The control of PDEs: some basic concepts, recent results and open problems

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ABSTRACT. These Notes deal with the control of systems governed by some PDEs. I will mainly consider time-dependent problems. The aim is to present some fundamental results, some applications and some open problems related to the optimal control and the controllability properties of these systems.

In Chapter 1, I will review part of the existing theory for the optimal control of partial differential systems. This is a very broad subject and there have been so many contributions in this field over the last years that we will have to limit considerably the scope. In fact, I will only analyze a few questions concerning some very particular PDEs. We shall focus on the Laplace, the stationary Navier-Stokes and the heat equations. Of course, the existing theory allows to handle much more complex situations.

Chapter 2 is devoted to the controllability of some systems governed by linear time-dependent PDEs. I will consider the heat and the wave equations. I will try to explain which is the meaning of controllability and which kind of controllability properties can be expected to be satisfied by each of these PDEs. The main related results, together with the main ideas in their proofs, will be recalled.

Finally, Chapter 3 is devoted to present some controllability results for other time-dependent, mainly nonlinear, parabolic systems of PDEs. First, we will revisit the heat equation and some extensions. Then, some controllability results will be presented for systems governed by stochastic PDEs. Finally, I will consider several nonlinear systems from fluid mechanics: Burgers, Navier-Stokes, Boussinesq, micropolar, etc.

Along these Notes, a set of questions (some of them easy, some of them more intrincate or even difficult) will be stated. Also, several open problems will be mentioned. I hope that all this will help to understand the underlying basic concepts and results and to motivate research on the subject.

Contents

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CHAPTER 1

Optimal control of systems governed by PDEs

In this Lecture, I will review part of the existing theory for the optimal control of partial differential systems. This is a very broad subject and there have been so many contributions in this field over the last years that we will have to limit considerably the scope. In fact, I will only analyze a few questions concerning some very particular PDEs. We shall focus on the Laplace, the stationary Navier-Stokes and the heat equations. Of course, the existing theory allows to handle much more complex situations. The optimal control of (elliptic, parabolic and hyperbolic) partial differential systems was addressed in [**76**]. Many other details can be found for instance in [**32, 59, 72, 75**] and the references therein. Along the text, several questions have been stated. They are of different nature and level of difficulty and it is highly recommended to the interested reader to try to answer them.

1.1. Some examples

It will be assumed that $\Omega \subset \mathbb{R}^N$ is a bounded, regular and connected open set, with boundary $\Gamma = \partial\Omega$.

The first example concerns the optimal control of a *capacitor*.

Let $\omega \subset\subset \Omega$ be a non-empty open set. For each $u \in L^2(\omega)$, we consider the state system

$$\begin{cases} -\Delta y = 1_\omega u & \text{in} \quad \Omega, \\ y = 0 & \text{on} \quad \Gamma, \end{cases} \tag{1.1}$$

where 1_ω is the characteristic function of ω.

The solution $y = y(x)$ to (1.1) can be interpreted as the *electric potential* of a capacitor to which a *density of charge* $1_\omega v$ is applied; $E = -\nabla y$ is the associated electric field.

In practice, it may be important to know how to choose v in a subset $\mathcal{U}_{\text{ad}} \subset L^2(\omega)$ in order to obtain a potential y as close as possible to a prescribed function y_d without too much effort. For instance, $\mathcal{U}_{\text{ad}}$ can be a ball in $L^2(\omega)$. It can also be a set of the form

$$\mathcal{U}_{\text{ad}} = \{\, u \in L^2(\omega) : \underline{u} \le u(x) \le \overline{u} \ \text{ a.e.}\,\}, \tag{1.2}$$

where $\underline{u}, \overline{u} \in \mathbb{R}$.

Thus, let us fix $y_d \in L^2(\Omega)$ and let us introduce the *cost functional* J, with

$$J(u) = \frac{a}{2}\int_\Omega |y - y_d|^2\, dx + \frac{b}{2}\int_\omega |u|^2\, dx \tag{1.3}$$

where $a, b > 0$. The optimal control problem we want to solve is then:

PROBLEM P1: *To find* $\hat{u} \in \mathcal{U}_{\rm ad}$ *such that* $J(\hat{u}) \leq J(u)$ *for all* $u \in \mathcal{U}_{\rm ad}$ *, where* J *is given by* (1.3).

We will see below that this problem can be solved. We will also see the way the solution (the optimal control) can be characterized by an appropriate *optimality system.* Additionally, we will present some generalizations and variants.

In our second problem, the control is performed through the coefficients of the system.

Assume that Ω is composed of two *dielectric materials* whose properties and prices are different. We want to build a *nonhomogeneous plate* with these two materials in such an optimal way. Here, the word optimal means that, under an applied density of charge (fixed and known), the associated potential is as close as possible to a prescribed state y_d .

Let α and β be the *permeability coefficients* of the first and the second material, respectively. We assume that $0 < \alpha < \beta$. Let $\{G_1, G_2\}$ be a *partition* of Ω (G_1 and G_2 are measurable sets) and set

$$a(x) = \begin{cases} \alpha & \text{if } x \in G_1 \,, \\ \beta & \text{if } x \in G_2 \,. \end{cases} \tag{1.4}$$

Then the *electrostatic potential* $y = y(x)$ corresponding to this distribution of the materials is the solution of the system

$$\begin{cases} -\nabla \cdot (a(x) \nabla y) = f(x) & \text{in} \quad \Omega, \\ y = 0 & \text{on} \quad \Gamma, \end{cases} \tag{1.5}$$

where $f \in H^{-1}(\Omega)$ (for instance) is given. In this example, the coefficient $a = a(x)$ is the control and y is the state.

Let us put

$$j(a) = \frac{1}{2} \int_\Omega |y - y_d|^2 \, dx \quad \forall a \in \mathcal{A}_{\rm ad} \,, \tag{1.6}$$

where $y_d \in L^2(\Omega)$ and, by definition, we have

$$\mathcal{A}_{\rm ad} = \{ \, a \in L^\infty(\Omega) : \; a(x) = \alpha \text{ or } a(x) = \beta \text{ a.e.} \} \tag{1.7}$$

The second problem we want to consider in this Section is then:

PROBLEM P2: *To find* $\hat{a} \in \mathcal{A}_{\rm ad}$ *such that* $j(\hat{a}) \leq j(a)$ *for all* $a \in \mathcal{A}_{\rm ad}$ *, where* j *is given by* (1.6).

It is well known that, in general, this problem has no solution and a "generalized" or "relaxed" version has to be introduced in order to describe the limitting behavior of the minimizing sequences. This is in fact typical in control problems where the control enters in the system through its coefficients and, specially, in the principal part of the operator. Phenomena of this kind have led to a very rich development of the theory. We will see later what can be done and which is the physical interpretation of the "generalized" or "relaxed solution".

The third example is an *optimal design* problem.

We will assume that Ω is filled with a viscous incompressible fluid and we will try to find the optimal shape of a body travelling at constant velocity in Ω.

Thus, assume that $B \subset \Omega$ is a non-empty closed subset whose shape is in principle unknown. We will assume that B is the closure of a connected open set and ∂B is piecewise Lipschitz-continuous. Let us choose a reference system fixed with respect to B. We will consider the following Navier-Stokes system in $\Omega \setminus B$:

$$
\left\{
\begin{array}{lll}
-\nu\Delta y + (y\cdot\nabla)y + \nabla\pi = 0, \quad \nabla\cdot y = 0 & \text{in} & \Omega\setminus B, \\
y = y_\infty & \text{on} & \Gamma, \\
y = 0 & \text{on} & \partial B.
\end{array}
\right.
\tag{1.8}
$$

Here, (y, π) is the state (the velocity field and the pressure of the fluid). The positive coefficient ν is the viscosity of the fluid. We have assumed that the velocity of the fluid particles on the *exterior* boundary Γ, that is, far from the body, is y_∞ (a constant vector). We have also imposed the usual *no-slip condition* on ∂B. These boundary conditions in mean that the body travels with velocity $-y_\infty$ and the fluid particles on ∂B adhere to the body.

For each B in a family $\mathcal{B}_{\rm ad}$ of *admissible bodies*, the state system (1.8) possesses at least one *weak solution* (y, π), with $y \in H^1(\Omega;\mathbb{R}^2)$ and $\pi \in L^2(\Omega)$. Now, we can associate to each solution the quantity

$$
T(B, y) = 2\nu \int_\Omega |Dy|^2\, dx,
\tag{1.9}
$$

where

$$
Dy = \frac{1}{2}(\nabla y + \nabla y^t)
$$

is the symmetric part of the gradient ∇y. It can be seen that $T(B, y)$ is in fact the *hydrodynamical drag* of the fluid, that is

$$
T(B, y) = -y_\infty \cdot \int_{\partial B} \big(-\pi\, I + \nu D(y)\big)\cdot n\, ds
$$

(the projection in the direction of the velocity of the body of the force exerted by the fluid particles).

Our third problem is the following:

> PROBLEM P3: *To find* $\hat B \in \mathcal{B}_{\rm ad}$ *such that the corresponding system* (1.8) *possesses a solution* $(\hat y, \hat\pi)$ *satisfying* $T(\hat B, \hat y) \le T(B, y)$ *whenever* (y, π) *is a solution to* (1.8) *and* $B \in \mathcal{B}_{\rm ad}$.

We will see below that, unless the family $\mathcal{B}_{\rm ad}$ satisfies particular and in some sense artificial conditions, it is not possible to prove an existence result for Problem P3.

Besides existence, another interesting question is to analyze the way $T(B, y)$ depends on B. In fact, we will show that, at least when y_∞ is small, the mapping $B \mapsto T(B, y)$ is well-defined and in some sense of class C^∞. We will also indicate how to compute its "derivative".

We will now consider an optimal control problem for a parabolic system with origin in biomedical science. As shown below, the control is oriented to the determination of cancer therapy strategies.

The state system is nonlinear and reads:

$$
\left\{
\begin{array}{lll}
c_t - \nabla \cdot (D(x)\nabla c) = f(c) - F(c,\beta) & \text{in} & Q = \Omega \times (0,T), \\
\beta_t - \mu\Delta\beta = -h(\beta) - H(c,\beta) + v 1_\omega & \text{in} & Q = \Omega \times (0,T), \\
c = 0 & \text{on} & \Sigma = \partial\Omega \times (0,T), \\
\beta = 0 & \text{on} & \Sigma = \partial\Omega \times (0,T), \\
c(x,0) = c_0(x) & \text{in} & \Omega, \\
\beta(x,0) = \beta_0(x) & \text{in} & \Omega.
\end{array}
\right.
\tag{1.10}
$$

We assume that Ω is an organ, where we find a population of cancer cells with density $c = c(x,t)$ and a distribution of inhibitors (or antibodies), of density $\beta = \beta(x,t)$. The antibodies are generated through a therapy process, determined by the control v and localized in a small open set $\omega \subset \Omega$. This can be used to model the evolution of a glioblastoma, i.e. a brain tumor, under radiotherapy, see [**103, 104**].

The functions f and h define the proliferation and death rates of c and β, respectively. On the other hand, F and H determine the way c and β interact. In the simplest cases we just take

$$
f(c) = \rho c, \quad h(\beta) = -m\beta, \quad F(c,\beta) = Rc\beta, \quad H(c,\beta) = Mc\beta,
\tag{1.11}
$$

for some positive constants ρ, m, R and M.

For a large family of functions f, h, F and H, for any $v \in L^2(\omega \times (0,T))$ there exists at least one solution (c,β) to (1.10).

Obviously, in order to make the problem realistic, we have to impose constraints on v. Thus, we will assume that $v \in \mathcal{V}_{\rm ad}$, where $\mathcal{V}_{\rm ad}$ is a bounded, closed and convex set of $L^2(\omega \times (0,T))$. A natural choice is the following:

$$
\mathcal{V}_{\rm ad} = \{\, v \in L^2(\omega \times (0,T)) : 0 \le v \le A, \quad \int_0^T v\,dt \le B, \quad v = 0 \ \text{ for } \ t \notin \mathcal{I} \,\},
$$

where $\mathcal{I}$ is a (small) closed set of times where the therapy is applied.

There are different possible choices for the cost function. A reasonable (but maybe nor the best) choice is the following:

$$
K(c,\beta,v) = \frac{a}{2}\int_\Omega |c(x,T)|^2\,dx + \frac{b}{2}\int_{\omega\times(0,T)} |v|^2\,dx\,dt.
\tag{1.12}
$$

The fourth considered problem is then:

> PROBLEM P4: *To find* $\hat v \in \mathcal{V}_{\rm ad}$ *such that the corresponding system* (1.10) *possesses a solution* $(\hat c, \hat\beta)$ *satisfying* $K(\hat c, \hat\beta, \hat v) \le K(c,\beta,v)$ *whenever* (c,β) *is a solution to* (1.10) *and* $v \in \mathcal{V}_{\rm ad}$.

Under very general conditions, we will give below an existence result for Problem P4. We will also find the *optimality system* for this problem.

1.2. Existence, uniqueness and optimality results

Our first result is the following:

THEOREM 1.1. *Assume that* $\mathcal{U}_{\rm ad}$ *is a non-empty closed convex set of* $L^2(\omega)$. *Then, Problem* P1 *possesses exactly one solution.*

PROOF: For the proof we only have to check that $u \mapsto J(u)$ is a strictly convex, coercive and weakly lower semicontinuous function on $L^2(\omega)$.

But this is very easy to verify. In fact, $u \mapsto J(u)$ can be written in the form

$$J(u) = \frac{1}{2}\, a_0(u,u) + a_1(u) + a_2 \quad \forall u \in \mathcal{U}_{\rm ad}\,, \tag{1.13}$$

where $a_0(\cdot\,,\cdot)$ is a continuous and coercive bilinear form on $L^2(\omega)$, $a_1(\cdot)$ is a continuous linear form on $L^2(\omega)$ and $a_2 \in \mathbb{R}$.

The forms $a_0(\cdot\,,\cdot)$ and $a_1(\cdot)$ are given as follows:

$$a_0(u,v) = a \int_\Omega yz\,dx + b \int_\omega uv\,dx$$

and

$$a_1(u) = -a \int_\Omega y_d y\,dx,$$

where y (resp. z) is the solution to (1.1) (resp. (1.1) with u replaced by v). On the other hand,

$$a_3 = \frac{a}{2} \int_\Omega |y_d|^2\,dx.$$

Hence, the usual arguments of the *direct method of the Calculus of Variations* lead to the existence and uniqueness of solution, as assserted. □

QUESTION 1: *What can be said if, in* (1.3)*, we assume that* $b = 0$*? Which interpretation can be given to the corresponding optimal control problem?*

We will now be concerned with the computation of $J'(u)$ and the obtention of an optimality system. Our result is the following:

THEOREM 1.2. *Assume that* $\mathcal{U}_{\rm ad} \subset L^2(\omega)$ *is a non-empty closed convex set and let* $\hat u$ *be the solution to Problem* P1*. Then there exists* $\hat y$ *and* $\hat p$ *such that the following optimality system is satisfied:*

$$\begin{cases} -\Delta \hat y = \hat u 1_\omega & in \quad \Omega, \\ \hat y = 0 & on \quad \Gamma, \end{cases} \tag{1.14}$$

$$\begin{cases} -\Delta \hat p = \hat y - y_d & in \quad \Omega, \\ \hat p = 0 & on \quad \Gamma, \end{cases} \tag{1.15}$$

$$\int_\omega (a\hat p + b\hat u)(u - \hat u)\,dx \ge 0 \quad \forall u \in \mathcal{U}_{\rm ad}\,. \tag{1.16}$$

PROOF: For the proof, we argue as follows. Since $\hat u$ is the solution to Problem P1, we must have

$$\langle J'(\hat u), u - \hat u\rangle \ge 0 \quad \forall u \in \mathcal{U}_{\rm ad}\,, \quad \hat u \in \mathcal{U}_{\rm ad}\,. \tag{1.17}$$

Here, $\langle\cdot\,,\cdot\rangle$ denotes the scalar product in $L^2(\omega)$. Taking into account (1.13), this can be written as follows:

$$a_0(\hat u, u - \hat u) + a_1(u - \hat u) \ge 0$$

that is to say,

$$a\int_\Omega (\hat y - y_d)(y-\hat y)\,dx + b\int_\omega \hat u\,(u-\hat u)\,dx \geq 0 \tag{1.18}$$

for all $u \in \mathcal{U}_{\rm ad}$. Of course, in (1.18) y is the solution to (1.1) and $\hat y$ is the solution to (1.1) with u replaced by $\hat u$.

Let $\hat p$ be the solution to (1.15), the *adjoint system*. It is then clear that

$$\int_\Omega (\hat y - y_d)(y-\hat y)\,dx = \int_\Omega \nabla\hat p\cdot\nabla(y-\hat y)\,dx = \int_\omega \hat p\,(u-\hat u)\,dx.$$

Consequently, (1.18) is equivalent to (1.16). This proves that the optimality system (1.14) – (1.16) must hold. □

REMARK 1.3. In this particular case, we also have the reciprocal or theorem 1.2: If $\hat u \in \mathcal{U}_{\rm ad}$ and there exist $\hat y$ and $\hat p$ such that (1.14) – (1.16) holds, then $\hat u$ is the unique solution to Problem P1. □

It is usual to say that $\hat p$ is the *adjoint state* associate to the optimal control $\hat u$. In fact, in view of the previous argument, for each $u \in \mathcal{U}_{\rm ad}$, we have

$$\langle J'(u), v\rangle = \int_\omega (ap+bu)\,v\,dx \quad \forall v \in \mathcal{U}_{\rm ad}, \tag{1.19}$$

where p is the adjoint state associate to u, i.e. the solution to

$$\begin{cases} -\Delta p = y - y_d & \text{in} \quad \Omega, \\ p = 0 & \text{on} \quad \Gamma. \end{cases} \tag{1.20}$$

This provides a very useful technique to compute the derivative $J'(u)$ for a given u. From the practical viewpoint this is very important, since a method to compute $J'(u)$ permits the use of *descent methods* in order to determine the optimal control $\hat u$.

QUESTION 2: *The optimality system in theorem 1.2 suggests the following iterative method for the computation of $\hat u$:*

$$\begin{cases} -\Delta y^n = u^{n-1}1_\omega & \textit{in} \quad \Omega, \\ y^n = 0 & \textit{on} \quad \Gamma, \end{cases} \tag{1.21}$$

$$\begin{cases} -\Delta p^n = y^n - y_d & \textit{in} \quad \Omega, \\ p^n = 0 & \textit{on} \quad \Gamma, \end{cases} \tag{1.22}$$

$$\int_\omega (ap^n + bu^n)(u-u^n)\,dx \geq 0 \quad \forall u \in \mathcal{U}_{\rm ad}. \tag{1.23}$$

What can be said on the convergence of these iterates?

QUESTION 3: *In view of* (1.19) – (1.20)*, how can we apply (for instance) the fixed-step gradient method to produce a sequence $\{u^n\}$ of controls converging to the optimal control $\hat u$? What about the optimal-step gradient method? What about the fixed-step and optimal-step conjugate gradient methods?*

The previous ideas can be generalized in several directions. We will present a generalization involving nonlinear elliptic state systems and nonquadratic cost functionals.

Thus, let us introduce the system

$$\begin{cases} Ay + f(y) = 1_\omega u & \text{in} \quad \Omega, \\ y = 0 & \text{on} \quad \Gamma, \end{cases} \tag{1.24}$$

where A is a linear second order partial differential operator given by

$$Ay = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial y}{\partial x_j} \right) + \sum_{j=1}^{2} b_j(x) \frac{\partial y}{\partial x_j} + c(x) y \tag{1.25}$$

and $f : \mathbb{R} \mapsto \mathbb{R}$ is (for instance) a nondecreasing C^1 function satisfying

$$|f(s)| \le C(1 + |s|) \quad \forall s \in \mathbb{R}. \tag{1.26}$$

We will assume that the coefficients a_{ij} , b_i and c satisfy:

$$\begin{aligned} & a_{ij}\,, b_i\,, c \in L^\infty(\Omega), \quad c \ge 0, \\ & \sum_{i,j=1}^{2} a_{ij}(x) \xi_i \xi_j \ge \alpha |\xi|^2 \quad \forall \xi \in \mathbb{R}^2 \quad \text{a.e. in } \Omega, \quad \alpha > 0. \end{aligned} \tag{1.27}$$

For each $u \in L^2(\omega)$, the corresponding system (1.24) possesses exactly one solution $y \in H_0^1(\Omega)$. Let $\mathcal{U}_{\rm ad} \subset L^2(\omega)$ be a family of admissible controls. We will now set

$$J(u) = \int_\Omega F(x, y(x), u(x))\, dx \quad \forall u \in \mathcal{U}_{\rm ad}\,, \tag{1.28}$$

where $F = F(x, s, v)$ is assumed to be a *Carathéodory function*, defined for $(x, s, v) \in \Omega \times \mathbb{R} \times \mathbb{R}$. We consider the following generalization of Problem P1:

PROBLEM P1$'$: *To find* $\hat u \in \mathcal{U}_{\rm ad}$ *such that* $J(\hat u) \le J(u)$ *for all* $u \in \mathcal{U}_{\rm ad}$ *, where* J *is given by* (1.24),(1.28).

Among all possible results that can be established in this context, let us indicate the following, that has been taken from [**16**]:

THEOREM 1.4. *Assume that* $\mathcal{U}_{\rm ad}$ *is a closed convex subset of* $L^2(\omega)$. *Also, assume that* F *is of the form*

$$F(x, s, v) = F_0(x, s) + F_1(x, v)\, 1_\omega(x),$$

where F_0 *and* F_1 *are Carathéodory functions satisfying:*

$$\begin{cases} |F_0(x, s)| \le C(1 + |s|^2) \quad \forall (x, s) \in \Omega \times \mathbb{R}, \\ a|v|^2 \le F_1(x, v) \le C(1 + |v|^2) \quad \forall (x, v) \in \omega \times \mathbb{R}, \quad a > 0, \\ F_1(x, \cdot) \text{ is convex for each } x \in \omega. \end{cases} \tag{1.29}$$

Then Problem P1$'$ *possesses at least one solution* $\hat u$.

The proof relies on arguments similar to those above but technically more involved. It will not be given here; see [**16**] for the details.

QUESTION 4: *What can be said if, in* (1.29), *we have* $a = 0$*?*

Notice that, in the previous result, the convexity hypothesis on $F_1(x,\cdot)$ is essential. Indeed, let us consider the particular case in which the state system is

$$\left\{ \begin{array}{lll} -\Delta y = u & \text{in} & \Omega, \\ y = 0 & \text{on} & \Gamma, \end{array} \right. \tag{1.30}$$

the set $\mathcal{U}_{\rm ad}$ is

$$\mathcal{U}_{\rm ad} = \{\, u \in L^2(\Omega) : |u| \leq 1 \quad \text{a.e. in } \Omega \,\} \tag{1.31}$$

and the cost functional is given by

$$J(u) = \int_\Omega \left(|u|^2 - 1\right)^2 \, dx + \frac{1}{2}\int_\Omega |y|^2 \, dx \quad \forall u \in \mathcal{U}_{\rm ad}\,. \tag{1.32}$$

Then, it can be shown that

$$\inf_{u\in\mathcal{U}_{\rm ad}} J(u) = 0$$

and however

$$J(u) > 0 \quad \forall u \in \mathcal{U}_{\rm ad}\,,$$

whence the optimal control problem associate to (1.30), (1.31) and (1.32) has no solution.

To end this Subsection, let us recall a result concerning the optimality system for Problem P1′. We will need the adjoint operator A^*, which is given as follows:

$$A^* p = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial p}{\partial x_i} + b_j(x)p \right) + c(x)p. \tag{1.33}$$

Then, one has:

Theorem 1.5. *Assume that F is as above, that F_0 and F_1 possess bounded partial derivatives and, also, that* (1.29) *is satisfied. Let $\hat u$ be a solution to Problem* P1′. *Then there exist $\hat y$ and $\hat p$ such that the following optimality system is satisfied:*

$$\left\{ \begin{array}{lll} A\hat y + f(\hat y) = \hat u 1_\omega & \textit{in} & \Omega, \\ \hat y = 0 & \textit{on} & \Gamma, \end{array} \right. \tag{1.34}$$

$$\left\{ \begin{array}{lll} A^*\hat p + f'(\hat y)\hat p = \dfrac{\partial F_0}{\partial s}(x,\hat y) & \textit{in} & \Omega, \\ \hat p = 0 & \textit{on} & \Gamma, \end{array} \right. \tag{1.35}$$

$$\int_\omega \left(\hat p + \frac{\partial F_1}{\partial v}(x,\hat u) \right)(u - \hat u)\, dx \geq 0 \quad \forall u \in \mathcal{U}_{\rm ad}\,. \tag{1.36}$$

As before, the method of proof of this result provides an expression for the derivative $J'(u)$ of J at each u. More precisely, one finds that

$$\langle J'(u), v\rangle = \int_\omega \left(p + \frac{\partial F_1}{\partial v}(x,u) \right) v\, dx \quad \forall v \in \mathcal{U}_{\rm ad}\,, \tag{1.37}$$

where p is the adjoint state associate to u, i.e. the solution to

$$\left\{ \begin{array}{lll} A^* p + f'(y)p = \dfrac{\partial F_0}{\partial s}(x,y) & \text{in} & \Omega, \\ p = 0 & \text{on} & \Gamma \end{array} \right. \tag{1.38}$$

and y is the state, i.e. the solution to (1.24).

For other similar results, see for instance [**14**] and [**17**].

QUESTION 5: *Is there a way to use the optimality system in theorem 1.5 to prove a uniqueness result?*

QUESTION 6: *The optimality system in theorem 1.5 also suggests a "natural" iterative method for the computation of $\hat{u}$. Which one? What can be said on the convergence of the iterates?*

QUESTION 7: *In view of* (1.37) – (1.38)*, how can we apply gradient and conjugate gradient method to produce a sequence of controls that converge to an optimal control?*

1.3. Control on the coefficients, nonexistence and relaxation

In this Section we assume for simplicity that $N = 2$ and we consider Problem P2.

We will try to show the complexity of the problems in which the control is applied through coefficients in the principal part of the operator. We will first see that, in general, there exists no solution to this problem.

The following notation is needed. For given α and β with $\alpha, \beta > 0$, let us denote by $\mathcal{A}(\alpha, \beta)$ the family of 2×2 matrices A with components $A_{ij} \in L^\infty(\Omega)$ such that

$$A(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad (A(x))^{-1}\xi \cdot \xi \geq \frac{1}{\beta}|\xi|^2 \quad \forall \xi \in \mathbb{R}^2, \quad x \text{ a.e. in } \Omega. \tag{1.39}$$

It will be useful to recall the concept of *H-convergence*, which was introduced by F. Murat in 1978 (see [**84**],[**85**] and [**88**]):

DEFINITION 1.6. *Assume that $A^n \in \mathcal{A}(\alpha, \beta)$ for each $n \geq 1$ and that $A^0 \in \mathcal{A}(\alpha, \beta)$. It will be said that A^n H-converges to A^0 in Ω if, for any non-empty open set $\mathcal{O} \subset \Omega$ and any $g \in H^{-1}(\mathcal{O})$, the solution y^n of the elliptic problem*

$$\begin{cases} -\nabla \cdot (A^n(x)\nabla y) = g & \text{in} \quad \mathcal{O}, \\ y = 0 & \text{on} \quad \partial\mathcal{O}, \end{cases} \tag{1.40}$$

satisfies

$$y^n \to y^0 \quad \text{weakly in } H_0^1(\mathcal{O})$$

and

$$A^n \nabla y^n \to A^0 \nabla y^0 \quad \text{weakly in } L^2(\mathcal{O}),$$

where y^0 is the unique solution of the problem

$$\begin{cases} -\nabla \cdot (A^0(x)\nabla y) = g & \text{in} \quad \mathcal{O}, \\ y = 0 & \text{on} \quad \partial\mathcal{O}. \end{cases} \tag{1.41}$$

It can be seen that the family $\mathcal{A}(\alpha, \beta)$ is *closed* for the H-convergence. The following is also true:

THEOREM 1.7. *The family $\mathcal{A}(\alpha, \beta)$ is compact for the H-convergence. In other words, any sequence in $\mathcal{A}(\alpha, \beta)$ possesses subsequences that H-converge in $\mathcal{A}(\alpha, \beta)$.*

A key point is that we can have all the A^n of the form

$$A^n = a^n I \quad \forall n \geq 1,$$

while the H-limit A^0 can have extra-diagonal terms. In fact, explicit examples can be constructed and, in particular, we can find $A^0 \in \mathcal{A}(\alpha, \beta)$ and $f^0 \in H^{-1}(\Omega)$ with the following two properties:

(a) A^0 is the H-limit of a sequence of the form $a^n I$, with $a^n(x) = \alpha$ or $a^n(x) = \beta$ a.e.
(b) Let y^0 be the solution to (1.41) with g replaced by f^0. Then there is no function a with $a(x) = \alpha$ or $a(x) = \beta$ a.e. such that y^0 solves (1.5) with f replaced by f^0.

We are now ready to prove that Problem P2 has no solution in general. Let us take $f = f^0$ and $y_d = y^0$, where y^0 is the solution of (1.41) with g replaced by f^0. In view of the properties of A^0, it is clear that

$$\inf_{a \in \mathcal{A}_{\text{ad}}} j(a) = 0$$

(recall that $\mathcal{A}_{\text{ad}}$ is given by (1.7)). However, in view of the properties of f^0, we also have

$$j(a) > 0 \quad \forall a \in \mathcal{A}_{\text{ad}}\,.$$

As a consequence, we must modify the definition of *optimal material*. Note that minimizing sequences do exist and that, in fact, they "describe" the optimal behavior. Consequently, it seems natural to adopt a new formulation in which the limits of minimizing sequences are distinguished material configurations. A satisfactory strategy consists of introducing a *relaxed problem*.

Relaxation is a useful tool in Optimization. Roughly speaking, to *relax* an extremal problem, say (P), is to introduce a second one, denoted by (Q), satisfying the following three conditions:

(a) (Q) possesses at least one solution.
(b) Any solution to (Q) can be written as the limit (in some sense) of a minimizing sequence for (P).
(c) Conversely, any minimizing sequence for (P) contains a subsequence that converges (in the same sense) to a solution of (Q).

For an overview on the role of the notion of relaxation in control problems, see [**67**] and [**93**]. We will only present here an intuitive and very simple argument which leads to a relaxed problem for P2.

The main point is to determine the "closure" in $\mathcal{A}(\alpha, \beta)$ of the family formed by the matrices of the form aI, with $a \in \mathcal{A}_{\text{ad}}$. The answer is given by the following result:

THEOREM 1.8. *Let $\tilde{\mathcal{A}}_{\text{ad}}$ be the family of all $A \in \mathcal{A}(\alpha, \beta)$ with the following two properties:*

(a) *$A(x)$ is symmetric for x a.e. in Ω.*
(b) *For almost all x, the eigenvalues $\lambda_1(x)$ and $\lambda_2(x)$ of the matrix $A(x)$ satisfy:*

$$\alpha \leq \lambda_1(x) \leq \lambda_2(x) \leq \beta, \quad \frac{\alpha\beta}{\alpha + \beta - \lambda_2(x)} \leq \lambda_1(x). \tag{1.42}$$

Then, if A is given in $\mathcal{A}(\alpha,\beta)$, one has $A \in \tilde{\mathcal{A}}_{\rm ad}$ if and only if A can be written as the H-limit of a sequence of matrices of the form $a^n I$, with $a^n \in \mathcal{A}_{\rm ad}$ for all n.

This is proved in [**106**] (see also [**88**]). At this respect, it is worth mentioning that, in a similar N-dimensional situation with $N \geq 3$, the determination of the set of H-limits of the matrices of the form aI with $a \in \mathcal{A}_{\rm ad}$ is an open problem.

The previous result permits to introduce a new control problem which is nothing but the relaxation of Problem P2.

Namely, for each $A \in \tilde{\mathcal{A}}_{\rm ad}$, let us consider the (relaxed) state system

$$\begin{cases} -\nabla\cdot(A(x)\nabla Y) = f(x) & \text{in} \quad \Omega, \\ Y = 0 & \text{on} \quad \Gamma \end{cases} \tag{1.43}$$

and let us set

$$k(A) = \frac{1}{2}\int_\Omega |Y - y_d|^2\,dx \quad \forall A \in \tilde{\mathcal{A}}_{\rm ad}\,. \tag{1.44}$$

The relaxed problem is then:

PROBLEM P2′: *To find $\hat{A} \in \tilde{\mathcal{A}}_{\rm ad}$ such that $k(\hat{A}) \leq k(A)$ for all $A \in \tilde{\mathcal{A}}_{\rm ad}$, where $\tilde{j}$ is given by* (1.44).

Indeed, the following can be proved:

THEOREM 1.9. *Assume that $f \in H^{-1}(\Omega)$ and $y_d \in L^2(\Omega)$ are given. Then, there exists at least one solution $\hat{A}$ to Problem* P2′. *This can be written as the H-limit of a minimizing sequence for Problem* P2. *Furthermore, any minimizing sequence for Problem* P2 *contains a subsequence that H-converges to a solution of Problem* P2′.

The proof of this result is not difficult, taking into account the definition of H-convergence and the fact that $\tilde{\mathcal{A}}_{\rm ad}$ is the H-closure of $\mathcal{A}_{\rm ad}$.

From a physical viewpoint, we see that the “generalized” solution to the original problem is a *composite material.* In general, it is anisotropic, i.e. $\hat{A}_{ij}(x)$ may be $\neq 0$ for $i \neq j$.

QUESTION 8: *Is it possible to deduce an optimality system for the solutions to Problem* P2′*? Which one? Does this optimality system lead to convergent iterates?*

QUESTION 9: *Is it possible to compute $k'(A)$ easily and use this computation to apply gradient and/or conjugate gradient methods in the context of Problem* P2′*?*

The reader is referred to [**82**] and the references therein for more details on the control of coefficients, the generation of composite materials and other related topics.

1.4. Optimal design and domain variations

We will now consider Problem P3.

This is an *optimal design* problem. The feature is that, now, the control is a geometric datum in (1.8) (the set B). Accordingly, we have to minimize a function over a set $\mathcal{B}_{\rm ad}$ where there is no vector structure at our disposal. It is thus reasonable to expect a higher level of difficulty than for other optimal control problems.

As mentioned above, the existence of a solution to Problem P3 is not clear at all. To simplify our arguments, let us introduce two non-empty open sets D_0 and D_1, with

$$D_0 \subset\subset D_1 \subset\subset \Omega$$

and let us first assume that $\mathcal{B}_{\rm ad}$ is the family of the non-empty closed sets B with piecewise Lipschitz-continuous boundary that satisfy

$$\overline{D_0} \subset B \subset \overline{D_1}\,. \tag{1.45}$$

Also, assume that $|y_\infty|$ is small enough (depending on ν and Ω). Then, for each $B \in \mathcal{B}_{\rm ad}$, the state system (1.8) possesses exactly one solution (y, π) (the pressure π is unique up to an additive constant). Consequently, we can assign to B a *drag* $D(B) = T(B, y)$, given by (1.9).

In other words, in this case the function $B \mapsto D(B)$ is well-defined and Problem P3 reads:

To find $\hat{B} \in \mathcal{B}_{\rm ad}$ *such that* $D(\hat{B}) \le D(B)$ *for all* $B \in \mathcal{B}_{\rm ad}$.

Let $\{B^n\}$ be a minimizing sequence. For each $n \ge 1$, let us denote by y^n the velocity field associated to B^n by (1.8). Then, it is clear that y^n is uniformly bounded in the H^1-norm. More precisely, the extensions-by-zero of y^n to the whole domain Ω, that we denote by $\tilde{y}^{\,n}$, are uniformly bounded in $H^1(\Omega;\mathbb{R}^2)$. We can thus assume that $\tilde{y}^{\,n}$ converges weakly in $H^1(\Omega;\mathbb{R}^2)$, strongly in $L^2(\Omega;\mathbb{R}^2)$ and a.e. to a function $\tilde{y}^{\,0}$. This is a consequence of the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$; see for instance [**1**].

On the other hand, since $\{B^n\}$ is a sequence of closed sets of Ω, we can also assume that B^n converges in the sense of the *Haussdorf distance* d_H to a closed set B^0. This is a consequence of the fact that the family of closed subsets of Ω is compact for d_H; see [**25**].

At this respect, recall that, when B and B' closed sets in $\mathbb{R}^2$, the Haussdorf distance $d_H(B, B')$ is given by

$$d_H(B, B') = \max\{\rho(B, B'), \rho(B', B)\},$$

where

$$\rho(B, B') = \sup_{x \in B} d(x, B') \quad \text{and} \quad d(x, B') = \inf_{x' \in B'} |x - x'| \quad \text{for all } B \text{ and } B'$$

and a similar definition holds for $\rho(B', B)$.

The set B^0 satisfies (1.45). However, the uniform bound in the H^1 norm does not give enough regularity for B^0 and it is not clear whether the restriction of $\tilde{y}^{\,0}$ to the limit set $\Omega \setminus B^0$ is, together with some π^0, the solution of (1.8) with B replaced by B^0.

We can overcome this difficulty by introducing a more restrictive family $\mathcal{B}_{\rm ad}$.

For instance, let us now assume that $\mathcal{B}_{\rm ad}$ is the family of the non-empty closed sets B satisfying (1.45) whose boundaries are *uniformly Lipschitz-continuous* with constant $L > 0$. By this we mean that the boundary ∂B of any $B \in \mathcal{B}_{\rm ad}$ can be written in the form

$$\partial B = \{\, x(\theta) : \theta \in [0, 1] \,\}, \tag{1.46}$$

where the function $\theta \mapsto x(\theta)$ satisfies $x(0) = x(1)$ and is Lipschitz-continuous on $[0, 1]$ with Lipschitz constant L. Obviously, $\mathcal{B}_{\rm ad}$ is non-empty if L is large enough.

It is clear that we can argue as before and find a limit set B^0 and a vector field $\tilde{y}^0$, defined in Ω. In this particular case, the set B^0 belongs to $\mathcal{B}_{\rm ad}$, that is, its boundary is also of the form (1.46), see [**21**]. In view of this regularity property for B^0, it can also be proved that the restriction of $\tilde{y}^0$ to $\Omega \setminus B^0$ is, together with an appropriate π^0, the solution of (1.8) with $B = B^0$.

QUESTION 10: *How can this be proved?*

Unfortunately, this new definition of the admissible set $\mathcal{B}_{\rm ad}$ can be too restrictive.

Actually, this is a common fact for optimal design problems: either we choose the apparently natural definition of $\mathcal{B}_{\rm ad}$ (and then existence is not known) or we make it more restrictive (and then the problem can become unrealistic). For more details on these and other similar results, see [**95, 96, 60**].

We will now study the behavior of the function $B \mapsto D(B)$. Let $\hat{B}$ be a reference shape for the body (arbitrary in $\mathcal{B}_{\rm ad}$ but fixed). The body variations are described by a field $u = u(x)$ and we search for a formula of the kind

$$D(\hat{B} + u) = D(\hat{B}) + D'(\hat{B}; u) + o(u), \tag{1.47}$$

where the modified fluid domain is

$$(\Omega \setminus \hat{B}) + u = \Omega \setminus (\hat{B} + u) = \{\, x \in \mathbb{R}^2 : x = (I + u)(\xi),\ \xi \in \Omega \setminus \hat{B} \,\}$$

and

$$o(u) \|u\|_{W^{1,\infty}}^{-1} \to 0 \quad \text{as} \quad \|u\|_{W^{1,\infty}} \to 0.$$

We are thus led to an analysis of the differentiability of the function $u \mapsto D(\hat{B} + u)$.

A lot of work has been made for the definition and computation of the variations with respect to a domain of functionals defined through the solutions to boundary value problems. The reader is referred to [**102**] and the references therein.

We will recall briefly a variant of a general method introduced by F. Murat and J. Simon in [**86**] and [**87**][1]. This is taken from [**10**]. Notice that some formal computations of the derivative were previously carried out by O. Pironneau in [**94**] (see also [**96**]), using "normal" variations.

We will choose fields $u \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ such that $u = 0$ on Γ. This includes many interesting situations in which $\partial(\Omega \setminus (\hat{B} + u))$ possesses "corner" points. Furthermore, the equality $u = 0$ on Γ expresses the fact that the outer boundary limiting the fluid is fixed.

We will also assume that $\|u\|_{W^{1,\infty}} \leq \eta$, with η being small enough to ensure that the boundary of $\Omega \setminus (\hat{B} + u)$ is Lipschitz-continuous and also that $\hat{B} + u$ is included in a fixed open set D_2 satisfying

$$\hat{B} \subset\subset D_2 \subset\subset \Omega$$

[1] The general method in [**86**] and [**87**] cannot be directly applied to the Stokes and Navier-Stokes cases. This is due to the incompressibility condition.

(such a constant $\eta > 0$ exists, see [**10**] for a proof).

For the sequel, we introduce

$$\mathcal{W} = \{\, u \in W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) : \ \|u\|_{W^{1,\infty}} \le \eta, \quad u = 0 \text{ on } \partial\Omega \,\}.$$

Now, we choose g satisfying

$$\nabla \cdot g = 0, \quad g = y_\infty \text{ in a neighborhood of } \partial\Omega, \quad g = 0 \text{ in a neighborhood of } D_2$$

(such a function g always exists; see for instance [**51**]). If $u \in \mathcal{W}$, one has $g = 0$ in a neighborhood of $\partial\hat{B} + u$. After normalization of the pressure, the Navier-Stokes problem in $\Omega \setminus (\hat{B} + u)$ can be written as follows:

(1.48)
$$\begin{cases} -\nu\Delta y(u) + (y(u)\cdot\nabla)y(u) + \pi(u) = 0, \quad \nabla\cdot y(u) = 0 \quad \text{in } \Omega\setminus(\hat{B}+u), \\ y(u) - g \in H^1_0(\Omega\setminus(\hat{B}+u);\mathbb{R}^2), \\ \pi(u) \in L^2(\Omega\setminus(\hat{B}+u)), \quad \displaystyle\int_{\Omega\setminus\hat{B}} \pi(u)\circ(I+u)\,dx = 0. \end{cases}$$

The drag associated to $\hat{B} + u$ can be defined and is given by

$$D(\hat{B}+u) = T(\hat{B}+u, y(u)) = 2\nu \int_{\Omega\setminus(\hat{B}+u)} |Dy(u)|^2\,dx, \tag{1.49}$$

where $Dy(u) = \frac{1}{2}(\nabla y(u) + \nabla y(u)^t)$.

Under these conditions, it is proved in [**10**] that the equality (1.47) is satisfied, with the first order term $D'(\hat{B};u)$ given by

$$D'(\hat{B};u) = 4\nu \int_{\Omega\setminus\hat{B}} Dy\cdot\left(D\dot{y}(u) - E(u,y) + \frac{1}{2}(\nabla\cdot u)Dy\right)dx.$$

Here, we have introduced the following notation:

(a) $(\dot{y}(u), \dot{\pi}(u))$ is the unique solution to the linear problem

$$\begin{cases} -\nu\Delta\dot{y}(u) + (y\cdot\nabla)\dot{y}(u) + (\dot{y}(u)\cdot\nabla)y + \dot{\pi}(u) = G(u,y,\pi), \ \nabla\cdot\dot{y}(u) = 0 \text{ in } \Omega\setminus\hat{B}, \\ \dot{y}(u) \in H^1_0(\Omega\setminus\hat{B};\mathbb{R}^2), \\ \dot{\pi}(u) \in L^2(\Omega\setminus\hat{B}), \quad \displaystyle\int_{\Omega\setminus\hat{B}} \dot{\pi}(u)\,dx = 0, \end{cases}$$

where

$$G(u,y,\pi) = -\nu\Delta((u\cdot\nabla)y) + \big(((u\cdot\nabla)y)\cdot\nabla\big)y + (y\cdot\nabla)((u\cdot\nabla)y) + \nabla(u\cdot\nabla\pi).$$

(b) $E(u,y)$ is the 2×2 tensor whose (i,j)-th component is given by

$$E_{ij}(u,y) = \frac{1}{2}\sum_k \left(\frac{\partial u_k}{\partial x_i}\frac{\partial y_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\frac{\partial y_i}{\partial x_k}\right).$$

(c) $y = y(0)$ and $\pi = \pi(0)$, i.e. (y,π) is the solution to (1.48) for $u = 0$.

It can also be proved that, if B and Ω are $W^{2,\infty}$ domains and $u \in W^{2,\infty}(\mathbb{R}^2;\mathbb{R}^2)$, then $y \in H^2(\Omega;\mathbb{R}^2)$, $\pi \in H^1(\Omega)$ and

$$D'(\hat{B};u) = \int_{\partial\hat{B}} \left(\frac{\partial w}{\partial n} - \frac{\partial y}{\partial n}\right)\cdot\frac{\partial y}{\partial n}\,(u\cdot n)\,d\sigma, \tag{1.50}$$

with (w, q) being the unique solution to the "adjoint" problem
(1.51)

$$\begin{cases} -\nu\Delta w_i + \sum_j \partial_i y_j \, w_j - \sum_j y_j \partial_j w_i + \partial_i q = -2\nu\, \Delta y_i \ (1 \le i \le 2), \ \nabla \cdot w = 0, \\ w \in H_0^1(\Omega \setminus \hat{B}; \mathbb{R}^2) \cap H^2(\Omega \setminus \hat{B}; \mathbb{R}^2), \\ q \in H^1(\Omega \setminus \hat{B}), \ \displaystyle\int_{\Omega \setminus \hat{B}} q \, dx = 0, \end{cases}$$

Notice that, in order to compute the derivative of the drag in several directions u, it is interesting to use the identity (1.50). Indeed, it suffices to solve (1.8) and (1.51) only once. Then, to determine $D'(\hat{B}; u)$ for a given u, we will only have to compute one integral on $\partial\hat{B}$.

QUESTION 11: *Assume that* $\mathcal{B}_{\rm ad}$ *is the family of the non-empty closed sets* B *satisfying* (1.45) *whose boundaries are uniformly Lipschitz-continuous with Lipschitz constant* L. *How can* (1.50) *be used to produce a sequence* $\{B^n\}$ *"converging" to a solution to Problem* P3*?*

To end this Section, let us state another result from [**10**]:

THEOREM 1.10. *There exists* $\alpha > 0$ *such that, if* $|y_\infty| \le \alpha\nu$, *then* $u \mapsto D(\hat{B}+u)$ *is a* C^∞ *mapping in the set* $\mathcal{W}$.

One can also obtain expressions for the derivatives of higher orders. This must be made with caution; indeed, $D''(\hat{B}; \cdot, \cdot)$ (i.e. the second derivative at 0 of $u \mapsto D(\hat{B} + u)$) does not coincide with $(D'(\hat{B}; \cdot)'; \cdot)$ (i.e. the derivative at 0 of the mapping $u \mapsto D'(\hat{B} + u; \cdot)$), see [**101**].

1.5. Optimal control for a system modelling tumor growth

This Section deals with Problem P4. For simplicity, we will assume that the functions f, h, F and H are given by (1.11), where ρ, m, R and M are positive constants. We will also assume that the initial data in (1.10) satisfy:

$$c_0\,, \beta_0 \in L^\infty(\Omega) \cap H_0^1(\Omega), \quad c_0\,, \beta_0 \ge 0.$$

For each $v \in L^2(\omega \times (0, T))$ with $v \ge 0$, there exists at least one solution (c, β) to (1.10), with

$$c \in L^\infty(Q), \quad c_t\,, \frac{\partial c}{\partial x_i}\,, \frac{\partial^2 c}{\partial x_i \partial x_j} \in L^2(Q)$$

and the same properties for β.

QUESTION 12: *Why is this true? What about uniqueness?*

Then the following results can be proved:

THEOREM 1.11. *Assume that* $\mathcal{V}_{\rm ad}$ *is a non-empty closed convex set of* $L^2(\omega)$ *and all* $v \in \mathcal{V}_{\rm ad}$ *satisfy* $v \ge 0$. *Then Problem* P4 *possesses at least one solution.*

THEOREM 1.12. *Let the assumptions of theorem 1.11 be satisfied and let $\hat{u}$ be a solution to Problem* P4. *Then there exists $(\hat{c}, \hat{\beta})$ and $(\hat{p}, \hat{\eta})$ such that*

$$\left\{\begin{array}{lll} \hat{c}_t - \nabla\cdot(D(x)\nabla\hat{c}) = \rho\hat{c} - R\hat{c}\hat{\beta} & \textit{in} & Q = \Omega\times(0,T), \\ \hat{\beta}_t - \mu\Delta\hat{\beta} = -m\hat{\beta} - M\hat{c}\hat{\beta} + v1_\omega & \textit{in} & Q = \Omega\times(0,T), \\ \hat{c} = 0 & \textit{on} & \Sigma = \partial\Omega\times(0,T), \\ \hat{\beta} = 0 & \textit{on} & \Sigma = \partial\Omega\times(0,T), \\ \hat{c}(x,0) = c_0(x) & \textit{in} & \Omega, \\ \hat{\beta}(x,0) = \beta_0(x) & \textit{in} & \Omega, \end{array}\right. \tag{1.52}$$

$$\left\{\begin{array}{lll} -\hat{p}_t - \nabla\cdot(D(x)\nabla\hat{p}) = \rho\hat{p} - R\hat{\beta}\hat{p} - M\hat{\beta}\hat{\eta} & \textit{in} & Q = \Omega\times(0,T), \\ -\hat{\eta}_t - \mu\Delta\hat{\eta} = -m\hat{\eta} - R\hat{c}\hat{p} - M\hat{c}\hat{\eta} & \textit{in} & Q = \Omega\times(0,T), \\ \hat{p} = 0 & \textit{on} & \Sigma = \partial\Omega\times(0,T), \\ \hat{\eta} = 0 & \textit{on} & \Sigma = \partial\Omega\times(0,T), \\ \hat{p}(x,T) = \hat{c}(x,T) & \textit{in} & \Omega, \\ \hat{\eta}(x,T) = 0 & \textit{in} & \Omega, \end{array}\right. \tag{1.53}$$

$$\iint_{\omega\times(0,T)} (a\hat{p} + b\hat{u})(u - \hat{u})\,dx\,dt \geq 0 \quad \forall u \in \mathcal{V}_{\rm ad}. \tag{1.54}$$

For the proofs, the arguments are not too different from those in Section 1.2.

Again, it is common to say that $(\hat{p}, \hat{\eta})$ is the *adjoint state* associate to the optimal control $\hat{u}$. Also,

$$\langle J'(u), v\rangle = \iint_{\omega\times(0,T)} (ap + bu)\,v\,dx\,dt \quad \forall v \in \mathcal{V}_{\rm ad}\,, \tag{1.55}$$

where (p, η) is the adjoint state associate to u, i.e. the solution to

$$\left\{\begin{array}{lll} -p_t - \nabla\cdot(D(x)\nabla p) = \rho p - R\beta p - M\beta\eta & \text{in} & Q = \Omega\times(0,T), \\ -\eta_t - \mu\Delta\eta = -m\eta - Rcp - Mc\eta & \text{in} & Q = \Omega\times(0,T), \\ p = 0 & \text{on} & \Sigma = \partial\Omega\times(0,T), \\ \eta = 0 & \text{on} & \Sigma = \partial\Omega\times(0,T), \\ p(x,T) = c(x,T) & \text{in} & \Omega, \\ \eta(x,T) = 0 & \text{in} & \Omega. \end{array}\right.$$

Once more, this provides very useful techniques to compute, for any control u, the associate $J'(u)$.

QUESTION 13: *Can the optimality system in theorem 1.12 be used to prove a uniqueness result for Problem* P4*?*

QUESTION 14: *Again, a "natural" iterative method for the computation of $\hat{u}$ is suggested by the optimality system in theorem 1.12. Which is this method? What can be said on the convergence of the iterates?*

QUESTION 15: *How can we apply gradient and conjugate gradient method to produce a sequence of controls that converge to an optimal control in he context of Problem* P4*?*

This optimal control problem has been solved numerically in [**28**]; more results will be given in a forthcoming paper.

CHAPTER 2

Controllability of the linear heat and wave PDEs

This Lecture is devoted to the controllability of some systems governed by linear time-dependent PDEs. I will consider the heat and the wave equations. I will try to explain which is the meaning of controllability and which kind of controllability properties can be expected to be satisfied by each of these PDEs. The main related results, together with the main ideas in their proofs, will be recalled.

2.1. Introduction

Let us first make some very general considerations on the following abstract problem:

$$\begin{cases} y_t - Ay = Bv, & t \in (0,T), \\ y(0) = y^0, \end{cases} \tag{2.1}$$

where A and B are linear operators, $v = v(t)$ is the control and $y = y(t)$ is the state.

For fixed $T > 0$, we choose y^0 and y^1 in the space of states (the space where y "lives") and we try to answer the following question:

Can one find a control v such that the solution y associated to v and y^0 takes the value y^1 at $t = T$?

This is an *exact controllability* problem. The control requirement $y(T) = y^1$ can be relaxed in various ways, leading to other notions of controllability.

Of course, the solvability of problems of this kind depends very much on the nature of the system under consideration; in particular, the following features may play a crucial role: time reversibility, regularity of the state, structure of the set of admissible controls, etc.

The controllability of partial differential equations has been the object of intensive research since more than 30 years. However, the subject is older than that. In 1978, D.L. Russell [**99**] made a rather complete survey of the most relevant results that were available in the literature at that time. In that paper, the author described a number of different tools that were developed to address controllability problems, often inspired and related to other subjects concerning partial differential equations: multipliers, moment problems, nonharmonic Fourier series, etc. More recently, J.-L. Lions introduced the so called *Hilbert Uniqueness Method* (H.U.M.; see [**77, 78**]). That was the starting point of a fruitful period for this subject.

It would be impossible to present here all the important results that have been proved in this area. I will thus only consider some model examples where the most interesting difficulties are found.

Several important related topics, like numerical computation and simulation in controllability problems, stabilizability, connections with finite dimensional controllability theory, etc. have been left out. However, some useful references for these issues have been included; see [**23, 24, 57, 58, 59, 111**].

2.2. Basic results for the linear heat equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain ($N \geq 1$), with boundary Γ of class C^2. Let ω be an open and non-empty subset of Ω. Let $T > 0$ and consider the linear controlled heat equation in the cylinder $Q = \Omega \times (0, T)$:

$$\begin{cases} y_t - \Delta y = v 1_\omega & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{2.2}$$

In (2.2), $\Sigma = \Gamma \times (0, T)$ is the lateral boundary of Q, 1_ω is the characteristic function of the set ω, $y = y(x,t)$ is the state and $v = v(x,t)$ is the control. Since v is multiplied by 1_ω , the action of the control is limitted to $\omega \times (0, T)$.

We assume that $y^0 \in L^2(\Omega)$ and $v \in L^2(\omega \times (0,T))$, so that (2.2) admits a unique solution

$$y \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).$$

We will set $R(T; y^0) = \{\, y(\cdot, T) : v \in L^2(\omega \times (0,T)) \,\}$. Then:

(a) It is said that system (2.2) is *approximately controllable* (at time T) if $R(T; y^0)$ is dense in $L^2(\Omega)$ for all $y^0 \in L^2(\Omega)$.

(b) It is said that (2.2) is *exactly controllable* if $R(T; y^0) = L^2(\Omega)$ for all $y^0 \in L^2(\Omega)$.

(c) Finally, it is said that (2.2) is *null controllable* if $0 \in R(T; y^0)$ for all $y^0 \in L^2(\Omega)$.

It will be seen below that approximate and null controllability hold for every non-empty open set $\omega \subset \Omega$ and every $T > 0$.

On the other hand, it is clear that exact controllability cannot hold, except possibly in the case in which $\omega = \Omega$. Indeed, due to the regularizing effect of the heat equation, the solutions of (2.2) at time $t = T$ are smooth in $\Omega \setminus \overline{\omega}$. Therefore, if $\omega \neq \Omega$, $R(T; y^0)$ is strictly contained in $L^2(\Omega)$ for all $y^0 \in L^2(\Omega)$.

Our first remark is that null controllability implies that the whole range of the semigroup generated by the heat equation is reachable too. Let us make this statement more precise.

Let us denote by $S(t)$ the semigroup generated by the heat equation (2.2) without control, i.e. with $v = 0$. Then, if null controllability holds, it follows that for any $y^0 \in L^2(\Omega)$ and any $y^1 \in S(T)(L^2(\Omega))$ there exists $v \in L^2(\omega \times (0,T))$ such that the solution of (2.2) satisfies $y(x,T) \equiv y^1(x)$. In other words,

$$S(T)(L^2(\Omega)) \subset R(T; y^0) \quad \forall y^0 \in L^2(\Omega).$$

QUESTION 1: *Why is this true?*

The space $S(T)(L^2(\Omega))$ is dense in $L^2(\Omega)$. Therefore, null controllability implies approximate controllability. Observe however that the reachable states we obtain by this argument are smooth, due to the regularizing effect of the heat equation.

Notice that proving that null controllability implies approximate controllability requires the use of the density of $S(T)(L^2(\Omega))$ in $L^2(\Omega)$. In the case of the linear heat equation this is easy to check developing solutions in Fourier series. However, if the equation contains time or space-time dependent coefficients, this is true but not so immediate. In those cases, the density of the range of the "semigroup", can be reduced by duality to a backward uniqueness property, in the spirit of J.-L. Lions and B. Malgrange [**81**].

Our first main result is the following:

THEOREM 2.1. *System* (2.2) *is approximately controllable for any non-empty open set* $\omega \subset \Omega$ *and any* $T > 0$.

PROOF: This is an easy consequence of Hahn-Banach theorem. For completeness, we will reproduce the argument here.

Let us fix ω and $T > 0$. Then, it is clear that (2.2) is approximately controllable if and only if $R(T;0)$ is dense in $L^2(\Omega)$. But this is true if and only if any φ^0 in the orthogonal complement $R(T;0)^\perp$ is necessarily zero.

Let $\varphi^0 \in L^2(\Omega)$ be given and assume that it belongs to $R(T;0)^\perp$. Let us introduce the following backwards in time system:

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 & \text{on} \quad \Sigma, \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{2.3}$$

Then, if $v \in L^2(\omega \times (0,T))$ is given and y is the solution to (2.2) with $y^0 = 0$, we have

$$\iint_{\omega\times(0,T)} \varphi v \, dx \, dt = \int_\Omega \varphi^0(x) y(x,T)\, dx = 0.$$

Consequently, approximate controllability holds if and only if the following uniqueness property is true:

If φ *solves* (2.3) *and* $\varphi = 0$ *in* $\omega \times (0,T)$*, then necessarily* $\varphi \equiv 0$*, i.e.* $\varphi^0 = 0$.

But this is a well known uniqueness property for the heat equation, a consequence of the fact that the solutions to (2.3) are analytic in space.

This proves that approximate controllability holds for (2.2). □

Following the variational approach in [**80**], we can also determine the way the "good" control can be constructed. First of all, observe that it is sufficient to consider the particular case $y^0 = 0$. Then, let us fix $y^1 \in L^2(\Omega)$ and $\varepsilon > 0$ and let us introduce the following functional on $L^2(\Omega)$:

$$J_\varepsilon(\varphi^0) = \frac{1}{2} \iint_{\omega\times(0,T)} |\varphi|^2 \, dx\, dt + \varepsilon \|\varphi^0\|_{L^2} - \int_\Omega \varphi^0 y^1 \, dx, \tag{2.4}$$

where for each φ^0 we have denoted by φ the solution to the corresponding problem (2.3).

The functional J_ε is continuous and strictly convex in $L^2(\Omega)$. On the other hand, in view of the unique continuation property above, it can be proved that

$$\liminf_{\|\varphi^0\|_{L^2}\to\infty} \frac{J_\varepsilon(\varphi^0)}{\|\varphi^0\|_{L^2}} \geq \varepsilon. \tag{2.5}$$

Hence, J_ε admits a unique minimizer $\hat\varphi^0$ in $L^2(\Omega)$. The control $u = \hat\varphi|_{\omega\times(0,T)}$, where $\hat\varphi$ solves (2.3) with $\hat\varphi^0$ as final data is such that the solution of (2.2) (with $y^0 = 0$) satisfies

$$\|y(\cdot,T) - y^1\|_{L^2} \leq \varepsilon. \tag{2.6}$$

QUESTION 2: *Why is* (2.5) *true? How can we prove* (2.6) *for this control?*

With a slight change in the definition of J_ε, we are also able to build *bang-bang* controls. Indeed, it suffices to consider the new functional

$$\tilde J_\varepsilon(\varphi^0) = \frac{1}{2}\left(\iint_{\omega\times(0,T)} |\varphi|\,dx\,dt\right)^2 + \varepsilon\|\varphi^0\|_{L^2} - \int_\Omega \varphi^0 y^1\,dx. \tag{2.7}$$

Then $\tilde J_\varepsilon$ is continuous and convex in $L^2(\Omega)$ and satisfies the coercivity property (2.5) too.

Let $\hat\varphi^0$ be a minimizer of $\tilde J_\varepsilon$ in $L^2(\Omega)$ and let $\hat\varphi$ be the corresponding solution of (2.3). Let us set

$$u = \left(\iint_{\omega\times(0,T)} |\hat\varphi|\,dx\,dt\right) \operatorname{sgn}(\hat\varphi)|_{\omega\times(0,T)}, \tag{2.8}$$

where sgn is the multivalued sign function: $\operatorname{sgn}(s) = 1$ if $s > 0$, $\operatorname{sgn}(0) = [-1,1]$ and $\operatorname{sgn}(s) = -1$ when $s < 0$. Again, the control u given by (2.8) is such that the solution to (2.2) with zero initial data satisfies (2.6).

Due to the regularizing effect of the heat equation, the zero set of nontrivial solutions of (2.3) is of zero $(n+1)-$dimensional Lebesgue measure. Thus, the control u in (2.8) belongs to $L^\infty(Q)$ and is of *bang-bang* form, i.e. $u = \pm\lambda$ a.e. in $\omega\times(0,T)$, where

$$\lambda = \iint_{\omega\times(0,T)} |\hat\varphi|\,dx\,dt.$$

In fact, it can be proved that u minimizes the L^∞-norm in the set of all controls such that (2.6) is satisfied (we refer to [**31**] for a proof of this assertion).

Following [**110**], we can improve the previous argument and show that, for any ω, any $T > 0$ and any finite-dimensional subspace $E \subset L^2(\Omega)$, (2.2) is E-approximate controllable. This means that, for arbitrary $y^0, y^1 \in L^2(\Omega)$ and any $\varepsilon > 0$, there exists a control $v \in L^2(\omega\times(0,T))$ such that the corresponding solution to (2.2) satisfies:

$$\|y(\cdot,T) - y^1\|_{L^2} \leq \varepsilon, \quad \pi_E(y(\cdot,T)) = \pi_E(y^1). \tag{2.9}$$

Here, $\pi_E : L^2(\Omega) \mapsto E$ stands for the usual orthogonal projector on E.

Indeed, it suffices to modify J_ε (or $\tilde{J}_\varepsilon$) and use instead the functional J_ε^E, where

$$J_\varepsilon^E(\varphi^0) = \frac{1}{2}\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt + \varepsilon\|(I-\pi_E)\varphi^0\|_{L^2} - \int_\Omega \varphi^0 y^1\,dx. \tag{2.10}$$

As before, J_ε^E is continuous, strictly convex and coercive in $L^2(\Omega)$. Once again, let us denote by $\hat{\varphi}^0$ its unique minimizer and let us set $u = \hat{\varphi}|_{\omega\times(0,T)}$. Then the associate state satisfies (2.9).

QUESTION 3: *Which is in this case the argument leading to* (2.9)*? Is the hypothesis "E is finite-dimensional" essential?*

Let us now analyze the null controllability of (2.2).

The null controllability property for system (2.2), together with a L^2- estimate of the control, is equivalent to the following observability inequality for the adjoint system (2.3):

$$\|\varphi(\cdot,0)\|_{L^2}^2 \le C\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt \quad \forall\varphi^0\in L^2(\Omega). \tag{2.11}$$

QUESTION 4: *Which is the proof of this assertion?*

Due to the regularizing effect of the heat equation, the norm in the left hand side of (2.11) is very weak. However, the irreversibility of the system makes (2.11) difficult to prove. For instance, multiplier methods do not apply in this context.

Thus, we see that the approximate (resp. null) controllability of (2.2) is related to the unique continuation property (resp. the observability) of (2.3).

Historically, it seems that the first null controllability results established for the heat equation involved boundary controls. They were given in [**99**] in the one-dimensional case, using moment problems and classical results on the linear independence in $L^2(0,T)$ of families of real exponentials. Later, in [**100**], a deep general result was proved. Roughly speaking, the following was shown:

> *If the wave equation is controllable for some $T>0$ with controls supported in ω, then the heat equation* (2.2) *is null controllable for every $T>0$ with controls supported in ω.*

In view of the controllability results in Section 2.3, according to this principle, it follows that the heat equation (2.2) is null controllable for all $T>0$ provided ω satisfies a specific geometric control condition. However, this geometric condition does not seem to be natural in the context of the heat equation and, therefore, this result is not completely satisfactory.

More recently, the following was shown by G. Lebeau and L. Robbiano [**70**]:

THEOREM 2.2. *System* (2.2) *is null controllable for any non-empty open set $\omega\subset\Omega$ and any $T>0$.*

SKETCH OF THE PROOF: A slightly simplified proof of this result was given in [**74**]. The main ingredient is an observability estimate for the eigenfunctions of the

Dirichlet-Laplace operator:

$$\begin{cases} -\Delta w_j = \lambda_j w_j & \text{in} \quad \Omega, \\ w_j = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.12}$$

Recall that the eigenvalues $\{\lambda_j\}$ form a nondecreasing sequence of positive numbers such that $\lambda_j \to \infty$ as $j \to \infty$ and the associated eigenfunctions $\{w_j\}$ form an orthonormal basis in $L^2(\Omega)$.

The following holds:

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. For any open set $\omega \subset \Omega$, there exist positive constants $C_1, C_2 > 0$ such that

$$\sum_{\lambda_j \le \mu} |a_j|^2 \le C_1 e^{C_2\sqrt{\mu}} \int_\omega \Big| \sum_{\lambda_j \le \mu} a_j w_j(x) \Big|^2 dx \tag{2.13}$$

whenever $\{a_j\} \in \ell^2$ and $\mu > 0$.

This result was implicitly used in [**70**] and is proved in [**74**]. A consequence is that the observability inequality (2.11) holds for the solutions to (2.3) with initial data in

$$E_\mu = \text{span}\{\, \varphi_j : \lambda_j \le \mu \,\},$$

the constant being of the order of $\exp\big(C\sqrt{\mu}\big)$.

This shows that the projection on E_μ of the solution of (2.3) can be controlled to zero with a control of size $\exp\big(C\sqrt{\mu}\big)$. Thus, when controlling the frequencies $\lambda_j \le \mu$, one increases the L^2-norm of the high frequencies $\lambda_j > \mu$ by a multiplicative factor of the order of $\exp\big(C\sqrt{\mu}\big)$.

However, it was observed in [**70**] that any solution of the heat equation (2.2) with $v = 0$ such that the projection on E_μ of $y(\cdot, 0)$ vanishes decays in $L^2(\Omega)$ at a rate of the order of $\exp(-\mu t)$.

Consequently, if we divide the time interval $[0, T]$ in two parts $[0, T/2]$ and $[T/2, T]$, we control to zero the frequencies $\lambda_j \le \mu$ in the interval $[0, T/2]$ and then allow the equation to evolve without control in the interval $[T/2, T]$, it follows that, at time $t = T$, the projection of the solution y over E_μ vanishes and the norm of the high frequencies does not exceed the norm of the initial data.

This argument allows to control to zero the projection over E_μ for any $\mu > 0$, but not the whole solution. To do that, an iterative argument is needed. Thus, we decompose the interval $[0, T)$ in disjoint subintervals of the form $[T_j, T_{j+1})$ for $j \in \mathbb{N}$, with a suitable choice of the sequence $\{T_j\}$. In each interval $[T_j, T_{j+1}]$, we control to zero the frequencies $\lambda_k \le 2j$. By letting $j \to \infty$, we obtain a control $v \in L^2(\omega \times (0, T))$ such that the solution of (2.2) satisfies

$$y(x, T) \equiv 0. \tag{2.14}$$

□

Once it is known that (2.2) is null controllable, one can obtain the control with minimal L^2-norm satisfying (2.14). It suffices to minimize the functional

$$J(\varphi^0) = \frac{1}{2} \iint_{\omega\times(0,T)} |\varphi|^2 \, dx \, dt + \int_\Omega \varphi(x, 0) y^0(x) \, dx$$

over the Hilbert space

$$H = \{ \varphi^0 : \text{ the solution } \varphi \text{ of (2.3) satisfies } \iint_{\omega\times(0,T)} |\varphi|^2 \, dx\, dt < \infty \}.$$

As a consequence of this theorem, we also have the null boundary controllability of the heat equation, with controls in an arbitrarily small open subset of the boundary. See [**70**] for more details.

QUESTION 5: *Why does theorem 2.2 imply null boundary controllability?*

The previous controllability results also hold for linear parabolic equations with lower order terms depending on time and space.

For instance, the following system can be considered:

$$\begin{cases} y_t - \Delta y + a(x,t)y = v1_\omega & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{2.15}$$

Here, we assume that $a \in L^\infty(Q)$. In this case, the adjoint system is

$$\begin{cases} -\varphi_t - \Delta \varphi + a(x,t)\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 & \text{on} \quad \Sigma, \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{2.16}$$

Again, the null controllability of (2.15), together with a L^2- estimate of the control, is equivalent to an observability inequality. Hence, in order to obtain a null controllability result for (2.15), what we have to do is to prove the estimate (2.11) for the solutions to (2.16).

The controllability properties of systems of this kind have been analyzed by several authors. Among them, let us mention the work of A.V. Fursikov and O.Yu. Imanuvilov (for instance, see [**19, 45, 48, 46, 47, 64**]; more complicate linear heat equations involving first-order terms of the form $B(x,t)\cdot\nabla y$ have recently been considered in [**66**]). Their approach to the controllability problem is different and more general than the previous one and relies on appropriate (global) *Carleman inequalities.*

A general global Carleman inequality is an estimate of the form

$$\iint_{\Omega\times(0,T)} \rho^{-2} |\varphi|^2 \, dx\, dt \le C \iint_{\omega\times(0,T)} \rho^{-2} |\varphi|^2 \, dx\, dt, \tag{2.17}$$

where $\rho = \rho(x,t)$ is continuous, strictly positive and bounded from below. For an appropriate ρ that depends on Ω, ω, T and $\|a\|_{L^\infty(Q)}$, it is possible to deduce (2.17) and, consequently, also estimates of the form

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2 \, dx\, dt \le C \iint_{\omega\times(0,T)} |\varphi|^2 \, dx\, dt. \tag{2.18}$$

This, together with the properties of the solutions of (2.16), leads to (2.11) and, therefore, implies the null controllability property for (2.15); see also [**66, 43, 26**] for some improved estimates.

QUESTION 6: *How can* (2.11) *be proved from* (2.18)*?*

Thus, at present we can affirm that, as in the case of the classical heat equation, (2.15) is both approximately and null controllable for any ω and any $T > 0$. Once more, null controllability implies approximate controllability for (2.15); this has been shown in [**43**].

An interesting question analyzed in [**43**] deals with explicit estimates of the *cost* in $L^2(Q)$ of the approximate, E-approximate (E is a finite-dimensional space) and null controllability of (2.15).

For instance, let us recall the results concerning the costs of approximate and null controllability. In the remainder of this Section, it will be assumed that C is a generic positive constant that only depends on Ω and ω.

Let us consider the linear state equation (2.15), where $a \in L^\infty(Q)$. For each $y^0 \in L^2(\Omega)$, $y^1 \in L^2(\Omega)$ and $\varepsilon > 0$, let us introduce the corresponding *set of admissible controls*

$$\mathcal{U}_{\rm ad}(y^0, y^1; \varepsilon) := \{\, v \in L^2(Q) : \text{ the solution of (2.15) satisfies (2.6)} \,\} \tag{2.19}$$

and the following quantity, which measures the *cost of approximate controllability* or, more precisely, the cost of achieving (2.6):

$$\mathcal{C}(y^0, y^1; \varepsilon) := \inf_{v \in \mathcal{U}_{ad}(y^0, y^1; \varepsilon)} \|v\|_{L^2(Q)} \,. \tag{2.20}$$

Then, the question is: can we obtain “explicit” upper bounds for $\mathcal{C}(y^0, y^1; \varepsilon)$?

Taking into account that system (2.15) is linear, one can assume, without loss of generality, that $y^0 = 0$. Indeed,

$$\mathcal{C}(y^0, y^1; \varepsilon) = \mathcal{C}(0, z^1; \varepsilon) \,, \tag{2.21}$$

where $z^1 = y^1 - z(\cdot, T)$ and z is the solution of (2.15) with $v \equiv 0$.

Let us denote by $\|\cdot\|_\infty$ the usual norm in $L^\infty(Q)$. Then the following is satisfied:

THEOREM 2.3. *For any $y^1 \in H^2(\Omega) \cap H^1_0(\Omega)$, $\varepsilon > 0$, $T > 0$ and $a \in L^\infty(Q)$, one has:*

$$\mathcal{C}(0, y^1; \varepsilon) \le \exp\left[C\left[1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3} + \frac{\|a\|_\infty \|y^1\|_{L^2} + \|\Delta y^1\|_{L^2}}{\varepsilon}\right]\right] \|y^1\|_{L^2} \,. \tag{2.22}$$

Notice that (2.22) is only of interest when

$$\frac{\|\Delta y^1\|_{L^2}}{\lambda_1} > \varepsilon \,,$$

with λ_1 being the first eigenvalue of the Dirichlet Laplacian $-\Delta$. Otherwise, we would have $\|y^1\|_{L^2} \le \varepsilon$ and then, taking $v \equiv 0$ in (2.15) for $y^0 = 0$, we would trivially obtain $y \equiv 0$ and

$$\|y(\cdot, T) - y^1\|_{L^2} \le \varepsilon \,.$$

In other words,

$$\mathcal{C}(0, y^1; \varepsilon) = 0 \ \text{ if } \ \frac{\|\Delta y^1\|_{L^2}}{\lambda_1} \le \varepsilon \,.$$

Furthermore, if instead of assuming $y^1 \in D(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$ we assume that $y^1 \in D((-\Delta)^{\gamma/2})$ with $0 < \gamma \leq 2$, other estimates similar to (2.22) can be established. See [**43**] for the details.

For the proof of (2.22), we first have to obtain sharp bounds on the cost of controlling to zero. Recall that (2.16) is the adjoint system of (2.15). Then we have the following explicit observability estimate:

LEMMA 2.4. *For any solution of* (2.16) *and for any* $a \in L^\infty(Q)$*, one has*

$$\|\varphi(\cdot,0)\|_{L^2}^2 \leq \exp\left(C\left(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3}\right)\right)\iint_{\omega\times(0,T)}|\varphi|^2\,dx\,dt\,. \tag{2.23}$$

The proof of (2.23) relies on global Carleman inequalities as in [**47**], but paying special attention to the constants arising in the integrations by parts. Once (2.23) is known, (2.22) can be proved easily.

QUESTION 7: *How can* (2.22) *be proved from* (2.23)*?*

As we have already seen, (2.23) implies the null controllability of (2.15). But it also provides an estimate for the associated cost $\mathcal{C}(y^0,0)$. More precisely, one has:

THEOREM 2.5. *For each* $y^0 \in L^2(\Omega)$*, the set* $\mathcal{U}_{\rm ad}(y^0,0)$ *is non-empty. Moreover, the associated cost* $\mathcal{C}(y^0,0)$ *satisifes:*

$$\mathcal{C}(y^0,0) \leq \exp\left(C\left(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3}\right)\right)\|y^0\|_{L^2}\,. \tag{2.24}$$

QUESTION 8: *How can* (2.24) *be proved from* (2.23)*?*

In the particular case in which $a \equiv$ Const, (2.22) can be improved. More precisely, we can obtain a bound of the cost of approximate controllability of the order of $\exp(1/\sqrt{\varepsilon})$. Furthermore, it can be proved that this estimate is optimal in an appropriate sense; see [**43**] for the details.

REMARK 2.6. We can be more explicit on the way the constants C in (2.22) and (2.24) depend on Ω and ω: there exist "universal" constants $C_0 > 0$ and $m \geq 1$ such that C can be taken of the form

$$C = \exp\left(C_0\|\psi\|_{C^2}^m\right),$$

where $\psi \in C^2(\overline{\Omega})$ is any function satisfying $\psi > 0$ in Ω, $\psi = 0$ on $\partial\Omega$ and $\nabla\psi \neq 0$ in $\overline{\Omega}\setminus\omega$. All this is a consequence of the particular form that must have ρ in order to ensure (2.17). □

The results of this Section can be extended to more general equations of the form

$$\begin{cases} y_t - \Delta y + \nabla\cdot(yB(x,t)) + a(x,t)y = v1_\omega & \text{in } Q,\\ y = 0 & \text{on } \Sigma,\\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{2.25}$$

where $a \in L^\infty(Q)$ and $B \in L^\infty(Q;\mathbb{R}^N)$.

To do that, it is sufficient to obtain suitable observability estimates for the solutions of adjoint systems of the form

$$
\left\{\begin{array}{lll} -\varphi_t - \Delta\varphi - B(x,t)\cdot\nabla\varphi + a(x,t)\varphi = 0 & \text{in} & Q, \\ \varphi = 0 & \text{on} & \Sigma, \\ \varphi(x,T) = \varphi^0(x) & \text{in} & \Omega. \end{array}\right. \tag{2.26}
$$

More precisely, we can deduce that

$$
\|\varphi(\cdot,0)\|_{L^2}^2 \le \exp\left(C\left(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3}+T^2\|B\|_\infty^2\right)\right)\iint_{\omega\times(0,T)}|\varphi|^2\,dx\,dt \tag{2.27}
$$

for any solution of (2.26) and for all $a\in L^\infty(Q)$, $B\in L^\infty(Q;\mathbb{R}^N)$. Then, arguments similar to those above lead to an estimate of the cost of approximate controllability.

The situation is more complicate when the state equation is of the form

$$
\left\{\begin{array}{lll} y_t - \Delta y + B(x,t)\cdot\nabla y + a(x,t)y = 0 & \text{in} & Q \\ y = 0 & \text{on} & \Sigma \\ y(x,0) = y^0(x) & \text{in} & \Omega\,. \end{array}\right. \tag{2.28}
$$

Indeed, if B is only assumed to be in $L^\infty(Q;\mathbb{R}^N)$, the adjoint systems take the form

$$
\left\{\begin{array}{lll} -\varphi_t - \Delta\varphi - \nabla\cdot(\varphi B(x,t)) + a(x,t)\varphi = 0 & \text{in} & Q \\ \varphi = 0 & \text{on} & \Sigma, \\ \varphi(x,T) = \varphi^0(x) & \text{in} & \Omega \end{array}\right. \tag{2.29}
$$

and, therefore, the usual Carleman inequalities do not suffice. These questions have been considered and solved in [**26**], using some ideas from [**66**]. We omit the details.

To end this Section, let us make some comments on the convergence rate of algorithms devised to construct "good" controls.

It is rather natural to build approximate controls by *penalizing* a suitable optimal control problem. This has been done systematically, for instance, in the works by R. Glowinski [**55**] and R. Glowinski et al. [**56**]. This method has also been used to prove the approximate controllability for some linear and semilinear heat equations in [**79**] and [**33**], respectively.

Let us briefly describe the procedure in the case of the linear heat equation. First of all, without loss of generality, we set $y^0 = 0$. Given $y^1\in L^2(\Omega)$, we introduce the functional F_k , with

$$
F_k(v) = \frac{1}{2}\iint_{\omega\times(0,T)}|v|^2\,dx\,dt + \frac{k}{2}\|y(\cdot,T)-y^1\|_{L^2}^2 \quad \forall v\in L^2(\omega\times(0,T)), \tag{2.30}
$$

where y is the solution of (2.2) with $y^0 = 0$.

It was proved in [**79**] that F_k has a unique minimizer $v_k\in L^2(\omega\times(0,T))$ for all $k>0$ and that the associated states y_k satisfy

$$
y_k(\cdot,T)\to y^1 \text{ in } L^2(\Omega) \text{ as } k\to\infty. \tag{2.31}
$$

In view of (2.31), in order to compute a control satisfying (2.6), it suffices to take $v = v_k$ for a sufficiently large $k = k(\varepsilon)$.

Using the results above, it is easy to get explicit estimates of the rate of convergence in (2.31) (we refer to [**43**] for the details of the proof):

THEOREM 2.7. *Under the previous conditions, there exists $C > 0$ such that*

$$\|y_k(\cdot, T) - y^1\| \leq \frac{C}{\log k} \tag{2.32}$$

and

$$\|v_k\|_{L^2(Q)} \leq \frac{C\sqrt{k}}{\log k} \tag{2.33}$$

as $k \to \infty$.

QUESTION 9: *How can* (2.32) *and* (2.33) *be proved?*

Notice that (2.32) provides logarithmic (and therefore very slow) convergence rates. This fact agrees with the extremely high cost (exponentially depending on $1/\varepsilon$) of approximate controllability.

The methods of this Section can also be applied to obtain estimates on the cost of controllability when the control acts on a non-empty open subset of $\partial\Omega$.

2.3. Basic results for the linear wave equation

Let us now consider the linear controlled wave equation

$$\begin{cases} y_{tt} - \Delta y = v 1_\omega & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x) & \text{in} \quad \Omega. \end{cases} \tag{2.34}$$

In (2.34), we have used the same notation as in Section 2.2. Again, $y = y(x,t)$ is the state and $v = v(x,t)$ is the control. For any $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and any $v \in L^2(\omega \times (0,T))$, (2.34) possesses exactly one solution $y \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$.

Roughly speaking, the *controllability* problem for (2.34) consists on *describing the set of reachable final states*

$$R(T; y^0, y^1) := \{\, (y(\cdot, T), y_t(\cdot, T)) : v \in L^2(\omega \times (0,T)) \,\}.$$

As in the case of the heat equation, we may distinguish several degrees of controllability:

(a) It is said that (2.34) is *approximately controllable* at time T if $R(T; y^0, y^1)$ is dense in $H_0^1(\Omega) \times L^2(\Omega)$ for every $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

(b) It is said that (2.34) is *exactly controllable* at time T if $R(T; y^0, y^1) = H_0^1(\Omega) \times L^2(\Omega)$ for every $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

(c) Finally, it is said that (2.34) is *null controllable* at time T if $(0,0) \in R(T; (y^0, y^1))$ for every $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

The previous controllability properties can also be formulated in other function spaces in which the wave equation is well posed.

Since we are now dealing with solutions to the wave equation, for any of these properties to hold, the control time T has to be sufficiently large, due to the finite speed of propagation. On the other hand, since (2.34) is linear and reversible in time, null and exact controllability are equivalent notions. As we have seen, the situation is completely different in the case of the heat equation.

QUESTION 10: *Why do we need large T for any kind of controllability of the wave equation? Why are null controllability and exact controllability equivalent properties?*

Clearly, every exactly controllable system is approximately controllable too. However, (2.34) may be approximately but not exactly controllable.

Let us now briefly discuss the *approximate controllability problem* for the wave equation.

Again, it is easy to see that approximate controllability is equivalent to a specific *unique continuation property.* More precisely, let us introduce the *adjoint system*

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 & \text{on} \quad \Sigma, \\ \varphi(x,T) = \varphi^0(x), \quad \varphi_t(x,T) = \varphi^1(x) & \text{in} \quad \Omega. \end{cases} \tag{2.35}$$

Then, (2.34) is approximately controllable with controls that depend continously on the data if and only if the following unique continuation property is fulfilled:

> *If φ solves* (2.35) *and $\varphi = 0$ in $\omega\times(0,T)$, then necessarily $\varphi \equiv 0$, i.e.* $(\varphi^0, \varphi^1) = (0,0)$.

In fact, that the previous uniqueness property implies approximate controllability can be checked at least in two ways:

(a) Applying the Hahn-Banach theorem; see [**78**].
(b) Using the variational approach developed in [**80**].

Both approaches have been considered in the context of the heat equation. They will not be revisited here, for reasons of space.

QUESTION 11: *Which are the detailed arguments?*

In view of a well known consequence of *Holmgren's uniqueness theorem*, it can be easily seen that, for any non-empty open set $\omega \subset \Omega$, the previous unique continuation property holds if T is large enough (depending on Ω and ω). We refer to Chapter 1 in [**78**] and [**18**] for a discussion on this problem.

Therefore, the following result holds:

Theorem 2.8. *Let $\omega \subset \Omega$ be a non-empty open set. There exists $T_1 > 0$, only depending on Ω and ω, such that, for any $T > T_1$, the linear system* (2.34) *is approximately controllable at time T.*

When approximate controllability holds, the following (apparently stronger) property is also satisfied:

> *Let E be a finite dimensional subspace of $H_0^1(\Omega) \times L^2(\Omega)$ and let us denote by $\pi_E : H_0^1(\Omega) \times L^2(\Omega) \mapsto E$ the corresponding orthogonal projector. Then, for any $(y^0, y^1), (z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and any $\varepsilon > 0$, there exists $v \in L^2(\omega \times (0,T))$ such that*

the solution of (2.34) *satisfies*

$$\|(y(\cdot,T)-z^0, y_t(\cdot,T)-z^1)\|_{H_0^1\times L^2} \le \varepsilon,\ \pi_E(y(\cdot,T),y_t(\cdot,T)) = \pi_E(z^0,z^1). \tag{2.36}$$

In other words, if $T>0$ is large enough to ensure approximate controllability, for any finite dimensional subspace $E\subset H_0^1(\Omega)\times L^2(\Omega)$ we also have E-approximate controllability.

QUESTION 12: *Why does approximate controllability imply E-approximate controllability for any finite-dimensional space $E\subset H_0^1(\Omega)\times L^2(\Omega)$?*

The previous results hold for wave equations with analytic coefficients too. However, the problem is not completely solved in the frame of the wave equation with lower order potentials $a\in L^\infty(Q)$ of the form

$$y_{tt}-\Delta y + a(x,t)y = v1_\omega \text{ in } Q.$$

We refer to [**3, 105, 98**] for some deep results in this direction.

Let us now consider the *exact controllability problem.*

It was shown by J.-L. Lions in [**78**] using the so called H.U.M. that exact controllability holds (with controls $v\in L^2(\omega\times(0,T))$) if and only if

$$\|(\varphi(\cdot,0),\varphi_t(\cdot,0))\|^2_{L^2\times H^{-1}} \le C\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt \tag{2.37}$$

for any solution φ to the adjoint system (2.35).

This is an observability inequality, playing in this context the role played by (2.11) in Section 2.2. It provides an estimate of the *total energy* of the solution (2.35) by means of a measurement in the control region $\omega\times(0,T)$.

Notice that the energy

$$E(t)=\|(\varphi(\cdot,t),\varphi_t(\cdot,t))\|^2_{L^2\times H^{-1}}$$

of any solution to (2.35) is conserved. Thus, (2.37) is equivalent to the so called *inverse inequality*

$$\|(\varphi^0,\varphi^1)\|^2_{L^2\times H^{-1}} \le C\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt. \tag{2.38}$$

QUESTION 13: *Why is* (2.37) *equivalent to the exact controlability of* (2.34)*?*

When (2.37) holds, one can minimize the functional W, with

$$W(\varphi^0,\varphi^1)=\frac{1}{2}\iint_{\omega\times(0,T)}|\varphi|^2\,dx\,dt + \langle(\varphi(\cdot,0),\varphi_t(\cdot,0)),(y^1,-y^0)\rangle, \tag{2.39}$$

in the space $L^2(\Omega)\times H^{-1}(\Omega)$. Indeed, the following result is easy to prove:

LEMMA 2.9. *Assume that* (2.37) *holds and* $(y^0,y^1)\in H_0^1(\Omega)\times L^2(\Omega)$ *is given. Then W possesses a unique minimizer* $(\hat\varphi^0,\hat\varphi^1)$ *in* $L^2(\Omega)\times H^{-1}(\Omega)$*. The control* $v=\hat\varphi 1_\omega$*, where $\hat\varphi$ is the solution to* (2.35) *corresponding to the final data* $(\hat\varphi^0,\hat\varphi^1)$*, is such that the associated state satisfies*

$$y(x,T)\equiv y_t(x,T)\equiv 0. \tag{2.40}$$

QUESTION 14: *How can we prove lemma 2.9?*

As a consequence, the exact controllability problem is reduced to the analysis of the inequality (2.38). Let us now indicate what is known about this inequality:

• Using multipliers techniques in the spirit of C. Morawetz, L.F. Ho proved in [**63**] that, for any subset of Γ of the form

$$\Gamma(x^0) = \{\, x \in \Gamma : (x - x^0) \cdot n(x) > 0 \,\}$$

with $x^0 \in \mathbb{R}^N$ ($n(x)$ is the outward unit normal to Ω at $x \in \Gamma$) and any sufficiently large T, the following boundary observability inequality holds:

$$\|(\varphi(\cdot,0), \varphi_t(\cdot,0))\|^2_{H_0^1 \times L^2} \leq C \iint_{\Gamma(x^0)\times(0,T)} \left|\frac{\partial \varphi}{\partial n}\right|^2 d\Gamma\, dt \tag{2.41}$$

for every couple $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

This is the observability inequality that is required to solve a boundary controllability problem similar to the one we are considering here.

Later, (2.41) was proved in [**77, 78**] for any

$$T > T(x^0) = 2\|x - x^0\|_{L^\infty}\,. \tag{2.42}$$

In fact, this is the optimal observability time that one may obtain by means of multipliers.

Proceeding as in Vol. 1 of [**78**], one can easily prove that (2.41) implies (2.37) when ω is a neighborhood of $\Gamma(x^0)$ in Ω and $T > T(x^0)$. Consequently, the following result holds:

THEOREM 2.10. *Assume that $x^0 \in \mathbb{R}^N$, ω is a neighborhood of $\Gamma(x^0)$ in Ω and* (2.42) *is satisfied. Then* (2.34) *is exactly controllable at time T.*

More recently, A. Osses has introduced in [**89**] a new multiplier which is essentially a rotation of the one in [**78**]. In this way, he proved that the class of subsets of the boundary for which observability holds is considerably larger.

• C. Bardos, G. Lebeau and J. Rauch [**9**] proved that, in the class of C^∞ domains, the observability inequality (2.37) holds if and only if the couple (ω, T) satisfies the following *geometric control condition* in Ω:

> *Every ray of geometric optics that begins to propagate in Ω at time $t = 0$ and is reflected on its boundary Γ enters ω at a time $t < T$.*

This result was proved with *microlocal analysis techniques*. Recently, the microlocal approach has been greatly simplified by N. Burq [**15**] by using the microlocal defect measures introduced by P. Gerard [**50**]. In [**15**], the geometric control condition was shown to be sufficient for exact controllability for domains Ω of class C^3 and equations with C^2 coefficients.

Therefore, one has:

THEOREM 2.11. *Let Ω be of class C^3, let $\omega \subset \Omega$ be a non-empty open set and let us assume that the couple (ω, T) satisfies the previous geometric condition. Then* (2.34) *is exactly controllable at time T.*

• Let us finally indicate that other methods have also been developed to address controllability problems for wave equations: Moment problems, the use of fundamental solutions, controllability via stabilization, Carleman estimates, etc. We will not present them here; for more details, we refer to the survey paper by D.L. Russell [**99**] and also to the works of J.-P. Puel [**97**] and X. Zhang [**107**].

As in the case of the heat equation, it is also natural to study the cost of the approximate controllability of the wave equation or, in other words, the minimal size of a control needed to reach the ε-neighborhood of a final state. The same can be said in the context of null controllability. These questions were considered by G. Lebeau in [**69**], with techniques which are not the same we used in Section 2.2.

CHAPTER 3

Controllability results for other time-dependent PDEs

This Lecture is devoted to present some controllability results for several time-dependent, mainly nonlinear, parabolic systems of PDEs. First, we will revisit the heat equation and some extensions. Then, some controllability results will be presented for systems governed by stochastic PDEs. Finally, we will consider several nonlinear systems from fluid mechanics: Burgers, Navier-Stokes, Boussinesq, micropolar, etc. Along this Lecture, several open questions will be stated.

3.1. Introduction. Recalling general ideas

Let us first recall some general ideas, many of them already mentioned in the previous Lecture.

Suppose that we are considering an abstract *state equation* of the form

$$\left\{ \begin{array}{l} y_t - A(y) = Bv, \quad t \in (0,T), \\ y(0) = y^0, \end{array} \right. \tag{3.1}$$

which governs the behavior of a physical system. It is assumed that

- $y : [0,T] \mapsto H$ is the *state*, i. e. the variable that serves to identify the physical properties of the system,
- $v : [0,T] \mapsto U$ is the *control*, i.e. the variable we can choose (for simplicity, we assume that U and H are Hilbert spaces),
- $A : D(A) \subset H \mapsto H$ is a (generally nonlinear) operator with $A(0) = 0$, $B \in \mathcal{L}(U;H)$ and $y^0 \in H$.

Suppose that (3.1) is well-posed in the sense that, for each $y^0 \in H$ and each $v \in L^2(0,T;U)$, it possesses exactly one solution. Then the *null controllability* problem for (3.1) can be stated as follows:

> *For each $y^0 \in H$, find $v \in L^2(0,T;U)$ such that the corresponding solution of* (3.1) *satisfies $y(T) = 0$.*

More generally, the *exact controllability to the trajectories* problem for (3.1) is the following:

> *For each free trajectory $\overline{y} : [0,T] \mapsto H$ and each $y^0 \in H$, find $v \in L^2(0,T;U)$ such that the corresponding solution of* (3.1) *satisfies $y(T) = \overline{y}(T)$.*

Here, by a *free* or *uncontrolled* trajectory we mean any (sufficiently regular) function $\overline{y} : [0,T] \mapsto H$ satisfying $\overline{y}(t) \in D(A)$ for all t and

$$\overline{y}_t - A(\overline{y}) = 0, \quad t \in (0,T).$$

Notice that the exact controllability to the trajectories is a very useful property from the viewpoint of applications: if we can find a control such that $y(T) = \overline{y}(T)$, then after time T we can switch off the control and let the system follow the "ideal" trajectory $\overline{y}$.

For each system of the form (3.1), these problems lead to several interesting questions. Among them, let us indicate the following:

- First, are there controls v such that $y(T) = 0$ and/or $y(T) = \overline{y}(T)$?
- Then, if this is the case, which is the *cost* we have to pay to drive y to zero and/or $\overline{y}(T)$? In other words, which is the minimal norm of a control $v \in L^2(0,T;U)$ satisfying these properties?
- How can these controls be computed?

As indicated in Lecture 2, the controllability of differential systems is a very relevant area of research and has been the subject of a lot of work the last years. In particular, in the context of PDEs, the null controllability problem was first analyzed in [**99, 100, 77, 78, 64, 70**]. For semilinear systems of this kind, the first contributions have been given in [**68, 109, 30, 47**].

In this Lecture, I will consider several linear and nonlinear parabolic PDEs. First, we will recall the results satisfied by the classical heat equation in a bounded N-dimensional domain, complemented with appropriate initial and boundary-value conditions. Secondly, we will deal with similar stochastic PDEs. We will then consider the viscous Burgers equation. We will see that, for this PDE, the null controllability problem (with distributed and locally supported control) is well understood.[1] We will also consider the Navier-Stokes and Boussinesq equations and some other systems from mechanics.

3.2. The heat equation. Observability and Carleman estimates

Let us consider the following control system for the heat equation:

$$\begin{cases} y_t - \Delta y = v1_\omega, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = y^0(x), & x \in \Omega. \end{cases} \tag{3.2}$$

Here, we conserve the notation of Lecture 2. In particular, $\Omega \subset \mathbb{R}^N$ is a nonempty regular and bounded domain, $\omega \subset\subset \Omega$ is a (small) nonempty open subset (1_ω is the characteristic function of ω) and $y^0 \in L^2(\Omega)$.

It is well known that, for every $y^0 \in L^2(\Omega)$ and every $v \in L^2(\omega \times (0,T))$, there exists a unique solution y to (3.2), with $y \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$.

In view of the results in Lecture 2, (3.2) is approximately, E-approximately and null controllable.

Also, if we introduce for each $\varphi^0 \in L^2(\Omega)$ the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = 0, & (x,t) \in \Omega \times (0,T), \\ \varphi(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \varphi(x,T) = \varphi^0(x), & x \in \Omega, \end{cases} \tag{3.3}$$

[1] More precisely, let us denote by $T^*(r)$ the minimal time needed to drive any initial state with L^2 norm $\leq r$ to zero. Then we will show that $T^*(r) > 0$, with explicit sharp estimates from above and from below.

we know that the null controllability of (3.2) is equivalent to the *observability* of (3.3), that is, to the following estimate:

$$\|\varphi(\cdot,0)\|_{L^2}^2 \le C \iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt \quad \forall \varphi^0 \in L^2(\Omega) \tag{3.4}$$

(where C only depends on Ω, ω and T).

We have already seen that the estimates (3.4) are implied by the so called global Carleman inequalities. These have been introduced in the context of the controllability of PDEs by Fursikov and Imanuvilov; see [**64, 47**]. When they are applied to the solutions to the adjoint system (3.3), they take the form

$$\iint_{\Omega\times(0,T)} \rho^{-2}\,|\varphi|^2\,dx\,dt \le K \iint_{\omega\times(0,T)} \rho^{-2}\,|\varphi|^2\,dx\,dt \quad \forall \varphi^0 \in L^2(\Omega), \tag{3.5}$$

where $\rho = \rho(x,t)$ is an appropriate weight depending on Ω, ω and T and the constant K only depends on Ω and ω.[2] Combining (3.5) and the dissipativity of the backwards heat equation (3.3), it is not difficult to deduce (3.4) for some C only depending on Ω, ω and T.

Since (3.2) is linear, null controllability is equivalent in this case to *exact controllability to the trajectories.* This means that, for any uncontrolled solution $\overline{y}$ and any $y^0 \in L^2(\Omega)$, there exists $v \in L^2(\omega\times(0,T))$ such that the associated state y satisfies

$$y(x,T) = \overline{y}(x,T) \ \text{ in } \ \Omega.$$

REMARK 3.1. Notice that the null controllability of (3.2) holds for *any* ω and T. This is a consequence of the fact that, in a parabolic equation, the transmission of information is instantaneous. Recall that this was not the case for the wave equation. Again, this is not the case for the transport equation. Thus, let us consider the control system

$$\begin{cases} y_t + y_x = v 1_\omega, & (x,t) \in (0,L)\times(0,T), \\ y(0,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,L), \end{cases} \tag{3.6}$$

with $\omega = (a,b) \subset\subset (0,L)$. Then, if $0 < T < a$, null controllability does not hold, since the solution always satisfies

$$y(x,T) = y^0(x-T) \quad \forall x \in (T,a),$$

independently of the choice of v; see [**23**] for more details and similar results concerning other control systems for the wave, Schrödinger and Korteweg-De Vries equations. □

There are many generalizations and variants of the previous argument that provide the null controllability of other similar linear (parabolic) state equations:

- Time-space dependent (and sufficiently regular) coefficients can appear in the equation, other boundary conditions can be used, boundary control (instead of distributed control) can be imposed, etc.; see [**47**]; see also [**36**] for a review of related results.

[2] In order to prove (3.5), we have to use a weight ρ that blows up as $t \to 0$ and also as $t \to T$, for instance exponentially.

- The null controllability of Stokes-like systems of the form

$$y_t - \Delta y + (a \cdot \nabla)y + (y \cdot \nabla)b + \nabla p = v1_\omega, \quad \nabla \cdot y = 0, \tag{3.7}$$

where a and b are regular enough, can also be analyzed with these techniques. See for instance [**39**]; see also [**29**] for other controllability properties. We will come back in Section 3.5 to systems of this kind.
- Other linear parabolic (non-scalar) systems can also be considered, etc.

However, there are several interesting problems related to the controllability of linear parabolic systems that remain open. Let us mention some of them.

First, let us consider the controlled system

$$\begin{cases} y_t - \nabla \cdot (a(x)\nabla y) = v1_\omega, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = y^0(x), & x \in \Omega, \end{cases} \tag{3.8}$$

where y^0 and v are as before and the coefficient a is assumed to satisfy

$$a \in L^\infty(\Omega), \quad 0 < a_0 \le a(x) \le a_1 < +\infty \quad \text{a.e.} \tag{3.9}$$

It is natural to consider the null controllability problem for (3.8). Of course, this is equivalent to the observability of the associated adjoint system

$$\begin{cases} -\varphi_t - \nabla \cdot (a(x)\nabla \varphi) = 0, & (x,t) \in \Omega \times (0,T), \\ \varphi(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y\varphi(x,T) = \varphi^1(x), & x \in \Omega, \end{cases} \tag{3.10}$$

that is to say, to the fact that an inequality like (3.4) holds for the solutions to (3.10).

To our knowledge, it is at present unknown whether (3.8) is null controllable. In fact, it is also unknown whether approximate controllability holds.

Recently, some partial results have been obtained in this context.

Thus, when $N = 1$, the null controllability of (3.8) has been established in [**2**] for general a satisfying (3.9). The techniques in the proof rely on the theory of quasi-conformal complex mappings and can be applied only to the one-dimensional case, with a independent of t. Furthermore, they only serve to apply directly the Lebeau-Robbiano method (recall the proof of theorem 2.2 in Lecture 2), that is, they do not lead to a Carleman estimate of the form (3.5).

When $N \ge 2$, it is known that (3.8) is null controllable under the following assumption

$$\exists \textit{ smooth open set } \Omega_0 \subset\subset \Omega \textit{ such that } a \textit{ is } C^1 \textit{ in } \overline{\Omega_0} \textit{ and } \overline{\Omega \setminus \Omega_0}. \tag{3.11}$$

This has been proved in [**73**]. A slight improvement has been performed in [**13**], where Ω_0 is allowed to touch the boundary of Ω. Again, the proofs use that a is independent of t in an essential way and do not clarify whether (3.5) holds.

In fact, it is an open question whether a Carleman estimate like (3.5) holds for the solutions to (3.10) even if $N = 1$ or (3.11) holds.

In order to have (3.5), we apparently need more regularity for a; see [**12**] for a proof when $N = 1$, a satisfies (3.9) and

$$a \in BV(\Omega); \tag{3.12}$$

see also [**27**] for a proof when $N \ge 2$, a is piecewise C^1 and satisfies (3.9) and some additional "sign" conditions.

At present, the following questions are open:

- Is (3.8) is null controllable when $N \geq 2$ and a satisfies (3.9) and (3.12)? Is (3.5) satisfied in this case?
- Is (3.5) satisfied when $N = 1$ and a only satisfies (3.9)?

QUESTION 1: *Assume that $N = 1$ and a is piecewise constant and satisfies* (3.9). *Is* (3.8) *approximately controllable?*

A similar question can be asked when $N \geq 2$. Which is the rigorous question and which is the answer?

Let us now consider the non-scalar system

$$\begin{cases} y_t - D\Delta y = My + Bv1_\omega, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = y^0(x), & x \in \Omega, \end{cases} \tag{3.13}$$

where $y = (y_1, \ldots, y_n)$ is the state, $v = (v_1, \ldots, v_m)$ is the control and D, M and B are constant matrices, with $D, M \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$. It is assumed that $n \geq 2$ and D is definite positive, that is,

$$D\xi \cdot \xi \geq d_0 |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad d_0 > 0. \tag{3.14}$$

When D is diagonal (or similar to a diagonal matrix), the null controllability problem for (3.13) is well understood. In view of the results in [**4**], (3.13) is null controllable if and only if

$$\text{rank}\, [(-\lambda_i D + M); B] = n \quad \forall i \geq 1, \tag{3.15}$$

where the λ_i are the eigenvalues of the Dirichlet-Laplace operator and, for any matrix $H \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$, $[H; B]$ stands for the $n \times nm$ matrix

$$[H; B] := [B|HB| \cdots |H^{n-1}B].$$

Therefore, it is natural to search for (algebraic) conditions on D, M and B that ensure the null controllability of (3.13) in the general case. But, to our knowledge, this is unknown.

The results in [**4**] have been extended recently to the case of any D having no eigenvalue of geometric multiplicity > 4; see [**35**].

QUESTION 2: *Under which conditions the system*

$$\begin{cases} y_t - D\Delta y = M(x,t)y + Bv1_\omega, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = y^0(x), & x \in \Omega, \end{cases} \tag{3.16}$$

where D is a diagonal matrix satisfying (3.14)*, $M \in L^\infty(Q; \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n))$ and $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$, is null controllable?*

REMARK 3.2. As we have said, global Carleman estimates are the main tool we can use to establish the observability property (3.4). These open questions can be viewed, at least in part, as a confirmation of the limitations of Carleman estimates:

first, they need sufficiently regular coefficients; then, they are actually well-suited only for *scalar* equations. □

3.3. Some remarks on the controllability of stochastic PDEs

In this Section, we deal briefly with a system governed by a linear stochastic partial differential equation:

$$\begin{cases} y_t - \Delta y = v 1_\omega + B(t)\, \dot{w}_t & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{3.17}$$

Here, v is again the control and $\dot{w}_t$ is a *Gaussian random field* (white noise in time). For instance, it can be regarded as the distributional time derivative of a Wiener process w_t. The equations are required to be satisfied $P-$a.e., i.e. P-almost surely, in a given probability space $\{\Lambda, \mathcal{F}, P\}$.

In the sequel, we are going to see that, for general y^0, y^1 and $B = B(t)$, one can obtain final states $y(\cdot, T)$ arbitrarily close to y^1 in quadratic mean by choosing v appropriately (an approximate controllability result). We will also see that, if B is not random and in some sense small, then one can also choose v such that $y(\cdot, T) = 0$ (amnull controllability result).

3.3.1. Some basic results from probability calculus. In order to present the results without too much ambiguity, we will first recall some basic definitions and results.

Thus, assume that a *complete probability space* $\{\Lambda, \mathcal{F}, P\}$ is given. If X is a Banach space and $f \in L^1(\Lambda, \mathcal{F}; X)$, we will denote by Ef the expectation of f:

$$Ef = \int_\Lambda f(\lambda)\, dP(\lambda).$$

Assume that a separable Hilbert space K and a *Wiener process* w_t on $\{\Lambda, \mathcal{F}, P\}$ with values in K are given. This means that

$$w_t = \sum_{k=1}^{\infty} \beta_t^k e_k \quad \forall t \geq 0,$$

where $\{e_k\}$ is an orthonormal basis in K and the β_t^k are mutually independent *real Wiener processes* satisfying

$$E|\beta_t^k|^2 = \mu_k^2 t, \qquad \sum_{k=1}^{\infty} \mu_k^2 < +\infty. \tag{3.18}$$

A normalized real Wiener process β_t is a measurable function $(\lambda, t) \mapsto \beta_t(\lambda)$ which is defined $P-$a.s. in Λ for all $t \in \mathbb{R}_+$ and satisfies the following:

(a) $\beta_0 = 0$,
(b) For each t, β_t is *normally distributed,* with mean 0 and variance t, i.e.

$$E\beta_t = 0, \quad E|\beta_t|^2 = t.$$

(c) $E(\beta_t\, \beta_s) = \sqrt{t}\sqrt{s}$ for all $t, s \geq 0$.

For other equivalent definitions and basic properties of real Wiener processes, see [**6**]. Recall that, in particular, the real processes β_t^k and the K-valued process w_t have Hölder-continuous sample paths $t \mapsto \beta_t^k(\lambda)$ and $t \mapsto w_t(\lambda)$.

In the sequel, we put

$$\mathcal{F}_t := \sigma(\, w_s,\ 0 \le s \le t\,)$$

($\mathcal{F}_t$ is the σ-algebra spanned by w_s for $0 \le s \le t$, completed with the negligeable sets in $\mathcal{F}$). Obviously, $\{\mathcal{F}_t\}$ is an increasing family of sub σ-algebras of $\mathcal{F}$ and, among other things, one has:

$$\mathcal{F}_t = \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right) \quad \forall t > 0. \tag{3.19}$$

Let H be a Hilbert space. For any $f \in L^1(\Lambda, \mathcal{F}; H)$, we denote by $E[f|\mathcal{F}_t]$ the *conditional expectation* of f with respect to $\mathcal{F}_t$, i.e. the unique element in $L^1(\Lambda, \mathcal{F}_t; H)$ such that

$$\int_A E[f|\mathcal{F}_t]\, dP = \int_A f\, dP \quad \forall A \in \mathcal{F}_t\,.$$

The existence and uniqueness of $E[f|\mathcal{F}_t]$ is implied by the celebrated Radon-Nykodim theorem. For the main properties of the conditional expectation, see for instance [**91**]. In particular, recall that, if $f \in L^2(\Lambda, \mathcal{F}; H)$, then $E[f|\mathcal{F}_t] \in L^2(\Lambda, \mathcal{F}_t; H)$ and icoincides with the orthogonal projection of f in $L^2(\Lambda, \mathcal{F}_t; H)$.

Let X be a Banach space. We denote by $I^2(0,T;X)$ the space formed by all stochastic processes $\Phi \in L^2(\Lambda \times (0,T), dP \otimes dt; X)$ which are $\mathcal{F}_t$-adapted a.e. in $(0,T)$, i.e. such that

$$\lambda \mapsto \Phi(\lambda, t) \text{ is } \mathcal{F}_t\text{-measurable for almost all } t \in (0,T)$$

In the case $X = \mathcal{L}(K;H)$, measurability will be understood in the *strong* sense, i.e. the measurability of $\lambda \mapsto \Phi(\lambda,t)w$ for each $w \in K$. Then, $I^2(0,T;X)$ is a closed subspace of $L^2(\Lambda \times (0,T), dP \otimes dt; X)$.

Recall that, for any $b \in I^2(0,T;\mathbb{R})$, any real-valued Wiener process β_t and any fixed $t \in [0,T]$, we can introduce a random variable $I_t(f) : \Lambda \mapsto \mathbb{R}$ known as the Ito stochastic integral in $[0,t]$:

$$I_t(f) = \int_0^t f(s)\, d\beta_s\,.$$

The stochastic process $(\lambda, t) \mapsto I_t(f)(\lambda)$ again belongs to $I^2(0,T;\mathbb{R})$ and, among other properties, satisfies the following:

$$E \int_0^t f(s)\, d\beta_s = 0$$

and

$$E \left| \int_0^t f(s)\, d\beta_s \right|^2 = \int_0^t E|f(s)|^2\, ds$$

for all $t \in [0,T]$.

Now, assume that a stochastic process B is given, with

$$B \in I^2(0,T;\mathcal{L}(K;H)) \tag{3.20}$$

(H is a Hilbert space). Then the stochastic integral of B with respect to w_t is defined by the formula

$$\int_0^t B(s)\,dw_s = \sum_{k=1}^{\infty} \int_0^t B(s)e_k\,d\beta_s^k \qquad \forall t \in [0,T].$$

Here, the convergence of the series is understood in the sense of $L^2(\Lambda, \mathcal{F}_t; H)$. The stochastic integrals in the right hand side are defined by the equalities

$$(\int_0^t B(s)e_k\,d\beta_s^k, h) = \int_0^t (B(s)e_k, h)\,d\beta_s^k \qquad \forall h \in H,$$

where the latter are usual *Ito stochastic integrals* with respect to the real-valued processes β_t^k; see [**6**] for more details.

3.3.2. The controllability results. In the remainder of this Section, H and V will denote the Hilbert spaces $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively.

Assume we are given an arbitrary but fixed initial state

$$y^0 \in H, \tag{3.21}$$

a Wiener process w_t with values in the separable Hilbert space K and a stochastic process $B \in I^2(0,T;\mathcal{L}(K;H))$. Let $A = -\Delta$ be the usual Laplace-Dirichlet operator in Ω, with domain $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. For each control $v \in I^2(0,T;H)$, there exists exactly one solution y to the state system

$$\left\{\begin{array}{l} y \in I^2(0,T;V) \cap L^2(\Lambda; C^0([0,T];H)), \\ y(\cdot,t) = y^0 + \displaystyle\int_0^t \{-Ay(\cdot,s) + 1_\omega v(\cdot,s)\}\,ds + \int_0^t B(s)\,dw_s \quad \forall t \in [0,T]. \end{array}\right. \tag{3.22}$$

In (3.22), the equalities have to be understood $P-$a.s. in V'.

Notice that we choose $\mathcal{F}_t$-adapted controls to govern the state system. This is a natural assumption from the stochastic viewpoint since, once w_t is given, only $\mathcal{F}_t$-adapted processes can be regarded as *statistically observable.*

Let $S(t)$ be the semigroup generated in H by A. Then, in accordance with the results in [**24, 90**], one has:

$$\left\{\begin{array}{l} y(\cdot,t) = S(t)y^0 + \displaystyle\int_0^t S(t-s)(1_\omega v(\cdot,s))\,ds + \int_0^t S(t-s)B(s)\,dw_s \\ \forall t \in [0,T] \end{array}\right. \tag{3.23}$$

Our first result deals with approximate controllability:

THEOREM 3.3. *The linear manifold $Y_T = \{\, y(\cdot,T) : v \in I^2(0,T;H)\,\}$ is dense in $L^2(\Lambda, \mathcal{F}_T; H)$. In other words: for any $y^1 \in L^2(\Lambda, \mathcal{F}_T; H)$ and any $\varepsilon > 0$, there exists a control $v \in I^2(0,T;H)$ such that the associated solution to* (3.22) *satisfies:*

$$E\|y(\cdot,T) - y^1\|_{L^2}^2 \le \varepsilon.$$

Accordingly, it is said that (3.22) *is approximately controllable in quadratic mean.*

PROOF: We will argue as in the deterministic case. In view of (3.23), it will suffice to check that, if $f \in L^2(\Lambda, \mathcal{F}_T; H)$ and

$$E(\int_0^T S(T-s)(1_\omega v(\cdot,s))\,ds,\ f)_{L^2} = 0 \quad \forall v \in I^2(0,T;H), \tag{3.24}$$

then necessarily $f = 0$.

Let f be a function in $L^2(\Lambda, \mathcal{F}_T; H)$ satisfying (3.24) and assume that $\phi \in I^2(0,T;H)$ is defined pathwise by

$$\begin{cases} -\phi_t + A\phi = 0 & \text{in} \quad Q, \\ \phi = 0 & \text{on} \quad \Sigma, \\ \phi(x,T) = f(x) & \text{in} \quad \Omega, \end{cases}$$

i.e. $\phi(\cdot,t) = S(T-t)f$ for all t. It will be sufficient to prove that

$$E[\phi(\cdot,t)|\mathcal{F}_t] = 0 \quad \forall t \in (0,T). \tag{3.25}$$

Indeed, this and the continuity property (3.19) of the family $\{\mathcal{F}_t\}$ clearly imply that

$$f = E[\phi(\cdot,T)|\mathcal{F}_T] = 0.$$

QUESTION 3: *Why do* (3.25) *and* (3.19) *imply that* $f = 0$*?*

We know that

$$E\int_0^T (v(\cdot,s),\ 1_\omega\phi(\cdot,s))_{L^2}\,ds = 0 \qquad \forall v \in I^2(0,T;H).$$

Thus, $1_\omega E[\phi(\cdot,t)|\mathcal{F}_t]$ is a stochastic process in $I^2(0,T;H)$ such that

$$E\int_0^T (v(\cdot,s),\ 1_\omega E[\phi(\cdot,s)|\mathcal{F}_s])\,ds = \int_0^T E(v(\cdot,s),\ 1_\omega\phi(\cdot,s))\,ds = 0$$

for all $v \in I^2(0,T;H)$ and, consequently,

$$1_\omega E[\phi(\cdot,t)|\mathcal{F}_t] = 0. \tag{3.26}$$

For each $t \in (0,T)$, $E[\phi(\cdot,t)|\mathcal{F}_t] = S(T-t)E[f|\mathcal{F}_t]$ is real analytic in the variable $x \in \Omega$. Hence, one must necessarily have $E[\phi(\cdot,t)|\mathcal{F}_t] = 0$ for all $t \in (0,T)$ and the result is proved. □

A consequence of this theorem is that, for any $y^1 \in L^2(\Lambda, \mathcal{F}_T; H)$, $\varepsilon > 0$ and $\delta > 0$, a control v can be found such that

$$P\{\,\|y(\cdot,T) - y^1\|_{L^2} < \varepsilon\,\} \geq 1 - \delta.$$

However, the existence of a control $v \in I^2(0,T;H)$ such that $P\{\,\|y(\cdot,T) - y^1\|_{L^2} < \varepsilon\,\} = 1$ is an interesting open question.

The approximate controllability in quadratic mean remains true for systems governed by more general linear equations. More precisely, the following result is proved in [**42**]:

THEOREM 3.4. *Assume that, in* (3.22), A *is an operator of the form*

$$Ay = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial y}{\partial x_j}\right) + \sum_{j=1}^{N} b_j \frac{\partial y}{\partial x_j} + cy, \tag{3.27}$$

where the coefficients satisfy

$$a_{ij} \in C^1(\overline{\Omega}), \quad b_j, c \in L^\infty(\Omega)$$

and the usual ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_j\xi_i \geq \alpha|\xi|^2 \quad \forall\lambda \in \mathbb{R}^N, \quad \forall x \in \Omega, \quad \alpha > 0.$$

Then the corresponding $Y_T = \{\, y(\cdot, T) : v \in I^2(0, T; H)\,\}$ *is dense in* $L^2(\Lambda, \mathcal{F}_T; H)$.

We will now recall a null controllability result for (3.22) from [**42**]. Again, this is the analog of a deterministic result.

THEOREM 3.5. *Let us set* $\gamma(t) := t(T-t)$. *Assume that* B *is not random,* $B \in C^1([0, T]; \mathcal{L}(K; H))$ *and, also, that the support of* $B(t)w$ *does not intersect* ω *for any* t *and* $w \in K$. *Then there exists a positive function* $\beta = \beta(x)$ *such that, if*

$$\iint_Q t\left(\gamma(t)^{-1}\|B\|^2_{\mathcal{L}(K;H)} + \gamma(t)^3\|B_t\|^2_{\mathcal{L}(K;H)}\right) e^{2\beta(x)/\gamma(t)}\,dx\,dt < +\infty, \tag{3.28}$$

for each $y^0 \in H$ *there exists* $v \in I^2(0, T; H)$ *satisfying* $y(x, T) \equiv 0$, *i.e.* (3.22) *is null controllable.*

As in the deterministic case, the proof relies on an observability estimate for the solution of the adjoint system.

The situation is more complicate in the case of a *multiplicative noise*, that is, for systems of the form

$$\begin{cases} y \in I^2(0, T; V) \cap L^2(\Omega; C^0([0, T], H)), \\ y(\cdot, t) = y^0 + \displaystyle\int_0^t \{-Ay(\cdot, s) + 1_\omega v(\cdot, s)\}\,ds + \int_0^t By(\cdot, s)\,dw_s \quad \forall t \in [0, T]. \end{cases} \tag{3.29}$$

Here, B is given by $(By)(x) = b(x)y(x)$ for some $b \in W^{1,\infty}(\Omega)$ and (for simplicity) w_t is a real Wiener process.

From the theory of stochastic partial differential equations, it follows in particular that, for each $v \in I^2(0, T; H)$, there exists exactly one solution y to (3.29), see [**90**].

The approximate controllability in quadratic mean of (3.29) is equivalent to the unique continuation property for the following backward (adjoint) stochastic system:

$$\begin{cases} p \in I^2(0, T; V) \cap L^2(\Omega; C^0([0, T]; H)), \quad q \in I^2(0, T; H), \\ p(\cdot, t) = f + \displaystyle\int_t^T \{A^*p(\cdot, s) + Bq(\cdot, s)\}\,ds - \int_t^T q(\cdot, s)\,dw_s \quad \forall t \in [0, T]. \end{cases} \tag{3.30}$$

In [**8**], a global Carleman estimate has been established for this system when $A = -\Delta$ and $b \in C^2(\overline{\Omega})$. Of course, this implies unique continuation for (3.30) and, consequently, approximate controllability in quadratic mean for (3.29) in this particular case.

On the other hand, an appropriate unique continuation property for (3.30) has been proved in [**34**] in a more general case. As a consequence, one has approximate controllability in quadratic mean for (3.29). In fact, when b is a constant, and Γ is of class C^∞, we can also prove approximate controllability in all spaces $L^r(\Lambda, \mathcal{F}_T; L^q(\mathcal{O}))$ with $1 \le r, q < +\infty$.

The previous analysis can also be made for stochastic Stokes systems; see [**41**].

For more results concerning the approximate and null controllability of stochastic PDEs, see the recent paper [**108**].

3.4. Positive and negative results for the Burgers equation

In this Section, we will be concerned with the null controllability of the following system for the viscous Burgers equation:

$$\begin{cases} y_t - y_{xx} + yy_x = v1_\omega\,, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,1). \end{cases} \tag{3.31}$$

Recall that some controllability properties of (3.31) have been studied in [**47**] (see Chapter 1, theorems 6.3 and 6.4). There, it is shown that, in general, a stationary solution of (3.31) with large L^2-norm cannot be reached (not even approximately) at any time T. In other words, with the help of one control, the solutions of the Burgers equation cannot go anywhere at any time.

For each $y^0 \in L^2(0,1)$, let us introduce

$$T(y^0) = \inf\{\, T > 0 : (3.31) \text{ is null controllable at time } T\,\}.$$

Then, for each $r > 0$, let us define the quantity

$$T^*(r) = \sup\{\, T(y^0) : \|y^0\|_{L^2} \le r\,\}.$$

Our main purpose is to show that $T^*(r) > 0$, with explicit sharp estimates from above and from below. In particular, this will imply that (global) null controllability at any positive time does not hold for (3.31).

More precisely, let us set $\phi(r) = (\log \frac{1}{r})^{-1}$. We have the following result from [**37**]:

THEOREM 3.6. *One has*

$$C_0\phi(r) \le T^*(r) \le C_1\phi(r) \quad \textit{as} \quad r \to 0, \tag{3.32}$$

for some positive constants C_0 and C_1 not depending of r.

REMARK 3.7. The same estimates hold when the control v acts on system (3.31) through the boundary *only* at $x = 1$ (or only at $x = 0$). Indeed, it is easy to transform the boundary controlled system

$$\begin{cases} y_t - y_{xx} + yy_x = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = 0, \quad y(1,t) = w(t), & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,1) \end{cases} \tag{3.33}$$

into a system of the kind (3.31). The boundary controllability of the Burgers equation with *two* controls (at $x = 0$ and $x = 1$) has been analyzed in [**54**]. There, it is shown that even in this more favorable situation null controllability does not hold for small time. It is also proved in that paper that exact controllability does not hold for large time.[3] □

REMARK 3.8. It is proved in [**20**] that the Burgers equation is *globally* null controllable when we act on the system through two boundary controls and an additional right hand side only depending on t. In other words, for any $y^0 \in L^2(0,1)$, there exist w_1 , w_2 and h in $L^2(0,T)$ such that the soution to

$$\begin{cases} y_t - y_{xx} + yy_x = h(t), & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = w_1(t), \quad y(1,t) = w_2(t), & t \in (0,T), \\ y(x,0) = y^0(x), & x \in (0,1) \end{cases} \tag{3.34}$$

satisfies

$$y(x,T) = 0 \ \text{ in } \ (0,1).$$

However, it is unknown whether this global property is conserved when one of the boundary controls w_1 or w_2 is eliminated. □

The proof of the estimate from above in (3.32) can be obtained by solving the null controllability problem for (3.31) via a (more or less) standard fixed point argument, using global Carleman inequalities to estimate the control and energy inequalities to estimate the state and being very careful with the role of T in these inequalities.

The proof of the estimate from below is inspired by the arguments in [**5**] and is implied by the following property: there exist positive constants C_0 and C_0' such that, for any sufficiently small $r > 0$, we can find initial data y^0 and associated states y satisfying $\|y^0\|_{L^2} \le r$ and

$$|y(x,t)| \ge C_0' r \ \text{ for some } x \in (0,1) \text{ and any } t \text{ satisfying } 0 < t < C_0\phi(r).$$

For more details, see [**37**].

3.5. The Navier-Stokes and Boussinesq systems

There is a lot of more realistic nonlinear equations and systems from mechanics that can also be considered in this context. First, we have the well known Navier-Stokes equations:

$$\begin{cases} y_t + (y \cdot \nabla)y - \Delta y + \nabla p = v 1_\omega, \quad \nabla \cdot y = 0, & (x,t) \in Q, \\ y = 0, & (x,t) \in \Sigma, \\ y(x,0) = y^0(x), & x \in \Omega. \end{cases} \tag{3.35}$$

Here and below, $N = 2$ or $N = 3$ and (again) $\omega \subset \Omega$ is a nonempty open set.

In (3.35), (y,p) is the state (the velocity field and the pressure distribution) and v is the control (a field of external forces applied to the fluid particles located at ω). To our knowledge, the best results concerning the controllability of this system

[3] Let us remark that the results in [**54**] do not allow to estimate $T(r)$; in fact, the proofs are based in contradiction arguments.

have been given in [**39**] and [**40**].[4] Essentially, these results establish the local exact controllability of the solutions of (3.35) to bounded uncontrolled trajectories.

In order to be more specific, let us recall the definition of some usual spaces in the context of Navier-Stokes equations:

$$V := \left\{ y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega \right\}$$

and

$$H := \{ y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \partial\Omega \}.$$

Of course, it will be said that (3.35) is *exactly controllable to the trajectories* if, for any trajectory $(\overline{y}, \overline{p})$, i.e. any solution of the uncontrolled Navier-Stokes system

$$\left\{ \begin{array}{ll} \overline{y}_t + (\overline{y} \cdot \nabla)\overline{y} - \Delta\overline{y} + \nabla\overline{p} = 0, \quad \nabla \cdot \overline{y} = 0, & (x,t) \in Q, \\ \overline{y} = 0, & (x,t) \in \Sigma \end{array} \right. \tag{3.36}$$

and any $y^0 \in H$, there exist controls $v \in L^2(\omega \times (0,T))^N$ and associated solutions (y,p) such that

$$y(x,T) = \overline{y}(x,T) \text{ in } \Omega. \tag{3.37}$$

At present, we do not know any global result concerning exact controllability to the trajectories for (3.35). However, the following local result holds:

THEOREM 3.9. *Let* $(\overline{y}, \overline{p})$ *be a strong solution of* (3.36)*, with*

$$\overline{y} \in L^\infty(Q)^N, \quad \overline{y}(\cdot\,,0) \in V. \tag{3.38}$$

Then, there exists $\delta > 0$ *such that, for any* $y^0 \in H \cap L^{2N-2}(\Omega)^N$ *satisfying* $\|y^0 - \overline{y}^0\|_{L^{2N-2}} \le \delta$*, we can find a control* $v \in L^2(\omega \times (0,T))^N$ *and an associated solution* (y,p) *to* (3.35) *such that* (3.37) *holds.*

In other words, the local exact controllability to the trajectories holds for (3.35) in the space $X = L^{2N-2}(\Omega)^N \cap H$ Similar questions were addressed (and solved) in [**46**] and [**45**]. The fact that we consider here Dirichlet boundary conditions and locally supported distributed control increases a lot the mathematical difficulty of the control problem.

REMARK 3.10. It is clear that we cannot expect exact controllability for the Navier-Stokes equations with an arbitrary target function, because of the dissipative and irreversible properties of the system. On the other hand, approximate controllability is still an open question for this system. Some results in this direction have been obtained in [**22**] for different boundary conditions (Navier slip boundary conditions) and in [**29**] with a different nonlinearity. However, the notion of approximate controllability does not appear to be optimal from a practical viewpoint. Indeed, even if we could reach an arbitrary neighborhood of a given target y^1 at time T by the action of a control, the question of what to do afterwards to stay in the same neighbourhood would remain open. □

[4] The main ideas come from [**49, 65**]; some similar results have been given more recently in [**52**].

The proof of theorem 3.9 can be obtained as an application of *Liusternik's inverse mapping theorem* in an appropriate framework.

A key point in the proof is a related null controllability result for the linearized Navier-Stokes system at $(\overline{y}, \overline{p})$, that is to say:

$$\begin{cases} y_t + (\overline{y}\cdot\nabla)y + (y\cdot\nabla)\overline{y} - \Delta y + \nabla p = v1_\omega, & (x,t)\in Q,\\ \nabla\cdot y = 0, & (x,t)\in Q,\\ y = 0, & (x,t)\in\Sigma,\\ y(x,0) = y^0(x), & x\in\Omega. \end{cases} \tag{3.39}$$

This is implied by a global Carleman inequality of the kind (3.5) that can be established for the solutions to the adjoint of (3.39), which is the following:

$$\begin{cases} -\varphi_t - (\nabla\varphi + \nabla\varphi^t)\,\overline{y} - \Delta\varphi + \nabla\pi = g, & (x,t)\in Q,\\ \nabla\cdot\varphi = 0, & (x,t)\in Q,\\ \varphi = 0, & (x,t)\in\Sigma,\\ \varphi(x,T) = \varphi^0(x), & x\in\Omega. \end{cases} \tag{3.40}$$

The details can be found in [**39**].

Similar results have been given in [**53**] for the Boussinesq equations

$$\begin{cases} y_t + (y\cdot\nabla)y - \Delta y + \nabla p = v1_\omega + \theta\,e_N, \quad \nabla\cdot y = 0 & (x,t)\in Q,\\ \theta_t + y\cdot\nabla\theta - \Delta\theta = h1_\omega, & (x,t)\in Q,\\ y = 0, \quad \theta = 0, & (x,t)\in\Sigma,\\ y(x,0) = y^0(x), \quad \theta(x,0) = \theta^0(x), & x\in\Omega. \end{cases} \tag{3.41}$$

Here, the state is the triplet (y,p,θ) (θ is interpreted as a temperature distribution) and the control is (v,h) (as before, v is a field of external forces; h is an external heat source).

QUESTION 4: *Can we deduce from theorem 3.9 a null controllability result for* (3.35) *for large T? What about* (3.41)*?*

QUESTION 5: *Does local null controllability imply local exact controllability to the trajectories in the context of* (3.35)*? What about* (3.41)*?*

An interesting question concerning both (3.35) and (3.41) is whether we can still get local exact controllability to the trajectories with a reduced number of scalar controls. This is partially answered in [**40**], where the following results are proved:

THEOREM 3.11. *Assume that the following property is satisfied:*

$$\exists x^0\in\partial\Omega,\ \exists\varepsilon>0 \ \text{ such that } \overline{\omega}\cap\partial\Omega\supset B(x^0;\varepsilon)\cap\partial\Omega. \tag{3.42}$$

Here, $B(x^0;\varepsilon)$ is the ball centered at x^0 of radius ε. Then, for any $T>0$, (3.35) *is locally exactly controllable at time T to the trajectories satisfying* (3.38) *with controls $v\in L^2(\omega\times(0,T))^N$ having one component identically zero.*

THEOREM 3.12. *Assume that ω satisfies* (3.42) *with $n_k(x^0) \neq 0$ for some $k < N$. Then, for any $T > 0$,* (3.41) *is locally exactly controllable at time T to the trajectories $(\overline{y}, \overline{p}, \overline{\theta})$ satisfying* (3.38) *and*

$$\overline{\theta} \in L^\infty(Q), \quad \overline{\theta}(\cdot\,, 0) \in H_0^1(\Omega), \tag{3.43}$$

with controls $v \in L^2(\omega \times (0,T))^N$ and $h \in L^2(\omega \times (0,T))$ such that $v_k \equiv v_N \equiv 0$. In particular, if $N = 2$, we have local exact controllability to these trajectories with controls $v \equiv 0$ and $h \in L^2(\omega \times (0,T))$.

The proofs of theorems 3.11 and 3.12 are similar to the proof of theorem 3.9. We have again to rewrite the controllability property as a nonlinear equation in a Hilbert space. Then, we have to check that the hypotheses of Liusternik's theorem are fulfilled.

Again, a crucial point is to prove the null controllability of certain linearized systems, this time with *modified* controls. For instance, when dealing with (3.35), the task is reduced to prove that, for some appropriate weights ρ, ρ_0 and some $K > 0$, the solutions to (3.40) satisfy the following Carleman-like estimates:

$$\iint_{\Omega\times(0,T)} \rho^{-2}|\varphi|^2 \, dx\, dt \leq K \iint_{\omega\times(0,T)} \rho_0^{-2}(|\varphi_1|^2 + |\varphi_2|^2) \, dx\, dt \quad \forall \varphi^0 \in H. \tag{3.44}$$

This inequality can be proved using the assumption (3.42) and the incompressibility identity $\nabla \cdot \varphi = 0$; see [**40**].

3.6. Some other nonlinear systems from mechanics

The previous arguments can be applied to other similar partial differential systems arising in mechanics. For instance, this is done in [**38**] in the context of micropolar fluids.

To fix ideas, let us assume that $N = 3$. The behavior of a micropolar three-dimensional fluid is governed by a system which has the form

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = \nabla \times w + v 1_\omega, \quad \nabla \cdot y = 0, & (x,t) \in Q, \\ w_t + (y \cdot \nabla) w - \Delta w - \nabla(\nabla \cdot w) = \nabla \times y + u 1_\omega, & (x,t) \in Q, \\ y = 0, \quad w = 0 & (x,t) \in \Sigma, \\ y(x,0) = y^0(x), \quad w(x,0) = w^0(x) & x \in \Omega. \end{cases} \tag{3.45}$$

Here, the state is (y, p, w) and the control is (v, u). As usual, y and p stand for the velocity field and pressure and w is the microscopic velocity of rotation of the fluid particles.

The following result holds:

THEOREM 3.13. *Let $(\overline{y}, \overline{p}, \overline{w})$ be such that*

$$\overline{y},\, \overline{w} \in L^\infty(Q) \cap L^2(0,T; H^2(\Omega)), \quad \overline{y}_t,\, \overline{w}_t \in L^2(Q) \tag{3.46}$$

and

$$\begin{cases} \overline{y}_t - \Delta \overline{y} + (\overline{y} \cdot \nabla)\overline{y} + \nabla \overline{p} = \nabla \times \overline{w}, \quad \nabla \cdot \overline{y} = 0, & (x,t) \in Q, \\ \overline{w}_t + (\overline{y} \cdot \nabla)\overline{w} - \Delta \overline{w} - \nabla(\nabla \cdot \overline{w}) = \nabla \times \overline{y}, & (x,t) \in Q, \\ \overline{y} = 0, \quad \overline{w} = 0 & (x,t) \in \Sigma. \end{cases} \tag{3.47}$$

Then, for each $T > 0$, (3.45) is locally exactly controllable to $(\overline{y}, \overline{p}, \overline{w})$ at time T. In other words, there exists $\delta > 0$ such that, for any initial data $(y^0, w^0) \in (H^2(\Omega) \cap V) \times H_0^1(\Omega)$ satisfying

$$\|(y^0, w^0) - (\overline{y}(\cdot\,,0), \overline{w}(\cdot\,,0))\|_{H^2 \times H_0^1} \leq \delta, \tag{3.48}$$

there exist L^2 controls u and v and associated solutions (y, p, w) satisfying

$$y(x,T) = \overline{y}(x,T), \quad w(x,T) = \overline{w}(x,T) \quad \text{in} \ \ \Omega. \tag{3.49}$$

Notice that this case involves a nontrivial difficulty. Indeed, w is a non-scalar variable and the equations satisfied by its components w_i are coupled through the second-order terms $\partial_i(\nabla \cdot w)$. This is a serious inconvenient. An appropriate strategy has to be applied in order to deduce the required Carleman estimates.

Let us also mention [**7, 61, 62**], where the controllability of the MHD and other related equations has been analyzed.

For all these systems, the proof of the controllability can be achieved arguing as in the first part of the proof of theorem 3.9. This is the general structure of the argument:

- First, rewrite the original controllability problem as a nonlinear equation in a space of admissible "state-control" variables.
- Then, prove an appropriate global Carleman inequality and a regularity result and deduce that the linearized equation possesses at least one solution. This provides a controllability result for a related linear problem.
- Fianlly, check that the hypotheses of a suitable implicit function theorem are satisfied and deduce a local result.

Remark 3.14. Recall that an alternative strategy was introduced in [**109**] in the context of the semilinear wave equation:

- First, consider a linearized similar problem and rewrite the original controllability problem in terms of a fixed point equation.
- Then, prove a global Carleman inequality and deduce an observability estimate for the adjoint system and a controllability result for the linearized problem.
- Finally, prove appropriate estimates for the control and the state (this usually needs some kind of *smallness* of the data), prove an appropriate compactness property of the state and deduce that there exists at least one fixed point.

This method has been used in [**30**] and [**44**] in the context of semilinear heat equations and in [**52**] to prove a result similar to theorem 3.9. □

Remark 3.15. Observe that all these results are positive, in the sense that they provide local controllability properties. At present, no negative result is known to hold for these nonlinear systems (except for the already considered one-dimensional Burgers equation). □

To end this Section, let us mention two systems from fluid mechanics, apparently not much more complex than (3.35), for which local controllability to the trajectories is an open question.

The first system is the following:

$$(3.50)\qquad \begin{cases} y_t + (y\cdot\nabla)y - \nabla\cdot(\nu(|Dy|)Dy) + \nabla p = v1_\omega\,, & (x,t)\in Q,\\ \nabla\cdot y = 0, & (x,t)\in Q,\\ y = 0, & (x,t)\in\Sigma,\\ y(x,0) = y^0(x), & x\in\Omega. \end{cases}$$

Here, $Dy = \frac{1}{2}(\nabla y + \nabla y^t)$ and $\nu : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a regular function (for example, we can take $\nu(s) \equiv a + bs^{r-1}$ for some $a, b > 0$ and some $r > 1$). This models the behavior of a *quasi-Newtonian* fluid; for a mathematical analysis, see [**11, 83**].

In view of the new nonlinear diffusion term $\nabla\cdot(\nu(|Dy|)Dy)$, the control properties of (3.51) are much more difficult to analyze than for (3.35). In particular, it is unknown whether the local approximate and the local null controllability properties hold for (3.50).

For the second system, we suppose that $N = 2$. It reads:

$$(3.51)\qquad \begin{cases} \theta_t + (y\cdot\nabla)\theta - \Delta\theta = v1_\omega\,, & (x,t)\in Q,\\ y = \nabla\times\left((-\Delta)^{-a}\theta\right), & (x,t)\in Q,\\ \theta = 0, & (x,t)\in\Sigma,\\ \theta(x,0) = \theta^0(x), & x\in\Omega, \end{cases}$$

where $a \in [1/2, 1]$. We are now modelling the behavior of a *quasi-geostrophic* fluid. The state variables θ and y may be viewed as a *generalized vorticity* and velocity field, respectively (notice that, for $a = 1$, we find again the Navier-Stokes system written in terms of y and $\nabla\times y$; see for instance [**92**]).

It is possible to prove a local null controllability result for (3.51). However, to our knowledge, the local approximate controllability and the local exact controllability to the trajectories are open problems.

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