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Locally Quasi-Homogeneous Free Divisors Are Koszul Free[1]

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Let X be a complex analytic manifold and $D \subset X$ be a free divisor. If D is locally quasi-homogeneous, then the logarithmic de Rham complex associated to D is quasi-isomorphic to $\mathbf{R}j_*(\mathbb{C}_{X\setminus D})$, which is a perverse sheaf. On the other hand, the logarithmic de Rham complex associated to a *Koszul*-free divisor is perverse. In this paper, we prove that every locally quasi-homogeneous free divisor is Koszul free.

1. INTRODUCTION

Let X be a complex analytic manifold. For a divisor $D \subset X$, let us write $j\colon U = X \setminus D \hookrightarrow X$ for the corresponding open inclusion and $\Omega^\bullet(*D)$ for the meromorphic de Rham complex with poles along D. In [5], Grothendieck proved that the canonical morphism $\Omega^\bullet(*D) \to \mathbf{R}j_*(\mathbb{C}_U)$ is an isomorphism (in the derived category). This result is usually known as (a version of) *Grothendieck's Comparison Theorem*.

In [9], K. Saito introduced the subcomplex $\Omega^\bullet_X(\log D)$ of $\Omega^\bullet(*D)$, which he called a *logarithmic de Rham complex* associated to D, generalizing the well-known case of normal crossing divisors (see [4]). In the same paper, K. Saito also introduced the important notion of *free divisor*.

In [3], it is proved that the logarithmic de Rham complex $\Omega^\bullet_X(\log D)$ computes the cohomology of the complement U if D is a locally quasi-homogeneous free divisor (we say that D satisfies the *logarithmic comparison theorem*). In other words, the canonical morphism $\Omega^\bullet_X(\log D) \to \mathbf{R}j_*(\mathbb{C}_U)$ is an isomorphism, or, using Grothendieck's result, the inclusion $\Omega^\bullet_X(\log D) \hookrightarrow \Omega^\bullet(*D)$ is a quasi-isomorphism. In fact, in [2] it is proved that, in the case of $\dim X = 2$, D is locally quasi-homogeneous if and only if it satisfies the logarithmic comparison theorem.

Since the derived direct image $\mathbf{R}j_*(\mathbb{C}_U)$ is a perverse sheaf (it is the de Rham complex of the holonomic module of meromorphic functions with poles along D [7, II, Theorem 2.2.4]), we deduce that the logarithmic comparison theorem for a free divisor D implies that the logarithmic de Rham complex associated to D is a perverse sheaf.

On the other hand, the first author proved in [1] the following results. Let $D \subset X$ be a Koszul-free divisor (see Definition 2.3) and $\mathcal{I}$ be the left ideal of the ring $\mathcal{D}_X$ of differential operators on X generated by the logarithmic vector fields with respect to D. Then,

1) The left $\mathcal{D}_X$-module $\mathcal{D}_X/\mathcal{I}$ is holonomic.

2) There is a canonical isomorphism in the derived category

$$\Omega^\bullet_X(\log D) \simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{I}, \mathcal{O}_X).$$

As a consequence of these results, the logarithmic de Rham complex associated to a Koszul-free divisor is a perverse sheaf.

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In this paper, we prove the following result, suggested by the previous ones: every locally quasi-homogeneous free divisor is Koszul free (see Theorem 3.2).

At the end, we study some examples in dimensions two and three.

2. PRELIMINARY RESULTS

Let X be an n-dimensional complex analytic manifold. We denote by $\pi\colon T^*X \to X$ the cotangent bundle, by $\mathcal{O}_X$ the sheaf of holomorphic functions on X, by $\mathcal{D}_X$ the sheaf of linear differential operators on X (with holomorphic coefficients), by $\mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$ the graded ring associated to the filtration by the order, and by $\sigma(P)$ the principal symbol of a differential operator P. We will denote by $\mathcal{O} = \mathcal{O}_{X,x}$, $\mathcal{D} = \mathcal{D}_{X,x}$, and $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)_x$ the respective stalks at x, with a point x in X. Let $D \subset X$ be a hypersurface. We denote by $\mathcal{D}\mathrm{er}(\log D)$ the $\mathcal{O}_X$-module of the logarithmic vector fields with respect to D [9].

Definition 2.1. A divisor D is *Euler homogeneous* at x if there are a local equation h for D around x and a germ of logarithmic vector field δ such that $\delta(h) = h$.

The set of points where a divisor is Euler homogeneous is open.

Definition 2.2 (see [3])**.** A divisor D in an n-dimensional complex manifold X is *locally quasi-homogeneous* if, at each point $q \in D$, there are local coordinates $(U; x_1, \dots, x_n)$ centered at q (i.e., with $x_i(q) = 0$ for $i = 1, \dots, n$) with respect to which $D \cap U$ has a weighted homogeneous defining equation (with strictly positive weights).

Obviously, a locally quasi-homogeneous divisor is Euler homogeneous at every point.

Definition 2.3 [1, Definition 4.1.1]**.** Let $D \subset X$ be a divisor. We say that D is a *Koszul-free* divisor at x if there exists a basis $\{\delta_1, \dots, \delta_n\}$ of $\mathcal{D}\mathrm{er}(\log D)_x$ such that the sequence of symbols $\{\sigma(\delta_1), \dots, \sigma(\delta_n)\}$ is regular in $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)_x$. If D is a Koszul-free divisor at each point of D, we simply say that it is a Koszul-free divisor.

Remark 2.4. The ideal $I_{D,x} = \mathrm{Gr}_{F^\bullet}(\mathcal{D})\mathcal{D}\mathrm{er}(\log D)_x$ is generated by the elements of any basis of $\mathcal{D}\mathrm{er}(\log D)_x$. Since D is Koszul free at x if and only if $\mathrm{depth}(I_{D,x}, \mathrm{Gr}_{F^\bullet}(\mathcal{D})) = n$ (see [6, Corollary 16.8]), it is clear that the definition of a Koszul-free divisor does not depend on the choice of a particular basis. By the coherence of $\mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$, if a divisor is Koszul free at a point, then it is Koszul free near this point.

We have not found a reference for the following well-known proposition (see [6, Theorem 17.4] for the local case).

Proposition 2.5. *Let $\mathbb{C}\{x\}$ be the ring of convergent power series in the variables $x = (x_1, \dots, x_n)$, and let G be the graded ring of polynomials in the variables $\xi_1, \dots, \xi_t$ with coefficients in $\mathbb{C}\{x\}$. A sequence $\sigma_1, \dots, \sigma_s$ of homogeneous polynomials in G is regular if and only if the set of zeros $V(I)$ of the ideal I generated by $\sigma_1, \dots, \sigma_s$ has dimension $n + t - s$ in $U \times \mathbb{C}^t$ for some open neighborhood U of 0 (then, each irreducible component has dimension $n + t - s$).*

Proof. Let $\mathbb{C}\{x, \xi\}$ be the ring of convergent power series in the variables $x_1, \dots, x_n, \xi_1, \dots, \xi_t$. Since the σ_i are homogeneous and the ring $\mathbb{C}\{x, \xi\}$ is a flat extension of G, the σ_i are a regular sequence in G if and only if they are a regular sequence in $\mathbb{C}\{x, \xi\}$. But the last condition is equivalent to the equality [6, Theorem 17.4]

$$\dim_{(0,0)}(V(I)) = \dim(\mathbb{C}\{x, \xi\}/I) = n + t - s.$$

Finally, since all σ_i are homogeneous in the variables ξ, the local dimension of $V(I)$ at $(0,0)$ coincides with its dimension in $U \times \mathbb{C}^t$ for some neighborhood U of 0. □

Corollary 2.6. *Let $D \subset X$ be a free divisor. Let J be the ideal in $\mathcal{O}_{T^*X}$ generated by $\pi^{-1}\mathcal{D}\mathrm{er}(\log D)$. Then, D is Koszul free if and only if the set $V(J)$ of zeros of J has dimension n (in this case, each irreducible component of $V(J)$ has dimension n).*

Proposition 2.7. *Let X be a complex manifold of dimension n and let $D \subset X$ be a divisor. Then, the following assertions are valid.*

1. *Let $X' = X \times \mathbb{C}$ and $D' = D \times \mathbb{C}$. The divisor $D \subset X$ is Koszul free if and only if $D' \subset X'$ is Koszul free.*

2. *Let Y be another complex manifold of dimension r and let $E \subset Y$ be a divisor. Then,*

   a) *the divisor $(D \times Y) \cup (X \times E)$ is free if $D \subset X$ and $E \subset Y$ are free*; *and*

   b) *the divisor $(D \times Y) \cup (X \times E)$ is Koszul free if $D \subset X$ and $E \subset Y$ are Koszul free.*

Proof. 1. It is a consequence of [3, Lemma 2.2(iv)] and the fact that $\sigma_1, \dots, \sigma_n$ is a regular sequence in $\mathcal{O}_{X,p}[\xi_1, \dots, \xi_n]$ if and only if $\xi_{n+1}, \sigma_1, \dots, \sigma_n$ is a regular sequence in $\mathcal{O}_{X',(p,t)}[\xi_1, \dots, \xi_n, \xi_{n+1}]$.

2. a) It is an immediate consequence of Saito's Criterion (see [3, Lemma 2.2(v)]).

b) It is a consequence of a) and Corollary 2.6. □

Example 2.8. Let us consider examples of Koszul-free divisors.

1) Nonsingular divisors.

2) Normal crossing divisors.

3) Plane curves: If $\dim_{\mathbb{C}} X = 2$, we know that every divisor $D \subset X$ is free [9, Corollary 1.7]. Let $\{\delta_1, \delta_2\}$ be a basis of $\mathcal{D}\mathrm{er}(\log D)_x$. Their symbols $\{\sigma_1, \sigma_2\}$ are obviously linearly independent over $\mathcal{O}$, and, by Saito's Criterion [9, 1.8], they are relatively prime in $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{O}[\xi_1, \xi_2]$. So, they form a regular sequence in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$, and D is Koszul free (see [1, Corollary 4.2.2]).

4) Proposition 2.7 gives a way to obtain Koszul-free divisors in any dimension.

5) There are irreducible Koszul-free divisors Y in dimensions greater than two, which are not normal crossing and do not have nontrivial factors [8]; for example, $X = \mathbb{C}^3$ and $Y \equiv \{f = 0\}$, with

$$f = 2^8 z^3 - 2^7 x^2 z^2 + 2^4 x^4 z + 2^4 3^2 x y^2 z - 2^2 x^3 y^2 - 3^3 y^4.$$

A basis of $\mathcal{D}\mathrm{er}(\log f)$ is $\{\delta_1, \delta_2, \delta_3\}$, with

$$\delta_1 = 6y\partial_x + (8z - 2x^2)\partial_y - xy\partial_z,$$

$$\delta_2 = (4x^2 - 48z)\partial_x + 12xy\partial_y + (9y^2 - 16xz)\partial_z,$$

$$\delta_3 = 2x\partial_x + 3y\partial_y + 4z\partial_z,$$

and the sequence $\{\sigma(\delta_1), \sigma(\delta_2), \sigma(\delta_3)\}$ is $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular.

3. MAIN RESULTS

Proposition 3.1. *Let D be a free divisor in an analytic manifold X and let $\Sigma \subset D$ be a discrete set of points. If D is Koszul free at every point $x \in D \setminus \Sigma$, then D is Koszul free* (*at every point of D*).

Proof. Let $p \in \Sigma$ and let $\{\delta_1, \dots, \delta_n\}$ be a basis of the logarithmic derivations of D at p. By Corollary 2.6, we have to prove that the symbols $\sigma_i = \sigma(\delta_i)$ define an analytic set $V = V(\sigma_1, \dots, \sigma_n) \subset \pi^{-1}(U)$ of dimension $n = \dim X$ for some open neighborhood $U \subset X$ of p. Let U be an open neighborhood of p such that $U \cap \Sigma = \{p\}$. By hypothesis, we know that D is Koszul free in $U \setminus \{p\}$, and so (Corollary 2.6) the dimension of $V \cap \pi^{-1}(U \setminus \{p\}) = V \setminus T_p^* X$ is n. Now, let W be an irreducible component of V. It has, at least, dimension n. If W is contained in $T_p^* X$, then it must be equal to $T_p^* X$, and $\dim W = n$. If not, $\dim W = \dim(W \setminus T_p^* X) \le \dim(V \setminus T_p^* X) = n$. So, we conclude that V has dimension n. □

Theorem 3.2. *Every locally quasi-homogeneous free divisor is Koszul free.*

Proof. We proceed by induction on the dimension t of the ambient manifold X. For $t = 1$, the theorem is trivial, and, for $t = 2$, the theorem is directly proved in Example 2.8, 3). Now, we suppose that the result is true for $t < n$, and let D be a locally quasi-homogeneous free divisor of a complex analytic manifold X of dimension n. Let $p \in D$, and let $\{\delta_1, \ldots, \delta_n\}$ be a basis of the logarithmic derivations of D at p.

Thanks to [3, Proposition 2.4 and Lemma 2.2(iv)], there is an open neighborhood U of p such that, for each $q \in U \cap D$ with $q \neq p$, the germ of pair (X, D, q) is isomorphic to a product $(\mathbb{C}^{n-1} \times \mathbb{C}, D' \times \mathbb{C}, (0, 0))$, where D' is a locally quasi-homogeneous free divisor. The induction hypothesis implies that D' is a Koszul-free divisor at 0. Then, by assertion 1 of Proposition 2.7, D is a Koszul-free divisor at q too. We have then proved that D is a Koszul-free divisor in $U \setminus \{p\}$. We conclude by using Proposition 3.1. □

Corollary 3.3. *Every free divisor that is locally quasi-homogeneous at the complement of a discrete set is Koszul free.*

In particular, the last corollary gives rise to a new proof of the fact that every divisor in dimension two is Koszul free (see Example 2.8, 3)).

4. EXAMPLES

We know several (related) kinds of free divisors:

[LQH] Locally quasi-homogeneous (Definition 2.2).

[EH] Euler homogeneous (Definition 2.1).

[LCT] Free divisors satisfying the logarithmic comparison theorem.

[KF] Koszul free (Definition 2.3).

[P] Free divisors such that the complex $\Omega^\bullet_X(\log D)$ is a perverse sheaf.

We have then the following implications:

$$\begin{aligned} &[\text{LQH}] \Rightarrow [\text{EH}] \ \ \text{(obvious)}, \\ &[\text{LQH}] \Rightarrow [\text{LCT}] \ \ \text{by [3, Theorem 1.1]}, \\ &[\text{LCT}] \Rightarrow [\text{P}] \ \ \text{by [7, II, Theorem 2.2.4]}, \\ &[\text{KF}] \Rightarrow [\text{P}] \ \ \text{by [1, Theorem 4.2.1]}, \\ &[\text{LQH}] \Rightarrow [\text{KF}] \ \ \text{by Theorem 3.2.} \end{aligned}$$

Example 4.1 (free divisors in dimension two)**.** We recall Theorem 3.9 from [2]. Let X be a complex analytic manifold of dimension two and $D \subset X$ be a divisor. The following conditions are equivalent:

1. D is Euler homogeneous.
2. D is locally quasi-homogeneous.
3. The logarithmic comparison theorem holds for D.

Consequently, in dimension two we have

$$[\text{LQH}] \Leftrightarrow [\text{EH}] \Leftrightarrow [\text{LCT}]$$

and [KF] and [P] always hold (see Example 2.8, 3)). In particular,

$$[\text{KF}] \not\Rightarrow [\text{LQH}], [\text{EH}], [\text{LCT}].$$

Examples of plane curves not satisfying logarithmic comparison theorem are, for instance, the curves of the family (see [2])

$$x^q + y^q + xy^{p-1} = 0, \qquad p \geq q + 1 \geq 5.$$

Example 4.2 (an example in dimension three). Let us consider $X = \mathbb{C}^3$ and $D = \{f = 0\}$, with $f = xy(x+y)(y+zx)$ [1]. A basis of $\mathcal{D}\mathrm{er}(\log D)$ is $\{\delta_1, \delta_2, \delta_3\}$ with

$$\delta_1 = x\partial_x + y\partial_y,$$

$$\delta_2 = x^2\partial_x - y^2\partial_y - z(x+y)\partial_z,$$

$$\delta_3 = (xz + y)\partial_z,$$

the determinant of the coefficients matrix being $-f$ and

$$\delta_1(f) = 4f, \qquad \delta_2(f) = (2x - 3y)f, \qquad \delta_3(f) = xf.$$

In particular, D is Euler homogeneous and satisfies the logarithmic comparison theorem [2]. Let $I \subset \mathcal{O}_{T^*X}$ be the ideal generated by the symbols $\{\sigma_1, \sigma_2, \sigma_3\}$ of the basis of $\mathcal{D}\mathrm{er}(\log D)$. By Corollary 2.6, D is not Koszul free, because the dimension of $V(I)$ at $((0,0,\lambda),0) \in T^*X (\lambda \neq 0)$ is greater than three. So, D is not locally quasi-homogeneous either.

Thus,

$$[\mathrm{LCT}] \not\Rightarrow [\mathrm{KF}], [\mathrm{LQH}], \qquad [\mathrm{EH}] \not\Rightarrow [\mathrm{KF}], [\mathrm{LQH}].$$

Finally, for the only relation that we have not solved, we quote the following conjecture from [2]:

Conjecture 4.3. *If the logarithmic comparison theorem holds for D, then D is Euler homogeneous.*

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