KYUNGPOOK Math. J. 39(1999), 465-476

A Contact Version of B.-Y. Chen's Inequality and Its Applications to Slant Immersions

Alfonso Carriazo
Departamento de Geometría y Topología, Facultad de Matemáticas. Universidad de Sevilla, Apartado de Correos 1160, 41080-Sevilla, Spain
e-mail: acarri@cica.es

(1991 Mathematics Subject Classification : 53C15, 53C40)

We establish a version of B.-Y. Chen's inequality for a submanifold of a Sasakian-space-form, tangent to the structure vector field of the ambient space. We obtain some applications and we study this inequality for slant submanifolds. We also characterize 3–dimensional slant submanifolds satisfying the equality case.

1. Introduction

Given a Riemannian manifold M, for each point $p \in M$, put

$$(\inf K)(p) \;=\; \inf\{K(\pi) : \text{ plane sections } \pi \subset T_pM\},$$

where $K(\pi)$ denotes the sectional curvature of M associated with π. Let

$$\delta_M(p) \;=\; \tau(p) - \inf K(p), \tag{1.1}$$

being τ the scalar curvature of M. Then, δ_M is a well–defined Riemannian invariant, which was recently introduced by B.-Y. Chen [4, 5].

For submanifolds M in a real-space-form $\widetilde{R}^m(c)$ of constant sectional curvature c, Chen gave the following basic inequality involving the intrinsic invariant δ_M and the squared mean curvature of the immersion,

$$\delta_M \le \frac{n^2(n-2)}{2(n-1)}|H|^2 + \frac{1}{2}(n+1)(n-2)c, \tag{1.2}$$

where n denotes the dimension of M and H is the mean curvature vector. On the other hand, it was remarked in [8] that the exact proof of (1.2) given in [4] yields the same inequality for totally real submanifolds in a complex-space-form $\widetilde{M}^m(4c)$ with constant holomorphic sectional curvature $4c$.

Later, Chen generalized the above situation by establishing an inequality for an arbitrary submanifold of dimension greater than 2 in a complex-space-form [6]. By

(Received : May 14, 1999)
The author is partially supported by the PAI project (Junta de Andalucía, Spain 1998).

applying this inequality, he showed that (1.2) holds for arbitrary submanifolds in the complex hyperbolic space $\mathbb{C}H^m(4c)$ $(c < 0)$ as well. He also stated a formula for a submanifold in the complex projective space $\mathbb{C}P^m(4c)$.

In contact geometry, Defever, Mihai and Verstraelen obtained an inequality similar to (1.2) for C-totally real submanifolds of a Sasakian-space-form with constant ϕ-sectional curvature c [11]:

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)}|H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3}{4}. \tag{1.3}$$

Several authors have studied the equality cases of the above inequalities (see, for instances, [6, 7, 9, 10, 11, 12, 13]).

C-totally real submanifolds have the structure vector field ξ of the ambient space as a normal vector field (and so, they are anti-invariant submanifolds if that ambient space is, at least, a contact metric manifold).

The purpose of the present paper is to establish a general inequality, similar to that of [6], for submanifolds tangent to the structure vector field of a Sasakian-space-form. We are specially interested in applying the obtained results to slant immersions in contact geometry (see, for references, [2, 3, 15]).

Thus, in Section 2, we review basic formulas and definitions for almost contact metric manifolds and their submanifolds, which we shall use later. In Section 3, we establish the mentioned inequality and we adapt our procedures to ξ-tangent situation by introducing a new invariant $\delta_M^{\mathcal{D}}$, closely related to δ_M. Finally, we show some applications in Sections 4 and 5, by paying a special attention to slant immersions. For example, we characterize 3-dimensional slant submanifolds satisfying our equality case.

When this paper was finished, the author learned that in [14], Y.H. Kim and D.-S. Kim obtained a basic inequality for δ_M for submanifolds in a Sasakian-space-form. Moreover, they apply it to get a characterization of an odd-dimensional great sphere of an odd-dimensional sphere. On the other hand, they do not study slant immersions and so, they do not modify that inequality in order to consider non-invariant submanifolds satisfying the equality case. Hence, even though $\delta_M^{\mathcal{D}} \leq \delta_M$, neither our main pinching result given in Theorem 3.5, nor the applications shown in Section 5, can be obtained from Theorem 3.3 of [14].

2. Preliminaries

Let $(\widetilde{M}, g)$ be an odd–dimensional Riemannian manifold and denote by $T\widetilde{M}$ the Lie algebra of vector fields in $\widetilde{M}$.

Let ϕ be a $(1,1)$ tensor field, ξ a global unit vector field (*structure vector field*), and η a 1–form on $\widetilde{M}$. If we have $\phi^2 X = -X + \eta(X)\xi$, $g(X,\xi) = \eta(X)$ and $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$, for any $X, Y \in T\widetilde{M}$, then $\widetilde{M}$ is said to have an almost contact metric structure (ϕ, ξ, η, g) and it is called an *almost contact metric manifold.*

Let Φ denote the *fundamental 2–form* in $\widetilde{M}$, given by $\Phi(X,Y) = g(X,\phi Y)$ for all $X, Y \in T\widetilde{M}$. If $\Phi = \mathrm{d}\eta$, then $\widetilde{M}$ is said to be a *contact metric manifold*. Moreover, if ξ is a Killing vector field with respect to g, the contact metric structure is called a *K–contact structure*.

The structure of $\widetilde{M}$ is said to be *normal* if $[\phi,\phi]+2\mathrm{d}\eta\otimes\xi = 0$, where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ. A *Sasakian manifold* is a normal contact metric manifold. Every Sasakian manifold is a K–contact manifold.

Given a Sasakian manifold $\widetilde{M}$, a plane section π in $T_p\widetilde{M}$ is called a *ϕ–section* if it is spanned by X and ϕX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\widetilde{K}(\pi)$ of a ϕ–section π is called *ϕ–sectional curvature*. If a Sasakian manifold $\widetilde{M}$ has constant ϕ–sectional curvature c, $\widetilde{M}$ is called a *Sasakian-space-form* and it is denoted by $\widetilde{M}(c)$. For more details and background, we refer to the standard reference [1].

Now, let M be a submanifold immersed in $(\widetilde{M},\phi,\xi,\eta,g)$. We also denote by g the induced metric on M. Let TM be the Lie algebra of vector fields in M and $T^{\perp}M$ the set of all vector fields normal to M. We denote by σ the second fundamental form of M and by A_V the Weingarten endomorphism associated with any $V \in T^{\perp}M$. We put $\sigma^r_{ij} = g(\sigma(e_i,e_j),e_r)$, for any $e_i, e_j \in TM$ and $e_r \in T^{\perp}M$.

The mean curvature vector H is defined by $H = (1/\dim M)$ trace σ. M is said to be minimal if H vanishes identically.

From now on, we denote by $n+1$ (resp. m) the dimension of M (resp. $\widetilde{M}$). We consider $n \geq 2$. We also suppose that the structure vector field ξ is tangent to M. Hence, if we denote by $\mathcal{D}$ the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = \mathcal{D} \oplus <\xi>$.

For any $X \in TM$, we write $\phi X = TX + NX$, where TX (resp. NX) is the tangential (resp. normal) component of ϕX. If $\widetilde{M}$ is a K-contact manifold, it is well-known that

$$\sigma(X,\xi) = -NX, \tag{2.1}$$

for any $X \in TM$.

Given a local orthonormal frame $\{e_1,\ldots,e_n\}$ of $\mathcal{D}$, we can define the squared norms of T and N by

$$|T|^2 = \sum_{i,j=1}^{n} g^2(e_i,Te_j), \quad |N|^2 = \sum_{i=1}^{n} |Ne_i|^2, \tag{2.2}$$

respectively. It is easy to show that both $|T|^2$ and $|N|^2$ are independent of the choice of the above orthonormal frame.

The submanifold M is said to be *invariant* if N is identically zero, that is, $\phi X \in TM$, for any $X \in TM$. On the other hand, M is said to be an *anti–invariant* submanifold if T is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

For each nonzero vector X tangent to M at p, such that X is not proportional to ξ_p, we denote by $\theta(X)$ the angle between ϕX and T_pM. Then, M is said to be *slant* [15] if the angle $\theta(X)$ is a constant, which is independent of the choice of $p \in M$ and $X \in T_pM - < \xi_p >$. The angle θ of a slant immersion is called the *slant angle* of the immersion. Invariant and anti–invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$ respectively. A slant immersion which is not invariant nor anti–invariant is called a *proper* slant immersion.

In [3] we have proved that a θ-slant submanifold M of an almost contact metric manifold $\widetilde{M}$ satisfies

$$g(TX, TY) \;=\; \cos^2\theta(g(X,Y) - \eta(X)\eta(Y)), \tag{2.3}$$

$$g(NX, NY) \;=\; \sin^2\theta(g(X,Y) - \eta(X)\eta(Y)). \tag{2.4}$$

for any $X, Y \in TM$. On the other hand, Lemma 2.3.8 of [2] implies

$$\sum_{j=1}^{n} g^2(e_i, \phi e_j) \;=\; \cos^2\theta, \tag{2.5}$$

for any $i = 1, \ldots, n$, where $\{e_1, \ldots, e_n, \xi\}$ is a local orthonormal frame of TM.

It is well-known that the curvature tensor R of a submanifold M of a Sasakian-space-form $\widetilde{M}(c)$ satisfies

$$\begin{aligned} R(X,Y;Z,W) \;=\; & g(\sigma(X,W),\sigma(Y,Z)) - g(\sigma(X,Z),\sigma(Y,W)) + \\ & + \frac{c+3}{4}(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)) + \frac{c-1}{4}(\eta(X)\eta(Z)g(Y,W) - \\ & -\eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z) + \\ & + g(\phi X,W)g(\phi Y,Z) - g(\phi X,Z)g(\phi Y,W) + 2g(X,\phi Y)g(\phi Z,W)), \end{aligned} \tag{2.6}$$

for any $X, Y, Z, W \in TM$.

For an orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ of the tangent space T_pM, $p \in M$, the scalar curvature τ at p is defined by

$$\tau \;=\; \sum_{i<j} K(e_i \wedge e_j), \tag{2.7}$$

where $K(e_i \wedge e_j)$ denotes the sectional curvature of M associated with the plane section spanned by e_i, e_j. In particular, if we put $e_{n+1} = \xi_p$, then (2.7) implies:

$$2\tau \;=\; \sum_{i\neq j}^{n} K(e_i \wedge e_j) + 2\sum_{i=1}^{n} K(e_i \wedge \xi). \tag{2.8}$$

From (2.2), (2.6) and (2.8), we obtain the following relation between the scalar curvature and the mean curvature of M,

$$2\tau \;=\; (n+1)^2|H|^2 - |\sigma|^2 + n(n+1)\frac{c+3}{4} + 2n + \frac{3(c-1)}{4}|T|^2, \tag{2.9}$$

where $|\sigma|$ denotes the norm of the second fundamental form σ.

3. Chen's inequality in Sasakian-space-forms

Let M^{n+1} be a submanifold of $\widetilde{M}^m(c)$, tangent to the structure vector field ξ, and $\pi \subset \mathcal{D}_p$ a plane section at $p \in M$, orthogonal to ξ_p. Then,

$$\Phi^2(\pi) \;=\; g^2(e_1, \phi e_2) \tag{3.1}$$

is a real number in $[0,1]$ which is independent of the choice of the orthonormal basis $\{e_1, e_2\}$ of π. Denote by τ and $K(\pi)$ the scalar curvature of M and the sectional curvature of M associated with π, respectively.

We first recall an algebraic lemma from [4]:

Lemma 3.1. *Let $a_1, \ldots, a_k, c$ be $k+1$ ($k \geq 2$) real numbers such that:*

$$\left(\sum_{i=1}^{k} a_i\right)^2 \;=\; (k-1)\left(\sum_{i=1}^{k} a_i^2 + c\right).$$

Then $2a_1a_2 \geq c$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_k$.

Now, we can prove the following contact version of Theorem 3 of [6]:

Theorem 3.2. *Let $\varphi : M^{n+1} \to \widetilde{M}^m(c)$ be an isometric immersion from a Riemannian $(n+1)$–manifold into a Sasakian-space-form $\widetilde{M}^m(c)$, such that $\xi \in TM$. Then, for any point $p \in M$ and any plane section $\pi \subset \mathcal{D}_p$, we have:*

$$\tau - K(\pi) \leq \frac{(n+1)^2(n-1)}{2n}|H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3}{4} +$$

$$+ n + \frac{3}{2}|T|^2\frac{c-1}{4} - 3\Phi^2(\pi)\frac{c-1}{4}. \tag{3.2}$$

Equality in (3.2) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ of T_pM and an orthonormal basis $\{e_{n+2}, \ldots, e_m\}$ of $T_p^{\perp}M$ such that (a) *$e_{n+1} = \xi_p$,* (b) *π is spanned by e_1, e_2 and* (c) *the shape operators $A_r = A_{e_r}$, $r = n+2, \ldots, m$, take the following forms:*

$$A_{n+2} \;=\; \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}, \tag{3.3}$$

$$A_r \;=\; \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 \\ \sigma_{12}^r & -\sigma_{11}^r & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}, \quad r = n+3, \ldots, m. \tag{3.4}$$

Proof. Let M^{n+1} be a submanifold of $\widetilde{M}^m(c)$. Put:

$$\varepsilon \;=\; 2\tau - \frac{(n+1)^2(n-1)}{n}|H|^2 - (n+1)(n-2)\frac{c+3}{4} - 2n - \frac{3(c-1)}{4}|T|^2. \quad (3.5)$$

Then, (2.9) and (3.5) yield:

$$(n+1)^2|H|^2 \;=\; n|\sigma|^2 + n\left(\varepsilon - \frac{2(c+3)}{4}\right). \quad (3.6)$$

Let $\pi \subset \mathcal{D}_p$ be a plane section. We choose an orthonormal frame $\{e_1, \ldots, e_{n+1}\}$ of T_pM and an orthonormal basis $\{e_{n+2}, \ldots, e_m\}$ of $T_p^\perp M$ such that $e_{n+1} = \xi_p$, π is spanned by e_1, e_2 and e_{n+2} is in the direction of the mean curvature vector H. Hence, (3.6) gives

$$\left(\sum_{i=1}^{n+1}\sigma_{ii}^{n+2}\right)^2 = n\left\{\sum_{i=1}^{n+1}(\sigma_{ii}^{n+2})^2 + \sum_{i\neq j}(\sigma_{ij}^{n+2})^2 + \sum_{r=n+3}^{m}\sum_{i,j}(\sigma_{ij}^r)^2 + \varepsilon - \frac{2(c+3)}{4}\right\},$$

and so, by applying Lemma 3.1, we obtain:

$$2\sigma_{11}^{n+2}\sigma_{22}^{n+2} \geq \sum_{i\neq j}(\sigma_{ij}^{n+2})^2 + \sum_{r=n+3}^{m}\sum_{i,j}(\sigma_{ij}^r)^2 + \varepsilon - \frac{2(c+3)}{4}. \quad (3.7)$$

On the other hand, from (2.6) we find:

$$\begin{aligned} K(\pi) \;=\; & \sigma_{11}^{n+2}\sigma_{22}^{n+2} - (\sigma_{12}^{n+2})^2 + \sum_{r=n+3}^{m}(\sigma_{11}^r\sigma_{22}^r - (\sigma_{12}^r)^2) + \\ & + \frac{c+3}{4} + \frac{3(c-1)}{4}g^2(e_1, \phi e_2). \end{aligned} \quad (3.8)$$

Then, from (3.7) and (3.8) we get:

$$\begin{aligned} K(\pi) \geq & \sum_{r=n+2}^{m}\sum_{j>2}\{(\sigma_{1j}^r)^2 + (\sigma_{2j}^r)^2\} + \frac{1}{2}\sum_{i\neq j>2}(\sigma_{ij}^{n+2})^2 + \frac{1}{2}\sum_{r=n+3}^{m}\sum_{i,j>2}(\sigma_{ij}^r)^2 + \\ & + \frac{1}{2}\sum_{r=n+3}^{m}(\sigma_{11}^r + \sigma_{22}^r)^2 + \frac{\varepsilon}{2} + \frac{3(c-1)}{4}g^2(e_1, \phi e_2) \geq \frac{\varepsilon}{2} + \frac{3(c-1)}{4}g^2(e_1, \phi e_2). \end{aligned} \quad (3.9)$$

Finally, combining (3.1), (3.5) and (3.9), we obtain (3.2).

If the equality in (3.2) holds, then the inequalities in (3.7) and (3.9) become equalities. Thus, we have:

$$\begin{aligned} & \sigma_{1j}^{n+2} = \sigma_{2j}^{n+2} = \sigma_{ij}^{n+2} = 0, \quad i \neq j > 2; \\ & \sigma_{1j}^r = \sigma_{2j}^r = \sigma_{ij}^r = 0, \quad r = n+3, \ldots, m; \quad i, j = 3, \ldots, n+1; \\ & \sigma_{11}^{n+3} + \sigma_{22}^{n+3} = \cdots = \sigma_{11}^m + \sigma_{22}^m = 0. \end{aligned}$$

Furthermore, we may choose e_1, e_2 such that $\sigma_{12}^{n+2} = 0$. Moreover, by applying Lemma 3.1 and (2.1), we also have:

$$\sigma_{11}^{n+2} + \sigma_{22}^{n+2} = \sigma_{33}^{n+2} = \cdots = \sigma_{n+1\ n+1}^{n+2} = 0.$$

Therefore, with respect to the chosen orthonormal basis $\{e_1, \ldots, e_m\}$, the shape operators of M take the forms (3.3) and (3.4).

The converse follows from a direct calculation.

Now, for each point $p \in M$, we define:

$$(\inf_{\mathcal{D}} K)(p) \;=\; \inf\{K(\pi) : \text{ plane sections } \pi \subset \mathcal{D}_p\}.$$

Then, $\inf_{\mathcal{D}} K$ is a well–defined function on M. Let $\delta_M^{\mathcal{D}}$ denote the difference between the scalar curvature and $\inf_{\mathcal{D}} K$, i.e.:

$$\delta_M^{\mathcal{D}}(p) \;=\; \tau(p) - \inf_{\mathcal{D}} K(p). \tag{3.10}$$

From (1.1) and (3.10), it is clear that:

$$\delta_M^{\mathcal{D}} \leq \delta_M. \tag{3.11}$$

If $c = 1$, then we obtain directly from (3.2) and (3.10) the following result:

Corollary 3.3. *Let $\varphi : M^{n+1} \to \widetilde{M}^m(1)$ be an isometric immersion from a Riemannian $(n+1)$-manifold into a Sasakian-space-form with constant ϕ-sectional curvature 1, such that $\xi \in TM$. Then, we have:*

$$\delta_M^{\mathcal{D}} \leq \frac{(n+1)^2(n-1)}{2n}|H|^2 + \frac{1}{2}(n+2)(n-1). \tag{3.12}$$

Note that, in fact, (3.12) also follows from (3.11) and (1.2) with $c = 1$, since a Sasakian-space-form with constant ϕ-sectional curvature 1 is a real-space-form of constant sectional curvature 1. This seems to point out that (3.2) may be a natural contact version of (1.2). Nevertheless, by using (2.1), (3.3) and (3.4), we can state the following result:

Corollary 3.4. *If equality in (3.2) holds at any $p \in M$, then φ is an invariant immersion.*

Now, we are going to modify (3.2) in order to consider non-invariant submanifolds (for example, proper slant submanifolds) satisfying a similar equality. We can prove the following theorem:

Theorem 3.5. *Let $\varphi : M^{n+1} \to \widetilde{M}^m(c)$ be an isometric immersion from a Riemannian $(n+1)$–manifold into a Sasakian-space-form $\widetilde{M}^m(c)$, such that $\xi \in TM$. Then, for any point $p \in M$ and any plane section $\pi \subset \mathcal{D}_p$, we have:*

$$\tau - K(\pi) \leq \frac{(n+1)^2(n-1)}{2n}|H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3}{4} +$$

$$+n+\frac{3}{2}|T|^2\frac{c-1}{4}-3\Phi^2(\pi)\frac{c-1}{4}-|N|^2. \tag{3.13}$$

Equality in (3.13) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1,\dots,e_{n+1}\}$ of T_pM and an orthonormal basis $\{e_{n+2},\dots,e_m\}$ of $T_p^{\perp}M$ such that (a) *$e_{n+1}=\xi_p$,* (b) *π is spanned by e_1, e_2 and* (c) *the shape operators $A_r = A_{e_r}$, $r=n+2,\dots,m$, take the following forms:*

$$A_{n+2} = \begin{pmatrix} a & 0 & 0 & \mu_1^{n+2} \\ 0 & -a & 0 & \vdots \\ 0 & 0 & 0_{n-2} & \mu_n^{n+2} \\ \mu_1^{n+2} & \cdots & \mu_n^{n+2} & 0 \end{pmatrix}, \tag{3.14}$$

$$A_r = \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 & \mu_1^r \\ \sigma_{12}^r & -\sigma_{11}^r & 0 & \vdots \\ 0 & 0 & 0_{n-2} & \mu_n^r \\ \mu_1^r & \cdots & \mu_n^r & 0 \end{pmatrix}, \quad r=n+3,\dots,m, \tag{3.15}$$

where $\mu_i^r = g(\phi e_i, e_r)$, for any $i=1,\dots,n$; $r=n+2,\dots,m$.

Proof. We follow the first steps of the proof of Theorem 3.2 and we state equations (3.5)-(3.9). Then, inequality (3.9) can now be written as:

$$K(\pi) \geq \sum_{r=n+2}^{m}\sum_{j=3}^{n}\{(\sigma_{1j}^r)^2+(\sigma_{2j}^r)^2\}+\frac{1}{2}\sum_{i\neq j>2}^{n}(\sigma_{ij}^{n+2})^2+\frac{1}{2}\sum_{r=n+3}^{m}\sum_{i,j=3}^{n}(\sigma_{ij}^r)^2+$$

$$+\frac{1}{2}\sum_{r=n+3}^{m}(\sigma_{11}^r+\sigma_{22}^r)^2+\frac{\varepsilon}{2}+\frac{3(c-1)}{4}g^2(e_1,\phi e_2)+\sum_{r=n+2}^{m}\sum_{i=1}^{n}(\sigma_{i\ n+1}^r)^2 \geq$$

$$\geq \frac{\varepsilon}{2}+\frac{3(c-1)}{4}g^2(e_1,\phi e_2)+\sum_{r=n+2}^{m}\sum_{i=1}^{n}(\sigma_{i\ n+1}^r)^2. \tag{3.16}$$

But, from (2.1) and (2.2) we find:

$$\sum_{r=n+2}^{m}\sum_{i=1}^{n}(\sigma_{i\ n+1}^r)^2 = |N|^2. \tag{3.17}$$

Hence, combining (3.1), (3.5), (3.16) and (3.17), we obtain (3.13).

If the equality in (3.13) holds, then the inequalities in (3.7) and (3.16) become equalities. By using this fact, (2.1) and Lemma 3.1, we have:

$$\begin{aligned}
&\sigma_{1j}^{m+2}=\sigma_{2j}^{n+2}=\sigma_{ij}^{n+2}=0, \quad 2<i\neq j<n;\\
&\sigma_{1j}^{r}=\sigma_{2j}^{r}=\sigma_{ij}^{r}=0, \quad r=n+3,\dots,m; \quad i,j=3,\dots,n;\\
&\sigma_{11}^{n+3}+\sigma_{22}^{n+3}=\cdots=\sigma_{11}^{m}+\sigma_{22}^{m}=0;\\
&\sigma_{11}^{n+2}+\sigma_{22}^{n+2}=\sigma_{33}^{n+2}=\cdots=\sigma_{n+1\ n+1}^{n+2}=0.
\end{aligned}$$

Hence, if we also choose e_1, e_2 such that $\sigma_{12}^{n+2} = 0$, then we obtain (3.14) and (3.15). As in the proof of Theorem 3.2, the converse can be verified by straight-forward computation.

Moreover, it is obvious that (3.2) follows from (3.13), since $|N|^2 \geq 0$.

On the other hand, it is also clear that, if φ is an anti-invariant immersion, then $|T|^2 = 0$, $|N|^2 = n$ and $\Phi^2(\pi) = 0$, for any plane section π orthogonal to ξ. Hence, from (3.13) we obtain:

Corollary 3.6. *Let M^{n+1} be an anti-invariant submanifold of a Sasakian-space-form $\widetilde{M}^m(c)$, such that $\xi \in TM$. Then, we have:*

$$\delta_M^{\mathcal{D}} \leq \frac{(n+1)^2(n-1)}{2n}|H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3}{4}. \tag{3.18}$$

Note that inequality (3.18) is the ξ-tangent version of (1.3), with the logical differences about the dimensions.

4. Some applications

By using Theorem 3.5, we can find some general pinching results for $\delta_M^{\mathcal{D}}$ if either $c > 1$ or $c < 1$.

Theorem 4.1. *Let $\varphi : M^{n+1} \to \widetilde{M}^m(c)$ be an isometric immersion from a Riemannian $(n+1)$–manifold $(n > 2)$ into a Sasakian-space-form $\widetilde{M}^m(c)$, with $c > 1$, such that $\xi \in TM$. Then:*

$$\delta_M^{\mathcal{D}} \leq \frac{(n+1)^2(n-1)}{2n}|H|^2 + \frac{1}{2}(n^2+2n-2)\frac{c+3}{4} - \frac{n}{2}. \tag{4.1}$$

Equality in (4.1) holds identically if and only if n is even and M^{n+1} is immersed as an invariant, totally geodesic submanifold of $\widetilde{M}^m(c)$.

Theorem 4.2. *Let $\varphi : M^{n+1} \to \widetilde{M}^m(c)$ be an isometric immersion from a Riemannian $(n+1)$–manifold $(n > 2)$ into a Sasakian-space-form $\widetilde{M}^m(c)$, with $c < 1$, such that $\xi \in TM$. Then:*

$$\delta_M^{\mathcal{D}} \leq \frac{(n+1)^2(n-1)}{2n}|H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3}{4} + n - |N|^2. \tag{4.2}$$

Equality in (4.2) holds at a point p of M if and only if there exist an orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ of T_pM and an orthonormal basis $\{e_{n+2}, \ldots, e_m\}$ of $T_p^{\perp}M$ such that (a) *$e_{n+1} = \xi_p$,* (b) *the subspace spanned by $e_3, \ldots, e_{n+1}$ is anti-invariant,* (c) *$K(e_1 \wedge e_2) = \inf_{\mathcal{D}} K$ at p, and* (d) *the shape operators $A_r = A_{e_r}$, $r = n+2, \ldots, m$, take the forms (3.14) and (3.15).*

Theorems 4.1 and 4.2 can be proved by following the same steps as in the proofs of Theorems 3 and 4 of [6], respectively.

5. Applications to slant immersions

Now, we are going to study inequality (3.13) when M is a slant submanifold. We first note down that, if M^{n+1} is a θ–slant submanifold of an almost contact metric manifold, then, (2.2), (2.4) and (2.5) imply:

$$|T|^2 \;=\; n\cos^2\theta, \quad |N|^2 \;=\; n\sin^2\theta. \tag{5.1}$$

Thus, from (3.13) and (5.1) we have:

Theorem 5.1. *Let $\varphi : M^{n+1} \to \widetilde{M}^m(c)$ be a θ-slant immersion of a Riemannian $(n+1)$–manifold into a Sasakian-space-form $\widetilde{M}^m(c)$. Then, for any point $p \in M$ and any plane section $\pi \subset \mathcal{D}_p$, we have:*

$$\tau - K(\pi) \leq \frac{(n+1)^2(n-1)}{2n}|H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3}{4} +$$

$$+ n\cos^2\theta + \frac{3(c-1)}{4}\left(\frac{n}{2}\cos^2\theta - \Phi^2(\pi)\right). \tag{5.2}$$

In particular, we can state the following result for 3–dimensional slant submanifolds:

Corollary 5.2. *In the above conditions, if $n = 2$ then,*

$$\delta_M^{\mathcal{D}} \leq \frac{9}{4}|H|^2 + 2\cos^2\theta, \tag{5.3}$$

with equality holding if and only if M is minimal.

Proof. If $n = 2$, then it is clear that

$$\delta_M^{\mathcal{D}} \;=\; \tau - K(\mathcal{D}) \tag{5.4}$$

and $\Phi^2(\mathcal{D}) = \cos^2\theta$. Thus, (5.3) follows directly from (5.2).

On the other hand, it is easy to see that

$$\tau - K(\mathcal{D}) \;=\; 2\cos^2\theta, \tag{5.5}$$

since M is a 3-dimensional slant submanifold of a Sasakian manifold. Hence, (5.4) and (5.5) imply the condition for the equality case in (5.3).

Note that, if we had chosen inequality (3.2) as our starting point, then, by following the same steps as in Theorem 5.1 and Corollary 5.2, we would have obtained, for 3-dimensional slant submanifolds, the inequality

$$\delta_M^{\mathcal{D}} \leq \frac{9}{4}|H|^2 + 2,$$

with equality holding if and only if the submanifold is invariant.

Thus, the converse of Corollary 3.4 holds for 3-dimensional slant submanifolds.

Finally, we can restrict our study to some special plane sections, orthogonal to ξ. Let M^{n+1} be a submanifold of a Sasakian-space-form $\widetilde{M}^m(c)$, such that $\xi \in TM$. Given a point $p \in M$, we say that a plane section $\pi \subset T_pM$ is a T-*section* if there exists a tangent vector $X \in \mathcal{D}_p$ such that π is spanned by X and TX.

For each point $p \in M$, we can define $(\inf_T K)(p) = \inf\{K(\pi) : T\text{–sections } \pi\}$ and $\delta_M^T(p) = \tau(p) - \inf_T K(p)$.

Since every T–section is orthogonal to ξ, it is clear that $\delta_M^T \leq \delta_M^{\mathcal{D}}$. In the case of slant submanifolds we have the following inequality for δ_M^T:

Theorem 5.3. *Let $\varphi : M^{n+1} \to \widetilde{M}^m(c)$ be a non-anti-invariant θ-slant immersion of a Riemannian $(n+1)$-manifold into a Sasakian-space-form $\widetilde{M}^m(c)$. Then:*

$$\delta_M^T \leq \frac{(n+1)^2(n-1)}{2n}|H|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3}{4} + n\cos^2\theta + \frac{1}{2}(n-2)\frac{3(c-1)}{4}\cos^2\theta.$$

Proof. Given a T-section π, we can choose two tangent vectors e_1, e_2 such that π is spanned by e_1 and e_2, being $e_2 = \sec\theta T e_1$. Then, (2.3) implies $\Phi^2(\pi) = \cos^2\theta$. The proof ends by applying (5.2).

Note that, if $n = 2$, then $\delta_M^T = \delta_M^{\mathcal{D}}$ and so, Corollary 5.2 also follows from Theorem 5.3.

References

[1] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics **509**, Springer-Verlag, New York, 1976.

[2] A. Carriazo, *Subvariedades Slant en Variedades de Contacto*, Tesis Doctoral, Universidad de Sevilla, 1998.

[3] J.L. Cabrerizo, A. Carriazo, L.M. Fernández, and M. Fernández, *Slant submanifolds in Sasakian manifolds*, to appear in Glasgow Mathematical Journal **42**(2000).

[4] B. Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. **60**(1993), 568-578.

[5] B. Y. Chen, *A Riemannian invariant and its applications to submanifold theory*, Results in Mathematics **27**(1995), 17-26.

[6] B. Y. Chen, *A general inequality for submanifolds in complex-space-forms and its applications*, Arch. Math. **67**(1996), 519-528.

[7] B. Y. Chen, F. Dillen, L. Verstraelen, and L. Vrancken. *Totally real submanifolds of $\mathbb{C}P^n$ satisfying a basic equality*, Arch. Math. **63**(1994), 553-564.

[8] B. Y. Chen, F. Dillen, L. Verstraelen, and L. Vrancken, *An exotic totally real minimal immersion of* S^3 *in* $\mathbb{C}P^3$ *and its characterization*, Proc. Roy. Soc. Edinburgh Sect. A **126**(1996), 153-165.

[9] B. Y. Chen and L. Vrancken, *CR-submanifolds of complex hyperbolic spaces satisfying a basic equality*, to appear in Israel Journal of Mathematics.

[10] M. Dajczer and A. Florit, *On Chen's basic equality*, Illinois J. Math. **42**(1)(1998), 97-106.

[11] F. Defever, I. Mihai, and L. Verstraelen, *B.-Y. Chen's inequality for* C*-totally real submanifolds of Sasakian-space-forms*, Bollettino U.M.I. (7)**11-B**(1997), 365-374.

[12] F. Dillen, M. Petrovic, and L. Verstraelen, *Einstein, conformally flat and semi-symmetric submanifolds satisfying Chen's equality*, Israel J. Math. **100**(1997), 163-169.

[13] F. Dillen and L. Vrancken, *Totally real submanifolds in* $S^6(1)$ *satisfying Chen's equality*, Trans. Amer. Math. Soc. **348**(4) (1996), 1633-1646.

[14] Y.H. Kim and D.-S. Kim, *A basic inequality for submanifolds in Sasakian space forms*, to appear in Houston J. Math.

[15] A. Lotta, *Slant submanifolds in contact geometry*, Bull. Math. Soc. Roumanie **39**(1996), 183-198.