:
```

Now, we save the output of the routine `law` in the variable `Eq`, changing the format of the subindexes of the basis vectors. After that, we evaluate the sentence

```
> DGsetup(law(z_1,z_2,n));
```

From this point, we can operate over the Lie algebra associated with the output given by `law`. This algebra is denoted by `Alg1`. Now, we execute

```
Alg1 > L:=LieAlgebraData(Eq,[seq(e[i],i=1..n)],Alg2);
> DGsetup(L);
```

From here on, we are working over the Lie algebra `Alg2`. With the command `LieAlgebraData`, we can evaluate the following sentence

```
Alg2 > TF,EQ,SOLN,AlgList:=Query(coefficients(z_1,z_2,n),"Jacobi");
```

This sentence provides us the conditions given by the Jacobi identities in terms of equations for the filiform Lie algebra. Additionally, `AlgList` shows us a list with all the nonzero brackets.

Finally, for each element in the previous output (that is a list), we have to write the following commands

```
Alg2 > DGsetup(AlgList[i]);
Alg2_i > f:=proc() if BracketOfSubspaces([ez_1],[en])=[] then return "It is not a filiform
Alg2_i > Lie algebra"; end if;
Alg2_i > if BracketOfSubspaces([ez_2],[ez_2+1])=[] then return "It is not a filiform Lie
Alg2_i >  algebra"; else return Alg2_i > AlgList[i],SOLN[i]; fi;
Alg2_i > end proc:
Alg2_i > f()
```

With these sentences, we program a routine which studies if the Lie algebra given in `AlgList[i]` is compatible with the definition of the invariants z_1 and z_2. If so, the output is the set of conditions for the structure constants given by the routine `coefficients` and the nonzero brackets of the Lie algebra. Otherwise, the output is the message “It is not a filiform Lie algebra”.

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Table 1: Triples for non-model filiform Lie algebras of dimension 13.

$\beta(\mathfrak{g})$	**Triples**	**Restrictions**
7	(k,8,13) $k \in [4,8]$	$\nexists$ algebra
8	(4,9,13)	$\nexists$ algebra
	(5,9,13)	$\{\alpha_i, \alpha_{i+1}^i = 0\}$, for $1 \leq i \leq 4$
	(6,9,13)	$\nexists$ algebra
	(7,9,13)	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_2^1, \alpha_3^2 = 0, \alpha_4 = -2\alpha_4^3\}$
		$\{\alpha_1, \alpha_2, \alpha_2^1 = 0\}$
		$\{\alpha_1, \alpha_4^3 = 0, \alpha_2 = \frac{-5\pm3\sqrt{65}}{28}\alpha_2^1,$ $\alpha_3^2 = -\frac{\alpha_3}{5}\frac{(297\pm3\sqrt{65})}{(-5\pm3\sqrt{65})}\}$
	(8,9,13)	$\{\alpha_1, \alpha_2^1 = 0\}, \{\alpha_1 = 0, \alpha_2 = -\frac{9}{5}\}$
	(9,9,13)	None
9	(4,10,13)	$\nexists$ algebra
	(5,10,13)	$\nexists$ algebra
	(6,10,13)	$\{\alpha_1, \alpha_2, \alpha_2^1 = 0, \alpha_3 = \pm\frac{\sqrt{5}}{3}\alpha_3^2\}$
	(7,10,13)	$\{\alpha_1 = 0, \alpha_2 = -\frac{5}{9}\alpha_2^1\}$, $\{\alpha_1 = 0, \alpha_2 = \alpha_2^1\}$
	(8,10,13)	$\{\alpha_1 = 0\}$
	(9,10,13)	None
	(10,10,13)	None
10	(4,11,13)	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 0\}$
	(5,11,13)	$\{\alpha_1, \alpha_2, \alpha_3 = 0\}$
	(6,11,13)	$\{\alpha_1, \alpha_2 = 0\}$
	(7,11,13)	$\{\alpha_1 = 0\}$
	(k,11,13) $k \in [8,11]$	None
11	(k,12,13) $k \in [4,12]$	None