Existence and asymptotic behavior of solutions for neutral stochastic partial integrodifferential equations with infinite delays

Mamadou Abdoul Diop*
Université Gaston Berger de Saint-Louis, UFR SAT
Département de Mathématiques, B.P234, Saint-Louis, Sénégal

Tomás Caraballo†
Dpto. Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla, Spain

Mahamat Mahamat Zene‡
Université Gaston Berger de Saint-Louis, UFR SAT, Département de Mathématiques
B.P234, Saint-Louis, Sénégal

Abstract

In this work we study the existence, uniqueness and asymptotic behavior of mild solutions for neutral stochastic partial integrodifferential equations with infinite delays. To prove the results, we use the theory of resolvent operators as developed by R. Grimmer [12], as well as a version of the fixed point principle. We establish sufficient conditions ensuring that the mild solutions are exponentially stable in pth-moment. An example is provided to illustrate the abstract results.

Keys words: Resolvent operators, C_0-semigroup, neutral stochastic partial integrodifferential equations, Wiener process, infinite delay, mild solutions, exponential stability.

1 Introduction

The study and importance of nonlinear stochastic delay partial differential equations delay are motivated by the fact that when one wants to model some evolution phenomena arising in Physics, Biology, Engineering, etc., some hereditary characteristics such as aftereffect, time lag, memory, and time delay can appear in the variables of the problem. Typical examples arise from the researches of materials with thermal memory, biochemical reactions, population models, etc. (see, for instance, Hale and Lunel [15], Murray [20], Ruess [24, 25], Wu [28], Caraballo et al. [4, 5, 6], Caraballo and Real [7], and references therein).

The existence, uniqueness and asymptotic behavior of solutions of stochastic partial differential equations have been considered by many authors (see for example [1, 2, 3, 8, 9, 10, 11, 14, 17, 26, 27]). Caraballo and Liu

*E-mail address: mamadou-abdoul.diop@ugb.edu.sn. Corresponding author
†E-mail address: caraball@us.es
‡E-mail address: mht.mhtzene@gmail.com

[8], Liu and Mao [19] and Taniguchi [26] discussed the exponential stability of the strong and mild solutions, by imposing some kind of coercivity condition, using the Lyapunov method and by a direct estimate of solutions, respectively. In particular, the Lyapunov direct method has some difficulties with the theory and application to specific problems when discussing the asymptotic behavior of solutions in stochastic differential equations. More recently, Luo [18] has studied the asymptotic stability of mild solutions of stochastic partial differential equations with finite delays using a fixed point approach which shows that some of these difficulties can be overcome with this fixed point theory. Moreover, systems with infinite delay deserve a study because they describe several interesting problems which are present in the real world. Therefore, it is interesting to study the stability problems for stochastic systems with infinite delays. However, to the best of our knowledge, no work has been reported on the existence of solutions and stability problems for stochastic integrodifferential equations with infinite delays. Motivated by the above considerations, in this paper we will establish sufficient conditions ensuring the existence and asymptotic stability of mild solutions to the following stochastic partial integrodifferential equations with infinite delays,

$$\begin{cases} d\left[x(t)+G(t,x(t-\rho(t)))\right] = A\left[x(t)+G(t,x(t-\rho(t)))\right]dt \\ \qquad\qquad + \displaystyle\int_0^t B(t-s)\left[x(s)+G(s,x(s-\rho(s)))ds\right]dt \\ \qquad\qquad\quad + b(t, \displaystyle\int_{-\infty}^0 g(\theta,x(t+\theta))d\theta)dt + h(t,\int_{-\infty}^0 \sigma(\theta,x(t+\theta))d\theta)dw(t), \quad t\geq 0, \\ x_0=\varphi\in\mathcal{B}, \end{cases} \tag{1.1}$$

here, the state $x(\cdot)$ takes values in a separable real Hilbert spaces H with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$, A is the infinitesimal generator of a strongly continous semigroup of bounded linear operators $S(t), t\geq 0$ on H, with domain $D(A)\subset H$, and $B(t)$, $t\geq 0$ is a closed linear operator on H. The history $x_t:(-\infty,0]\to H$, $x_t(\theta)=x(t+\theta)$, for $t\geq 0$, belongs to some abstract phase space $\mathcal{B}$ which will be described axiomatically in Section 2. Let K be another separable Hilbert spaces with inner product $\langle\cdot,\cdot\rangle_K$ and norm $\|\cdot\|_K$. Suppose that $\{w(s): 0\leq s\leq t\}$ is a given K-valued Wiener process with covariance operator $Q\geq 0$ defined on a complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is generated by the Wiener process $w(\cdot)$. We are also using the same notation $\|\cdot\|$ for the norm $\mathcal{L}(K;H)$, where $\mathcal{L}(K;H)$ denotes the space of all bounded linear operator from K into H.

We assume that $G,b:[0,+\infty)\times H\to H$, $h:[0,+\infty)\times H\to\mathcal{L}_2^0(K,H)$, $g,\sigma:(-\infty,0]\times H\to H$ are all Borel measurable, $\rho(t):[0,+\infty)\to[0,r]$ is continuous. Here $\mathcal{L}_2^0=\mathcal{L}_2(K_0;H)$ denotes the space of all Q-Hilbert-Schmidt operators (see [22]) from K_0 to H with the norm

$$|\xi|^2_{\mathcal{L}_2^0} := tr(\xi Q\xi^*) < \infty, \quad \xi\in\mathcal{L}(K,H).$$

The initial data $\varphi=\{\varphi(t):-\infty<t\leq 0\}$ is an $\mathcal{F}_0$-adapted, $\mathcal{B}$-valued random variable independent of the Wiener process w with finite second moment.

Our main results concerning (1.1), rely essentially on techniques based on the use of a strongly continuous family of operators $R(t), t\geq 0$ defined on the Hilbert space H and called their resolvent (the precise definition will be given below).

The paper is organized as follows: in Section 2 we recall some preliminaries which are used throughout this paper. In Section 3 we state the existence, uniqueness and asymptotic behavior of a mild solution. Finally, in Section 4, an example is given to illustrate our abstract results.

2 Preliminary Notes

2.1 Wiener process

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual conditions. We set $\mathcal{F}_t = \mathcal{F}_0$ for $t < 0$. We denote by $W = (W)_{t\geq 0}$ a K-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with covariance operator Q. That is,

$$E \langle w(t), x\rangle_K \langle w(t), y\rangle_K = (t \wedge s) \langle Qx, y\rangle_K , \ \forall x, y \in K,$$

where Q is a positive, self-adjoint trace class operator on K. For the construction of stochastic integral in Hilbert spaces, we refer to Da Prato and Zabczyk [22].

2.2 Partial integrodifferential equations in Banach spaces

In this section, we recall some fundamental facts needed to establish our results. Regarding the theory of resolvent operators we refer the reader to [12, 23]. Throughout the paper, H will denote a Banach space with norm $\|\cdot\|_H$, A and $B(t)$ are closed linear operators on H. Y represents the Banach space $D(A)$, the domain of operator A, equipped with the graph norm

$$\|y\|_Y := \|Ay\|_H + \|y\|_H \quad \text{for } \ y \in Y.$$

The notation $C([0, +\infty); Y)$ stands for the space of all continuous functions from $[0, +\infty)$ into Y. We then consider the following Cauchy problem

$$\begin{cases} v'(t) &=& Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds \ \text{ for } \ t \geq 0, \\ v(0) &=& v_0 \in H. \end{cases} \tag{2.1}$$

Definition 2.1. ([12]) A resolvent operator for equation (2.1) is a bounded linear operator valued function $R(t) \in \mathcal{L}(H)$ for $t \geq 0$, satisfying the following properties :

(i) $R(0) = I$ and $\|R(t)\| \leq Ne^{\beta t}$ for some constants N and β.

(ii) For each $x \in H$, $R(t)x$ is strongly continuous for $t \geq 0$.

(iii) For $x \in Y$, $R(\cdot)x \in C^1([0, +\infty); H) \cap C([0, +\infty); Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \qquad \text{for } \ t \geq 0. \end{aligned}$$

For additional details on resolvent operators, we refer the reader to [12, 23]. The resolvent operator plays an important role to study the existence of solutions and to establish a variation of constants formula for non–linear systems. For this reason, we need to know when the linear system (2.1) possesses a resolvent operator. Theorem 2.2 below provides a satisfactory answer to this problem.

In what follows we suppose the following assumptions:

(H1) A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ which is compact for $t > 0$ on H.

(H2) For all $t \geq 0$, $B(t)$ is a continuous linear operator from $(Y, \|\cdot\|_Y)$ into $(H, \|\cdot\|_H)$. Moreover, there exists an integrable function $c : [0,+\infty) \to \mathbb{R}^+$ such that for any $y \in Y$, $t \mapsto B(t)y$ belongs to $W^{1,1}([0,+\infty), H)$ and

$$\left\|\frac{d}{dt}B(t)y\right\|_H \leq c(t)\|y\|_Y \ \text{ for } \ y \in Y \ \text{ and } \ t \geq 0.$$

Theorem 2.2. *([12]) Assume that hypotheses* **(H1)** *and* **(H2)** *hold. Then equation* (2.1) *admits a resolvent operator* $(R(t))_{t\geq 0}$.

Theorem 2.3. *([16]) Assume that hypotheses* **(H1)** *and* **(H2)** *hold. Then, the corresponding resolvent operator $R(t)$ of the equation* (2.1) *is continuous for $t > 0$ in the operator norm, namely, for all $t_0 > 0$, it holds that* $\lim_{h\to 0}\|R(t_0+h) - R(t_0)\| = 0$.

In the sequel, we recall some results from [12] concerning the existence of solutions for the following integrodifferential equation

$$\begin{cases} v'(t) &=& Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds + q(t) \ \text{ for } \ t \geq 0, \\ v(0) &=& v_0 \in H, \end{cases} \tag{2.2}$$

where $q : [0,+\infty[\to H$ is a continuous function.

Definition 2.4. ([12]) A continuous function $v : [0,+\infty) \to H$ is said to be a strict solution of equation (2.2) if

(i) $v \in C^1([0,+\infty); H) \cap C([0,+\infty); Y)$,

(ii) v satisfies Eq. (2.2) for $t \geq 0$.

Remark 2.5. From this definition we deduce that $v(t) \in D(A)$, and the function $B(t-s)v(s)$ is integrable, for all $t > 0$ and $s \in [0,+\infty)$.

Theorem 2.6. *([12]) Assume that* ***(H1)-(H2)*** *hold. If v is a strict solution of the* (2.2)*, then the following variation of constants formula holds*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds \quad \textit{for} \quad t \geq 0. \tag{2.3}$$

Accordingly, we can establish the following definition.

Definition 2.7. ([12]) A function $v : [0,+\infty) \to H$ is called a mild solution of equation (2.2), for $v_0 \in H$, if v satisfies the variation of constants formula (2.3).

The next theorem provides sufficient conditions ensuring the regularity of solutions of the equation (2.2).

Theorem 2.8. *([12]) Let $q \in C^1([0,+\infty); H)$ and v be defined by* (2.3)*. If $v_0 \in D(A)$, then v is a strict solution of equation* (2.2).

In the sequel, we suppose that the phase space is axiomatically defined, and we use the approach proposed by Hale and Kato in [13]. To establish the axioms of the phase space $\mathcal{B}$ we follow the terminology used in Hino et al. [29]. The axioms of the phase space $\mathcal{B}$ are established for $\mathcal{F}_0$-measurable functions from $(-\infty, 0]$ into H, endowed with a seminorm and are the following:

(**A1**) If $x : (-\infty, T] \to H,\ T > 0$ is such that $x_0 \in \mathcal{B}$, then, for every $t \in [0, T]$, the following conditions hold:
(i) $x_t \in \mathcal{B}$,
(ii) $\|x(t)\| \leq L \|x_t\|_{\mathcal{B}}$,
(iii) $\|x_t\|_{\mathcal{B}} \leq u(t) \sup_{0 \leq s \leq t} \|x(s)\|_H + v(t) \|x_0\|_{\mathcal{B}}$,
where $L > 0$ is a constant; $u(\cdot), v(\cdot) : [0, +\infty) \to [1, +\infty)$, $u(.)$ is continuous, $v(\cdot)$ is locally bounded, and $L, u(\cdot), v(\cdot)$ are independent of $x(\cdot)$
(**A2**) The space $\mathcal{B}$ is complete.
(**A3**) For the functions $x(\cdot)$ in (**A1**), $t \to x_t$ is a $\mathcal{B}$-valued continuous function for $t \in [0, T]$.
(**A4**) If $(\varphi^n)_{n \in \mathbb{N}}$ is a sequence of continuous functions with compact support defined from $(-\infty, 0]$ into H, which converges to φ uniformly on compact subsets of $(-\infty, 0]$, then $\varphi \in \mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.
Before starting and proving the main results, we present the definition of the mild solution to (1.1).

Definition 2.9. An H-valued stochastic process $\{x(t), t \in (-\infty, T]\}$, $0 \leq T \leq \infty$, is called a mild solution of equation (1.1) if

(i) $x(t)$ is an $\mathcal{F}_t$-adapted, continuous process with $\int_0^T \|x(t)\|_H^p \, dt < \infty$ almost surely;

(ii) for $t \geq 0$, $x(t)$ satisfies the following integral equation:

$$\begin{aligned} x(t) &= R(t)[\varphi(0) + G(0, \varphi(-\rho(0)))] - G(t, x(t - \rho(t))) \\ &\quad + \int_0^t R(t-s) b(s, \int_{-\infty}^0 g(\theta, x(s+\theta)) d\theta) ds \\ &\quad + \int_0^t R(t-s) h(s, \int_{-\infty}^0 \sigma(\theta, x(s+\theta)) d\theta) dW(s), \end{aligned}$$

and $x_0 = \varphi \in \mathcal{B}$, i.e. $x(t) = \varphi(t)$ for $t \leq 0$.

Definition 2.10. Let $p \geq 2$ be an integer. The mild solution $x(t)$ of (1.1) with an initial value $\varphi \in \mathcal{B}$ is said to decay exponentially to zero in pth-moment if there exist some constants $M \geq 1$, $\eta > 0$ such that

$$E \|x(t)\|_H^p < M E \sup_{\theta \leq 0} \|\varphi(\theta)\|_H^p e^{-\eta t}, \quad t \geq 0.$$

3 Main Results

In this section we discuss the existence, uniqueness and asymptotic behavior of the mild solution to equation (1.1). In order to obtain our main result, we shall impose the following assumptions:

(H3) The resolvent operator given by Theorem 2.2 satisfies the following condition:

$$\|R(t)\| \leq e^{-\gamma t} \text{ for some constant } \quad \gamma > 0.$$

(H4) For $p \geq 2$, there exists a constant $K_G > 0$ such that for any $x, y \in H$, and $t \geq 0$,

$$\|G(t, x) - G(t, y)\|_H^p \leq K_G \|x - y\|_H^p.$$

(H5) The mappings $b : [0,+\infty) \times H \to H, h : [0,+\infty) \times H \to \mathcal{L}_2^0(K,H)$ satisfy Lipschitz conditions, i.e., there exist positive constants K_b, K_h such that, for any $x, y \in H$, and $t \in \mathbb{R}$,

$$\|b(t,x) - b(t,y)\|_H \leq K_b \|x-y\|_H, \|h(t,x) - h(t,y)\|_{\mathcal{L}_2^0} \leq K_h \|x-y\|_H.$$

Moreover, we assume that $\|b(t,0)\|_H = \|h(t,0)\|_{\mathcal{L}_2^0} = 0$.
(H6) There exist some positive constants $L_g,\ L_\sigma, \xi$, with $0 < \xi < \gamma$, and such that for all $t \in \mathbb{R},\ x, y \in H$,

$$\|g(t,x) - g(t,y)\|_H \leq L_g e^{-\xi|t|} \|x-y\|_H;$$
$$\|\sigma(t,x) - \sigma(t,y)\|_H \leq L_\sigma e^{-\xi|t|} \|x-y\|_H,$$

we further assume that $\|G(t,0)\|_H = \|\sigma(t,0)\|_H = \|g(t,0)\|_H = 0$.
Our main goal is to state and prove the following theorem.

Theorem 3.1. *Let $p \geq 2$ be an integer and assume that **(H1)-(H6)** are satisfied. Suppose also that*

$$3^{p-1}\left[K_G + K_b^p L_g^p (\xi\gamma)^{-p} + (2\gamma)^{\frac{-p}{2}} K_h^p L_\sigma^p \xi^{-p} C_p\right] < 1, \tag{3.1}$$

where $C_p = \left(\dfrac{p(p-1)}{2}\right)^{\frac{p}{2}}$. If an initial value $\varphi \in \mathcal{B}$ satisfies

$$E\|\varphi(t)\|_H^p \leq M_0 E\|\varphi(0)\|_H^p e^{-\mu t},\ t \leq 0,$$

for some $M_0 \geq 1$ and $0 < \mu < \xi$, then there exists a unique mild solution to (1.1) *associated to $\varphi(t)$ and decays exponentially to zero in pth moment.*

In order to prove the theorem, we first recall a useful lemma.

Lemma 3.2. *(Burkholder-Davis-Gundy inequality) ([22], p. 182) Let $l \geq 1$. Then for an arbitrary $\mathcal{L}_2^0$-valued predictable process $\Phi(t)$,*

$$\sup_{0\leq s\leq t} E\left\|\int_0^s \phi(u)dw(u)\right\|_H^{2l} \leq (l(2l-1))^l \left(\int_0^t \left(E\|\Phi(s)\|_{\mathcal{L}_2^0}^{2l}\right)^{\frac{1}{l}} ds\right)^l. \tag{3.2}$$

Proof of Theorem 3.1. Without loss of generality, we assume that $0 < \eta < \xi$. For the given initial datum $\varphi \in \mathcal{B}$, we denote by $\mathbf{S}$ the subset of the Banach space of all $\mathcal{F}_t$-adapted continuous processes $x(\cdot) : \mathbb{R} \to H$ endowed with the norm $\|x\|_{\mathbf{S}} := \sup_{t\in\mathbb{R}} E\|x(t)\|_H^p$, such that $x(t) = \varphi(t)$ for $t \leq 0$, and there exist some constants $M^* \geq 1$, $\eta > 0$ and $\eta < \xi$ depending on $x(\cdot)$, satisfying

$$E\|x(t)\|_H^p < M^* E\sup_{\theta\leq 0}\|\varphi(\theta)\|_H^p e^{-\eta t},\ t \geq 0,$$

which is a complete metric space for the distance induced by this norm.
Define a mapping $\pi : \mathbf{S} \to \mathbf{S}$ by $\pi(x)(t) = \varphi(t)$ for $t \leq 0$ and

$$\begin{aligned}
\pi(x)(t) &= R(t)[\varphi(0) + G(0,\varphi(-\rho(0)))] - G(t, x(t-\rho(t))) \\
&+ \int_0^t R(t-s)b(s, \int_{-\infty}^0 g(\theta, x(s+\theta))d\theta)ds \\
&+ \int_0^t R(t-s)h(s, \int_{-\infty}^0 \sigma(\theta, x(s+\theta))d\theta)dW(s) \quad \text{for } t \geq 0 \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.3}$$

We need to prove that $\pi(\mathbf{S}) \subset \mathbf{S}$ and that is contractive.
Let us first prove the continuity of $(\pi x)(t)$ on $t \geq 0$. To this end, let $x \in \mathbf{S}, t_1 \geq 0$ and $|r| > 0$ be sufficiently small. Notice that

$$E \left\|(\pi x)(t_1 + r) - (\pi x)(t_1)\right\|_H^p \leq 4^{p-1} \sum_{i=1}^{4} E \left\|I_i(t_1 + r) - I_i(t_1)\right\|_H^p .$$

Applying Lemma 3.2 together with assumption **(H3)**, it follows that

$$\begin{aligned}
& E \left\|I_4(t_1 + r) - I_4(t_1)\right\|_H^p \\
&= E \left\| \int_0^{t_1+r} R(t_1 + r - s) h(s, \int_{-\infty}^{0} \sigma(\theta, x(s+\theta)) d\theta) dW(s) \right. \\
&\quad \left. - \int_0^{t_1} R(t_1 - s) h(s, \int_{-\infty}^{0} \sigma(\theta, x(s+\theta)) d\theta) dW(s) \right\|_H^p \\
&\leq 2^{p-1} C_p \left\{ \left[\int_0^{t_1} \left(E \left\| (R(t_1 + r - s) - R(t_1 - s)) h(s, \int_{-\infty}^{0} \sigma(\theta, x(s+\theta)) d\theta) \right\|_{\mathcal{L}_2^0}^p \right)^{\frac{2}{p}} ds \right]^{\frac{p}{2}} \right. \\
&\quad \left. + \left[\int_{t_1}^{t_1+r} \left(E \left\| R(t_1 + r - s) h(s, \int_{-\infty}^{0} \sigma(\theta, x(s+\theta)) d\theta) \right\|_{\mathcal{L}_2^0}^p \right)^{\frac{2}{p}} ds \right]^{\frac{p}{2}} \right\} \\
&\leq 2^{p-1} C_p \left\{ \left[\int_0^{t_1} \left(\left\| R(t_1 + r - s) - R(t_1 - s) \right\|^p E \left\| h(s, \int_{-\infty}^{0} \sigma(\theta, x(s+\theta)) d\theta) \right\|_{\mathcal{L}_2^0}^p \right)^{\frac{2}{p}} ds \right]^{\frac{p}{2}} \right. \\
&\quad \left. + \left[\int_{t_1}^{t_1+r} \left(\left\| R(t_1 + r - s) \right\|^p E \left\| h(s, \int_{-\infty}^{0} \sigma(\theta, x(s+\theta)) d\theta) \right\|_{\mathcal{L}_2^0}^p \right)^{\frac{2}{p}} ds \right]^{\frac{p}{2}} \right\}.
\end{aligned}$$

Noting that for any $s \in [0, T],\ 0 \leq T < \infty$, we have

$$\begin{aligned}
& E \left\| h(s, \int_{-\infty}^{0} \sigma(\theta, x(s+\theta)) d\theta) \right\|_{\mathcal{L}_2^0}^p \\
&\leq K_h^p E \left[\int_{-\infty}^{0} \left\| \sigma(\theta, x(s+\theta)) \right\|_H d\theta \right]^p \\
&\leq K_h^p L_\sigma^p \left(\int_{-\infty}^{s} e^{\xi(\tau - s)} d\tau \right)^{p-1} \int_{-\infty}^{s} e^{\xi(\tau - s)} E \left\|x(\tau)\right\|_H^p d\tau \\
&\leq K_h^p L_\sigma^p \xi^{1-p} \left[\int_{-\infty}^{0} e^{\xi(\tau - s)} M_0 E \left\|\varphi(0)\right\|_H^p e^{-\mu\tau} d\tau + \int_0^s e^{\xi(\tau - s)} M^* E \sup_{\theta \leq 0} \left\|\varphi(\theta)\right\|_H^p e^{-\eta\tau} d\tau \right] \\
&\leq K_h^p L_\sigma^p \xi^{1-p} \left(\frac{M^* E \sup_{\theta \leq 0} \left\|\varphi(\theta)\right\|_H^p}{\xi - \eta} e^{-\eta s} + \frac{M_0 E \sup_{\theta \leq 0} \left\|\varphi(\theta)\right\|_H^p}{\xi - \mu} e^{-\xi s} \right) \\
&\leq L,
\end{aligned}$$

where L is a positive constant. Using the norm continuity of $R(t)$ for $t > 0$ and applying Lebesgue's dominated convergence theorem, it follows that

$$E \|I_4(t_1 + r) - I_4(t_1)\|_H^p \to 0 \quad \text{as } \ r \to 0.$$

Similarly, it is not difficult to check that $E \|I_i(t_1 + r) - I_i(t_1)\|_H^p \to 0 \quad \text{as } r \to 0,\ i = 1, 2, 3.$
Next, we show that $\pi(\mathbf{S}) \subset \mathbf{S}$. Let $x \in \mathbf{S}$. From the definition of π, we have

$$\begin{aligned} E\,|\pi(x)(t)|_H^p \ &\leq\ 4^{p-1} E\,|R(t)[\varphi(0) + G(0,\varphi)]|_H^p + 4^{p-1} E\,|G(t, x(t-\rho(t)))|_H^p \\ &\quad + 4^{p-1} E \left\| \int_0^t R(t-s) b(s, \int_{-\infty}^0 g(\theta, x(s+\theta)) d\theta) ds \right\|_H^p \\ &\quad + 4^{p-1} E \left\| \int_0^t R(t-s) h(s, \int_{-\infty}^0 \sigma(\theta, x(s+\theta)) d\theta) dW(s) \right\|_H^p \\ &:=\ 4^{p-1}(I_1 + I_2 + I_3 + I_4). \end{aligned} \tag{3.4}$$

By assumption (**H4**), we obtain

$$\begin{aligned} E\,\|G(t, x(t-\rho(t)))\|_H^p \ &\leq\ E\,\|G(t, x(t-\rho(t))) - G(t, 0)\|_H^p \\ &\leq\ K_G E\,|x(t-\rho(t))|_H^p \\ &\leq\ K_G \left(M^* e^{\eta r} E \sup_{\theta \leq 0} \|\varphi(\theta)\|_H^p e^{-\eta t} + M_0 e^{\mu r} E\,\|\varphi(0)\|_H^p e^{\mu t} \right). \end{aligned} \tag{3.5}$$

For the term I_3, by an application of Hölder inequality and assumption **(H4)**, it follows that

$$\begin{aligned} I_3 \ &\leq\ E \left[\int_0^t \left\| R(t-s) b(s, \int_{-\infty}^0 g(\theta, x(s+\theta)) d\theta) \right\|_H ds \right]^p \\ &\leq\ K_b^p E \left[\int_0^t e^{-\gamma(t-s)} \left\| \int_{-\infty}^0 g(\theta, x(s+\theta)) d\theta \right\|_H ds \right]^p \\ &\leq\ K_b^p \left[\int_0^t e^{-\gamma(t-s)} ds \right]^{p-1} E \int_0^t \left\| e^{\frac{-\gamma(t-s)}{p}} \int_{-\infty}^0 g(\theta, x(s+\theta)) d\theta \right\|_H^p ds \\ &\leq\ K_b^p \gamma^{1-p} E \int_0^t \left[\int_{-\infty}^s L_g e^{\frac{-\gamma(t-s)}{p}} e^{\xi(\tau - s)} \|x(\tau)\|_H d\tau \right]^p ds \\ &\leq\ K_b^p \gamma^{1-p} L_g^p \int_0^t \left[\left(\int_{-\infty}^s e^{\xi(\tau-s)} d\tau \right)^{p-1} \int_{-\infty}^s e^{-\gamma(t-s)} e^{\xi(\tau-s)} E\,\|x(\tau)\|_H^p d\tau \right] ds \\ &\leq\ K_b^p \gamma^{1-p} L_g^p \xi^{1-p} \int_0^t \left[\int_{-\infty}^0 e^{-\gamma(t-s)} e^{\xi(\tau-s)} M_0 E\,\|\varphi(0)\|_H^p e^{-\mu\tau} d\tau \right] ds \\ &\quad + K_b^p \gamma^{1-p} L_g^p \xi^{1-p} \int_0^t \left[\int_0^s e^{-\gamma(t-s)} e^{\xi(\tau-s)} M^* E \sup_{\theta \leq 0} \|\varphi(\theta)\|_H^p e^{-\eta\tau} d\tau \right] ds \\ &\leq\ K_b^p \gamma^{1-p} L_g^p \xi^{1-p} \left[\frac{M^* E \sup_{\theta \leq 0} \|\varphi(\theta)\|_H^p}{(\gamma - \eta)(\xi - \eta)} e^{-\eta t} + \frac{M_0 E\,\|\varphi(0)\|_H^p}{(\gamma - \xi)(\xi - \mu)} e^{-\xi t} \right]. \end{aligned} \tag{3.6}$$

Taking into account Lemma 3.2 and assumption (**H5**), we obtain that

$$
\begin{aligned}
I_4 &= E\left\|\int_0^t R(t-s)h(s,\int_{-\infty}^0 \sigma(\theta,x(s+\theta))d\theta)dW(s)\right\|_H^p \\
&\leq C_p\left\{\int_0^t\left[E\left\|R(t-s)h(s,\int_{-\infty}^0 \sigma(\theta,x(s+\theta))d\theta)\right\|_{\mathcal{L}_2^0}^p\right]^{\frac{2}{p}} ds\right\}^{\frac{p}{2}} \\
&\leq K_h^p C_p\left\{\int_0^t e^{-2\gamma(t-s)}\left[E\left\|\int_{-\infty}^0 \sigma(\theta,x(s+\theta))d\theta\right\|_H^p\right]^{\frac{2}{p}} ds\right\}^{\frac{p}{2}},
\end{aligned}
$$

where $C_p = \left(\dfrac{p(p-1)}{2}\right)^{\frac{p}{2}}$.
By assumption **(H5)** and Hölder inequality, we deduce

$$
\begin{aligned}
I_4 &\leq K_h^p C_p L_\sigma^p\left\{\int_0^t e^{-2\gamma(t-s)}\left[E\left(\int_{-\infty}^s e^{\xi(\tau-s)}\|x(\tau)\|_H d\tau\right)^p\right]^{\frac{2}{p}} ds\right\}^{\frac{p}{2}} \\
&\leq K_h^p C_p L_\sigma^p\left\{\int_0^t e^{-2\gamma(t-s)}\left[\left(\int_{-\infty}^s e^{\xi(\tau-s)}d\tau\right)^{p-1}\int_{-\infty}^s e^{\xi(\tau-s)}E\|x(\tau)\|_H^p d\tau\right]^{\frac{2}{p}} ds\right\}^{\frac{p}{2}} \\
&\leq K_h^p C_p L_\sigma^p \xi^{1-p}\left\{\int_0^t e^{-2\gamma(t-s)}\left[\int_{-\infty}^s e^{\xi(\tau-s)}E\|x(\tau)\|_H^p d\tau\right]^{\frac{2}{p}} ds\right\}^{\frac{p}{2}}.
\end{aligned}
$$

Noting that $p \geq 2$, we then have

$$
I_4 \leq K_h^p C_p L_\sigma^p \xi^{1-p}\left\{\int_0^t e^{-2\gamma(t-s)}\left[\int_0^s e^{\xi(\tau-s)}M^* E\sup_{\theta\leq 0}\|\varphi(\theta)\|_H^p e^{-\eta\tau}d\tau + \int_{-\infty}^0 e^{\xi(\tau-s)}M_0 E\|\varphi(0)\|_H^p e^{-\mu\tau}d\tau\right]^{\frac{2}{p}} ds\right\}^{\frac{p}{2}}
$$

$$
\begin{aligned}
&\leq K_h^p C_p L_\sigma^p \xi^{1-p}\left\{\int_0^t e^{-2\gamma(t-s)}\left[\left(\frac{M^* E\sup_{\theta\leq 0}\|\varphi(\theta)\|_H^p}{\xi-\eta}e^{-\eta s}\right)^{\frac{2}{p}} + \left(\frac{M_0 E\|\varphi(0)\|_H^p}{\xi-\mu}e^{-\xi s}\right)^{\frac{2}{p}}\right] ds\right\}^{\frac{p}{2}} \quad (3.7) \\
&\leq 2^{\frac{p-2}{2}} K_h^p C_p L_\sigma^p \xi^{1-p}\left[\frac{M^* E\sup_{\theta\leq 0}\|\varphi(\theta)\|_H^p}{\xi-\eta}\left(\frac{p}{2p\gamma-2\eta}\right)^{\frac{p}{2}} e^{-\eta t} + \frac{M_0 E\|\varphi(0)\|_H^p}{\xi-\mu}\left(\frac{p}{2p\gamma-2\xi}\right)^{\frac{p}{2}} e^{-\xi t}\right].
\end{aligned}
$$

Recalling (3.3), from (3.4) to (3.7), we can deduce that there exists $M_1 \geq 1$ such that

$$
E\|(\pi x)(t)\|_H^p \leq M_1 E\sup_{\theta\leq 0}\|\varphi(\theta)\|_H^p e^{-\eta t}.
$$

Since each term of $(\pi x)(t)$ is $\mathcal{F}_t$-measurable then the $\mathcal{F}_t$-measurability of $(\pi x)(t)$ is easily verified. It follows that π is well defined.
Thus, we conclude that $\pi(\mathbf{S}) \subset \mathbf{S}$.
It remains to show that π has a unique fixed point. For any $x, y \in \mathbf{S}$, we have

$$E\left\|(\pi x)(t)-(\pi y)(t)\right\|_H^p \le 3^{p-1}\sum_{i=1}^{3} J_i. \tag{3.8}$$

We now estimate each J_i in (3.8). Noting that $x(s)=y(s)=\varphi(s)$ for $s\le 0$, by assumption (**H3**), we have

$$\begin{aligned}
J_1 &= E\left\|G(t,x(t-\rho(t)))-G(t,y(t-\rho(t)))\right\|_H^p \\
&\le K_G E\left|x(t-\rho(t))-y(t-\rho(t))\right|_H^p \\
&\le K_G \sup_{t\ge 0} E\left\|x(s)-y(s)\right\|_H^p .
\end{aligned}$$

By standard computations involving (**H6**) and Hölder inequality we obtain

$$\begin{aligned}
J_2 &= E\left\|\int_0^t R(t-s)\left(b(s,\int_{-\infty}^0 g(\theta,x(s+\theta)))d\theta)-b(s,\int_{-\infty}^0 g(\theta,y(s+\theta)))d\theta\right)ds\right\|_H^p \\
&\le K_b^p E\left[\int_0^t e^{-\gamma(t-s)}\left\|\int_{-\infty}^0 g(\theta,x(s+\theta))d\theta-\int_{-\infty}^0 g(\theta,y(s+\theta))d\theta\right\|_H ds\right]^p \\
&\le K_b^p E\left(\int_0^t e^{-\gamma(t-s)}ds\right)^{p-1}\int_0^t\left(\int_{-\infty}^0\left\|e^{\frac{-\gamma(t-s)}{p}}\Big(g(\theta,x(s+\theta))-g(\theta,y(s+\theta))\Big)\right\|_H d\theta\right)^p ds \\
&\le K_b^p\gamma^{1-p}L_g^p\int_0^t\left(\int_{-\infty}^s e^{\xi(\tau-s)}d\tau\right)^{p-1}\left(\int_{-\infty}^s e^{-\gamma(t-s)}e^{\xi(\tau-s)}E\left\|x(\tau)-y(\tau)\right\|_H^p d\tau\right)^p ds \\
&\le K_b^p\gamma^{-p}L_g^p\xi^{-p}\sup_{s\ge 0}E\left\|x(s)-y(s)\right\|_H^p ,
\end{aligned}$$

and

$$\begin{aligned}
J_3 &= E\left\|\int_0^t R(t-s)\left[h(s,\int_{-\infty}^0\sigma(\theta,x(s+\theta)))d\theta-h(s,\int_{-\infty}^0\sigma(\theta,y(s+\theta)))d\theta\right]dW(s)\right\|_H^p \\
&\le C_pK_h^p\left\{\int_0^t e^{-2\gamma(t-s)}\left[E\left\|\int_{-\infty}^0\left(\sigma(\theta,x(s+\theta))-\sigma(\theta,y(s+\theta))\right)d\theta\right\|_H^p\right]^{\frac{2}{p}}ds\right\}^{\frac{p}{2}} \\
&\le C_pK_h^pL_\sigma^p\left\{\int_0^t e^{-2\gamma(t-s)}\left[E\left(\int_{-\infty}^s e^{\xi(\tau-s)}\left\|x(\tau)-y(\tau)\right\|_H d\tau\right)^p\right]^{\frac{2}{p}}ds\right\}^{\frac{p}{2}} \\
&\le C_pK_h^pL_\sigma^p\left\{\int_0^t e^{-2\gamma(t-s)}\left[\left(\int_{-\infty}^s e^{\xi(\tau-s)}d\tau\right)^{p-1}\int_{-\infty}^s e^{\xi(\tau-s)}E\left\|x(\tau)-y(\tau)\right\|_H d\tau\right]^{\frac{2}{p}}ds\right\}^{\frac{p}{2}} \\
&\le C_pK_h^pL_\sigma^p\xi^{-p}\left[\int_0^t e^{-2\gamma(t-s)}ds\right]^{\frac{p}{2}}\sup_{s\ge 0}E\left\|x(s)-y(s)\right\|_H^p \\
&\le C_pK_h^pL_\sigma^p\xi^{-p}(2\gamma)^{\frac{-p}{2}}\sup_{s\ge 0}E\left\|x(s)-y(s)\right\|_H^p .
\end{aligned}$$

Consequently, we have

$$\sup_{s\ge 0}E\left\|(\pi x)(t)-(\pi y)(t)\right\|_H^p \le 3^{p-1}\left[K_G+K_b^pL_g^p(\xi\gamma)^{-p}+(2\gamma)^{\frac{-p}{2}}K_h^pL_\sigma^p\xi^{-p}C_p\right]\sup_{s\ge 0}E\left\|x(s)-y(s)\right\|_H^p ,$$

and by (3.1) it follows that π is contractive. Thus, the Banach fixed point principle implies that there exists a unique $x(\cdot)\in\mathcal{S}$ which solves (1.1) with $x(s)=\varphi(s)$ on $(-\infty,0]$, and furthermore, $x(t)$ decays exponentially to zero in pth-moment. The proof is therefore complete.

4 Example

In this section we make use of our previous existence result to study the existence, uniqueness and asymptotic behavior of mild solutions to concrete neutral stochastic partial integrodifferential equations with infinite delays. For that, let $\Omega \subset \mathbb{R}^2$ be an open subset whose boundary $\partial\Omega$ is sufficiently regular. Let $H = H_0^1(\Omega) \times L^2(\Omega)$ and consider the linear operator A whose domain is given by $D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ and

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \alpha(0)x'' - \beta(0)y \end{pmatrix}$$

where $\alpha(.)$, $\beta(.)$ are real-valued functions of class $\mathcal{C}^2$ on $[0,\infty)$ such that $\alpha(0) > 0$ and $\beta(0) > 0$.
In Chen [21] it is proved that A is the infinitesimal generator of a uniformly exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on $H_0^1(\Omega) \times L^2(\Omega)$. In what follows, we will assume $\tilde{M}$, γ are positive constants and that $\|T(t)\| \leq \tilde{M}e^{-\gamma t}$ for all $t > 0$. Let $B(t) = F(t)A$ where $F : H_0^1(\Omega) \times L^2(\Omega) \to H_0^1(\Omega) \times L^2(\Omega)$ is the operator family defined by

$$F = (F_{ij}) = \begin{pmatrix} 0 & 0 \\ -\beta'(t) + \beta(0)\dfrac{\alpha'(t)}{\alpha(0)} & \dfrac{\alpha'(t)}{\alpha(0)} \end{pmatrix}.$$

Assume that

$$\begin{aligned} \max\left\{ \left|\frac{\alpha'(t)}{\alpha(0)}\right|, \left|-\beta'(t) + \beta(0)\frac{\alpha'(t)}{\alpha(0)}\right| \right\} &\leq \frac{\gamma}{2\tilde{M}}e^{-\gamma t}, \qquad t \geq 0, \\ \max\left\{ \left|\frac{\alpha''(t)}{\alpha(0)}\right|, \left|-\beta''(t) + \beta(0)\frac{\alpha''(t)}{\alpha(0)}\right| \right\} &\leq \frac{\gamma^2}{4\tilde{M}^2}e^{-\gamma t}, \qquad t \geq 0. \end{aligned} \tag{4.1}$$

From Theorem 4.1 in Grimmer [12] we deduce that the abstract integro-differential system

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds,$$

possesses an associated uniformly exponentially stable resolvent of operators $(R(t))_{t\geq 0}$ on $H_0^1(\Omega) \times L^2(\Omega)$ with

$$\|R(t)\| \leq \tilde{M}e^{\frac{-\gamma}{2}t}, \text{ for } t \geq 0.$$

This integro-differential system was discussed by Grimmer to illustrate his result in (Theorem 4.1, Ref.[12]) about exponential stability for resolvent operators.
Here, we will use the phase space $\mathcal{B} := C_r \times L^p(z, H_0^1(\Omega) \times L^2(\Omega))$ $\quad r \geq 0, \quad 1 \leq p < \infty$. Let $z : (-\infty, -r) \to \mathbb{R}$ be a positive (Lebesgue) integrable function and assume that there exists a nonnegative and locally bounded function γ_1 on $(-\infty, 0]$ such that $z(\xi + \theta) \leq \gamma_1(u)z(\theta)$ for all $u \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$ where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero.
The space $C_r \times L^p(z, H_0^1(\Omega) \times L^2(\Omega))$ consists of the collection of all functions $\varphi : (-\infty, 0] \to H_0^1(\Omega) \times L^2(\Omega)$ such that is continuous on $[-r, 0]$, Lebesgue mesurable and $z\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $\|.\|_B$ is defined by

$$\|\varphi\|_B := \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} z(\theta)\|\varphi(\theta)\|^p d\theta\right)^{1/p}.$$

Under the previous assumptions, the phase space $\mathcal{B}$ verifies the axioms: (**A1**), (**A2**),(**A3**),(**A4**), see Theorem 1.3.8 in [29]. Moreover, when $r = 0$ we have that $L = 1$,

$$\begin{aligned} v(t) &= \gamma(-t)^{1/2}, \\ u(t) &= 1 + \left(\int_{-t}^{0} h(\theta) d\theta \right)^{1/2} \quad \text{for} \quad t \geq 0. \end{aligned}$$

Consider the neutral system

$$\begin{cases} \dfrac{\partial}{\partial t} \left[\beta(t,u) + \dfrac{\beta(t-\rho(t),u)}{1+|\beta(t-\rho(t),u)|} \right] = A \left[\beta(t,u) + \dfrac{\beta(t-\rho(t),u)}{1+|\beta(t-\rho(t),u)|} \right] \\ \qquad + \displaystyle\int_0^t F(t-s) A \left[\beta(s,u) + \dfrac{\beta(s-\rho(s),u)}{1+|\beta(s-\rho(s),u)|} \right] ds \\ \qquad + f_1 \left(t, \displaystyle\int_{-\infty}^{0} \alpha_2 e^{\xi\theta} \beta(t+\theta,u) d\theta \right) dt + f_2 \left(t, \displaystyle\int_{-\infty}^{0} \alpha_3 e^{\xi\theta} \beta(t+\theta,u) d\theta \right) dw(t), \quad \text{for } t \geq 0 \\ \\ \beta(\theta,u) = \beta_0(\theta,u) \quad \text{for} \quad \theta \in]-\infty, 0] \quad \text{and} \quad u \leq 0, \end{cases} \tag{4.2}$$

where $\xi, \alpha_i > 0, i = 1,2,3$, $w(t)$ denotes an $\mathbb{R}$-valued Brownian motion, $\rho : [0,+\infty) \to [0,r]$.
Let

$$G(t, \beta(t-\rho(t),u)) = \frac{\beta(t-\rho(t),u)}{1+|\beta(t-\rho(t),u)|},$$

$$b(t, \int_{-\infty}^{0} g(\theta, \beta(t+\theta,u)) d\theta) = f_1 \left(t, \int_{-\infty}^{0} \alpha_2 e^{\xi\theta} \beta(t+\theta,u) d\theta \right),$$

$$h(t, \int_{-\infty}^{0} \sigma(\theta, \beta(t+\theta,u)) d\theta) = f_2 \left(t, \int_{-\infty}^{0} \alpha_3 e^{\xi\theta} \beta(t+\theta,u) d\theta \right).$$

If we put

$$\begin{cases} x(t) = \beta(t,u) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad u \leq 0 \\ \varphi(\theta)(u) = \beta_0(\theta,u) \quad \text{for} \quad \theta \in]-\infty,0] \quad \text{and} \quad u \leq 0, \end{cases}$$

then equation (4.2) takes the following abstract form

$$\begin{cases} d\left[x(t) + G(t, x(t-\rho(t)))\right] = A\left[x(t) + G(t, x(t-\rho(t)))\right] dt \\ \qquad + \displaystyle\int_0^t B(t-s) \left[x(s) + G(s, x(s-\rho(s))) ds\right] dt \\ \qquad + b(t, \displaystyle\int_{-\infty}^{0} g(\theta, x(t+\theta)) d\theta) dt + h(t, \displaystyle\int_{-\infty}^{0} \sigma(\theta, x(t+\theta)) d\theta) dw(t), \quad t \geq 0, \\ x_0 = \varphi, \end{cases} \tag{4.3}$$

We assume that there exist some positive constants K_{f_i}, $i = 1,2$ such that for any $x, y \in H$ $t \geq 0$,

$$\|f_1(t,x) - f_1(t,y)\|_H \leq K_{f_1} \|x-y\|_H, \|f_2(t,x) - f_2(t,y)\|_H \leq K_{f_2} \|x-y\|_H.$$

Then it is obvious that the assumption (**H1**)-(**H6**) are satisfied with

$$K_G = 1, K_b = K_{f_1}, K_h = K_{f_2}, L_G = \alpha_2, L_\sigma = \alpha_3.$$

Thus, by Theorem 3.1, if

$$E\left\|\varphi(t)\right\|_H^p \leq M_0 E\left\|\varphi(0)\right\|_H^p e^{-\mu t},\ t \leq 0,$$

for some $M_0 \geq 1$ and $0 < \mu < \xi$, then there exists a unique mild solution of (4.3) and decays exponentially to zero in p-th moment provided

$$3^{p-1}\left[K_G + K_b^p L_g^p (\xi\gamma)^{-p} + (2\gamma)^{\frac{-p}{2}} K_h^p L_\sigma^p \xi^{-p} C_p\right] < 1.$$

Acknowledgements. The authors would like to thank the referee for the helpful comments and suggestions which allowed to improve the presentation of the paper.
This work has been partially supported by FEDER and the Spanish Ministerio de Economía y Competitividad project MTM2011-22411 and the Consejería de Innovación, Ciencia y Empresa (Junta de Andalucía) under grant 2010/FQM314 and Proyecto de Excelencia P12-FQM-1492.

References

[1] T. A. Burton. Fixed points, stability and exact linearization. Nonlinear Analysis, 61: 857-870, 2005.

[2] T. A. Burton and F. Tetsuo. Asymptotic behavior of solutions of functional differential equations by fixed point theorems. Dynamical Systems and Applications, 11: 499-521, 2002.

[3]] T. A. Burton and B. Zhang. Fixed points and stability of an integral equation: nonuniqueness. Appl. Math. Lett, 17:839-846, 2004.

[4] T. Caraballo, I.D. Chueshov, P. Marín-Rubio, and J. Real. Existence and asymptotic behaviour for stochastic heat equations with multiplicative noise in materials with memory. Discrete Contin. Dyn. Syst. Ser. A, 18:253-270, 2007.

[5] T. Caraballo, M.J. Garrido-Atienza and T. Taniguchi. The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion. Nonlinear Anal. 74(11): 3671-3684, 2011.

[6] T. Caraballo, P. Marín-Rubio and J. Valero. Autonomous and non-autonomous attractors for differential equations with delays. J. Differential Equations 208(1):9-41, 2005.

[7] T. Caraballo, and J. Real. Attractors for 2D-Navier-Stokes models with delays. J. Differential Equations 205(2):271-297, 2004.

[8] T. Caraballo and K. Liu. Exponential stability of mild solutions of stochastic partial differential equations with delays. Stochastic Analysis and Applications, 17(5):743-763, 1999. 13

[9] T. Caraballo and J. Real. Partial differential equations with delay and random perturbations: existence, uniqueness and stability of solutions. Stochastic Analysis and Applications, 11(5):497-511, 1993.

[10] Jing Cui and Litan Yan. Asymptotic behavior for neutral stochastic partial differential equations with infinite delays. Electron. Commun. Probab, 45:1-12, 2013.

[11] E.M.Hernandez, E. M. Henriquez H.R., and C.José Paulio. Existence results for abstract partial neutral integro-differential equation with unbounded delay. Electr, J. Qualitative Th.Diff, Equa., 29:1-23, 2009.

[12] R. Grimmer. Resolvent operators for integral equations in a banach space. Transactions of the American Mathematical Society, 273(1): 333-349, 1982.

[13] J.K. Hale and J. Kato. Phase spaces for retarded equations with infinite delay. Funkcial. Ekvac, 21:11-41, 1978.

[14] X. Sun J. Cui, L. Yan. Exponential stability for neutral stochastic partial differential equations with delays and poisson jumps. Statistics and Probability Letters, 81:1970-1977, 2011.

[15] J.K.Hale and S.M.Verdyn Lunel. Introduction to Functional Differential Equations. Springer-Verlag, 1990.

[16] J.Liang, J.H.Liu, and T.J.Xiao. Nonlocal problems for integrodifferential equations. Dynamics of Continuous, Discrete and Impulsive Systems, Series A, mathematical analysis, 15:815-824, 2008.

[17] J.Luo. Fixed points and stability of neutral stochastic delay differential equations. J. Math. Anal. Appl., 334:431-440, 2007.

[18] J.Luo. Fixed points and exponential stability of mild solutions of stochastic partial differential equations with delays. J. Math. Anal. Appl., 342:753-760, 2008.

[19] K. Liu and X. Mao. Exponential stability of non-linear stochastic evolution equations. Stochastic Process.Appl, 78:173-193, 1998.

[20] J.D. Murray. Mathematical Biology. Springer-Verlag, Berlin, 1993.

[21] G. Chen, Control and stabilization for the wave equation in a bounded domain, SIAM J. Control Optim. 17 (1) (1979) 66-81.

[22] G. Da Prato and J. Zabczyk. Stochastic Equations in Infinite Dimensions. Cambridge University Press Cambridge, 1992.

[23] J. Pruss. Evolutionary Integral Equations and Applications. Birkhauser, 1993.

[24] W. Ruess. Existence of solutions to partial functional differential equations with delay. A.G. Kartsatos (Ed.), Theory and Applications of onlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, pages 259-288, 1996.

[25] W. Ruess. Existence and stability of solutions to partial functional differential equations with delay. Adv. Differential Equations 4, 6:843-876, 1999.

[26] T.Taniguchi. The exponential stability for stochastic delay partial differential equations. J. Math. Anal. Appl, 331:191-205, 2007.

[27] T.Taniguchi. The existence and asymptotic behavior of mild solution to stochastic evolution equations with infinite delay dreven by poisson jumps. Stoch.Dyn, 9(2):217-229, 2009.

[28] J. Wu. W. Ruess, Existence and stability of solutions to partial functional differential equations with delay. Springer-Verlag, New York, 1996.

[29] Y.Hino, S. Murakami, and T. Naito. Functional differential equations with Infinite Delay, volume 1473. Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1991.

Received: Month xx, 20xx