Numerical solution of some geometric inverse problems

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joint work with

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Numerical Resolution for Inverse Problems
BCAM, Bilbao, 8-9 January 2015

Outline

We consider:

- Geometric inverse problems
- Wave equation and Lamé systems
- Motivation: Elastrography
  A non-invasive method of tumor detection: when a mechanical compression or vibration is applied, the tumor deforms less than the surrounding tissue

  A technique to detect elastic properties of tissue from acoustic wave generators (applications in Medicine)

Classical detection methods in mammography:

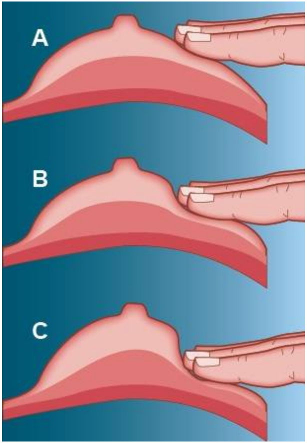


Figure: Palpation

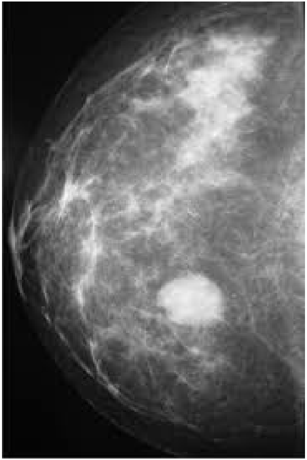


Figure: *x*-rays

Elastography (“imaging palpation”) is better suited than palpation and x-rays techniques:

— Tumors can be far from the surface
— or small
— or may have properties indistinguishable through palpation or x-rays

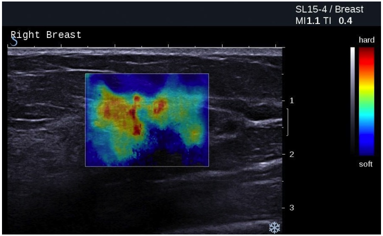


Figure: Stiffness is represented by a color spectrum, ranging from dark red (very stiff) through orange, yellow, and green, to blue (very soft).

(a) Direct problem:

Data: Ω, $T > 0$, φ, D and $\gamma \subset \partial\Omega$
Result: the solution u

$$(1) \quad \begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad (\Omega \setminus \overline{D}) \times (0, T) \\ u = \varphi & \text{on} \quad (\partial\Omega) \times (0, T) \\ u = 0 & \text{on} \quad (\partial D) \times (0, T) \\ u(x, 0) = u_0, \quad u_t(x, 0) = u_1 & \text{in} \quad \Omega \end{cases}$$

Information:

$$(2) \qquad \alpha = \frac{\partial u}{\partial n} \quad \text{on} \quad \gamma \times (0, T)$$

(b) Inverse problem:

(Partial) data: Ω, T, φ and $\gamma \subset \partial\Omega$
(Additional) information: α
Goal: Find D such that the solution to (1) satisfies (2)

$$\begin{cases} u^i_{tt} - \Delta u^i = 0 & \text{in } \Omega \setminus \overline{D^i} \times (0,T), \quad i = 0,1 \\ u^i = \varphi & \text{in } \partial\Omega \times (0,T) \\ u^i = 0 & \text{in } \partial D^i \times (0,T) \\ u^i(x,0) = 0, \quad u^i_t(x,0) = 0 & \text{in } \Omega \setminus \overline{D^i} \end{cases}$$

Theorem

$$\left.\begin{array}{l} T > T_*(\Omega,\gamma), \quad D^0, D^1 \textit{ are convex}, \quad \varphi \neq 0 \\ \dfrac{\partial u^0}{\partial n} = \dfrac{\partial u^1}{\partial n} \quad \textit{on} \quad \gamma \times (0,T) \end{array}\right\} \Longrightarrow \quad D^0 = D^1$$

Fundamental results: Hörmander, Lions

Attention: Weaker than the geometric condition (Only uniqueness, not observability!)

$$\begin{cases} u_{tt} - \nabla \cdot (\mu(x)(\nabla u + \nabla u^t) + \lambda(x)(\nabla \cdot u)\mathbf{Id.}) = 0 & \text{in } \Omega \setminus \overline{D} \times (0,T) \\ u = \varphi & \text{on } \partial\Omega \times (0,T) \\ u = 0 & \text{on } \partial D \times (0,T) \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega \setminus \overline{D} \end{cases}$$

Observation: $\sigma(u) \cdot n := \Big(\mu(x)(\nabla u + \nabla u^t) + \lambda(x)(\nabla \cdot u)\mathbf{Id.}\Big) \cdot n$ on $\gamma \times (0,T)$

Explanations:

- $u = (u_1, u_2, u_3)$ is the displacement vector
- Small displacements. Hence, linear elasticity
- Isotropy assumptions. The tissue is described by λ and μ

$$\begin{cases} u^i_{tt} - \nabla\cdot(\mu(x)(\nabla u^i + \nabla)(u^i)^t) + \lambda(x)(\nabla\cdot u^i)\mathbf{Id.}) = 0 & \text{in } \Omega\setminus\overline{D^i}\times(0,T) \\ u^i = \varphi & \text{on } \partial\Omega\times(0,T) \\ u^i = 0 & \text{on } \partial D^i\times(0,T) \\ u^i(0) = 0, \quad u^i_t(0) = 0 & \text{in } \Omega\setminus\overline{D^i} \end{cases}$$

Theorem (Constant coefficients)

$$\left.\begin{array}{l} T > T_*(\Omega,\gamma), \quad D^0, D^1 \textit{ are convex}, \quad \varphi\neq 0 \\ \sigma(u^0)\cdot n = \sigma(u^1)\cdot n \quad \textit{on} \quad \gamma\times(0,T) \end{array}\right\} \implies D^0 = D^1$$

For **uniqueness**, the key point is Unique continuation property (Imanuvilov–Yamamoto, 2008, complex conditions on μ, λ)

AD, E. Fernández-Cara, work in progress

($\exists$ Other unique continuation results for stationary problems: Lin–Wang, 2005; Escauriaza, 2005; Alessandrini–Morasi, 2001; Nakamura–Wang, 2006; Imanuvilov–Yamamoto, 2012)

Resolution of an optimization problem

Optimization problem: case of a ball

Given: $\widetilde{\alpha} = \widetilde{\alpha}(x,t)$.

Find x_0, y_0 and r such that $(x_0, y_0, r) \in X_b$ and

$$J(x_0, y_0, r) \leq J(x_0', y_0', r') \quad \forall\, (x_0', y_0', r') \in X_b$$

the function $J : X_b \mapsto \mathbb{R}$ is defined by

$$J(x_0, y_0, r) := \frac{1}{2} \iint_{\gamma \times (0,T)} |\alpha[x_0, y_0, r] - \widetilde{\alpha}|^2 \, ds\, dt$$

with

$$\alpha[x_0, y_0; r] := \frac{\partial u}{\partial n} \quad \text{on } \gamma \times (0,T)$$

and

$$X_b := \{\, (x_0, y_0, r) \in \mathbb{R}^3 \;:\; \overline{B}(x_0, y_0; r) \subset \Omega \,\}$$

The problem formulation contains inequality constraints

$$\begin{cases} \text{Minimize } f(x) \\ \text{Subject to } x \in X_0; \quad c_i(x) \geq 0,\ 1 \leq i \leq I \end{cases}$$

$$X_0 = \{x \in \mathbb{R}^m \ : \ \underline{x}_j \leq x_j \leq \overline{x}_j, \quad 1 \leq j \leq m\}$$

Optimization problem

$$\begin{cases} \text{Minimize } \mathcal{L}_A(x, \lambda^k; \mu_k) := f(x) - \sum_{i=1}^{I} \lambda_i^k (c_i(x) - s_i) + \frac{1}{2\mu_k} \sum_{i=1}^{I} (c_i(x) - s_i)^2 \\ \text{Subject to } x \in X_0;\ s_i \geq 0,\ 1 \leq i \leq I \end{cases}$$

λ_i^k : multipliers, μ_k : penalty parameters

Algorithm (Augmented Lagrangian: inequality constraints)

(a) *Fix* μ_1 *and starting points* x^0 *and* λ^1*;*

(b) *The, for given* $k \geq 1$, μ_k, x^{k-1}, λ^k*:*

(b.1) *Unconstrained optimization: Find an approximate minimizer* x^k *of* $\mathcal{L}_A(\cdot, \lambda^k; \mu_k)$*, starting at* x^{k-1}*:*

$$\begin{cases} \text{Minimize } \mathcal{L}_A(x, \lambda^k; \mu_k) \\ \text{Subject to } x \in X_0 \end{cases}$$

(b.2) *Update the Lagrange multipliers:*

$$\lambda_i^{k+1} = \max\left(\lambda_i^k - \frac{c_i(x_k)}{\mu_k}, 0\right), \quad 1 \leq i \leq I$$

(b.3) *Choose a new parameter and check whether a stopping convergence test is satisfied:*

$$\mu_{k+1} \in (0, \mu_k)$$

Subsidiary optimization algorithms for (b.1) (among others):

- **CRS2** is a gradient-free algorithm a version of Controlled Random Search (CRS) for global optimization
- **DIRECTNoScal** is variant of the DIviding RECTangles algorithm for global optimization

$$\begin{cases} \text{Minimize } f(x) \\ \text{Subject to } x \in G \end{cases} \quad (1)$$

where $G \subset \mathbb{R}^m$ is either a box or some other region easy to sample and $f : G \subset \mathbb{R}^m \mapsto \mathbb{R}$ is continuous.

CRS2: Main ideas

1. Large initial sample of random points
2. At each step:
   - The current worst point x_h is replaced by a new trial point $\widetilde{x}$ (generates from the current best point x_ℓ and other random points)
   - A stopping condition $f_h - f_\ell \leq \varepsilon$ is checked

DIRECT: Main ideas

(a) Normalize the domain to be the unit hyper-cube with center c^1

Find $f(c^1)$; set $f_{min} = f(c^1)$, $i = 0$, $k = 1$

Evaluate $f(c^1 \pm \frac{1}{3}e^i)$, $1 \leq i \leq m$, and divide the hyper-cube: $c^1 \pm \frac{1}{3}e^i$ are the centers of the new hyper-rectangles (see Figure)

(b) Then, for given $k \geq 1$:

(b.1) Identify the set S of all potentially optimal rectangles

(b.2) For each rectangle in S, identify the longest side(s), evaluate f at the center, divide in smaller rectangles and update f_{min}

Potentially optimal means:

- Best value at the center if the size is the same
- Optimal value at the center if the size is minimal

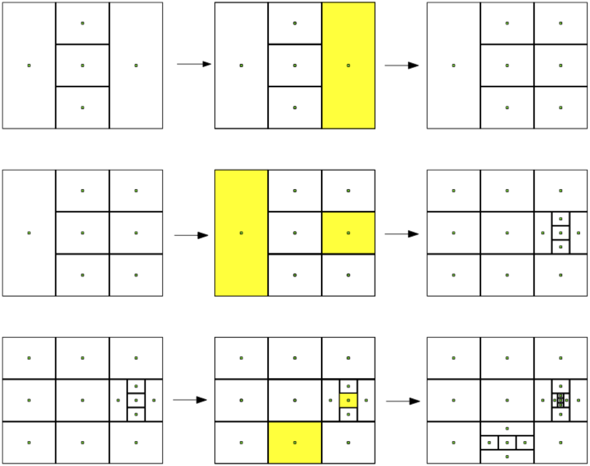


Figure: Some interactions of DIRECT algorithm

Test 1: $T = 5, \quad u_0 = 10x, \quad u_1 = 0, \quad \varphi = 10x$

`x0des = -3,  y0des = 0,  rdes = 0.4`
`x0ini = 0,   y0ini = 0,  rini = 0.6`

`NLopt (`**`AUGLAG + CRS2`**`), N°Iter = 1007, FreeFem++`:

`x0cal = -2.998645439,  y0cal = 0.000425214708`
`rcal = 0.4001667063`

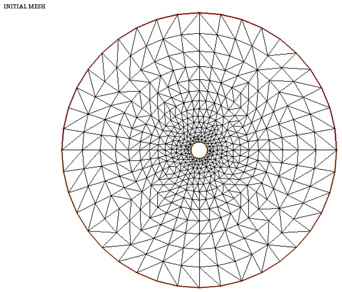


Figure: Initial mesh: triangles 992, vertices 526

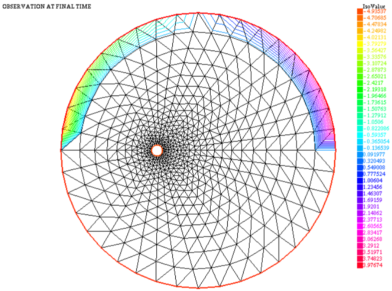


Figure: The desired center and radius of the ball

Numerical results: 2-D wave equation II

Case of a ball

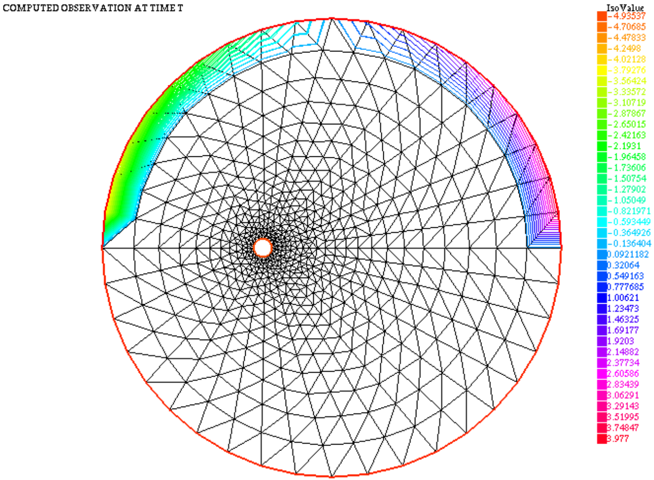


Figure: Computed center and radius: `AUGALG + CRS2`
`x0cal= -2.998645439, y0cal = 0.000425214708, rcal=0.4001667063`

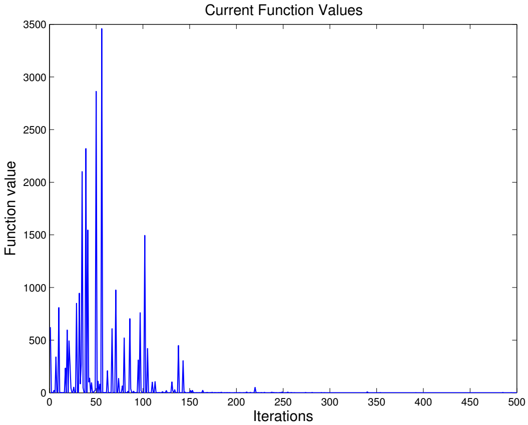


Figure: Evolution of J during the first 500 iterations of `CRS2`

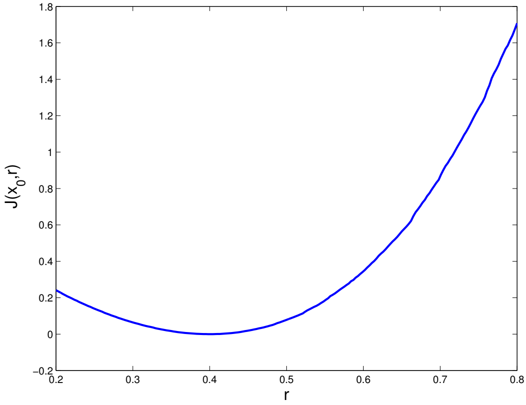


Figure: The functional J with respect to the variable r

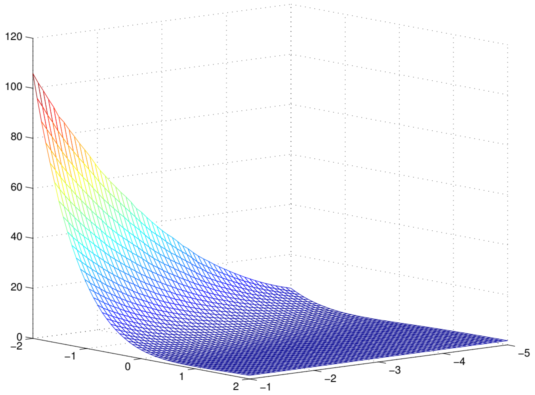


Figure: The functional J with respect to the variables x_0 and y_0

Test 2: $T = 5, \quad u_0 = 10x, \quad u_1 = 0, \quad \varphi = 10x$

```
x0des = -3,  y0des = 0,  rdes = 0.4
x0ini = 0,   y0ini = 0,  rini = 0.6
```

NLopt (**AUGLAG + DIRECT**), N°Iter = 1001, FreeFem++:

```
x0cal  =  -2.962962963
y0cal  =  -0.01219326322
 rcal  =  0.4220164609
```

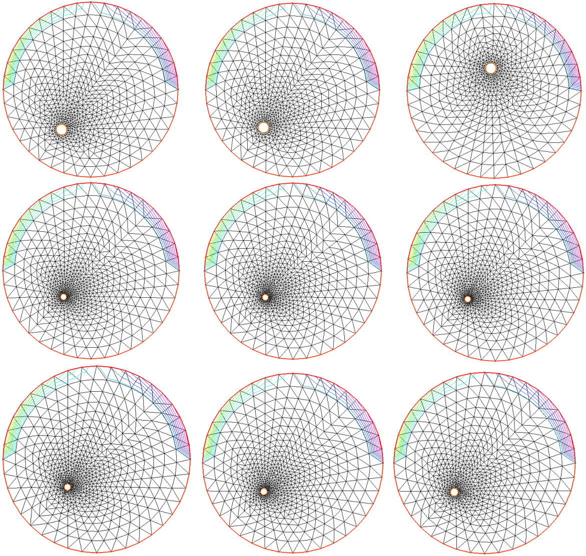


Figure: Some experiences with AUGLAG and DIRECT

Numerical results: 2-D wave equation VIII

Case of a ball

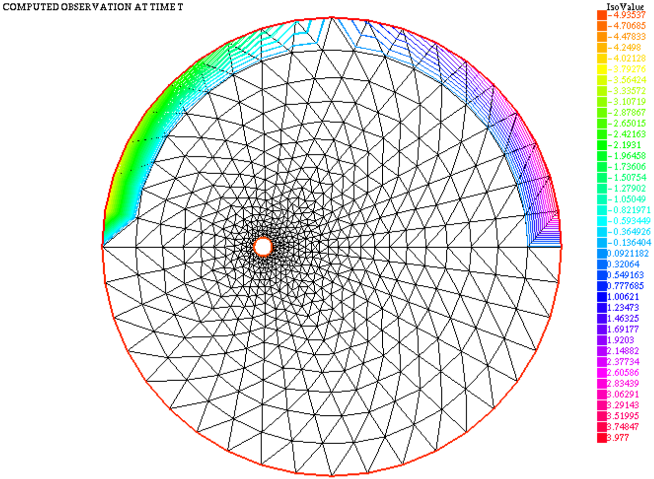


Figure: Desired and computed radius and centers of the ball

Optimization problem: case of an ellipse

Given: $\widetilde{\alpha} = \widetilde{\alpha}(x, t)$.

Find x_0, y_0 and θ and a, b such that $(x_0, y_0, \theta, a, b) \in X_e$ and

$$J(x_0, y_0, \theta, a, b) \leq J(x_0', y_0', \theta', a', b') \quad \forall (x_0', y_0', \theta', a', b') \in X_e, \qquad (2)$$

the function $J : X_e \mapsto \mathbb{R}$ is defined by

$$J(x_0, y_0, \theta, a, b) := \frac{1}{2} \iint_{\gamma \times (0,T)} |\alpha[x_0, y_0, \theta, a, b] - \widetilde{\alpha}|^2 \, ds \, dt$$

with

$$\alpha[x_0, y_0, \theta, a, b] = \frac{\partial u}{\partial n} \quad \text{on } \gamma \times (0, T)$$

$$X_e := \{ (x_0, y_0, \theta, a, b) \in \mathbb{R}^5 \ : \ \overline{E}(x_0, y_0, \theta, a, b) \subset \Omega \}$$

Resolution of an optimization problem. Now, $J = J(x_0, y_0, \theta, a, b)$

Test 3: $T = 5, \quad u_0 = 10x, \quad u_1 = 0, \quad \varphi = 10x$

x0des=-3, y0des=-3, sin(thetades)=0, ades=0.8, bdes=0.4

x0ini=-1, y0ini=-1, sin(thetaini)=0, aini=0.5, bini=0.5

NLopt (AUGLAG + DIRECTNoScal), N°Iter = 2001, FreeFem++

x0cal	=	-2.963301665
y0cal	=	-3.035106437
sin(thetacal)	=	0.112178021
acal	=	0.8446502058
bcal	=	0.4166666667

Numerical results: 2-D wave equation III

Case of an ellipse

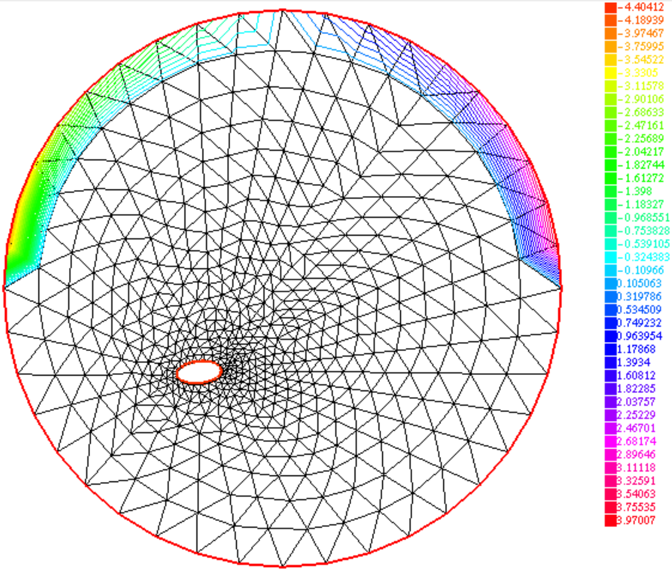


Figure: Computed center, radius, angle and semi-axis

$$\begin{cases} u_{tt} - \nabla \cdot \sigma(u) = 0 & \text{in } \Omega \setminus \overline{D} \times (0,T) \\ u = \varphi & \text{on } \partial\Omega \times (0,T) \\ u = 0 & \text{on } \partial D \times (0,T) \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega \setminus \overline{D} \end{cases}$$

$$\sigma(u) \cdot n := \left(\mu(x)(\nabla u + \nabla u^t) + \lambda(x)(\nabla \cdot u)\mathbf{Id.}\right) \cdot n = \widetilde{\sigma} \quad \text{on} \quad \gamma \times (0,T)$$

Optimization problem

Given: $\widetilde{\sigma} = \widetilde{\sigma}(x,t)$

Find x_0, y_0 and r $(x_0, y_0, r) \in X_b$ and

$$J(x_0, y_0, r) \le J(x_0', y_0', r') \quad \forall\, (x_0', y_0', r') \in X_b, \tag{3}$$

the function $J : X_b \mapsto \mathbb{R}$ is defined by

$$J(x_0, y_0, r) := \frac{1}{2} \iint_{\gamma \times (0,T)} |\sigma[x_0, y_0, r] - \widetilde{\sigma}|^2 \, ds\, dt.$$

Test 4: $T = 5, \quad u01 = 10x, \quad u02 = 10y, \quad u11 = 0, \quad u12 = 0,$
$\varphi_1 = 10x, \quad \varphi_2 = 10y$

x0des = -3, y0des = 0, rdes = 0.4

x0ini = 0, y0ini = 0, rini = 0.6

NLopt (AUGLAG + DIRECTNoScal), N° Iter = 1000, FreeFem++

x0cal	=	-3.000224338
y0cal	=	-0.0005268693985
rcal	=	0.4000228624

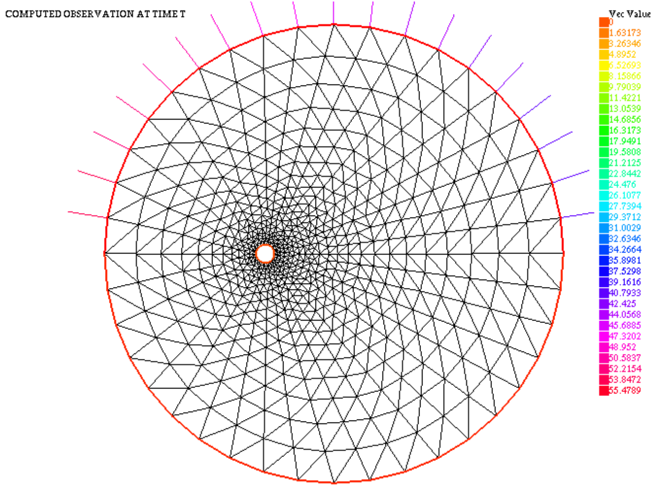


Figure: Computed center and radius

Test 5: $T = 5, \quad u01 = 10x, \quad u02 = 10y, \quad u11 = 0, \quad u12 = 0,$
$\varphi_1 = 10x, \quad \varphi_2 = 10y$

```
x0des=-3,   y0des=0,   sin(thetades)=0,   ades=0.8,   bdes=0.4
x0ini=-1,   y0ini=-1,  sin(thetaini)=0,   aini=0.5,   bini=0.5
```

NLopt (AUGLAG + DIRECTNoScal), N° Iter = 2001, FreeFem++:

x0cal	=	-3.002591068
y0cal	=	-3.001574963
sin(thetacal)	=	0.00548696845
acal	=	0.8036351166
bcal	=	0.400617284

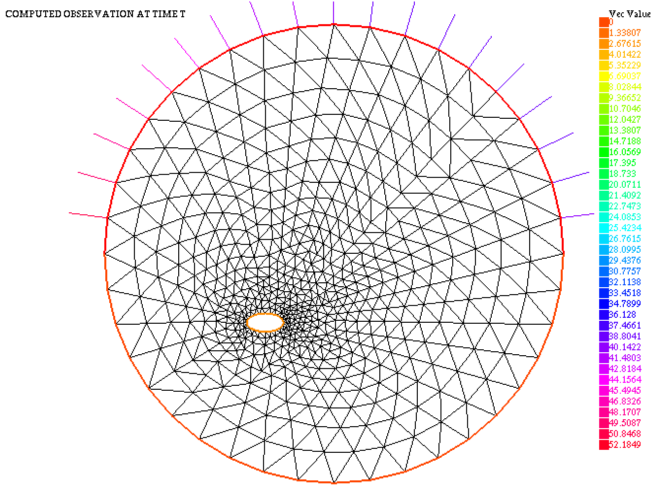


Figure: Computed center, angle and semi-axis

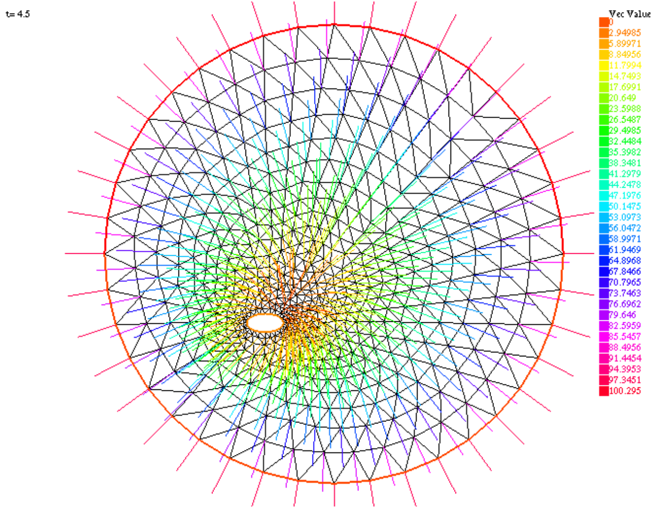


Figure: Computed solution at the final time

Test 6: $T = 5, \quad u_0 = 10x, \quad u_1 = 0, \quad \varphi = 10x$

x0des = −2, y0des = −2, z0des = −2, rdes = 1

x0ini = 0, y0ini = 0, z0des = 0, rini = 0.6

NLopt (AUGLAG + DIRECTNoScal), N° Iter = 438, FreeFem++

x0cal	=	−1.975308642
y0cal	=	−2.232383275
z0cal	=	−2.305542854
rcal	=	1.05

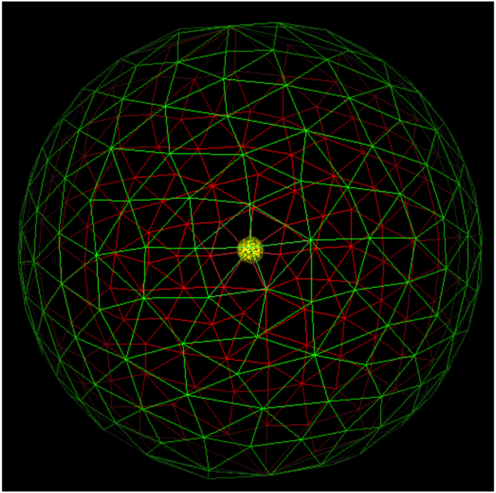


Figure: Initial mesh. Points: 829, tetrahedra: 4023, faces: 8406, edges: 5210, boundary faces: 720, boundary edges: 1080

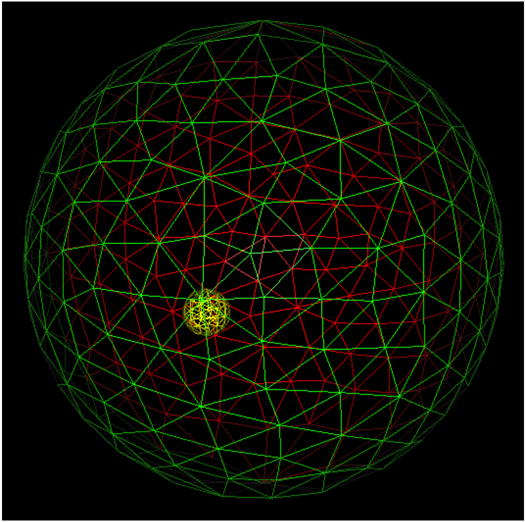


Figure: Desired configuration

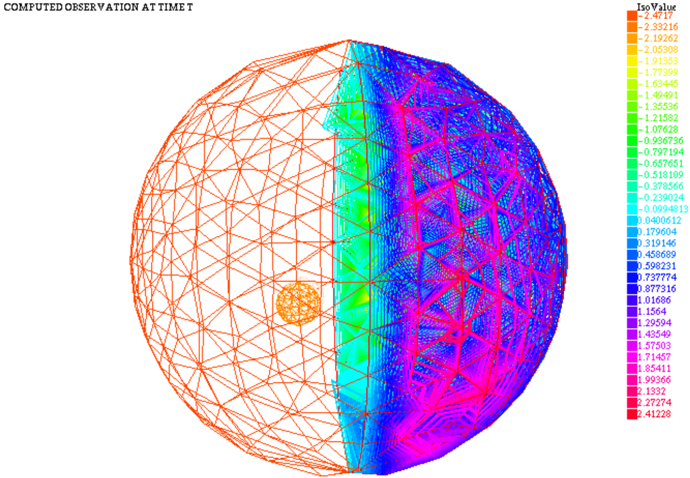


Figure: Computed observation and mesh

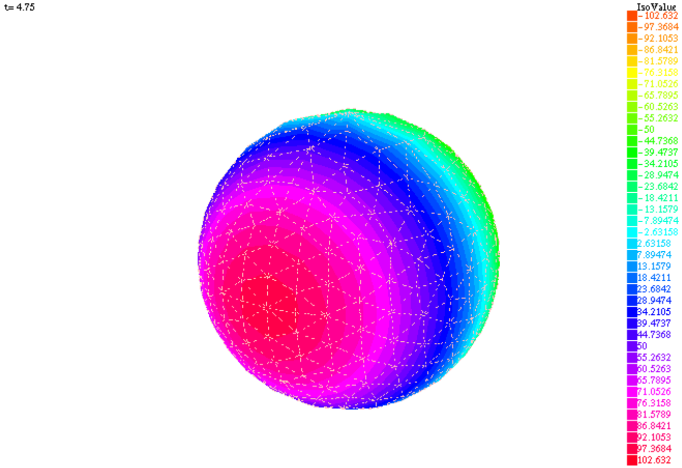


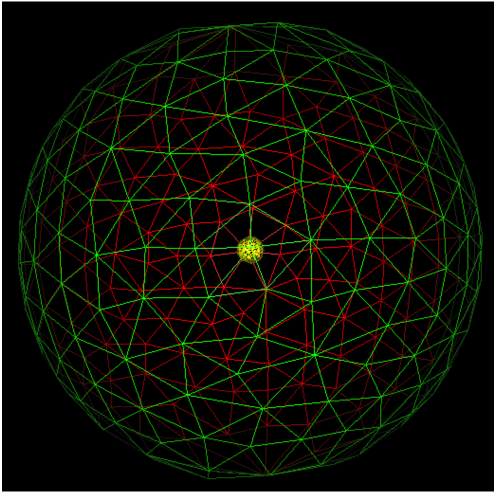
Figure: Solution of the wave equation corresponding to computed data

$$\textbf{Test 7}: T = 5, \quad \begin{array}{lll} u01 = 10x, & u02 = 10y, & u03 = 10z \\ u11 = 0, & u12 = 0, & u13 = 0 \\ \varphi_1 = 10x, & \varphi_2 = 10y, & \varphi_3 = 10z \end{array}$$

```
x0des = -2,  y0des = -2,  z0des = -2,  rdes = 1
x0ini = 0,   y0ini = 0,   z0des = 0,   rini = 0.6
```

NLopt (AUGLAG + DIRECTNoScal), N° Iter = 444 , FreeFem++:

```
x0cal  =  -1.981405274
y0cal  =  -2.225232904
z0cal  =  -2.148084171
 rcal  =  0.9504115226
```



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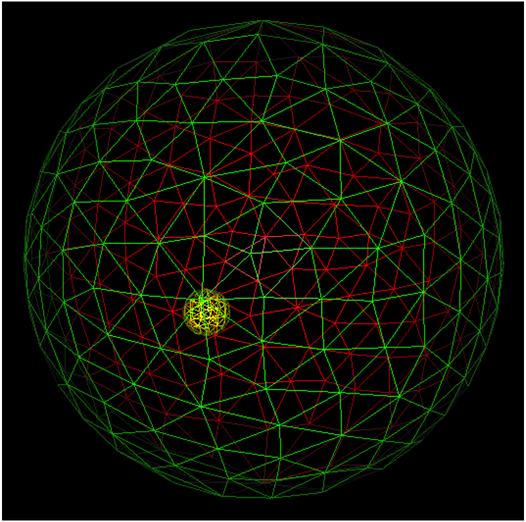
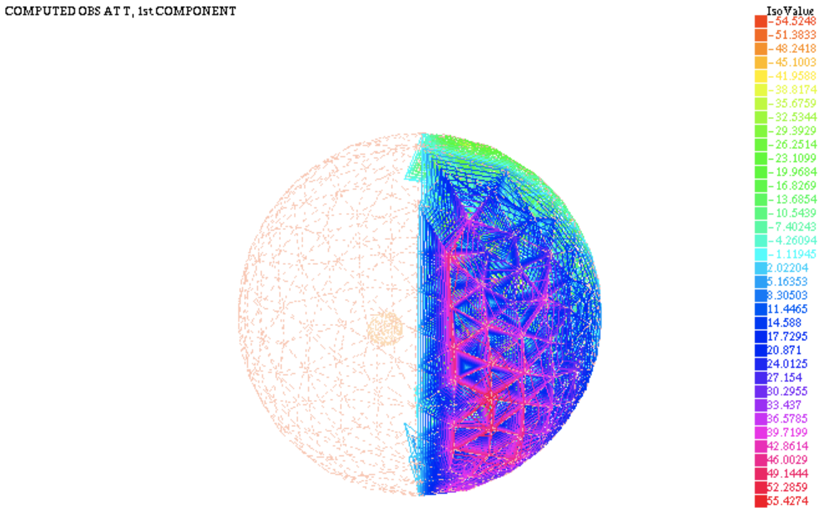


Figure: Desired configuration


COMPUTED OBS AT T, 1st COMPONENT
IsoValue
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-48.2418
-45.1003
-41.9588
-38.8174
-35.6759
-32.5344
-29.3929
-26.2514
-23.1099
-19.9684
-16.8269
-13.6854
-10.5439
-7.40243
-4.26094
-1.11945
2.02204
5.16353
8.30503
11.4465
14.588
17.7295
20.871
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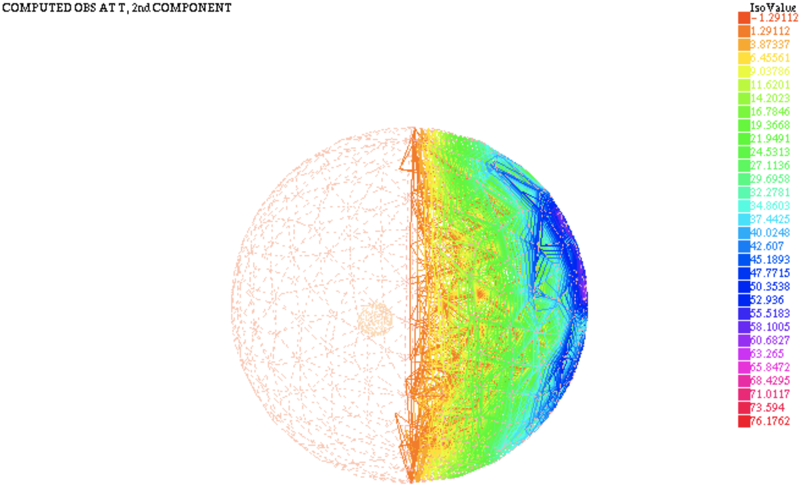
COMPUTED OBS AT T, 2nd COMPONENT
IsoValue
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11.6201
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19.3668
21.9491
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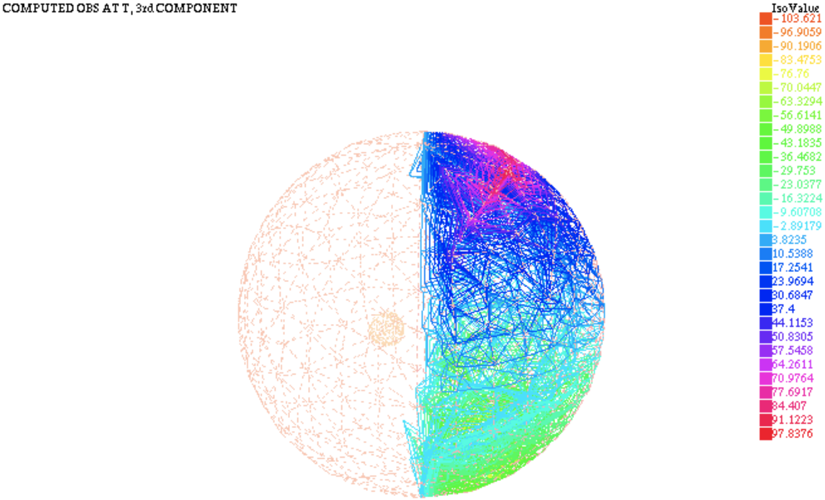
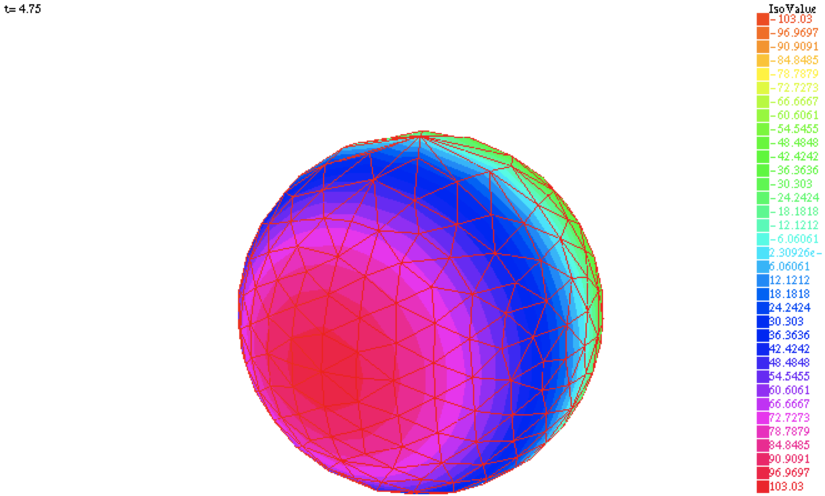
27.1136
29.6958
32.2781
34.8603
37.4425
40.0248
42.607
45.1893
47.7715
50.3538
52.936
55.5183
58.1005
60.6827
63.265
65.8472
68.4295
71.0117
73.594
76.1762


COMPUTED OBS AT T, 3rd COMPONENT
IsoValue
-103.621
-96.9059
-90.1906
-83.4753
-76.76
-70.0447
-63.3294
-56.6141
-49.8988
-43.1835
-36.4682
-29.753
-23.0377
-16.3224
-9.60708
-2.89179
3.8235
10.5388
17.2541
23.9694
30.6847
37.4
44.1153
50.8305
57.5458
64.2611
70.9764
77.6917
84.407
91.1223
97.8376

Numerical results: 3-D Lamé system VII
Case of a sphere



Work in progress:

1. Evolution elasticity system:

$$\begin{cases} -u_{tt} - \nabla \cdot \sigma(u) = 0 & \text{in } \Omega \setminus \overline{D} \times (0, T) \\ u = \varphi & \text{on } \partial\Omega \times (0, T) \\ u = 0 & \text{on } \partial D \times (0, T) \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega \setminus \overline{D} \end{cases}$$

$$\sigma_{kl}(u) = \sum_{i,j=1}^{3} a_{ijkl}\varepsilon_{ij}(u), \quad \varepsilon_{kl}(u) = \frac{1}{2}(\partial_k u_l + \partial_l u_k)$$

$$a_{ijkl} = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta jk), \quad 1 \le i, j, k, l \le 3$$

— Numerical results ?

2. Ellipsoids, other more complicated geometries

3. Internal observations ?: At present: new (emerging) techniques detect internal waves via non-invasive techniques (a very precise description)

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