A CHARACTERIZATION OF THE CLASSICAL ORTHOGONAL DISCRETE AND $q-$POLYNOMIALS

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Abstract. In this paper we present a new characterization for the classical discrete and $q-$classical (discrete) polynomials (in the Hahn's sense).

Introduction

The classical orthogonal polynomials are very interesting mathematical objects that have attracted the attention not only of mathematicians since their appearance at the end of the XVIII century connected with some physical problems. They are used in several branches of mathematical and physical sciences and they have a lot of useful properties: they satisfy a three-term recurrence relation (TTRR), they are the solution of a second order linear differential (or difference) equation, their derivatives (or finite differences) also constitute an orthogonal family, their generating functions can be given explicitly, among others (for a review see e.g. [2, 6, 15, 16] and the recent [3]). Among such properties, a fundamental role is played by the so-called characterization theorems, i.e., such properties that completely define and characterize the classical polynomials. Obviously not every property characterize the classical polynomials and as an example we can use the TTRR. It is well known that, under certain conditions —by the so-called Favard Theorem (for a review see [11])–, the TTRR characterizes the orthogonal polynomials (OP) but there exist families of OP that satisfy a TTRR but not a linear differential equation with polynomial coefficients, or a Rodrigues-type formula, etc. For a more complete review on this see e.g. [1, 7, 12, 14] or the more recent work [2]. In this paper we will complete the works [7, 14] proving a new characterization for the classical discrete [7, 10] and the q-classical [8, 14] polynomials.

1. Preliminaries

Let $\mathbb{P}$ be the linear space of polynomial functions in $\mathbb{C}$ with complex coefficients and $\mathbb{P}^*$ be its algebraic dual space, i.e., $\mathbb{P}^*$ is the linear space of

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all linear functionals $\mathbf{u}:\mathbb{P}\to\mathbb{C}$. In general, we will represent the action of a functional over a polynomial by

$$\langle \mathbf{u},\pi\rangle,\quad \mathbf{u}\in\mathbb{P}^*\ ,\ \pi\in\mathbb{P},$$

and therefore a functional is completely determined by a sequence of complex numbers $\langle \mathbf{u},x^n\rangle=u_n$, $n\geq 0$, the so-called moments of the functional.

Definition 1.1. *Let $(P_n)_{n\geq 0}$ be a basis sequence of $\mathbb{P}$ such that $degP_n=n$. We say that $(P_n)_{n\geq 0}$ is an orthogonal polynomial sequence (OPS), if and only if there exists a functional $\mathbf{u}\in\mathbb{P}^*$ such that*

$$\langle \mathbf{u},P_mP_n\rangle=k_n\delta_{mn},\quad k_n\neq 0,\quad n\geq 0\ ,$$

where δ_{mn} is the Kronecker delta. If the leading coefficient of P_n is equal to 1 for all n, i.e., $P_n(x)=x^n+\cdots$, we say that the sequence $(P_n)_{n\geq 0}$ is a monic orthogonal polynomial sequence (MOPS) and denote it by $(P_n)_{n\geq 0}=$ $\mathrm{mops}(\mathbf{u})$.

It is very well known that a such OPS exists if and only if the linear functional $\mathbf{u}$ is quasi-definite.

Next, we introduce the forward and backward difference operators defined on $\mathbb{P}$ by

$$\Delta:\mathbb{P}\mapsto\mathbb{P},\quad \Delta y(x)=y(x+1)-y(x),$$
$$\nabla:\mathbb{P}\mapsto\mathbb{P},\quad \nabla y(x)=y(x)-y(x-1).$$

For the Δ operator we have the property

$$\Delta[f(x)g(x)]=f(x)\Delta g(x)+g(x+1)\Delta f(x). \tag{1.1}$$

Also we will use the Jackson q-derivative operator $\mathcal{D}_q$ on $\mathbb{P}$ defined by

$$\mathcal{D}_q:\mathbb{P}\mapsto\mathbb{P},\quad \mathcal{D}_q\pi=\frac{\pi(qx)-\pi(x)}{(q-1)x}\ ,\quad |q|\neq 0,1, \tag{1.2}$$

Notice that in this case we have

$$\mathcal{D}_q(\pi(x)\rho(x))=\rho(x)\mathcal{D}_q\pi(x)+\pi(qx)\mathcal{D}_q\rho(x)=\rho(qx)\mathcal{D}_q\pi(x)+\pi(x)\mathcal{D}_q\rho(x)\ . \tag{1.3}$$

All the above operators are linear and

$$\Delta x^n=nx^{n-1}+\cdots\ ,\quad \mathcal{D}_qx^n=[n]_qx^{n-1},\quad n>0,\qquad \Delta 1=\mathcal{D}_q1=0,$$

i.e., $\mathcal{D}_q\pi,\Delta\pi\in\mathbb{P}$. Here, and throughout the paper, $[n]_q$, $n\in\mathbb{N}$, denotes the basic $q-$number n defined by

$$[n]_q:=\frac{q^n-1}{q-1}=1+q+\cdots+q^{n-1},\quad n>0,\quad [0]_q:=0\ . \tag{1.4}$$

Definition 1.2. *Let $\mathbf{u}\in\mathbb{P}^*$ and $\pi\in\mathbb{P}$. We define the action of the $\Delta-$difference operator Δ on $\mathbb{P}^*$ by $\Delta:\mathbb{P}^*\to\mathbb{P}^*$, $\langle\Delta\mathbf{u},\pi\rangle=-\langle\mathbf{u},\Delta\pi\rangle$. We define the action of the $q-$derivatives $\mathcal{D}_q$ on $\mathbb{P}^*$ by $\mathcal{D}_q:\mathbb{P}^*\to\mathbb{P}^*$, $\langle\mathcal{D}_q\mathbf{u},\pi\rangle=-\langle\mathbf{u},\mathcal{D}_q\pi\rangle$.*

Definition 1.3. *Let $\mathbf{u}\in\mathbb{P}^*$ and $\pi\in\mathbb{P}$. We define a polynomial modification of a functional $\mathbf{u}$, the functional $\pi\mathbf{u}$, by $\langle\pi\mathbf{u},\rho\rangle=\langle\mathbf{u},\pi\rho\rangle$, $\forall\,\rho\in\mathbb{P}$.*

From the above definition and the identities (1.1) and (1.3) it follows that

$$\Delta(\pi(x)\mathbf{u}) = \pi(x-1)\Delta\mathbf{u} + \Delta\pi(x-1)\,\mathbf{u}, \tag{1.5}$$

$$\mathcal{D}_q(\pi(x)\mathbf{u}) = \pi(x/q)\mathcal{D}_q\mathbf{u} + \mathcal{D}_q\pi(x/q)\,\mathbf{u}, \tag{1.6}$$

for the discrete and the q-case, respectively.

Given a basis sequence of polynomials $(B_n)_{n\geq 0}$ we define the so-called dual basis of $(B_n)_{n\geq 0}$ as a sequence of linear functionals $(\mathbf{b}_n)_{n\geq 0}$ such that

$$\langle \mathbf{b}_n, B_m\rangle = \delta_{nm}, \quad n,m \geq 0 \,,$$

Furthermore, if $(P_n)_{n\geq 0}$ is a MOPS associated to the quasi-definite functional $\mathbf{u} \in \mathbb{P}^*$, then their corresponding dual basis $(\mathbf{p}_n) \subset \mathbb{P}^*$ is given by

$$\mathbf{p}_n = k_n^{-1} P_n \mathbf{u}, \quad k_n = \langle \mathbf{u}, P_n^2\rangle, \quad n \geq 0 \,. \tag{1.7}$$

Definition 1.4. *Let $\mathbf{u} \in \mathbb{P}^*$ be a quasi-definite functional and $(P_n)_{n\geq 0} =$ $\mathrm{mops}(\mathbf{u})$. We say that $\mathbf{u}$ (respectively $(P_n)_{n\geq 0}$) is a Δ-classical functional (respectively a Δ-classical MOPS), if and only if the sequence $(\Delta P_{n+1})_{n\geq 0}$ is also orthogonal. We say that $\mathbf{u}$ (respectively $(P_n)_{n\geq 0}$) is a $q-$classical functional (respectively a $q-$classical MOPS), if and only if the sequence $(\mathcal{D}_q P_{n+1})_{n\geq 0}$ is also orthogonal.*

In the following $(Q_n)_{n\geq 0}$ will denote either the sequence of monic Δ-differences or $q-$derivatives of $(P_n)_{n\geq 0}$, i.e., either $Q_n = \frac{1}{n+1}\Delta P_{n+1}$, or $Q_n = \frac{1}{[n+1]_q}\mathcal{D}_q P_{n+1}$, respectively, for all $n \geq 0$. It is known that

Proposition 1.5. [7, 14] *Let $(P_n)_{n\geq 0} = \mathrm{mops}(\mathbf{u})$ and $(Q_n)_{n\geq 0}$ be the sequence of monic Δ-differences or $q-$derivatives. If $(Q_n)_{n\geq 0} = \mathrm{mops}(\mathbf{v})$, then $\mathbf{v} = \phi\mathbf{u}$ where $\phi \in \mathbb{P}$, $\deg\phi \leq 2$.*

In the next Theorem we collect the characterizations already known of the Δ-classical MOPS and the $q-$classical MOPS, respectively:

Theorem 1.6. *Let $\mathbf{u} \in \mathbb{P}^*$ be a quasi-definite functional and $(P_n)_{n\geq 0} =$ $\mathrm{mops}(\mathbf{u})$. The following statements are equivalent* [7]*:*

(a) $\mathbf{u}$ *and $(P_n)_{n\geq 0}$ are, respectively, a $\Delta-$classical functional and a $\Delta-$classical MOPS.*

(b) *There exist two polynomials ϕ and ψ, $\deg\phi \leq 2$, $\deg\psi = 1$, such that*

$$\Delta(\phi\mathbf{u}) = \psi\mathbf{u}\,. \tag{1.8}$$

(c) *There exist two polynomials ϕ and ψ, $\deg\phi \leq 2$, $\deg\psi = 1$, and $\lambda_n \in \mathbb{C}$, $\lambda_n \neq 0$, $n \geq 1$ and $\lambda_0 = 0$, such that*

$$\phi\Delta\nabla P_n + \psi\nabla P_n + \lambda_n P_n = 0\,, \quad n = 0,1,2,\ldots\,. \tag{1.9}$$

(d) *$(P_n)_{n\geq 0}$ satisfies the distributional Rodrigues formula, i.e., there exist a polynomial $\phi \in \mathbb{P}$, $\deg\phi \leq 2$ and a sequence of complex numbers r_n, $r_n \neq 0$, $n \geq 1$ such that*

$$P_n\mathbf{u} = r_n\Delta^n(\phi_{(n)}\mathbf{u})\,,\ n \geq 1 \quad \textit{where} \quad \phi_{(n)}(x) = \prod_{k=0}^{n-1}\phi(x+k)\,, \tag{1.10}$$

Whereas for the q-classical polynomials the following statements are equivalent [14]*:*

(i) $\mathbf{u}$ *and* $(P_n)_{n\geq 0}$ *are, respectively, a* $q-$*classical functional and a* $q-$*classical MOPS.*

(ii) *There exist two polynomials* ϕ *and* ψ*,* $\deg\phi \leq 2$*,* $\deg\psi = 1$*, such that*

$$\mathcal{D}_q(\phi\mathbf{u}) = \psi\mathbf{u}\,. \tag{1.11}$$

(iii) *There exist two polynomials* ϕ *and* ψ*,* $\deg\phi \leq 2$*,* $\deg\psi = 1$*, and* $\lambda_n \in \mathbb{C}$*,* $\lambda_n \neq 0$*,* $n \geq 1$ *and* $\lambda_0 = 0$*, such that*

$$\phi\mathcal{D}_q\mathcal{D}_{1/q}P_n + \psi\mathcal{D}_{1/q}P_n + \lambda_n P_n = 0\,, \quad n = 0, 1, 2, \dots\,. \tag{1.12}$$

(iv) $(P_n)_{n\geq 0}$ *satisfies the distributional Rodrigues formula, i.e., there exist a polynomial* $\phi \in \mathbb{P}$*,* $\deg\phi \leq 2$ *and a sequence of complex numbers* r_n*,* $r_n \neq 0$*,* $n \geq 1$ *such that*

$$P_n\mathbf{u} = r_n\mathcal{D}_q^n(\phi_{(n)}\mathbf{u})\;,\; n \geq 1 \quad \textit{where} \quad \phi_{(n)}(x) = \prod_{i=0}^{n-1}\phi(q^i x)\;, \tag{1.13}$$

Moreover, if

$$\phi(x) = Ax^2 + Bx + C, \qquad \psi(x) = Mx + M_1, \qquad M \neq 0, \tag{1.14}$$

and $\mathbf{u}$ *is quasi-definite then we have the regularity condition* $nA + M \neq 0$ *for the* Δ*-case and* $[n]_q A + M \neq 0$ *for the q-case, respectively.*

Later, we will use the next technical result:

Proposition 1.7. *Let* $(P_n)_{n\geq 0} = \operatorname{mops}\mathbf{u}$ *and* $(Q_n)_{n\geq 0}$ *be the sequence of their monic* Δ*-differences or* $q-$*derivatives, respectively. If* $\mathbf{u}$ *is either a* $\Delta-$*classical functional or a* $q-$*classical functional then,*

$$\Delta(Q_n\,\phi\mathbf{u}) = (M + nA)P_{n+1}\;\mathbf{u}, \quad n \geq 0, \tag{1.15}$$

$$\mathcal{D}_q(Q_n\,\phi\mathbf{u}) = q^{-n}(M + [n]_q A)P_{n+1}\,\mathbf{u}, \quad n \geq 0, \tag{1.16}$$

for the discrete and q-cases, respectively, where M*,* A *are as in (1.14).*

Proof. Using (1.7), the Lemmas 1.7 and 1.8 in [7] for the discrete case, and Corollary 2.3 in [14] for the q-case, and proposition 1.5 we find

$$\Delta(Q_n\,\phi\mathbf{u}) = -(n+1)\frac{k'_n}{k_{n+1}}P_{n+1}\;\mathbf{u}, \quad \mathcal{D}_q(Q_n\,\phi\mathbf{u}) = -[n+1]_q\frac{k'_n}{k_{n+1}}P_{n+1}\,\mathbf{u},$$

respectively, where $k_{n+1} = \langle \mathbf{u}, P_{n+1}^2 \rangle$, $k'_n = \langle \phi\mathbf{u}, Q_n^2 \rangle$. Next we compute k'_n/k_{n+1}. For the discrete case we have

$$\begin{aligned} k'_n := \langle \phi\mathbf{u}, Q_n^2 \rangle &= \frac{1}{n+1} \langle \phi\mathbf{u}, (x+1)^n \Delta P_{n+1} \rangle \\ &= \frac{1}{n+1} \langle \phi\mathbf{u}, \Delta(x^n P_{n+1}) - \Delta(x^n) P_{n+1}) \rangle \\ &= -\frac{1}{n+1} \left\{ \langle \Delta(\phi\mathbf{u}), x^n P_{n+1}) \rangle + \langle \phi\mathbf{u}, \Delta(x^n) P_{n+1}) \rangle \right\} \\ &= -\frac{1}{n+1} \left\{ \langle \mathbf{u}, M x^{n+1} P_{n+1}) \rangle + \langle \mathbf{u}, nA x^{n+1} P_{n+1}) \rangle \right\} \\ &= -\frac{M+nA}{n+1} k_{n+1}, \end{aligned}$$

where we use the equation (1.1) in the second equality and (1.8), (1.14) in the forth one. In the same way, but using (1.3) and (1.11) we find for the q-case

$$k'_n := \langle \phi\mathbf{u}, Q_n^2 \rangle = \frac{q^{-n}}{[n+1]_q} \langle \phi\mathbf{u}, (qx)^n \mathcal{D}_q P_{n+1} \rangle = -q^{-n} \frac{M + [n]_q A}{[n+1]_q} k_{n+1},$$

□

2. Main result

In this Section, we prove the characterization Theorem in both situations.

2.1. **Classical discrete polynomials.**

Theorem 2.1. *Let $\mathbf{u} \in \mathbb{P}^*$ be a quasi-definite functional and $(P_n)_{n\geq 0} =$ mops$(\mathbf{u})$. Then, $(P_n)_{n\geq 0}$ is a Δ-classical MOPS if and only if for every $n \geq 1$,*

$$P_n \mathbf{u} = \Delta(\alpha_{n-1} \phi \mathbf{u}), \tag{2.1}$$

where α_{n-1} is a polynomial of degree $n-1$ and ϕ is a polynomial of degree less or equal to 2.

Proof. $\Leftarrow$ Taking $n = 1$ we get

$$P_1 \mathbf{u} = \Delta(\alpha_0 \phi \mathbf{u}) = \alpha_0 \Delta(\phi \mathbf{u}),$$

thus by Theorem 1.6, $\mathbf{u}$ is a classical functional with $\psi = P_1/\alpha_0$.

$\Rightarrow$ If $\mathbf{u}$ is classical then, by Theorem 1.6 (d),

$$P_n \mathbf{u} = r_n \Delta^n (\phi_{(n)} \mathbf{u}) \ , \ n \geq 1 \quad \text{where} \quad \phi_{(n)}(x) = \prod_{k=0}^{n-1} \phi(x+k) \ .$$

Therefore, it is enough to show that $r_n \Delta^n(\phi_{(n)}\mathbf{u}) = \Delta(\alpha_{n-1}\phi\mathbf{u})$ being α_{n-1} a $(n-1)$-degree polynomial. For doing that, let n be a fixed positive integer, we prove that

$$r_n \Delta^n (\phi_{(n)} \mathbf{u}) = \Delta^{n-k} (\alpha_k \phi_{(n-k)} \mathbf{u}), \quad k = 0, 1, \ldots, n-1, \tag{2.2}$$

where α_k is a polynomial of degree k. Obviously the formula is correct for $k=0$ by taking $\alpha_0(x)=r_n$. Let suppose that it is true for $k=0,1,2,\dots,p$, $p<n-1$

$$r_n\Delta^n(\phi_{(n)}\mathbf{u})=\Delta^{n-p}(\alpha_p\phi_{(n-p)}\mathbf{u})=\Delta^{n-(p+1)}\Delta(\alpha_p\phi_{(n-p)}\mathbf{u}).$$

Let show that

$$\Delta(\alpha_p\phi_{(n-p)}\mathbf{u})=\alpha_{p+1}\phi_{(n-p-1)}\mathbf{u},\quad \deg\alpha_{p+1}=p+1.$$

For doing that, notice that $\phi_{(n-p)}(x)=\phi(x)\phi_{(n-p-1)}(x+1)$, and therefore using (1.5)

$$\begin{aligned}\Delta(\alpha_p(x)\phi_{(n-p)}(x)\mathbf{u})&=\Delta(\alpha_p(x)\phi_{(n-p-1)}(x+1)\phi(x)\mathbf{u})\\&=\alpha_p(x-1)\phi_{(n-p-1)}(x)\Delta(\phi(x)\mathbf{u})+\Delta(\alpha_p(x-1)\phi_{(n-p-1)}(x))\phi(x)\mathbf{u}\\&=\{\alpha_p(x-1)\phi_{(n-p-1)}(x)\psi(x)+\Delta(\alpha_p(x-1)\phi_{(n-p-1)}(x))\phi(x)\}\mathbf{u}\end{aligned}$$

To complete the proof it suffices to show that

$$\begin{aligned}\Lambda(x):=&\ \alpha_p(x-1)\phi_{(n-p-1)}(x)\psi(x)+\Delta(\alpha_p(x-1)\phi_{(n-p-1)}(x))\phi(x)\\=&\ \{\alpha_p(x-1)\psi(x)+\phi(x)\Delta\alpha_p(x-1)\}\phi_{(n-p-1)}(x)\\&+\alpha_p(x)\phi(x)\Delta\phi_{(n-p-1)}(x)\\=&\ \alpha_{p+1}(x)\phi_{(n-p-1)}(x),\end{aligned}$$

being α_{p+1} a polynomial of degree $p+1$. Using that

$$\Delta\phi_{(n-p-1)}(x)=\frac{\phi(x+n-p-1)-\phi(x)}{\phi(x)}\phi_{(n-p-1)}(x)$$

we finally obtain

$$\Lambda(x)=\{\alpha_p(x-1)\psi(x)+\alpha_p(x)\phi(x+n-p-1)-\alpha_p(x-1)\phi(x)\}\phi_{(n-p-1)}(x).$$

To prove that the above quotient is a polynomial of degree $p+1$ we substitute (1.14), $\alpha_p(x)=a_px^p+\cdots$, and equate the coefficients of the x^{p+1}. This gives $a_p\{M+A(2n-p-2)\}\neq 0$, due to the regularity condition (see Theorem (1.6)) and the fact that $a_p\neq 0$ since the polynomial α_p has degree equal to p. This prove (2.2). Now putting $k=n-1$, the result follows. □

To conclude this section notice that comparing (2.1) and (1.15) we obtain that

$$\alpha_{n-1}(x)=\frac{Q_{n-1}(x)}{M+A(n-1)}=\frac{\Delta P_n(x)}{n(M+A(n-1))}.$$

2.1.1. *Examples.* As examples we will take the monic polynomials of Hahn $h_n^{(\alpha,\beta)}(x,N)$, Meixner $m_n^{(\nu,\mu)}(x)$, Kravchuk $k_n^{(p)}(x,N)$, and Charlier $c_n^{(a)}(x)$, (see [4]). So we have

- Hahn case: Since $\phi(x)=(x+\beta+1)(N-x-1)$, $\psi(x)=-(\alpha+\beta+2)x-(\beta+1)(N-1)$, thus

$$\alpha_{n-1}(x)=\frac{\Delta h_n^{(\alpha,\beta)}(x,N)}{n(n+\alpha+\beta+1)}=\frac{h_{n-1}^{(\alpha+1,\beta+1)}(x,N-1)}{n+\alpha+\beta+1}.$$

- Meixner case: Now $\phi(x) = \mu x + \mu\nu$, $\psi(x) = -(1-\mu)x + \mu\nu$, and then
$$\alpha_{n-1}(x) = \frac{\Delta m_n^{(\nu,\mu)}(x)}{n(\mu-1)} = \frac{m_{n-1}^{(\nu+1,\mu)}(x)}{\mu-1}.$$
- Kravchuk case: Since $\phi(x) = -\frac{p}{1-p}(x-N)$, $\psi(x) = \frac{-x+Np}{1-p}$, thus
$$\alpha_{n-1}(x) = \frac{\Delta k_n^{(p)}(x,N))}{n(\mu-1)} = (p-1)k_{n-1}^{(p)}(x, N-1).$$
- Charlier case: Since $\phi(x) = a$, $\psi(x) = -x + a$, therefore
$$\alpha_{n-1}(x) = -\frac{\Delta c_n^{(a)}(x)}{n} = c_{n-1}^{(a)}(x).$$

2.2. **q-Classical polynomials.**

Theorem 2.2. *Let $\mathbf{u} \in \mathbb{P}^*$ be a quasi-definite functional and $(P_n)_{n\geq 0} =$ mops$(\mathbf{u})$. Then, $(P_n)_{n\geq 0}$ is a q-classical if and only if for every $n \geq 1$,*
$$P_n\mathbf{u} = \mathcal{D}_q(\alpha_{n-1}\phi\mathbf{u}), \tag{2.3}$$
where α_{n-1} is a polynomial of degree $n-1$ and ϕ is a polynomial of degree less or equal to 2.

Proof. $\Leftarrow$ Taking $n=1$ we get
$$P_1\mathbf{u} = \mathcal{D}_q(\alpha_0\phi\mathbf{u}) = \alpha_0\mathcal{D}_q(\phi\mathbf{u}),$$
thus by Theorem 1.6 (iv), $\mathbf{u}$ is a classical functional with $\psi = P_1/\alpha_0$.

$\Rightarrow$ If $\mathbf{u}$ is classical then, by Theorem 1.6 then
$$P_n\mathbf{u} = r_n\mathcal{D}_q^n(\phi_{(n)}\mathbf{u})\ ,\ n\geq 1 \quad \text{where} \quad \phi_{(n)}(x) = \prod_{k=0}^{n-1}\phi(q^k x)\ .$$
Therefore, it is enough to show that $r_n\mathcal{D}_q^n(\phi_{(n)}\mathbf{u}) = \mathcal{D}_q(\alpha_{n-1}\phi\mathbf{u})$ being α_{n-1} a $(n-1)$-degree polynomial. For doing that we prove that
$$r_n\mathcal{D}_q^n(\phi_{(n)}\mathbf{u}) = \mathcal{D}_q^{n-k}(\alpha_k\phi_{(n-k)}\mathbf{u}), \quad k = 0,1,\ldots,n-1, \tag{2.4}$$
where α_k is a polynomial of degree k. Obviously the formula is correct for $k=0$ just taking $\alpha_0(x) = r_n$. Let suppose that it is true for $k = 0,1,2,\ldots,p$, $p < n-1$
$$\mathcal{D}_q^n(\phi_{(n)}\mathbf{u}) = \mathcal{D}_q^{n-p}(\alpha_p\phi_{(n-p)}\mathbf{u}) = \mathcal{D}_q^{n-(p+1)}\mathcal{D}_q(\alpha_p\phi_{(n-p)}\mathbf{u}).$$
Let show that
$$\mathcal{D}_q(\alpha_p(x)\phi_{(n-p)}(x)\mathbf{u}) = \alpha_{p+1}(x)\phi_{(n-p-1)}(x)\mathbf{u}, \quad \deg\alpha_{p+1} = p+1.$$

For doing that, notice that $\phi_{(n-p)}(x) = \phi(x)\phi_{(n-p-1)}(qx)$, and therefore using (1.6)

$$\begin{aligned}\mathcal{D}_q(\alpha_p(x)\phi_{(n-p)}(x)\mathbf{u}) &= \mathcal{D}_q(\alpha_p(x)\phi_{(n-p-1)}(qx)\phi(x)\mathbf{u})\\ &= \alpha_p(x/q)\phi_{(n-p-1)}(x)\mathcal{D}_q(\phi(x)\mathbf{u}) + \mathcal{D}_q(\alpha_p(x/q)\phi_{(n-p-1)}(x))\phi(x)\mathbf{u}\\ &= \{\alpha_p(x/q)\phi_{(n-p-1)}(x)\psi(x) + \mathcal{D}_q(\alpha_p(x/q)\phi_{(n-p-1)}(x))\phi(x)\}\mathbf{u}\end{aligned}$$

To complete the proof let show that

$$\begin{aligned}\Lambda(x) &:= \alpha_p(x/q)\phi_{(n-p-1)}(x)\psi(x) + \mathcal{D}(\alpha_p(x/q)\phi_{(n-p-1)}(x))\phi(x)\\ &= \{\alpha_p(x/q)\psi(x) + \phi(x)\mathcal{D}_q\alpha_p(x/q)\}\phi_{(n-p-1)}(x) + \alpha_p(x)\phi(x)\mathcal{D}_q\phi_{(n-p-1)}(x)\\ &= \alpha_{p+1}(x)\phi_{(n-p-1)}(x),\end{aligned}$$

where α_{p+1} is a polynomial of degree $p+1$. Using that

$$\mathcal{D}_q\phi_{(n-p-1)}(x) = \frac{\phi(q^{n-p-1}x) - \phi(x)}{(q-1)x}\frac{\phi_{(n-p-1)}(x)}{\phi(x)}$$

we finally obtain

$$\Lambda(x) = \left\{\alpha_p(x/q)\psi(x) + \frac{\alpha_p(x)\phi(q^{n-p-1}x) - \alpha_p(x/q)\phi(x)}{(q-1)x}\right\}\phi_{(n-p-1)}(x).$$

First of all notice that Λ is a polynomial. To prove that the expression on the brackets is a polynomial of degree $p+1$ we substitute (1.14), $\alpha_p(x) = a_p x^p + \cdots$, and equate the coefficients of the x^{p+1}. This leads to the value $a_p q^{-p}\{M+A[2n-p-2]_q\}$, that is different from zero because of the regularity condition (see Theorem (1.6)) and the fact that the polynomial α_p has degree equal to p, and then $a_p \neq 0$. This prove (2.4). Now putting $k = n-1$, the result follows. □

Observe that comparing (2.3) and (1.16) we obtain that

$$\alpha_{n-1}(x) = \frac{q^n Q_{n-1}(x)}{M + A[n-1]_q} = \frac{q^n \mathcal{D}_q P_n(x)}{[n]_q(M + A[n-1]_q)}.$$

2.2.1. *Examples.* In this case we have 12 families of classical q-polynomials (see [4]). We will take two representatives examples corresponding to the big q-Jacobi polynomials $P_n(x,a,b,c;q)$ and the little q-Jacobi polynomials $p_n(x;a,b|q)$. So we have

- Big q-Jacobi polynomials: Since $\phi(x) = aq(x-1)(bx-c)$, $\psi(x) = \frac{1-abq^2}{(1-q)q}x + \frac{a(bq-1)+c(aq-1)}{1-q}$, therefore

$$\begin{aligned}\alpha_{n-1}(x) &= \frac{q^n}{[n]_q}\frac{\mathcal{D}_q p_n(x,a,b|q)}{\frac{1-abq^2}{(1-q)q} + abq[n-1]_q}\\ &= \frac{(1-q)q^{n+1}}{\frac{1-abq^2}{(1-q)q} + abq[n-1]_q}P_{n-1}(qx,qa,qb,qc;q).\end{aligned}$$

- Little q-Jacobi polynomials: In this case $\phi(x) = ax(bqx-1)$, $\psi(x) = \frac{1}{(1-q)q}\left\{(1-abq^2)x + aq - 1\right\}$, and thus

$$\alpha_{n-1}(x) = \frac{(1-q)q^{n+1}}{[n]_q}\frac{\mathcal{D}_q p_n(x,a,b,c;q)}{\frac{1-abq^2}{(1-q)q} + abq[n-1]_q}$$
$$= \frac{q^n}{\frac{1-abq^2}{(1-q)q} + abq[n-1]_q} p_{n-1}(x, qa, qb|q).$$

The other ten cases can be obtained in an analogous way or by taking appropriate limits (see e.g. [4, 9]).

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References

[1] W. A. Al-Salam, Characterization theorems for orthogonal polynomials. In: *Orthogonal Polynomials: Theory and Practice.* P. Nevai (Ed.) NATO ASI Series C, **Vol. 294**. Kluwer Acad. Publ., Dordrecht, 1990, 1-24.

[2] R. Álvarez-Nodarse, *Polinomios hipergemétricos y q-polinomios.* Monografías del Seminario Matemático "García de Galdeano" Vol. **26**. Prensas Universitarias de Zaragoza, Zaragoza, Spain, 2003. In Spanish.

[3] R. Álvarez-Nodarse, On characterizations of classical polynomials. *J. Comput. Appl. Math.* (2005) (in press)

[4] R. Álvarez-Nodarse and J. C. Medem, q−Classical polynomials and the q−Askey and Nikiforov-Uvarov Tableaus. *J. Comput. Appl. Math.* **135** (2001) 197-223.

[5] N. M. Atakishiyev, M. Rahman, and S. K. Suslov, On classical orthogonal polynomials. *Constr. Approx.* **11** (1995), 181-226.

[6] T. Chihara, *An Introduction to Orthogonal Polynomials.* Gordon and Breach, New York, 1978.

[7] A. G. García, F. Marcellán, and L. Salto, A distributional study of discrete classical orthogonal polynomials. *J. Comput. Appl. Math.* **57** (1995), 147-162.

[8] W. Hahn, Über orthogonalpolynomen die q-differentialgleichungen genügen. *Math. Nachr.* **2** (1949), 4-34.

[9] R. Koekoek and R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analog. *Reports of the Faculty of Technical Mathematics and Informatics* **No. 98-17**. Delft University of Technology, Delft, 1998.

[10] P. Lesky, Über Polynomsysteme, die Sturm-Liouvilleschen differenzengleichungen genügen. *Math. Zeit.* **78** (1962), 439-445.

[11] F. Marcellán and R. Álvarez-Nodarse, On the Favard Theorem and their extensions. *Jornal of Computational and Applied Mathematics* **127**, (2001) 231-254.

[12] F. Marcellán, A. Branquinho, and J. Petronilho, Classical orthogonal polynomials: A functional approach. *Acta Appl. Math.* **34** (1994), 283-203.

[13] F. Marcellán and J. Petronilho, On the solution of some distributional differential equations: existence and characterizations of the classical moment functionals. *Integral Transform. Special Funct.* **2** (1994) 185-218.

[14] J. C. Medem, R. Álvarez-Nodarse and F. Marcellán, On the q−polynomials: A distributional study. *J. Comput. Appl. Math.* **135** (2001), 157-196.

[15] A.F. Nikiforov and V.B. Uvarov, *Special Functions of Mathematical Physics.* Birkhäuser, Basel, 1988.

[16] G. Szegö, *Orthogonal Polynomials.* Amer. Math. Soc. Coll. Pub. **23** AMS, Providence, Rhode Island, 1975.

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