Games on concept lattices: Shapley value and core

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Abstract. We introduce cooperative TU-games on concept lattices, where a concept is a pair (S, S') with S being a subset of players or objects, and S' a subset of attributes. Any such game induces a game on the set of players/objects, which appears to be a TU-game whose collection of feasible coalitions is a lattice closed under intersection, and a game on the set of attributes. We propose a Shapley value for each type of game, axiomatize it, and investigate the geometrical properties of the core (nonemptiness, boundedness, pointedness, extremal rays).

Keywords: cooperative game, restricted cooperation, concept lattice, core, Shapley value

1 Introduction

Cooperative games with transferable utility (TU-games) have been widely studied and used in many domains of applications. N being a set of players, or more generally, a set of abstract objects, a TU-game $v : 2^N \to \emptyset$ assigns to every coalition or group $S \subseteq N$ a number representing its "worth" (monetary value: benefit created by the cooperation of the members of S, or cost saved by the common usage of a service by the members of S, power, importance, etc.).

Once the function v is determined, the main concern of cooperative game theory is to provide a rational scheme for distributing the total worth $v(N)$ of the cooperation among the members of N (or determining individual power/importance degrees, if $v(N)$ is not interpreted as a monetary value). The until now most popular methods to achieve this are the Shapley value [19] and the core [14]. The Shapley value yields a single distribution vector, satisfying a set of four natural axioms (Pareto optimality, symmetry, linearity, null player property), while the core is a set of distribution vectors that are Pareto optimal and satisfy coalitional rationality (i.e., a coalition receives at least equal its own worth). While the Shapley value always exists for any game, the core is a convex polyhedron, but may be empty.

In many situations, however, not all subsets of N can be realized as coalitions or are feasible, which means that the mapping v is defined on a subcollection $\mathcal{F}$ of 2^N only.

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Such games are said to have *restricted cooperation* [9]. $\mathcal{F}$ has been studied under many structural assumptions, such as distributive lattices (closed under union and intersection) [12], convex geometries [3, 4], antimatroids [2], union-stable systems [1] (a.k.a. weakly union-closed systems [11, 10]), etc. In this case, the study of the geometric properties of the core is challenging since the core may become unbounded or have no vertices (see a survey in [15]). Also the Shapley value has to be redefined, and its axiomatization may become difficult.

In most of the cases, the structural assumptions on $\mathcal{F}$ are not clearly motivated or are too restrictive. The aim of this paper is to study a structure for $\mathcal{F}$ which is both very general (a lattice of sets closed under intersection), and produced in a natural way, through a set of attributes possessed by the players or objects in N. In short, our framework is based on *concept lattices* [5, 6, 16], a notion which has lead to the now quite active field of *formal concept analysis* [13]. M being a set of attributes, a concept is a pair (S, S') with $S \subseteq N$ and $S' \subseteq M$ such that S' is the set of those attributes that are satisfied by all members of S. A remarkable result is that any (finite) lattice is isomorphic to a concept lattice, and that the lattice of extents (i.e., the lattice of concepts (S, S') limited to the first arguments S) is a set lattice closed under intersection, and moreover any such lattice arises that way. We define a game v on the lattice of concepts, dividing it into a game v_N on the lattice of extents (which corresponds to a game with restricted cooperation $(v, \mathcal{F})$ where $\mathcal{F}$ is a lattice closed under intersection), and a game v_M on the lattice of intents (which corresponds to a game on the set of attributes). For both types of games, we propose a Shapley value with its axiomatization. Moreover, we investigate in details the properties of the core. Our results can be seen to generalize many results of the literature on games with restricted cooperation.

The paper is organized as follows. Section 2 introduces the main notions needed in the paper: cooperative games, concept lattices and games on concept lattices. Section 3 proposes a definition for the Shapley value, which is a natural generalization of those values presented by Faigle and Kern [12], and Bilbao and Edelman [4], together with its axiomatization. Section 4 studies the properties of the core: nonemptiness, boundedness, pointedness, and extremal rays. Some interesting properties of balanced collections are also presented.

2 Framework

2.1 Cooperative games

Let $N = \{1, \ldots, n\}$ be a finite set of players. A *cooperative (TU) game* (or *game* for short) on N is a mapping $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Any subset $S \subseteq N$ is called a *coalition*. The quantity $v(S)$ represents the "worth" of the coalition, that is, depending on the application context, the benefit realized (or cost saved, etc.) by cooperation of the members of S.

We consider the general case where the cooperation is restricted, i.e., where the set $\mathcal{F}$ of all feasible coalitions might be a proper subset of 2^N. We denote the corresponding *game with restricted cooperation* as a pair $(\mathcal{F}, v)$, or simply v if there is no ambiguity.

Let us consider a cooperative game $(\mathcal{F}, v)$ with $N \in \mathcal{F}$. A payoff vector is a vector $x \in \mathbb{R}^n$. For any $S \subseteq N$, we denote by $x(S) = \sum_{i \in S} x_i$ the total payoff given by x to the coalition S. The payoff x is *efficient* if $x(N) = v(N)$. The core of a cooperative game is the set of efficient payoff vectors such that no coalition can achieve a better payoff by itself:

$$\mathrm{core}(\mathcal{F}, v) = \{x \in \mathbb{R}^n \mid x(S) \geq v(S)\ \forall S \in \mathcal{F},\ \text{and } x(N) = v(N)\}.$$

Note that $\mathrm{core}(\mathcal{F}, v)$ is a convex closed bounded polyhedron when $\mathcal{F} = 2^N$. In other cases, the core may be unbounded or non pointed, and its study becomes difficult (see [9] and a survey in [15]). We recall from the theory of polyhedra that a polyhedron defined by a set of inequalities $\mathbf{Ax} \geq \mathbf{b}$ is the Minkowski sum of its convex part and its conic part (the so-called recession cone), the latter being determined by the inequalities $\mathbf{Ax} \geq \mathbf{0}$, and being therefore independent of the righthand side $\mathbf{b}$. So the recession cone of $\mathrm{core}(\mathcal{F}, v)$ is the polyhedron $\mathrm{core}(\mathcal{F}, 0)$, which does not depend on v.

A collection $\mathcal{B} \subseteq \mathcal{F}$ of nonempty sets is said to be *balanced* if there exist positive weights $\lambda_S, S \in \mathcal{B}$ such that

$$\sum_{S \in \mathcal{B}, S \ni i} \lambda_S = 1 \quad \forall i \in N.$$

A game $(\mathcal{F}, v)$ is said to be *balanced* if $v(N) \geq \sum_{S \in \mathcal{B}} \lambda_S v(S)$ holds for every balanced collection $\mathcal{B}$ with weight system $(\lambda_S)_{S \in \mathcal{B}}$. It is well-known that the core of v is nonempty if and only if v is balanced [9].

2.2 Concept lattices

We begin by recalling that a *lattice* is a partially ordered set (poset) $(L, \preceq)$ such that for any two elements $x, y \in L$, a supremum $x \vee y$ and an infimum $x \wedge y$ exists. If no ambiguity occurs, the lattice is simply denoted by L. The *dual partial order* $\preceq^\partial$ is defined by $x \preceq^\partial y$ if and only if $y \preceq x$. The *dual* of the lattice $(L, \preceq)$ is the poset $(L, \preceq^\partial)$, denoted by L^∂ if no ambiguity occurs.

A *context* (see, e.g., [5, 6, 13, 16]) is a triple $\mathcal{C} = (N, M, I)$, where N is a finite nonempty set of objects, M is a finite set of attributes, and $I : N \times M \to \{0, 1\}$ is a binary relation defined by $I(i, a) = 1$ if object $i \in N$ satisfies attribute $a \in M$, and 0 otherwise. The binary relation can be represented as a matrix or table called the *incidence matrix (table).*

Let $\mathcal{C} = (N, M, I)$ be a context. The *intent* of a subset of objects $S \subseteq N$ is defined as the set of attributes satisfied by all objects in S:

$$S'_{\mathcal{C}} = \{a \in M \mid I(i, a) = 1,\ \forall i \in S\}.$$

Dually, the *extent* of any set of attributes $A \subseteq M$ is defined as the set of objects satisfying all attributes in A:

$$A'_{\mathcal{C}} = \{i \in N \mid I(i, a) = 1,\ \forall a \in A\}.$$

To avoid a heavy notation, we write simply S', A' for the intent of S and the extent of A, when the meaning is clear. A basic property of the extents and intents is the relation

$$(S')' \supseteq S, \quad (A')' \supseteq A \quad \forall S \subseteq N, A \subseteq M. \tag{1}$$

A *concept* in $\mathcal{C}$ is a pair (S, A) with $S \subseteq N$ and $A \subseteq M$ such that $S = A'$ and $A = S'$. Equivalently, a concept is a maximal rectangle of "1" in the incidence matrix, or it is $(N, \emptyset)$ if $N' = \emptyset$, or $(\emptyset, M)$ if $M' = \emptyset$.

We denote by $L_{\mathcal{C}}$ the set of all concepts in $\mathcal{C}$, and endow it with a partial order $\leq$ defined by

$$(S, A) \leq (T, B) \text{ if } S \subseteq T$$

(equivalently, if $B \supseteq A$). Then $(L_{\mathcal{C}}, \leq)$ is a lattice, called the *concept lattice*, with supremum and infimum given by

$$\begin{aligned}(S, A) \wedge (T, B) &= ((S \cap T), (S \cap T)')\\ (S, A) \vee (T, B) &= ((A \cap B)', (A \cap B)).\end{aligned}$$

The top and bottom elements of this lattice are (N, N') and (M', M) respectively. It is important to note that any finite lattice is isomorphic to a concept lattice.

Given a context $\mathcal{C}$ and its concept lattice $L_{\mathcal{C}}$, the *lattice of extents* $(L^N_{\mathcal{C}}, \subseteq)$ is defined by the set

$$L^N_{\mathcal{C}} = \{S \subseteq N \mid (S, S') \in L_{\mathcal{C}}\}.$$

Similarly, we define the *lattice of intents* $(L^M_{\mathcal{C}}, \subseteq)$ as the set

$$L^M_{\mathcal{C}} = \{A \subseteq M \mid (A', A) \in L_{\mathcal{C}}\}.$$

Clearly, the lattices $L_{\mathcal{C}}, L^N_{\mathcal{C}}, (L^M_{\mathcal{C}})^\partial$ are isomorphic.

Example 1. Consider $N = \{1, 2, 3, 4\}$, $M = \{a, b, c\}$, and the incidence table given in Figure 1. The lattices $L_{\mathcal{C}}$, $L^N_{\mathcal{C}}$ and $L^M_{\mathcal{C}}$ are shown on the right of the table. For ease of notation, sets like $\{2, 4\}$ and $\{b, c\}$ are denoted by 24 and bc.

	a	b	c
1	×	×	
2		×	×
3	×		
4			×

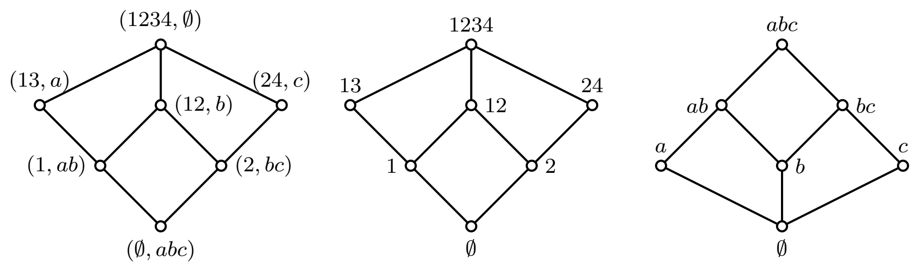


Fig. 1. From left to right: The incidence table of a context, its concept lattice, the lattices of extents, and the lattice of intents

2.3 Games on concept lattices

We consider a context $\mathcal{C} = (N, M, I)$, the lattice of concepts $L_{\mathcal{C}}$, the lattice of extents $L^N_{\mathcal{C}}$ and the lattice of intents $L^M_{\mathcal{C}}$. We assume that no attribute is superfluous, i.e., the top element of $L_{\mathcal{C}}$ is $(N, \emptyset)$ (no attribute is satisfied by all objects), however, the bottom element (M', M) may be with $M' \neq \emptyset$ (there are objects satisfying all attributes).

To each concept $(A, A') \in L_{\mathcal{C}}$, we assign a number $v(A, A') \in \mathbb{R}$ (its meaning could be benefit, cost, evaluation, certainty degree of occurrence, etc.). We call the pair $(\mathcal{C}, v)$ a *cooperative game on concepts* or *concept game* for short, and impose the restriction $v(\emptyset, M) = 0$ whenever $(\emptyset, M) \in L_{\mathcal{C}}$. We denote by CCG the set of all concept games.

We derive from v two mappings $v_N : L^N_{\mathcal{C}} \to \mathbb{R}$ and $v_M : L^M_{\mathcal{C}} \to \mathbb{R}$ defined by

$$
\begin{aligned}
v_N(S) &= v(S, S') - v(M', M) \qquad (S \in L^N_{\mathcal{C}}) \\
v_M(A) &= v(N, \emptyset) - v(A', A) \qquad (A \in L^M_{\mathcal{C}}).
\end{aligned}
$$

Note that $v_N(M') = 0$ and $v_M(\emptyset) = 0$ holds, i.e., these set functions vanish at the bottom of their respective lattices, and could thus be considered as cooperative games. Also, if v is monotone nondecreasing, then so are v_N and v_M (because $A \subseteq B$ implies $A' \supseteq B'$) (and similarly for monotone nonincreasing). We call v_N and v_M the *game on extents* and the *game on intents*, respectively.

Example 2. An immediate application of the above framework in cooperative game theory is: N is the set of players, and M is the set of attributes of players. Attributes can be thought of as any kind of property, or simply, as membership cards of any association, club, party, etc., that the players may possess. Now a coalition is feasible iff it corresponds to the extent of a concept. Indeed, if $S \subseteq N$ is not an extent in $L^N_{\mathcal{C}}$, then it is not stable in the sense that the players of S have common attributes S', but other players also satisfy these attributes in S', so they have an incentive to join S.

Example 3. Games on concepts can model the interplay between sellers and markets. Consider $N = \{1, 2, 3, 4, 5, 6\}$ where agent 1 has the patent of a product in a market, and the remaining agents are sellers who want to sell the product in this market. The market is divided into three submarkets $M = \{\alpha, \beta, \gamma\}$, but there are restrictions on which agent can sell in which market: only sellers 4, 5, 6 can sell in market α, only 2, 4, 6 can sell in β, and only 3 and 5 can sell in γ. In addition, we suppose that agent 1 is not a seller. We define a context $\mathcal{C} = (N, M, I)$ to represent this situation with the relation I defined by $I(i, a) = 1$ if i cannot sell in submarket a. Figure 2 gives the incidence table and concept lattice[1] of $\mathcal{C}$. Let us define a concept game as follows. We make the assumption that

	α	β	γ
1	×	×	×
2	×		×
3	×	×	
4			×
5		×	
6			×

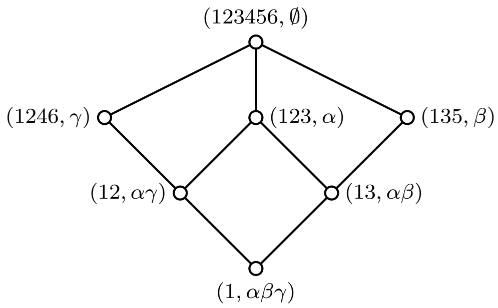


Fig. 2. The incidence table of $\mathcal{C}$ (left) and its concept lattice (right)

1) Profits obtained by a coalition of sellers depend on the submarkets where they can develop their activity, but 2) Sellers cannot ban others from these submarkets if they are the relation I makes them eligible. By a pair $(S, A) \in N \times M$, we represent a situation where S is a set of agents and A is a set of submarkets where they cannot sell. However,

[1] Note that this lattice is isomorphic to the one of Example 1, although the incidence tables are completely different.

not every pair (S, A) is admissible. Indeed, the agents in S cannot sell in the submarkets of S', thus $A \supseteq S'$. Moreover, a seller has no interest to be in a situation which reduces his sale domain, thus $A \subseteq S'$, and therefore $A = S'$. By (1), this yields $A' = (S')' \supseteq S$. Now, 2) implies that $A' = S$, so finally (S, A) must be a context. We may take for v the following values (omitting braces and commas):

$$v(1, M) = 6, \quad v(13, \alpha\beta) = 16. \quad v(12, \alpha\gamma) = 20$$
$$v(123, \alpha) = 32, \quad v(135, \beta) = 28, \quad v(1246, \gamma) = 30, \quad v(N, \emptyset) = 60.$$

The value $v(1, M)$ represents the fixed payoff obtained by the patent owner if the product is sold.

2.4 Set lattices and concept lattices

We investigate in this section the relation between concept lattices and set lattices on N, i.e., sublattices of $(2^N, \subseteq)$ (see, e.g., [16]). We begin by recalling some useful notions about finite lattices and posets. For x, y in a poset $(P, \preceq)$, we say that *x is covered by y*, or *y covers x*, denoted by $x \prec\cdot\, y$, if $x \preceq y$, and $x \preceq z \preceq y$ implies $x = z$ or $z = y$. For $x \preceq y$ in $(P, \preceq)$, a *maximal chain* from x to y is a sequence of elements $x = x_0, x_1, \ldots, x_p = y$ such that $x_0 \prec\cdot\, x_1 \prec\cdot\, \cdots \prec\cdot\, x_p$. Its *length* is p. The *height* of a lattice is the length of a longest maximal chain.

Given a poset $(P, \preceq)$, $x \in P$ is a *join-irreducible element* if it covers exactly one element. Dually, x is a *meet-irreducible element* if it is covered by exactly one element. We denote respectively by $\mathcal{J}(P, \preceq)$ and $\mathcal{M}(P, \preceq)$ the sets of join-irreducible and meet-irreducible elements. A subset $Q \subseteq P$ of a poset is a *downset* if $x \in Q$ and $y \in P$ such that $y \preceq x$ imply $y \in Q$. The set of all downsets of $(P, \preceq)$ is denoted by $\mathcal{O}(P, \preceq)$.

A lattice is *distributive* if $\vee, \wedge$ obey the algebraic distributivity law. A fundamental result due to Birkhoff says that a finite lattice $(L, \preceq)$ is isomorphic to $(\mathcal{O}(\mathcal{J}(L), \preceq), \subseteq)$ if and only if the lattice is distributive. This means that a distributive lattice can be reconstructed from its join-irreducible elements. The same statement with meet-irreducible elements holds as well, because L is distributive if and only if $(\mathcal{J}(L), \preceq)$ is isomorphic to $(\mathcal{M}(L), \preceq)$.

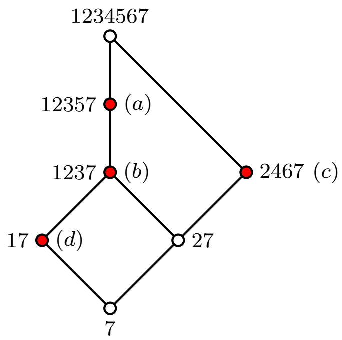
An application in game theory of the result of Birkhoff is the following: consider a set N of players endowed with a partial order $\preceq$. Then the set of downsets $\mathcal{O}(N, \preceq)$ forms a collection $\mathcal{F} \subseteq 2^N$ containing N and $\emptyset$, which is a distributive (set) lattice when ordered by inclusion, with supremum and infimum being union and intersection. Conversely, any collection $\mathcal{F} \subseteq 2^N$ of height n containing $N, \emptyset$ and closed under union and intersection arises that way (see Faigle and Kern [12]).

A *closure system on* N is a collection $\mathcal{F}$ of subsets of N which is closed under intersection and contains N, while a *dual closure system* is a collection closed under union and containing the empty set. Endowing a closure system (or a dual closure system) with inclusion order $\subseteq$, we obtain a poset with remarkable properties:

(i) Any lattice is isomorphic to a closure system, and to a dual closure system;
(ii) The lattice of extents of a context is a closure system, while the collection of complement sets of the lattice of intents, i.e., $\{A \in 2^M \mid A^c \in L^M_{\mathcal{C}}\}$, is a dual closure system. As a consequence, the lattices of extents and of intents are closed under intersection;

(iii) Conversely, any closure system $\mathcal{F}$ on N is the lattice of extents of some context. Specifically, the simplest context is $\mathcal{C} = (N, M, I)$, with M the set of meet-irreducible elements of $\mathcal{F}$, and $I(i, j) = 1$ iff $i \in j$, with $i \in N$ and $j \in M$.

Example 4. Take $N = \{1, 2, 3, 4, 5, 6, 7\}$ and the closure system represented on Figure 3 (left). Its meet-irreducible elements are (in red): 12357, 1237, 2467, 17, which we denote a, b, c, d, respectively. The corresponding table is given in the middle, and the context lattice on the right of the figure.



	a	b	c	d
1	×	×		×
2	×	×	×	
3	×	×		
4			×	
5	×			
6			×	
7	×	×	×	×

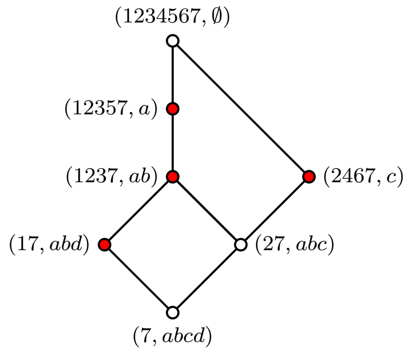


Fig. 3. From left to right: a closure system (meet-irreducible elements in red), the corresponding table and context lattice

We observe on the example that lines 4 and 6 are duplicate, line 7 is full, and line 3 is the intersection of lines 1 and 2. On the closure system, this corresponds respectively to the fact that 4 and 6 are always together in a concept, 7 is always present, and 3 is present whenever 1 and 2 are present. These situations are captured under the notion of macro-player and companion player (assuming N is the set of players).

Definition 1. *Let $\mathcal{F}$ be a closure system on N. A subset $K \subseteq N$, $|K| > 1$, is a macro-player in $\mathcal{F}$ if either $K \subseteq S$ or $K \cap S = \emptyset$ for every nonempty $S \in \mathcal{F}$ (equivalently, no $S \in \mathcal{F}$ "separates" K, i.e., $S \cap K \neq \emptyset$ and $K \setminus S \neq \emptyset$).*

Definition 2. *Let $\mathcal{F}$ be a closure system on N. A player $i \in N$ is a companion player of S, $S \subseteq N \setminus i$, if $S \cup i \in \mathcal{F}$, and for all $T \in \mathcal{F}$,*

$$T \ni i \text{ if and only if } S \subseteq T.$$

It is clear from the definition that macro-players arise as identical lines in the table, while a companion i of S corresponds to the situation where line i is the intersection of the lines in S. The following properties are noteworthy:

(i) If K, K' are maximal (w.r.t. inclusion) macro-players, then $K \cap K' = \emptyset$.
(ii) If M' (bottom of $\mathcal{F}$) is nonempty, then M' is a macro-player when $|M'| > 1$, and a companion player when $|M'| = 1$ (companion of $\emptyset$).
(iii) When $M' = \emptyset$, atoms which are not singletons are macro-players, but the converse is false. More precisely, a macro-player K is an atom if and only if $K \in \mathcal{F}$.
(iv) If i is a companion of $\{j\}$, then $\{i, j\}$ is a macro-player.
(v) If i and j are companion of the same S, then $\{i, j\}$ is a macro-player.

Consider a closure system $\mathcal{F}$ on N with $|N| = n$, and consider $\mathcal{J}(\mathcal{F})$ the set of its join-irreducible elements. The following is important to note.

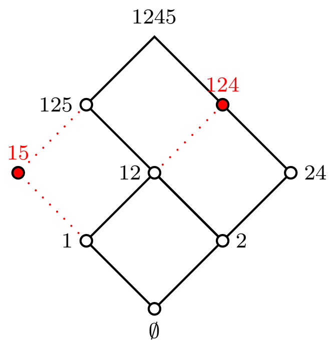
(i) Suppose $|\mathcal{J}(\mathcal{F})| < n$. Then there exist either companion players or macro-players. Indeed, each join-irreducible element corresponds to a line in the incidence table, and each additional line must not create a new maximal rectangle with new attributes. We have

$$n = |\mathcal{J}(L)| + \sum_{i=1}^{p} |K_i| - p + c$$

where $K_1, \ldots, K_p$ are the maximal macro-players and c is the number of companion players which do not belong to some macro-player (see Ex. 4).

(ii) Suppose $|\mathcal{J}(\mathcal{F})| = n$. In this case, $\mathcal{F}$ is an *irreducible closure system* since there is no redundant line in the incidence table. Moreover, $\mathcal{F} \subseteq \mathcal{O}(\mathcal{J}(\mathcal{F}))$ where "missing" sets (i.e., those not in $\mathcal{O}(\mathcal{J}(\mathcal{F}))$) are necessarily unions of sets in $\mathcal{F}$, since taking the closure under union of $\mathcal{F}$ would give $\mathcal{O}(\mathcal{J}(\mathcal{F}))$.

Example 5 (Example 4 continued). Let us make the closure system of Example 4 irredundant by suppressing the superfluous elements 3, 6 and 7, so as to have $n = |\mathcal{J}(\mathcal{F})| = 4$. This gives the closure system represented on Figure 4 (solid lines), to which we give a slightly different shape, in order to make it apparent as a sublattice of $\mathcal{O}(\mathcal{J}(\mathcal{F}))$ (additional links in red dotted lines). The two missing sets are 15 and 124 (in red).



	a	b	c	d
1	×	×		×
2	×	×	×	
4			×	
5	×			

Fig. 4. Irredundant version of Figure 3

Let L be any lattice, with $N = \{1, 2, \ldots\}$ the set of its join-irreducible elements, and $M = \{a, b, c, \ldots\}$ the set of its meet-irreducible elements. The *irreducible closure system associated to* L is the set lattice $C(L)$ on N defined by

$$C(L) = \{J(x) \mid x \in L\}, \text{with } J(x) = \{i \in N \mid i \leq x\}.$$

Its bottom element is $\emptyset$. The *irreducible dual closure system associated to* L is the set lattice $O(L)$ on M defined by

$$O(L) = \{M(x) \mid x \in L\}, \text{with } M(x) = \{j \in M \mid j \not\geq x\}.$$

Its top element is M. The *irreducible concept lattice* associated to L is given by the context $\mathcal{C} = (N, M, I)$ with $I(i, j) = 1$ iff $i \leq j$. Then

$$L^N_{\mathcal{C}} = C(L) \text{ and } L^M_{\mathcal{C}} = \{A \subseteq M \mid A^c \in O(L)\}.$$

Example 6. Take the lattice on Figure 5 (left), its join-irreducible elements are in red, the meet-irreducible elements are in blue. The irreducible closure and dual closure systems (ordered by $\subseteq$) are depicted in the middle and on the right of the figure. By comparing with Example 1, one can see the above identity.

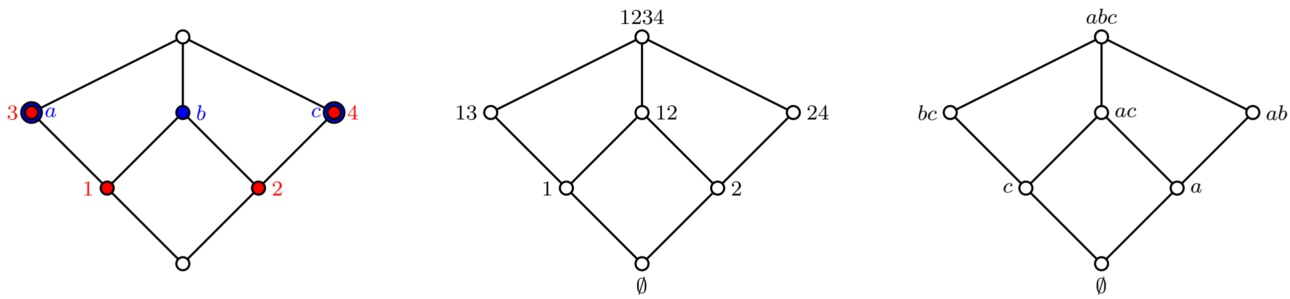


Fig. 5. From left to right: a lattice (in red: join-irreducible elements, in blue: meet-irreducible elements), and the corresponding irreducible closure and dual closure systems

3 The Shapley value

Given a game $(\mathcal{C}, v)$ on a context, we define the extent Shapley value and the intent Shapley value as the Shapley value for the games on extents and on intents respectively. We begin with the extent Shapley value.

3.1 The extent Shapley value

We consider the lattice of extents $L^N_{\mathcal{C}}$ of a context $\mathcal{C}$ and the game v_N defined on it.

Consider the set $CH(\mathcal{C})$ of all maximal chains from the bottom M' to the top N in $L^N_{\mathcal{C}}$ (equivalently, in $L_{\mathcal{C}}$), and denote its cardinality by $ch(\mathcal{C})$. Consider a given maximal chain $C \in CH(\mathcal{C})$, letting $C = M' = S_0 \subset S_1 \cdots \subset S_k = N$, and a player i. Denote by T^i_C and S^i_C respectively, the last set in the sequence which does not contain i, and the first set containing i.

The *extent Shapley value* of $(\mathcal{C}, v)$, denoted by $\Phi^{ex}(\mathcal{C}, v)$, is defined to be the Shapley value $\Phi(v_N)$ of the game on extents, given by

$$\Phi^{ex}(\mathcal{C}, v) = \Phi_i(v_N) = \begin{cases} \dfrac{1}{ch(\mathcal{C})} \displaystyle\sum_{C \in CH(\mathcal{C})} \dfrac{1}{|S^i_C \setminus T^i_C|}\big(v_N(S^i_C) - v_N(T^i_C)\big), & \text{if } i \notin M' \\ \dfrac{v_N(M')}{|M'|}, & \text{otherwise.} \end{cases} \tag{2}$$

This definition is a natural generalization of the values introduced by Faigle and Kern [12], and Bilbao and Edelman [4].

We formulate properties to axiomatize the extent Shapley value. Let F be any value over the set of concept games.

Taking into account that we consider $v(M', M)$ as a separable payoff to players in M', we propose:

Separable payoff axiom (SP): If $(\mathcal{C}, v) \in CCG$ and $\mathcal{C} = (N, M, I)$ then one has

$$\sum_{i \in M'} F_i(\mathcal{C}, v) = v(M', M).$$

As for the classical Shapley value, we look for efficient payoff vectors.

Efficiency axiom (E): For all $(\mathcal{C}, v) \in CCG$, one has

$$\sum_{i \in N} F_i(\mathcal{C}, v) = v(N, \emptyset).$$

All the agents in a macro-player are equivalent for the concept lattice, thus their worths should be the same.

Macro-player axiom (MP): If $(\mathcal{C}, v) \in CCG$ with $\mathcal{C} = (N, M, I)$ and K is a macro-player in $L^N_{\mathcal{C}}$, then

$$F_i(\mathcal{C}, v) = F_j(\mathcal{C}, v) \quad \forall i, j \in K.$$

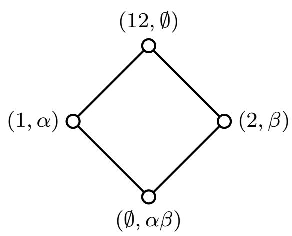
A context $\mathcal{C}_2 = (N_2, M_2, I_2)$ is *concatenable* to a context $\mathcal{C}_1 = (N_1, M_1, I_1)$ if $N_1 = (M_2)'_{\mathcal{C}_2}$ and $M_1 \cap M_2 = \emptyset$. The result of the concatenation of the two concatenable contexts is a new context $\mathcal{C}_2 * \mathcal{C}_1 = (N, M, I)$, where $N = N_2$, $M = M_1 \cup M_2$ and

$$I(i, a) = \begin{cases} I_1(i, a), & \text{if } i \in N_1, a \in M_1 \\ I_2(i, a), & \text{if } i \in N_2 \setminus N_1, a \in M_2 \\ 1, & \text{if } i \in N_1, a \in M_2 \\ 0, & \text{if } i \in N_2 \setminus N_1, a \in M_1. \end{cases}$$

As is easy to see, concatenation amounts to the concatenation of the two incidence tables and hence to the concatenation of the two concept lattices $\{(S, A \cup M_2) \mid (S, A) \in L_{\mathcal{C}_1}\}$ and $L_{\mathcal{C}_2}$.

Example 7. Consider $N_1 = \{1, 2\}$, $N_2 = \{1, 2, 3, 4, 5\}$, $M_1 = \{\alpha, \beta\}$ and $M_2 = \{a, b, c\}$. The two incidence tables and the concept lattices $L_{\mathcal{C}_1}, L_{\mathcal{C}_2}$ are given on Figure 6. The

	α	β
1	×	
2		×



	a	b	c
1	×	×	×
2	×	×	×
3	×		
4		×	
5			×

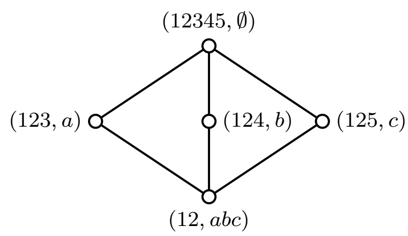


Fig. 6. Two contexts $\mathcal{C}_1, \mathcal{C}_2$ represented by their table and concept lattice

result of the concatenation $\mathcal{C}_2 * \mathcal{C}_1$ is shown on Figure 7.

	a	b	c	α	β
1	×	×	×	×	
2	×	×	×		×
3	×				
4		×			
5			×		

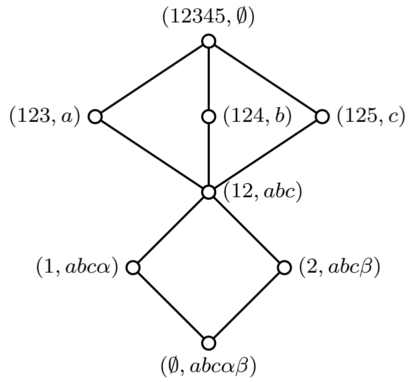


Fig. 7. The concatenation $\mathcal{C}_2 * \mathcal{C}_1$ of the two contexts of Figure 6

The concatenation of contexts should not change the payoffs of the players.

Concatenation axiom (C): For all $(\mathcal{C}_1, v_1)$, $(\mathcal{C}_2, v_2) \in CCG$ such that $\mathcal{C}_2$ is concatenable to $\mathcal{C}_1$, and $v_1(N_1, \emptyset) = v_2((M_2)'_{\mathcal{C}_2}, M_2)$, one has

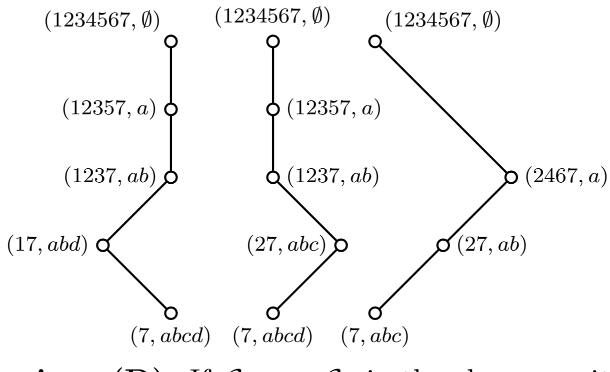
$$F_i(\mathcal{C}_2 * \mathcal{C}_1, v_2 * v_1) = \begin{cases} F_i(\mathcal{C}_2, v_2), & \text{if } i \in N_2 \setminus N_1 \\ F_i(\mathcal{C}_1, v_1), & \text{if } i \in N_1, \end{cases}$$

where $v_2 * v_1$ is a concept game on $\mathcal{C} = \mathcal{C}_2 * \mathcal{C}_1$, defined by

$$(v_2 * v_1)(S, A) = \begin{cases} v_1(S, A \setminus M_2), & \text{if } (A \setminus M_2)'_{\mathcal{C}} \subseteq N_1 \\ v_2(S, A), & \text{if } N_1 \subseteq S. \end{cases}$$

Let $\mathcal{C} = (N, M, I)$ be a concept, and consider the maximal chains $C_1, \ldots, C_q$ of its extent lattice $L^N_{\mathcal{C}}$. The *decomposition into maximal chains* of $\mathcal{C}$ is a collection of contexts $\mathcal{C}_1, \ldots, \mathcal{C}_q$ with set of objects N such that their extent lattices $L^N_{\mathcal{C}_1}, \ldots, L^N_{\mathcal{C}_q}$ are precisely the maximal chains $C_1, \ldots, C_q$. Note that the set of attributes for $\mathcal{C}_1, \ldots, \mathcal{C}_q$ may differ from M, as illustrated by the next example.

Example 8 (Example 4 continued). We decompose the context $\mathcal{C}$ into its three maximal chains as follows:



Decomposition axiom (D): If $\mathcal{C}_1, \ldots, \mathcal{C}_q$ is the decomposition of a context $\mathcal{C}$ in maximal chains, then

$$F(\mathcal{C}, v) = \frac{1}{ch(\mathcal{C})} \sum_{p=1}^{q} F(\mathcal{C}_p, v_p),$$

with $v_p(S, A) = v_N(S)$ for all $(S, A) \in L_{\mathcal{C}_p}$ and $p = 1, \ldots, q$.

We show now that the extent Shapley value satisfies all these axioms.

Theorem 1. *The extent Shapley value satisfies (SP), (E), (MP), (C) and (D).*

Proof. – Separable payoff: it is obviously satisfied.

– Efficiency: Let $(\mathcal{C}, v) \in CCG$ with $\mathcal{C} = (N, M, I)$. We have, using (2),

$$\sum_{i \in N} \Phi_i^{ex}(\mathcal{C}, v) = \sum_{i \in M'} \Phi_i^{ex}(\mathcal{C}, v) + \sum_{i \in N \setminus M'} \Phi_i^{ex}(\mathcal{C}, v)$$
$$= v(M', M) + \frac{1}{ch(\mathcal{C})} \sum_{C \in CH(\mathcal{C})} (v(N, \emptyset) - v(M', M)) = v(N, \emptyset).$$

– Macro-players: Let $(\mathcal{C}, v) \in CCG$ and K be a macro-player in $L_{\mathcal{C}}^N$. If $K = M'$, by definition, every two players in K receive the same payoff with Φ^{ex}. Suppose now $K \neq M'$, i.e., $K \subsetneq N \setminus M'$. In that case, for any $i, j \in K$ and for any $C \in CH(\mathcal{C})$, $S_C^i = S_C^j$ and $T_C^i = T_C^j$. Thus $\Phi_i^{ex}(\mathcal{C}, v) = \Phi_j^{ex}(\mathcal{C}, v)$ for every $i, j \in K$.

– Concatenation: We consider two concatenable contexts $\mathcal{C}_1 = (N_1, M_1, I_1)$, $\mathcal{C}_2 = (N_2, M_2, I_2)$ with $N_1 = (M_2)'_{\mathcal{C}_2}$. Observe that $ch(\mathcal{C}_2 * \mathcal{C}_1) = ch(\mathcal{C}_2)\, ch(\mathcal{C}_1)$. If $i \in (M_1)'_{\mathcal{C}_1}$ then

$$\Phi_i^{ex}(\mathcal{C}_2 * \mathcal{C}_1, v_2 * v_1) = \frac{(v_2 * v_1)\left((M_1)'_{\mathcal{C}_1}, M_1 \cup M_2\right)}{|(M_1)'_{\mathcal{C}_1}|} = \Phi_i^{ex}(\mathcal{C}_1, v_1).$$

If $i \in N_1 \setminus (M_1)'_{\mathcal{C}_1}$, then for any maximal chain C in $\mathcal{C}_2 * \mathcal{C}_1$ the transition T_C^i to S_C^i occurs $ch(\mathcal{C}_2)$ times as the same transition in C restricted to $\mathcal{C}_1$. Hence, we have

$$\Phi_i^{ex}(\mathcal{C}_2 * \mathcal{C}_1, v_2 * v_1) =$$
$$= \frac{1}{ch(\mathcal{C}_2 * \mathcal{C}_1)} \sum_{C \in CH(\mathcal{C}_2 * \mathcal{C}_1)} \frac{1}{|S_C^i \setminus T_C^i|} \left[(v_1 * v_2)\left(S_C^i, (S_C^i)'_{\mathcal{C}_2 * \mathcal{C}_1}\right) - (v_1 * v_2)\left(T_C^i, (T_C^i)'_{\mathcal{C}_2 * \mathcal{C}_1}\right)\right]$$
$$= \frac{1}{ch(\mathcal{C}_1)} \sum_{C \in CH(\mathcal{C}_1)} \frac{1}{|S_C^i \setminus T_C^i|} \left[v_1\left(S_C^i, (S_C^i)'_{\mathcal{C}_1}\right) - v_1\left(T_C^i, (T_C^i)'_{\mathcal{C}_1}\right)\right] = \Phi_i^{ex}(\mathcal{C}_1, v_1).$$

If $i \in N_2 \setminus N_1$, then for each maximal chain C in $\mathcal{C}_2 * \mathcal{C}_1$ the transition T_C^i to S_C^i occurs $ch(\mathcal{C}_1)$ times as the same transition in C restricted to $\mathcal{C}_2$. Hence

$$\Phi_i^{ex}(\mathcal{C}_2 * \mathcal{C}_1, v_2 * v_1) =$$
$$= \frac{1}{ch(\mathcal{C}_2 * \mathcal{C}_1)} \sum_{C \in CH(\mathcal{C}_2 * \mathcal{C}_1)} \frac{1}{|S_C^i \setminus T_C^i|} \left[(v_1 * v_2)\left(S_C^i, (S_C^i)'_{\mathcal{C}_2 * \mathcal{C}_1}\right) - (v_1 * v_2)\left(T_C^i, (T_C^i)'_{\mathcal{C}_2 * \mathcal{C}_1}\right)\right]$$
$$= \frac{1}{ch(\mathcal{C}_2)} \sum_{C \in CH(\mathcal{C}_2)} \frac{1}{|S_C^i \setminus T_C^i|} \left[v_2\left(S_C^i, (S_C^i)'_{\mathcal{C}_2}\right) - v_2\left(T_C^i, (T_C^i)'_{\mathcal{C}_2}\right)\right] = \Phi_i^{ex}(\mathcal{C}_2, v_2).$$

– Decomposition: consider the decomposition $(\mathcal{C}_1, \ldots, \mathcal{C}_q)$ in maximal chains of the context $\mathcal{C}$, for the concept game $(\mathcal{C}, v)$. By the definition of v_p, $p = 1, \ldots, q$, we have for every player i

$$\begin{aligned}\Phi_i^{ex}(\mathcal{C},v) &= \frac{1}{ch(\mathcal{C})}\sum_{C\in CH(\mathcal{C})}\frac{1}{|S_C^i\setminus T_C^i|}\left[v_N\left(S_C^i\right)-v_N\left(T_C^i\right)\right]\\ &= \frac{1}{ch(\mathcal{C})}\sum_{p=1}^{q}\frac{1}{|S_C^i\setminus T_C^i|}\left[v_p\left(S_C^i,(S_C^i)'_{\mathcal{C}_p}\right)-v_p\left(T_C^i,(T_C^i)'_{\mathcal{C}_p}\right)\right]\\ &= \frac{1}{ch(\mathcal{C})}\sum_{p=1}^{q}\Phi_i^{ex}(\mathcal{C}_p,v_p).\end{aligned}$$

□

Finally, we prove that the extent Shapley value is the only value for concept games that satisfies the above axioms.

Theorem 2. *The extent Shapley value is the only value over concept games satisfying (SP), (E), (MP), (C) and (D).*

Proof. We have already proved in Theorem 1 that the extent Shapley value satisfies all these axioms. It remains to show that they uniquely determine the value.

Let $(\mathcal{C},v)\in CCG$ be a concept game with $\mathcal{C}=(N,M,I)$ with $M=\{a\}$ (simple context). We have in that case $M'=\{i\in N\mid I(i,a)=1\}$, so that the concept lattice is reduced to $\{(M',\{a\}),(N,\emptyset)\}$. Observe that whenever $|M'|>1$, M' is a macro-player, and similarly $N\setminus M'$ is a macro-player as well if $|N\setminus M'|>1$. Suppose first that $M'=\emptyset$. The macro-player axiom imposes that $F_i(\mathcal{C},v)=F_j(\mathcal{C},v)$ for all $i,j\in N$, hence by the efficiency axiom, it follows that

$$F_i(\mathcal{C},v)=\frac{1}{|N|}v(N,\emptyset),\quad \forall i\in N,$$

so that F is uniquely determined for that type of game. Suppose now that M' is reduced to a singleton, say $\{i\}$. The separable payoff axiom imposes that $F_i(\mathcal{C},v)=v(M',M)$. If $|M'|>1$, M' is a macro-player, and by the macro-player axiom, it follows that for every $i,j\in M'$, $F_i(\mathcal{C},v)=F_j(\mathcal{C},v)$. Now, the separable payoff axiom implies

$$\sum_{j\in M'}F_j(\mathcal{C},v)=v(M',M),$$

so that finally $F_i(\mathcal{C},v)=\dfrac{v(M',M)}{|M'|}$, for all $i\in M'$. We can proceed similarly with the remaining players in $N\setminus M'$: applying (MP) and (E) finally yields

$$F_i(\mathcal{C},v)=\frac{1}{|N\setminus M'|}\big(v(N,\emptyset)-v(M',M)\big),\quad \forall i\in N\setminus M'.$$

As a conclusion, F is uniquely determined for any game $(\mathcal{C},v)$ with $\mathcal{C}=(N,\{a\},I)$.

Consider now any concept game $(\mathcal{C},v)$ and the decomposition $\mathcal{C}_1,\dots,\mathcal{C}_q$ of $\mathcal{C}$. The decomposition axiom implies that if F is uniquely determined on each $\mathcal{C}_1,\dots,\mathcal{C}_q$, then F is uniquely determined on $\mathcal{C}$. Hence we consider now $\mathcal{C}=(N,M,I)$ such that $L_{\mathcal{C}}^N=$

$\{S_1, \ldots, S_m\}$ with $S_1 = M'_{\mathcal{C}}$, $S_m = N$ and $S_{p-1} \subset S_p$. For each $p = 2, \ldots, m$ we define an attribute a_p, the context $\mathcal{C}_p = (S_p, \{a_p\}, I_p)$ where $I_p(i, a_p) = 0$ if $i \in S_{p-1}$ and $I_p(i, a_p) = 1$ otherwise, and the function

$$v_p(S_p, \emptyset) = v_N(S_p), \quad v_p(S_{p-1}, \{a_p\}) = v_N(S_{p-1}).$$

As $\mathcal{C}_p$ is a simple context, F is uniquely determined. It is easy to see that

$$\mathcal{C} = \mathcal{C}_m * (\mathcal{C}_{m-1} * (\cdots * (\mathcal{C}_2 * \mathcal{C}_1)))$$

and

$$v = v_m * (v_{m-1} * (\cdots * (v_2 * v_1))).$$

The concatenation axiom implies that F is uniquely determined. □

3.2 The intent Shapley value

We proceed in a similar way as for the extent value. We consider the lattice of intents $L^M_{\mathcal{C}}$ of a context $\mathcal{C}$ and the game v_M defined on it. We note that the set of maximal chains $CH(\mathcal{C})$ is isomorphic to the set of chains in $L^M_{\mathcal{C}}$. For a given maximal chain $C = \emptyset \subset S_1 \subset \cdots \subset S_k = M$ in $L^M_{\mathcal{C}}$ and an attribute $a \in M$, let B^a_C and A^a_C be respectively the last set in the sequence which does not contain a, and the first set containing a.

The *intent Shapley value* of $(\mathcal{C}, v)$, denoted by $\Phi^{in}(\mathcal{C}, v)$, is defined to be the Shapley value $\Phi(v_M)$ of the game of intents, given by

$$\Phi^{in}_a(\mathcal{C}, v) = \Phi_a(v_M) = \frac{1}{ch(\mathcal{C})} \sum_{C \in CH(\mathcal{C})} \frac{1}{|A^a_C \setminus B^a_C|}(v_M(A^a_C) - v_M(B^a_C)).$$

We formulate several properties. Let Ψ be any value over the set of concept games.

Efficiency axiom (E): $\sum_{a \in M} \Psi_a(\mathcal{C}, v) = v_M(M) = v(N, \emptyset) - v(M', M)$, for all $(\mathcal{C}, v) \in CCG$.

Let $K \subseteq M$, $|K| > 1$. We say that the set K is a *macro-attribute* if for any $S' \in L^M_{\mathcal{C}}$, $S' \neq \emptyset$, we have $K \subseteq S'$ or $K \cap S' = \emptyset$.

Macro-attribute axiom (MA): If $(\mathcal{C}, v) \in CCG$ and K is a macro-attribute in $L^M_{\mathcal{C}}$, then

$$\Psi_a(\mathcal{C}, v) = \Psi_b(\mathcal{C}, v) \qquad \forall a, b \in K.$$

Using the definition of the concatenation of contexts as given for the case of extent games, we introduce the following axiom.

Concatenation axiom (C): For all $(\mathcal{C}_1, v_1), (\mathcal{C}_2, v_2) \in CCG$ such that $\mathcal{C}_2$ is concatenable to $\mathcal{C}_1$, and $v_1(N_1, \emptyset) = v_2((M_2)'_{\mathcal{C}_2}, M_2)$, it holds

$$\Psi_a(\mathcal{C}_2 * \mathcal{C}_1, v_2 * v_1) = \begin{cases} \Psi_a(\mathcal{C}_2, v_2), & \text{if } a \in M_2 \\ \Psi_a(\mathcal{C}_1, v_1), & \text{if } a \in M_1 \setminus M_2 \end{cases}$$

where $v_2 * v_1$ is defined as for the extent value.

We consider a concept $\mathcal{C} = (N, M, I)$ and the maximal chains $C_1, \ldots, C_q$ of its intent lattice $L^M_{\mathcal{C}}$. The decomposition into maximal chains of $\mathcal{C}$ w.r.t. the intent lattice is a collection of contexts $\mathcal{C}_1, \ldots, \mathcal{C}_q$ with set of attributes M, such that their intent lattices $L^M_{\mathcal{C}_1}, \ldots, L^M_{\mathcal{C}_q}$ are precisely the maximal chains $C_1, \ldots, C_q$.

Decomposition axiom (D): If $\mathcal{C}_1, \ldots, \mathcal{C}_q$ is the decomposition of a context $\mathcal{C}$ in maximal chains w.r.t. the intent lattice, then

$$\Psi(\mathcal{C}, v) = \frac{1}{ch(\mathcal{C})} \sum_{p=1}^{q} \Psi(\mathcal{C}_p, v_p),$$

where $v_p(S, A) = v(S, A)$ for all $(S, A) \in L_{\mathcal{C}_p}$, and $p = 1, \ldots, q$.

Theorem 3. *The intent Shapley value is the unique value over the set of concept games which satisfy (E), (MA), (C) and (D).*

Proof is similar to the case of the extent Shapley value and is therefore omitted.

4 The core

Given a game $(\mathcal{C}, v)$ on a context, we consider the cores of the games on extents and intents:

$$\begin{aligned} \mathrm{core}(v_N) &= \{x \in \mathbb{R}^N \mid x(S) \geq v_N(S), S \in L^N_{\mathcal{C}} \text{ and } x(N) = v_N(N)\} \\ \mathrm{core}^*(v_M) &= \{y \in \mathbb{R}^M \mid y(A) \leq v_M(A), A \in L^M_{\mathcal{C}} \text{ and } y(M) = v_M(M)\}. \end{aligned}$$

(Note: $\mathrm{core}^*(v_M)$ is the anti-core, i.e., the set of vectors y such that $-y$ is in the core of $-v_M$). Let us call them for convenience the extent core and the intent core respectively.

We can write the intent core in a more convenient way. For any vector $y \in \mathrm{core}^*(v_M)$, we have

$$\begin{aligned} & y(A) \leq v_M(A) = v(N, \emptyset) - v(A', A), \forall A \in L^M_{\mathcal{C}}, \text{ and } y(M) = v_M(M) = v(N, \emptyset) \\ \Leftrightarrow\ & y(M) - y(M \setminus A) \leq v(N, \emptyset) - v(A', A), \forall A \in L^M_{\mathcal{C}}, \text{ and } y(M) = v(N, \emptyset) \\ \Leftrightarrow\ & y(M \setminus A) \geq v(A', A), \forall A \in L^M_{\mathcal{C}}, \text{ and } y(M) = v_M(M) = v(N, \emptyset) \end{aligned}$$

i.e., $y \in \mathrm{core}(\overline{v}_M)$, with $\overline{v}_M(A) = v((M \setminus A)', M \setminus A)$, for all $A \in \overline{L^M_{\mathcal{C}}}$, where $\overline{L^M_{\mathcal{C}}} = \{A \subseteq M \mid M \setminus A \in L^M_{\mathcal{C}}\}$ is the dual closure system associated to $L^M_{\mathcal{C}}$. This prove $\mathrm{core}^*(v_M) = \mathrm{core}(\overline{v}_M)$. Note that if $M' = \emptyset$, $\overline{v}$ coincide with the conjugate of v_M, that is, $\overline{v}_M(A) = v_M(M) - v_M(A^c)$.

Example 9 (Example 1 continued). Let us define the following game on the concept lattice of Figure 1:

(S, S')	$(1, ab)$	$(2, bc)$	$(13, a)$	$(12, b)$	$(24, c)$	$(1234, \emptyset)$
$v(S, S')$	10	20	50	40	40	100

We obtain

$$\mathrm{core}(v_N) = \begin{cases} x_1 & \geq 10 \\ x_2 & \geq 20 \\ x_1 + x_2 & \geq 40 \\ x_1 + x_3 & \geq 50 \\ x_2 + x_4 & \geq 40 \\ x_1 + x_2 + x_3 + x_4 & = 100 \end{cases}, \quad \mathrm{core}^*(v_M) = \begin{cases} y_c & \geq 10 \\ y_a & \geq 20 \\ y_b + y_c & \geq 50 \\ y_a + y_c & \geq 40 \\ y_a + y_b & \geq 40 \\ y_a + y_b + y_c & = 100 \end{cases}$$

which are not empty since $x = (20, 20, 30, 30) \in \mathrm{core}(v_N)$ and $y = (30, 30, 40) \in \mathrm{core}^*(v_M)$.

An important consequence of the above facts, we find that the study of the extent and intent cores amounts to the study of the core of games on closure systems (closed under intersection) and on dual closure systems (closed under union). In what follows we study in depth the structure of the extent core, especially its conic part. Results on the intent core will be obtained by duality. In the whole section, $\mathcal{F}$ denotes any collection of sets.

4.1 Nonemptiness

We ask when the cores are nonempty. As said in Section 2.1, the core of a game on a subcollection $\mathcal{F}$ of 2^N is nonempty if and only if the game is balanced in the usual sense. Hence, $\mathrm{core}(v_N)$ is nonempty if and only if v_N is balanced, and $\mathrm{core}^*(v_M)$ is nonempty if and only if $\overline{v}_M$ is balanced.

The case $M' \neq \emptyset$ deserves some attention, because then $\mathrm{core}(v_N)$ is never empty. Indeed, it is not difficult to see that the only balanced collection in $L^N_{\mathcal{C}}$ is $\{N\}$, whence any game on the lattice of extents is balanced. There is no such conclusion for $\mathrm{core}^*(v_M)$ because $N' = \emptyset$.

There seems to be no clear relation between the nonemptiness of the extent and intent cores. One could be empty while the other is not, or both could be empty or nonempty. The following example illustrates this.

Example 10 (Example 9 continued). Let us take the concept lattice of Examples 9 and 1, but without specific values for v. The minimal balanced collections for $L^N_{\mathcal{C}}$ are $\{1234\}$ and $\{13, 24\}$, while those for $L^M_{\mathcal{C}}$ are $\{abc\}$, $\{bc, ac, ab\}$, $\{c, ab\}$ and $\{bc, a\}$. Hence, by Bondareva-Shapley theorem, the extent core is nonempty if and only

$$v(13) + v(24) \leq v(1234)$$

while the intent core is nonempty if and only if

$$\begin{aligned} \frac{1}{2}v(13) + \frac{1}{2}v(12) + \frac{1}{2}v(24) &\leq v(1234) \\ v(1) + v(24) &\leq v(1234) \\ v(2) + v(13) &\leq v(1234). \end{aligned}$$

The nonemptiness of one the core does not imply the nonemptiness of the other one, unless some conditions on v are satisfied. For example, the nonemptiness of the intent core implies the nonemptiness of the extent core if $v(12) \leq v(1234)$, or if $v(1)+v(2) \leq v(1234)$.

4.2 Pointedness and boundedness of the extent core

We assume that core(v_N) is nonempty. The aim of this section is to study the question whether the core is unbounded and whether it contains a line, in which case it is not pointed (i.e., it has no vertices). The general condition to be pointed is that the system of linear equations

$$x(S) = 0, \forall S \in \mathcal{F}$$

has 0 as its unique solution (in which case we say that, following Derks and Reijnierse [8], $\mathcal{F}$ is *nondegenerate*). It is easy to see that $\mathcal{F}$ is degenerate if there exists a macro-player K in $\mathcal{F}$, because $\mathcal{F}$ contains the hyperplane $x(K) = 0$. A remarkable result with closure systems is that the converse is also true.

Theorem 4. *A closure system is nondegenerate if and only if it contains no macro-player.*

Proof. The "only if" part is obvious since the presence of a macro-player implies degeneracy.

Suppose that $\mathcal{F}$ is a closure system on N with bottom element M', which has no macro-player. We prove by induction on $n = |N|$ that it is nondegenerate. The assertion is easily checked for $n = 1$, with the two possible closure systems $\{\emptyset, \{1\}\}$ and $\{\{1\}\}$. Suppose the assertion holds till some value $n-1$ and let us prove it for n.

Claim: there exists $i \in N$ such that $\{i\} \in \mathcal{F}$.

Proof of the claim: Since $\mathcal{F}$ has no macro-player, we know that its bottom M' is either $\emptyset$ or some singleton. In the latter case, the claim is proved. Suppose then that $M' = \emptyset$. Then necessarily, every atom is a singleton. Indeed, suppose *per contra* that S is an atom, with $|S| > 1$. Since S is not a macro-player, there exists $T \in \mathcal{F}$ separating S, i.e., $j \in T \not\ni k$ for some $j, k \in S$. Since $\mathcal{F}$ is closed under intersection, it follows that $S \cap T \in \mathcal{F}$ and $\emptyset \neq S \cap T \subsetneq S$, a contradiction with the fact that S is an atom. □

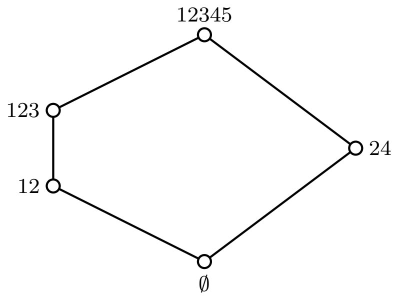
Consider then $\mathcal{F}^{-i} = \{S \subseteq N \setminus i \mid S \text{ or } S \cup i \in \mathcal{F}\}$ on $N \setminus i$, the collection of sets obtained from $\mathcal{F}$ by removing i in every set. Note that $\emptyset \in \mathcal{F}^{-i}$. We prove that $\mathcal{F}^{-i}$ is a closure system without macro-players.

- $\mathcal{F}^{-i} \ni N \setminus i$: clear since $N \in \mathcal{F}$.
- $\mathcal{F}^{-i}$ is closed under intersection: take $S, S' \in \mathcal{F}^{-i}$. Then three cases arise. If $S, S' \in \mathcal{F}$, then $S \cap S' \in \mathcal{F}$ and $i \notin S \cap S'$, hence $S \cap S' \in \mathcal{F}^{-i}$. If $S \in \mathcal{F}$ and $S' \cup i \in \mathcal{F}$, then $i \notin S \cap (S' \cup i) \in \mathcal{F}$, and therefore $S \cap (S' \cup i) = S \cap S' \in \mathcal{F}^{-i}$. Lastly, if $S \cup i, S' \cup i \in \mathcal{F}$, then $i \in (S \cup i) \cap (S' \cup i) \in \mathcal{F}$, therefore $((S \cup i) \cap (S' \cup i)) \setminus i = S \cap S' \in \mathcal{F}^{-i}$.
- $\mathcal{F}^{-i}$ has no macro-player: suppose $K \subseteq N \setminus i$ is a macro-player in $\mathcal{F}^{-i}$. Take $S \in \mathcal{F}^{-i}$. Then either $S \cap K = \emptyset$ or $S \supseteq K$. If $S \in \mathcal{F}$, then $S \cap K = \emptyset$ or $S \supseteq K$ remains true. If $S \cup i \in \mathcal{F}$, then $(S \cup i) \cap K = \emptyset$ or $S \cup i \supseteq K$ is true because $K \not\ni i$. Hence K is a macro-player in $\mathcal{F}$, a contradiction.

Then $\mathcal{F}^{-i}$ is a closure system without macro-player on $N \setminus i$, and by the induction hypothesis, $\mathcal{F}^{-i}$ is nondegenerate, i.e., the system of equations $x(S) = 0, S \in \mathcal{F}^{-i}$ has a unique solution $x = 0$. Finally, observe that the system $x(S) = 0, S \in \mathcal{F}$ differs from the previous one only by the adjunction of x_i in some lines. Since $\{i\} \in \mathcal{F}$, the line $x_i = 0$ makes the two systems equivalent. Therefore, $\mathcal{F}$ is nondegenerate. □

The next example shows that this result does not extend to arbitrary collections of sets.

Example 11. Take $n = 5$ and the collection $\mathcal{F}$ shown below.



$\mathcal{F}$ is not closed under intersection but has no macro-player. However it is degenerate (rank is 4 and $(1, -1, 0, 1, -1)$ is a vector of the null space).

This being established, we turn to the question whether the core is unbounded or not. The following result is useful [8].

Theorem 5. *The recession cone of a game on a collection of sets $\mathcal{F}$ is a linear subspace if and only if $\mathcal{F} \setminus \{\emptyset, N\}$ is a balanced collection.*

Corollary 1. *The core is bounded (equiv., the recession cone reduces to $\{0\}$) if and only if $\mathcal{F}$ is nondegenerate and $\mathcal{F} \setminus \{\emptyset, N\}$ is balanced.*

The situation is summarized by the following table.

	$M' = \emptyset$	$\lvert M'\rvert = 1$	$\lvert M'\rvert > 1$
pointed	if $\mathcal{F}$ has no macro-player	if $\mathcal{F}$ has no macro-player	no
bounded	if $\mathcal{F}$ balanced and no macro-player	no	no

Table 1. Boundedness and pointedness of the extent core

4.3 Some results on balanced collections

For any collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$, its *closure by intersection* denoted by $\overline{\mathcal{B}}$ is formed by all sets of $\mathcal{B}$, plus the intersection of any family of sets of $\mathcal{B}$, provided the intersection is nonempty[2]. Note that $\overline{(\cdot)}$ is a closure operator, in the sense that $\overline{\overline{\mathcal{B}}} = \overline{\mathcal{B}}$, $\mathcal{B} \subseteq \overline{\mathcal{B}}$, and $\mathcal{B} \subseteq \mathcal{B}'$ implies $\overline{\mathcal{B}} \subseteq \overline{\mathcal{B}'}$.

Theorem 6. *Suppose $\mathcal{B}$ is a balanced collection on N. Then $\overline{\mathcal{B}}$, its closure by intersection, is balanced.*

[2] Be careful that this does not mean that $\overline{\mathcal{B}}$ is closed under intersection, since $\overline{\mathcal{B}}$ does not contain the empty set, despite that it may contain disjoint sets.

Proof. We prove the result by induction on $|N|$. The result is trivially true when $|N| = 1$. Suppose that the property holds for all $N$of size at most n, and let us prove it for $|N| = n + 1$.

Suppose that there exists a macro-player K, and let $[k]$ be a (new) player replacing K. Then by considering $N' = (N \setminus K) \cup \{[k]\}$, we can define a balanced collection from $\mathcal{B}$ on N', and since $|N'| < n$, the result is proved by the induction hypothesis. We can therefore consider in the rest of the proof that no macro-player exists. This implies in particular that $|\mathcal{B}| > 1$ and that there is no set $S \in \mathcal{B}$, $|S| > 1$ which is disjoint from every other set.

Suppose first that $\mathcal{B} \ni S$ such that $S \cap T = \emptyset$ for all $T \in \mathcal{B}$, $T \neq S$, with $|S| = 1$. Then observe that $\mathcal{B} \setminus \{S\}$ is a balanced collection over $N \setminus S$. By induction hypothesis, $\overline{\mathcal{B} \setminus \{S\}}$ is balanced, and since $\overline{\mathcal{B}} = \overline{\mathcal{B} \setminus \{S\}} \cup \{S\}$, the result is proved in this case.

Suppose on the contrary that for every $S \in \mathcal{B}$, S intersects some $T \in \mathcal{B}$. We claim that the collection $\mathcal{B}_\cap = \{S \cap T \mid S, T \in \mathcal{B}, S \neq T\}$ is balanced. Since the union of balanced collections is balanced (see, e.g., Owen [17]), it follows that $\mathcal{B} \cup \mathcal{B}_\cap$ is balanced over N. Applying the same procedure over $\mathcal{B} \cup \mathcal{B}_\cap$ yields another balanced collection, and continuing like this eventually leads to $\overline{\mathcal{B}}$, proving that it is balanced.

Proof of the claim: Take $(\lambda_S)_{S \in \mathcal{B}}$ any system of balancing weights for $\mathcal{B}$. For each $i \in N$, consider the subcollection $\mathcal{B}^i = \{S_1, \ldots S_k\}$ of $\mathcal{B}$ of sets containing i. Observe that $\mathcal{B}^i \neq \emptyset$ since $\mathcal{B}$ is balanced, and $k > 1$. Indeed, $k = 1$ yields $\lambda_{S_1} = 1$. By assumption, there exists $T \in \mathcal{B}$ intersecting S_1, therefore there exists $j \neq i$, $j \in T \cap S_1$, and $\sum_{T' \ni j} \lambda_{T'} = 1$ forces $\lambda_T = 0$, which is impossible.

For each $\mathcal{B}^i$ we construct the collection of pairwise intersections $\mathcal{B}^i_\cap = \{S_j \cap S_\ell \mid S_j, S_\ell \in \mathcal{B}^i, S_j \neq S_\ell\}$, with the following weights:

$$\lambda^i_S = \sum_{\substack{S_j, S_\ell \in \mathcal{B}^i \\ S_j \neq S_\ell \\ S_j \cap S_\ell = S}} \frac{\lambda_{S_j} + \lambda_{S_\ell}}{k-1}, \text{ for every } S \in \mathcal{B}^i_\cap.$$

By construction, each λ^i_S is positive, and we have

$$\sum_{S \in \mathcal{B}^i_\cap} \lambda^i_S = \frac{1}{k-1} \sum_{j=1}^{k} \sum_{\ell=j+1}^{k} (\lambda_{S_j} + \lambda_{S_\ell}) = \sum_{j=1}^{k} \lambda_{S_j} = 1.$$

Clearly, $\mathcal{B}_\cap = \bigcup_{i \in N} \mathcal{B}^i_\cap$, however some sets may be present in two different $\mathcal{B}^i_\cap$, possibly with different weights. The last step consists in defining a unique weight for each $S \in \mathcal{B}_\cap$, while keeping the normalization condition. Observe that if S appears in, say, $\mathcal{B}^i_\cap$ and $\mathcal{B}^j_\cap$, then $S \supseteq \{i, j\}$. Since there is no macro-player, $\{i\} \in \mathcal{B}^i_\cap$, and $\{j\} \in \mathcal{B}^j_\cap$. Define

$$\lambda'_S = \min(\lambda^i_S, \lambda^j_S),$$

and assuming that $\lambda^i_S < \lambda^j_S$, define

$$\lambda'_{\{j\}} = \lambda^j_{\{j\}} + (\lambda^j_S - \lambda^i_S).$$

Observe that $\lambda'_S > 0$, $\lambda'_{\{j\}} > 0$. Putting for the other sets $\lambda'_T = \lambda^i_T$ for some i, the system of weights $(\lambda'_S)_{S \in \mathcal{B}_\cap}$ satisfies $\sum_{T \ni i} \lambda'_T = 1$ for all $i \in N$. □

Lemma 1. *Suppose $\mathcal{B}$ is a balanced collection on N. Then $\overline{\mathcal{B}}$ contains all singletons in N if and only if $\mathcal{B}$ has no macro-player*[3].

Proof. $\Rightarrow$) Clear.

$\Leftarrow$) Suppose $\mathcal{B}$ has no macro-player and for some $i \in N$, $\{i\} \notin \overline{\mathcal{B}}$. Then $\bigcap_{B\in\mathcal{B},B\ni i} B = S \ni i$, with $|S| > 1$. Since S is not a macro-player, there must exist $T \in \mathcal{B}$ such that $T \not\ni i$ and $T \cap S \neq \emptyset$. Take $j \in T \cap S$, and consider a balancing system $(\lambda_B)_{B\in\mathcal{B}}$ for $\mathcal{B}$. Then

$$1 = \sum_{B\in\mathcal{B},B\ni i} \lambda_B < \sum_{B\in\mathcal{B},B\ni i} \lambda_B + \lambda_T \leq \sum_{B\in\mathcal{B},B\ni j} \lambda_B = 1,$$

a contradiction. □

Note that $\Rightarrow$) holds also if $\mathcal{B}$ is not balanced. An immediate consequence is:

Corollary 2. *If $\mathcal{B}$ is a balanced collection on N, then $\overline{\mathcal{B}}$ contains all macro-players in $\mathcal{B}$ and all singletons in N not contained in macro-players.*

Proof. If there is no macro-player, just apply Lemma 1. Otherwise, replace each macro-player by a single player and apply Lemma 1. □

Remark 1. (i) If $\mathcal{B}$ is minimal balanced and contains no macro-player, then $\mathcal{B} \neq \overline{\mathcal{B}}$. This is clear from Lemma 1 and from the fact that a minimal balanced collection has at most n sets.

(ii) One may wonder if a dual version of Theorem 6 exists, i.e.: if $\mathcal{B}$ is balanced and $\mathcal{B} = \overline{\mathcal{B}}$, then its opening $\mathcal{B}^\circ$ (i.e., removing all sets being intersection of others) is balanced. This is not true as shown by the following example: take $N = \{1, 2, 3, 4\}$ and the balanced collection $\mathcal{B} = \{12, 23, 2, 134, 14, 34\}$ ($\lambda_S = \frac{1}{3}$ can be taken for any $S \in \mathcal{B}$). Its closure is

$$\overline{\mathcal{B}} = \{12, 23, 2, 134, 14, 34, 1, 3, 14, 4\}$$

and is balanced by Theorem 6. Now its opening is $(\overline{\mathcal{B}})^\circ = \{12, 23, 134, 14, 34\}$, but this is not a balanced collection, as it can be checked.

(iii) Observe that in general $\overline{\mathcal{B}_1} \cup \overline{\mathcal{B}_2} \subseteq \overline{\mathcal{B}_1 \cup \mathcal{B}_2}$ with possibly strict inclusion (e.g., take $\mathcal{B}_1 = \{1, 23\}$ and $\mathcal{B}_2 = \{12, 3\}$). It is not sure whether one can obtain any closed balanced collection as a union of the closure of minimal balanced collections.

We come to our final result.

Theorem 7. *Assume $M' = \emptyset$. Then the core of any balanced game is bounded if and only if $\mathcal{F}$ has no macro-player and $\mathcal{F} \setminus \{\emptyset, N\}$ arises as the closure under intersection of the union of some minimal balanced collections on N, i.e., $\mathcal{F} \setminus \{\emptyset, N\} = \overline{\bigcup \mathcal{B}_i}$.*

Proof. $\Leftarrow$) Since the union of balanced collections is balanced, by Theorem 6, $\mathcal{F} \setminus \{\emptyset, N\}$ is balanced. Since no macro-player exists, by Lemma 1, $\mathcal{F}$ contains all singletons, and is therefore nondegenerate. Hence $\mathrm{core}(0) = \{0\}$ by Corollary 1.

$\Rightarrow$) $\mathrm{core}(0) = \{0\}$ iff $\mathcal{F}$ is non-degenerate and $\mathcal{F} \setminus \{\emptyset, N\}$ is balanced. Since $\mathcal{F}$ is closed under intersection, it follows that $\mathcal{F} \setminus \{\emptyset, N\}$ is the closure of some balanced collection $\mathcal{B}$, with $\mathcal{F} \setminus \{\emptyset, N\} \supseteq \mathcal{B} \supseteq (\mathcal{F} \setminus \{\emptyset, N\})^\circ$. Since $\mathcal{B}$ is the union of some minimal balanced collections, $\mathcal{F} \setminus \{\emptyset, N\}$ has the required form. Finally, nondegeneracy implies that $\mathcal{F}$ does not contain any macro-player.

[3] We mean: there is no macro-player in $\mathcal{B}$. Note that K could be a macro-player in $\mathcal{B}$ but not in $\mathcal{F}$.

The theorem as well as Remark 1 (iii) gives us a means to derive any set system $\mathcal{F}$ closed under intersection such that $\mathcal{F} \setminus \{\emptyset, N\}$ is balanced. It suffices to generate all minimal balanced collections, except $\{N\}$ (see Peleg [18] for the description of an algorithm doing this), and build all possible unions of them, then take their closure under intersection. This gives $\mathcal{F} \setminus \{\emptyset, N\}$.

4.4 Extremal rays of the extent core

We recall the classical result which holds for the case $\mathcal{F} = \mathcal{O}(N, \preceq)$ (distributive lattice), where $\mathcal{O}(\cdot)$ indicates the set of downsets of some poset.

Lemma 2. *([7, 20]) If $\mathcal{F} = \mathcal{O}(N, \preceq)$, the extremal rays of* $\mathrm{core}(0)$ *are $1^i - 1^j$, for $i \prec\cdot\, j$ in $(N, \preceq)$.*

Let $\mathcal{F} = L^N_{\mathcal{C}}$ be a closure system on N, with bottom element M'. We deal for simplicity with the case where there is no macro-player nor companion player in $\mathcal{F}$ (irreducible closure system). In this case $M' = \emptyset$ and $|\mathcal{J}(\mathcal{F})| = n$, therefore $\mathcal{J}(\mathcal{F})$ can be assimilated to N. We write $(N, \preceq)$ for the poset on N isomorphic to $(\mathcal{J}(\mathcal{F}), \subseteq)$.

Theorem 8. *Suppose that $\mathcal{F}$ is an irreducible closure system. Then $1^i - 1^j$ is a ray of* $\mathrm{core}(\mathcal{F}, 0)$ *for every $i \prec\cdot\, j$ in the poset $(N, \preceq)$, not necessarily extremal. Moreover,* $\mathrm{core}(\mathcal{F}, 0) = \mathrm{core}(\mathcal{O}(N, \preceq), 0)$ *if and only if any $S \in \mathcal{O}(N, \preceq) \setminus \mathcal{F}$ can be written as a union of disjoint sets in $\mathcal{F}$.*

Proof. 1. Let $(N, \preceq)$ be the poset of join-irreducible elements of $\mathcal{F}$. Then $\mathcal{O}(N, \preceq)$ is a distributive lattice and by Lemma 2, $\mathrm{core}(\mathcal{O}(N, \preceq), 0)$ is generated by the rays $1^i - 1^j$, for $i \prec\cdot\, j$ in the poset $(N, \preceq)$. Observe that $\mathcal{F}$ is obtained from $\mathcal{O}(N, \preceq)$ by removing some subsets, therefore $\mathrm{core}(\mathcal{F}, 0) \supseteq \mathrm{core}(\mathcal{O}(N, \preceq), 0)$. Hence any extremal ray of the latter remains a ray in the former, although not necessarily extremal.

2. To prove the second assertion, equality of the recession cones amounts to show that the "missing" inequalities in $\mathrm{core}(\mathcal{F}, 0)$ are implied by the present ones.

2.1. If $S \in \mathcal{O}(N) \setminus \mathcal{F}$ can be written as a disjoint union of sets in $\mathcal{F}$, say $S_1, \ldots, S_k$, then clearly $x(S) \geq 0$ is implied by $x(S_1) \geq 0, \ldots, x(S_k) \geq 0$.

2.2. Conversely, suppose that $x(T) \geq v(T)$ with $T \in \mathcal{O}(N) \setminus \mathcal{F}$ is implied by the other inequalities. It means that there exist $\lambda_S \geq 0$, $S \in \mathcal{F} \setminus \{N\}$, and $\lambda_N \in \mathbb{R}$ such that

$$\sum_{S \in \mathcal{F}, S \neq N, T} \lambda_S 1^S + \lambda_N 1^N = 1^T. \tag{3}$$

2.2.1. Suppose first that for all $\lambda_S > 0$, we have $S \subseteq T$, and $\lambda_N = 0$. Then we have found a subcollection $\mathcal{B} = \{S \in \mathcal{F} : \lambda_S > 0\}$ in $\{S \in \mathcal{F} : S \subseteq T\} =: \mathcal{F}(T)$ which is a balanced collection over T. Since $\mathcal{F}$ is closed under intersection, $\mathcal{F}(T)$ contains $\overline{\mathcal{B}}$. By Lemma 1 and Corollary 2, it follows that $\overline{\mathcal{B}}$ (and hence $\mathcal{F}$) contains all macro-players of $\mathcal{B}$ and all singletons in T. Therefore, we have found a decomposition of T into disjoint sets of $\mathcal{F}$.

2.2.2. Suppose on the contrary that no set of coefficients λ_S, $S \in \mathcal{F}$ satisfy (3) with the condition given in 2.2.1. (i.e., no balanced collection over T exists in $\mathcal{F}$), but (3) can be satisfied provided $\lambda_S > 0$ for some $S \not\subseteq T$. Then there exist $j \notin T$ induced by the combination, i.e., $\sum_{S\in\mathcal{F},S\neq N}\lambda_S 1^S$ has a positive component for j. It follows that necessarily $\lambda_N < 0$, and we may put w.l.o.g. $\lambda_N = -1$. In order to have (3) satisfied, we must have

$$\sum_{S\in\mathcal{F},S\neq N}\lambda_S 1^S = 1^{N\setminus T} + 2\cdot 1^T.$$

This means that we must find a collection $\mathcal{B}$ in $\mathcal{F}$ such that $\{S\setminus T : S\in\mathcal{B}\}$ is a balanced collection over $N\setminus T$, and $\{S\cap T : S\in\mathcal{B}\}$ is a balanced collection over T with the same coefficients, and we must find also a balanced collection $\mathcal{B}'\subseteq\mathcal{F}(T)$ over T. But the latter does not exist by assumption. □

The above condition is easily violated as shown in the next example.

Example 12. Consider $n = 5$ and $\mathcal{F}\subset\mathcal{O}(N,\preceq)$ depicted on Figure 8 together with $(N,\preceq)$ (join-irreducible sets in red). Observe that $\mathcal{F}\setminus\mathcal{O}(N) = \{1234\}$, and that it is not possible

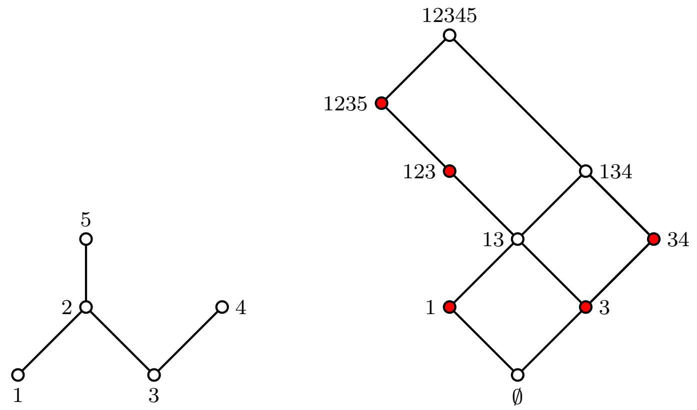


Fig. 8. A lattice (right) with the poset of its join-irreducible elements (left)

to write 1234 as a union of the two atoms 1 and 3. Hence $\mathrm{core}(\mathcal{F},0)\neq\mathrm{core}(\mathcal{O}(N),0)$. This can be verified as $r = (0,0,1,-1,0)$, which is extremal in $\mathrm{core}(\mathcal{O}(N),0)$, is no more an extremal ray of $\mathrm{core}(\mathcal{F},0)$. We can see this in two ways. First, the set of equalities satisfied by r is

$$\begin{aligned} x_1 &= 0\\ x_3 + x_4 &= 0\\ x_1 + x_3 + x_4 &= 0\\ x_1 + x_2 + x_3 + x_4 + x_5 &= 0\end{aligned}$$

Observe that the 3d equality is implied by the two first, hence the system determines a 2-dim space, not a ray. Also, it can be checked in the same way that $r_1 = (0,-1,1,-1,1)$ and $r_2 = (0,1,0,0,-1)$ are extremal rays, and that $r = r_1 + r_2$.

4.5 Properties of the intent core

Similar results for the intent core can be obtained easily from the previous results by duality.

Pointedness and boundedness As explained above, and since $\mathrm{core}^*(v_M) = \mathrm{core}(\overline{v}_M)$, all reduces to the study of the system of linear equations

$$y(S) = 0, \quad \forall S \in \overline{L^M_{\mathcal{C}}}.$$

Since $y(S) + y(M \setminus S) = y(M) = 0$ for any $S \in L^M_{\mathcal{C}}$, the above system is equivalent to

$$y(S) = 0, \quad \forall S \in L^M_{\mathcal{C}}.$$

Since $L^M_{\mathcal{C}}$ is closed under intersection, all previous results apply directly. Since $N' = \emptyset$, the situation is simpler than with the extent core, and we find:

(i) The intent core is pointed if and only if $L^M_{\mathcal{C}}$ has no macro-attribute (defined similarly as a macro-player);

(ii) The intent core is bounded if $L^M_{\mathcal{C}}$ is balanced and has no macro-attribute.

Extremal rays We can proceed similarly. The recession cone of the intent core is given by the system

$$\begin{aligned} y(S) &\geq 0, \quad \forall S \in \overline{L^M_{\mathcal{C}}} \\ y(M) &= 0. \end{aligned} \tag{4}$$

Since $y(M) = 0$, proceeding as above, the system is equivalent to

$$\begin{aligned} y(S) &\leq 0, \quad \forall S \in L^M_{\mathcal{C}} \\ y(M) &= 0. \end{aligned} \tag{5}$$

Again, since $L^M_{\mathcal{C}}$ is a closure system, we can benefit from previous results. First, we have the following lemma, similar to Lemma 2.

Lemma 3. *Suppose that $(L^M_{\mathcal{C}}, \subseteq)$ is a distributive lattice, and denote by $(M, \preceq)$ the poset of its join-irreducible elements. Then the extremal rays of* $\mathrm{core}^*(v_M)$ *are $1^b - 1^a$, for any $a \prec\cdot\, b$ in $(M, \preceq)$.*

Proof. Recall that the intent core is equal to $\mathrm{core}(\overline{L^M_{\mathcal{C}}}, \overline{v}_M)$, hence the recession cone of the intent core is simply $\mathrm{core}(\overline{L^M_{\mathcal{C}}}, 0)$, given by (4). It is equivalent to the system (5), hence $\mathrm{core}(\overline{L^M_{\mathcal{C}}}, 0) = -\mathrm{core}(L^M_{\mathcal{C}}, 0)$. Since $L^M_{\mathcal{C}}$ is a distributive lattice, it is generated by $(M, \preceq)$. It follows from Lemma 2 that extremal rays of $\mathrm{core}(L^M_{\mathcal{C}}, 0)$ are of the form $1^a - 1^b$, with $a \prec\cdot\, b$. Since $\mathrm{core}(\overline{L^M_{\mathcal{C}}}, 0) = -\mathrm{core}(L^M_{\mathcal{C}}, 0)$, the result follows.

As a consequence, and since $L^M_{\mathcal{C}}$ is a closure system, we obtain by application of Theorem 8 the main result of this section:

Theorem 9. *Suppose that $L^M_{\mathcal{C}}$ is irreducible. Then $1^b - 1^a$ is a ray of* $\mathrm{core}^*(0)$ *(recession cone of the intent core) for every $a \prec\cdot\, b$ in the poset $(M, \preceq)$, not necessarily extremal. Moreover,* $\mathrm{core}^*(0) = \mathrm{core}(\mathcal{O}(M, \preceq^\partial), 0)$ *if and only if any $S \in \mathcal{O}(M, \preceq^\partial) \setminus \overline{L^M_{\mathcal{C}}}$ can be written as the intersection of sets in $\overline{L^M_{\mathcal{C}}}$ whose union covers M.*

Note that the poset $(M, \preceq^\partial)$ is isomorphic to the poset of meet-irreducible elements of the concept lattice $L_{\mathcal{C}}$. We formulate the same result in terms of the core of games on dual closure systems.

Corollary 3. *Let $\mathcal{F}$ be a dual closure system on M, and $(M, \preceq)$ the poset of its join-irreducible elements. Then $1^a - 1^b$ is a ray of* $\text{core}(\mathcal{F}, 0)$ *for every $a \prec\cdot\, b$ in $(M, \preceq)$, not necessarily extremal. Moreover,* $\text{core}(\mathcal{F}, 0) = \text{core}(\mathcal{O}(M, \preceq), 0)$ *if and only if any $S \in \mathcal{O}(M, \preceq) \setminus \mathcal{F}$ can be written as the intersection of sets in $\mathcal{F}$ whose union covers M.*

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