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THE FREUDENTHAL COMPACTIFICATION OF TREE-LIKE GENERALIZED CONTINUA

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Abstract. Tree-likeness of generalized continua is defined by means of inverse limits of locally finite trees with proper bonding maps. The main theorem of this paper shows that the Freudenthal compactification preserves and reflects tree-likeness. Some consequences of interest are given.

1. Introduction

Classical continuum theory is a powerful branch of topology concerning compact spaces. However, the class of non-compact spaces is far from being irrelevant and it seems natural to explore a generalized continuum theory for locally compact spaces.

The proper category provides a very convenient framework for this task. Recall that a continuous map $f : X \to Y$ is said to be *proper* if for any compact subset $K \subset Y$, $f^{-1}(K)$ is compact in X. In particular, classes of spaces and maps of interest in continuum theory are extended to the proper category.

This paper is focused on the well-known class of tree-like spaces, usually described as inverse limits of sequences of compact trees. Compactness and connectedness of tree-like spaces readily follow from this description. Unfortunately, connectedness does not need to be preserved by inverse limits of non-compact spaces, and this requires connectedness in the definition of a tree-like space in the proper category; that is, a generalized

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continuum is said to be a *tree-like space* if it can be expressed as the limit of an inverse sequence of locally compact trees with proper bonding maps; see [10].

By use of compactifications, non-compact spaces are transformed into compact spaces, and an immediate question is to ask whether notions in the proper category remain under compactifications. In this paper we will show that, in fact, the tree-likeness of a generalized continuum is equivalent to the tree-likeness of its Freudenthal compactification (Theorem 4.1) or, equivalently, of some compactification of X with 0-dimensional remainder (Theorem 5.1). Some consequences are derived (Theorem 5.6 and Theorem 5.4).

2. The Freudenthal Compactification

Throughout this paper, a *continuum* (*generalized continuum*, respectively) is a connected compact (locally compact, respectively) metric space. By a *graph* we mean a locally finite 1-dimensional simplicial complex and *trees* are contractible graphs.

Concerning inverse limits, we will use the notation $X = \varprojlim_p \{X_n; g_n\}$ to represent inverse limits of sequences with proper bonding maps $g_n : X_{n+1} \to X_n$; see [10] for more details. This way a generalized continuum X is tree-like if $X = \varprojlim_p \{T_n, g_n\}$ where each T_n is a tree.

It follows from [5, Exercise 4.4. F.(c)] that any generalized continuum is separable and hence second countable and σ-compact [5, Corollary 4.1.16 and Exercise 3.8.C.(b)]. It is readily checked that local compactness, together with σ-compactness, yields the existence of *exhausting sequences* in a generalized continuum X; that is, increasing sequences of compact subsets $X_n \subset X$ with $X = \bigcup_{n=1}^{\infty} X_n$ and $X_n \subset \mathrm{int} X_{n+1}$.

Given an exhausting sequence $\{X_n\}_{n\geq 1}$ of the generalized continuum X, a *Freudenthal end of* X, $\varepsilon = (Q_n)_{n\geq 1}$, is a decreasing nested sequence of quasicomponents $Q_n \subset X - \mathrm{int} X_n$. Recall that the *quasicomponent* of a point x is defined to be the intersection of all open and closed sets containing x.

Let $\mathcal{F}(X)$ denote the set of all Freudenthal ends of X. The set $\widehat{X} = X \cup \mathcal{F}(X)$ admits a compact topology whose basis consists of all open sets of X together with the sets

$$\widehat{\Omega} = \Omega \cup \{(Q_n)_{n\geq 1};\ Q_n \subset \Omega \text{ for } n \text{ large enough}\}$$

where $\Omega \subset X$ is any open set with compact frontier. The space $\widehat{X}$ is called the *Freudenthal compactification of* X. Moreover, $\widehat{X}$ is metrizable and the subspace of Freudenthal ends $\mathcal{F}(X) \subset \widehat{X}$ turns out to be homeomorphic to a closed subset of the Cantor set; see [1], [11]. In particular, $\widehat{X}$ is the union of the σ-compact space X and the 0-dimensional space $\mathcal{F}(X)$ and then $\dim \widehat{X} = \dim X$; see [6, Corollary 1.5.4].

Recall that the Freudenthal compactification of X is maximal among the compactifications of X with 0-dimensional remainder; that is, if $\widetilde{X}$ is a metric compact space containing X as a dense open set and the difference $\widetilde{X} - X$ is 0-dimensional, then there is a continuous map $f : \widehat{X} \to \widetilde{X}$ which extends the identity on X; see [16, Example 3.9].

Any proper map $f : X \to Y$ between generalized continua extends to a continuous map $\widehat{f} : \widehat{X} \to \widehat{Y}$ with a continuous restriction $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ which maps $\varepsilon = (Q_n)_{n\geq 1}$ to the nested sequence $f_*(\varepsilon) = (Q'_n)_{n\geq 1}$ such that there is a subsequence $n_1 < n_2 \dots$ with $f(Q_{n_k}) \subset Q'_k$ for all $k \geq 1$. The following lemma is easily proved.

Lemma 2.1 ([10, Corollary 4.2])**.** *If f is onto, then f_*, and hence $\widehat{f}$, are onto.*

The following well-known lemma pinpoints a crucial feature of the Freudenthal compactification of any generalized continuum X.

Lemma 2.2. *Any generalized continuum X admits a metric and an exhausting sequence $\{X_k\}_{k\geq 1}$ such that $\widehat{X} - \mathrm{int}X_k = \bigsqcup_{j\in J_k} A_j^k$ $(k \geq 1)$ is a finite disjoint union of closed and open sets in $\widehat{X} - \mathrm{int}X_k$ with* diameter $A_j^k < \frac{1}{k}$ *for all $j \in J_k$. In particular,* $\mathrm{Fr}A_j^k \subset \mathrm{Fr}X_k$.

Proof. Recall that $\widehat{X}$ is metrizable so that we can find a metric d for the topology of $\widehat{X}$. For each $k \geq 1$, we write the 0-dimensional compact set of Freudenthal ends $\mathcal{F}(X) = \bigsqcup_{j\in J_k} F_j^k$ as a finite union of pairwise disjoint closed sets with diameter $F_j^k < \frac{1}{4k}$ for all $j \in J_k$. Moreover, for any $0 < \lambda_k < \frac{1}{4k}$ smaller than $\min\{\frac{1}{2}d(F_j^k, F_{j'}^k); j, j' \in J_k \text{ and } j \neq j'\}$, the closed sets

$$A_j^k = \{x \in \widehat{X}; d(x, F_j^k) \leq \lambda_k\}$$

are pairwise disjoint with diameter $A_j^k < \frac{1}{k}$ and $F_j^k \subset \mathrm{int}A_j^k$ for each $j \in J_k$. In particular, $\mathrm{Fr}A_j^k = \{x \in \widehat{X}; d(x, F_j^k) = \lambda_k\}$ misses $\mathcal{F}(X)$, and so the complement $X_k = \widehat{X} - \bigsqcup_{j\in J_k} \mathrm{int}A_j^k$ is a compact subset of X. Moreover, one readily checks that $\mathrm{Fr}X_k = \bigsqcup_{j\in J_k} \mathrm{Fr}A_j^k$. Thus, $\widehat{X} - \mathrm{int}X_k = \bigsqcup_{j\in J_k} A_j^k$ is a disjoint union of open and closed sets in $\widehat{X} - \mathrm{int}X_k$. Note that $X_k \subset \mathrm{int}X_{k+1}$. □

Remark 2.3. (1) Notice that in Lemma 2.2 for each $j \in J_k$ there exists a unique $j' \in J_{k-1}$ with $A_j^k \subset A_{j'}^{k-1}$ and that the Freudenthal ends of X are in $1-1$ correspondence with the sequences $(A_{j(k)}^k)_{k\geq 1}$ with $A_{j(k)}^k \subset A_{j(k-1)}^{k-1}$ for all $k \geq 2$.

(2) The proof of Lemma 2.2 also shows that the complement $\widehat{X}-X_k = \bigsqcup_{j\in J_k} B^k_j$ $(k\geq 1)$ is the finite disjoint union of the sets $B^k_j = \mathrm{int}A^k_j$ which are closed and open in $\widehat{X}-X_k$ with diameter$B^k_j < \frac{1}{k}$ for all $j\in J_k$.

We will also use the following simple fact; see [10, Proposition 4.1].

Lemma 2.4. *Let $X = \varprojlim_p\{X_n; g_n\}$ be a generalized continuum which is the inverse limit of generalized continua X_n with proper onto bonding maps g_n. Then there is a continuous surjection*

$$\varphi : \widehat{X} \longrightarrow L = \varprojlim\{\widehat{X}_n; \widehat{g}_n\}$$

such that $\varphi^{-1}(L_0) = \mathcal{F}(X)$ where $L_0 = \varprojlim\{\mathcal{F}(X_n), g_{n*}\} \subset L$ and $\varphi|X$ is the identity on $X = L - L_0$. In particular, φ induces a homeomorphism $\overline{\varphi} : \widehat{X}/\mathcal{F}(X) \cong L/L_0$.

Proof. If $p_i : X \to X_i$ are the canonical projections of the inverse limit, then the map $\varphi : \widehat{X} \to L$ is defined by the induced maps $\widehat{p}_i : \widehat{X} \to \widehat{X}_i$; that is, $\varphi(x) = (\widehat{p}_i(x))_{i\geq 1}$. Notice that each $\widehat{p}_i$ is onto by Lemma 2.1. Furthermore, the equalities $\widehat{g}_i^{-1}(\mathcal{F}(X_i)) = \mathcal{F}(X_{i+1})$ and $\widehat{p_i}^{-1}(\mathcal{F}(X_i)) = \mathcal{F}(X)$ for all $i \geq 1$ yield $\varphi^{-1}(L_0) = \mathcal{F}(X)$ and $X = L - L_0$ with $\varphi|X$ the identity on X. From this it readily follows that the induced map $\overline{\varphi} : \widehat{X}/\mathcal{F}(X) \to L/L_0$ is a continuous bijection and hence a homeomorphism. □

The following example shows that the map φ in the previous lemma does not have to be bijective.

Example 2.5. Consider the inverse sequence of one-ended graphs $\{G_n, f_n\}_{n\geq 1}$ where G_n is the graph depicted in Figure 1.

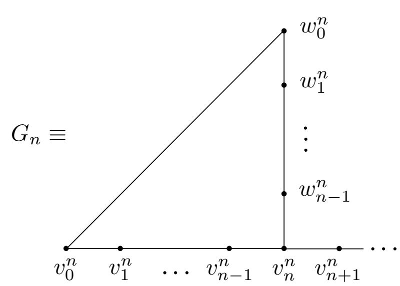


FIGURE 1

The map $f_n : G_{n+1} \to G_n$ is the linear extension on each edge of G_{n+1} of the map between vertex sets given by $f_n(v^{n+1}_i) = v^n_i$ for $i \geq 0$,

$f(w_j^{n+1}) = w_j^n$ for $j \leq n-1$ and $f(w_n^{n+1}) = v_n^n$. It is clear that each f_n is a proper map and the inverse limit $X = \varprojlim_p \{G_n, f_n\}$ is homeomorphic to the euclidean line; thus, $\widehat{X}$ is homeomorphic to an interval, while $L = \varprojlim \{\widehat{G}_n, \widehat{f}_n\}$ is homeomorphic to the circle.

3. The Case-Chamberlin Characterization of Tree-Likeness

In the proof of Theorem 4.1 the following purely homotopical characterization of tree-like continua due to J. H. Case and R. E. Chamberlin [2] will be crucial. Recall that a continuous map $f : X \to Y$ is termed *inessential* if it is homotopic to a constant map.

Theorem 3.1 ([2, Theorem 1])**.** *A* 1*-dimensional continuum X is tree-like if and only if, for every graph G, every continuous map $f : X \to G$ is inessential.*

As consequences of Theorem 3.1 we have the following two lemmas which will be used in the proof of Theorem 4.1. The first lemma is similar to [7, Lemma 2]. We give here a proof for the sake of completeness.

Lemma 3.2. *Let $Z = \varprojlim \{Z_n, f_n\}$ be a* 1*-dimensional continuum which is the inverse limit of contractible continua Z_n. Then Z is tree-like.*

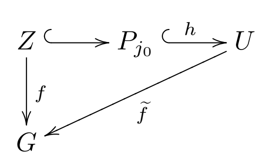
Proof. We have $Z = \bigcap_{j=1}^{\infty} P_j$ where

$$P_j = \{(x_n)_{n \geq 1};\ f_n(x_{n+1}) = x_n \text{ for } n \leq j\} \subset \Pi_{j=1}^{\infty} Z_j.$$

Moreover, each P_j is contractible since P_j is homeomorphic to $\Pi_{n=j+1}^{\infty} Z_n$ by the map $\alpha_j : \Pi_{n=j+1}^{\infty} Z_n \to P_j$ which carries $(x_n)_{n \geq 1}$ to $(\widetilde{x}_n)_{n \geq 1}$ where $\widetilde{x}_n = x_n$ if $n \geq j+1$ and $\widetilde{x}_{n-1} = f_n \circ \ldots \circ f_j(x_{j+1})$ if $n \leq j$.

Let $f : Z \to G$ be any continuous map into a graph G. As G is an ANR then there is an extension $\widetilde{f} : U \to G$ of f to some open neighborhood $Z \subset U$ in $\Pi_{j=1}^{\infty} Z_j$.

By compactness there is j_0 such that $P_j \subset U$ for $j \geq j_0$. In particular, we have a commutative diagram



where the horizontal arrows are inclusions and $\widetilde{f} \circ h$ is inessential since P_{j_0} is contractible. Hence, f is inessential and so Z is tree-like by Theorem 3.1. $\square$

The second lemma is part of the folklore of continuum theory.

Lemma 3.3. *Assume that the inverse limit* $Z = \varprojlim\{G_n, f_n\}$ *of graphs is a tree-like continuum. Then there is an increasing sequence* $\{n_s\}_{s\geq 0}$ *with* $n_0 = 1$ *such that the composite* $\rho_s = f_{n_s} \circ \ldots \circ f_{n_{s+1}-1} : G_{n_{s+1}} \to G_{n_s}$ *is inessential for all* $s \geq 0$.

Proof. Let $q_j : Z \to G_j$ $(j \geq 1)$ denote the canonical maps of the inverse limit which are the restrictions to Z of the projections $p_j : \prod_{n=1}^{\infty} G_n \to G_j$. We start by taking $n_0 = 1$ and observing that Theorem 3.1 yields that q_1 is inessential. Let $H : Z \times I \to G_1$ be a homotopy between q_1 and the constant map. As G_1 is an ANR, there is a neighborhood of Z, $U \subset \prod_{n=1}^{\infty} G_n$, such that the homotopy H extends to a homotopy $\widetilde{H} : U \times I \to G_1$ between $\widetilde{p}_1 = p_1|U$ and the constant map (see [4, Exercise IV.8.13(2)]). Hence, $\widetilde{p}_1$ is an inessential map. In addition, as we have seen in the proof of Lemma 3.2, we have $Z = \bigcap_{j=1}^{\infty} P_j$ with a homeomorphism

$$\alpha_j : \prod_{n=j+1}^{\infty} G_n \overset{\cong}{\to} P_j = \{(x_n)_{n\geq 1};\ f_n(x_{n+1}) = x_n \text{ for } n \leq j\}$$

for each $j \geq 1$. By compactness there exists n_1 such that $P_j \subset U$ for all $j \geq n_1$, and we can form the composite

$$\rho = \widetilde{p}_1 \circ k \circ \alpha_{n_1} \circ \beta : G_{n_1+1} \to G_1$$

where $k : P_{n_1} \hookrightarrow U$ is the inclusion and $\beta : G_{n_1+1} \to \prod_{n=n_1+1}^{\infty} G_n$ is given by $\beta(x) = (y_n)_{n\geq n_1+1}$, where $y_{n_1+1} = x$, and $y_n = *_n \in G_n$ is any fixed point for all $n > n_1 + 1$. It readily follows that ρ is inessential and it coincides with the composite $f_1 \circ \ldots \circ f_{n_1} : G_{n_1+1} \to G_1$. This argument is repeated for G_{n_1+1} in the role of G_1, and we obtain inductively the required subsequence. □

4. Main Theorem

This section is devoted to the proof of the main theorem establishing that tree-like generalized continua are exactly those generalized continua whose Freudenthal compactification is tree-like. More precisely, let $\mathbf{G}C$ be the category of generalized continua and proper maps and $\mathbf{C} \subset \mathbf{G}C$ be the full subcategory of continua. Let $\wedge : \mathbf{G}C \to \mathbf{C}$ denote the functor which carries X to its Freudenthal compactification $\widehat{X}$. We will prove the following characterization theorem.

Theorem 4.1. *The functor* $\wedge$ *preserves and reflects tree-likeness; that is, a generalized continuum is tree-like if and only if its Freudenthal compactification* $\widehat{X}$ *is a tree-like continuum.*

The proof of Theorem 4.1 splits into the two following propositions.

Proposition 4.2. *Let X be a tree-like generalized continuum. Then its Freudenthal compactification $\widehat{X}$ is a tree-like continuum.*

Proposition 4.3. *Let X be a generalized continuum such that its Freudenthal compactification $\widehat{X}$ is a tree-like continuum. Then X is tree-like as well.*

For the proof of Proposition 4.2 we will need the following lemma.

Lemma 4.4. *Let X be a continuum. Any continuous map $f : \widehat{X} \to G$ into a graph is homotopic to a map $f' : \widehat{X} \to G$ such that $f'(\mathcal{F}(X)) \subset$ Vert(G).*

Proof. Given the barycentric subdivision G' of G, let $\mathfrak{U} = \{S_v\}_{v \in Vert(G')}$ denote the open cover of G consisting of the open stars $S_v = st^\circ(v; G')$ of all vertices of G'. We choose a Lebesgue number $\delta > 0$ for the open cover $f^{-1}\mathfrak{U}$ of X and apply Lemma 2.2 to find a compact set $L \subset X$ such that $\widehat{X} - \text{int}L = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$ decomposes into a finite disjoint union of closed and open sets in $\widehat{X} - \text{int}L$ with diameter $A_i < \delta$. Hence, for each i there exists a vertex $v(i)$ for which $f(A_i) \subset S_{v(i)}$. Notice that each A_i is a closed set in $\widehat{X}$ which meets FrL. Here we use the connectedness of X.

At this point we observe that the Tietze extension theorem holds for each $S_{v(i)}$ (it is a retract of the open 2-disk) so that we can extend $g_i : (A_i \cap \text{Fr}L) \sqcup (\mathcal{F}(X) \cap A_i) \to S_{v(i)}$ given by $g(x) = f(x)$ for $x \in A_i \cap \text{Fr}L$ and $g(\varepsilon) = v_i$ for all $\varepsilon \in \mathcal{F}(X) \cap A_i$ to a continuous map $f'_i : A_i \to S_{v(i)}$.

Similarly, we find a homotopy $H^i : A_i \times I \to S_{v(i)}$ between $f|_{A_i}$ and f'_i. Here we apply the Tietze extension theorem to the map $\widetilde{H}^i : A_i \times \{0,1\} \cup (A_i \cap \text{Fr}L) \times I \to S_{(v(i)}$ where $\widetilde{H}^i(x,0) = f(x)$, $\widetilde{H}^i(x,1) = f'_i(x)$, and $\widetilde{H}^i(z,t) = f(z)$ for $z \in A_i \cap \text{Fr}L$.

Finally, let $f' : \widehat{X} \to G$ be the map $f'(x) = f(x)$ for $x \in L$ and $f'(y) = f_i(y)$ if $y \in A_i$. Moreover, $H : \widehat{X} \times I \to G$ given by $H(x,t) = f(x)$ if $x \in L$ and $H(y,t) = H^i(y,t)$ if $y \in A_i$ yields a homotopy between f and f' relative L. □

Proof of Proposition 4.2. We write $X = \varprojlim_p \{T_n, g_n\}$ as an inverse limit of trees with proper bonding maps and consider any map $f : \widehat{X} \to G$ to an arbitrary graph G. We can assume that $f(\mathcal{F}(X)) \subset \text{Vert}(G)$ by Lemma 4.4 and so $f(\mathcal{F}(X)) \subset T_G$ if $T_G \subset G$ is a maximal tree (i.e., a tree containing all vertices). Let $\overline{f} : \widehat{X}/\mathcal{F}(X) \to G/T_G$ denote the induced map. On the other hand, by Lemma 2.4 there is a continuous surjection $\varphi : \widehat{X} \longrightarrow L = \varprojlim\{\widehat{T}_n; \widehat{g}_n\}$ which carries $\mathcal{F}(X)$ to $L_0 = \varprojlim\{\mathcal{F}(T_n), g_{n*}\} \subset L$ and induces a homeomorphism $\overline{\varphi} : \widehat{X}/\mathcal{F}(X) \cong L/L_0$ fitting in the commutative diagram

$$\begin{array}{ccccc}
G & \xleftarrow{\ f\ } & \widehat{X} & \xrightarrow{\ \varphi\ } & L \\
\simeq \big\downarrow \pi & & p \big\downarrow & & \big\downarrow q \\
G/T_G & \xleftarrow{\ \overline{f}\ } & \widehat{X}/\mathcal{F}(X) & \xrightarrow[\cong]{\ \overline{\varphi}\ } & L/L_0
\end{array}$$

where π is a well-known homotopy equivalence. Moreover, since the Freudenthal compactification of a tree is a dendrite, each $\widehat{T}_n$ is contractible [8, Proposition 4 and Theorem 13] and so L is tree-like by Lemma 3.2. Thus, Theorem 3.1 yields that the composite $\rho = \overline{f}\overline{\varphi}^{-1}q : L \to G/T_G$ is homotopically trivial and so is $\pi \circ f = \rho \circ \varphi$, and hence f since π is a homotopy equivalence. The proof finishes by again applying Theorem 3.1. $\square$

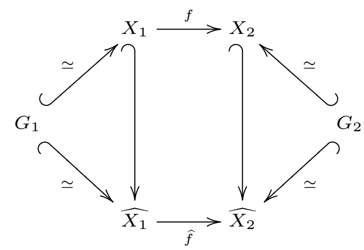
Next, we proceed to prove Proposition 4.3, and the proof of Theorem 4.1 will be accomplished. For this we start with the following definition. By a *ray-extension*, we mean a polyhedron $P = Q \cup T$ which is obtained by the attaching of a finite disjoint union of half-lines $T = \bigsqcup_{i=1}^{m} \mathbb{R}_+$ to a compact connected polyhedron Q. For ray-extensions we have the following improvement of the proper analogue of a classical theorem due to Freudenthal which can be found in [18, Proposition 13].

Theorem 4.5. *For any generalized continuum X with $1 \leq \dim X \leq n$, there exists a homeomorphism $h : X \cong \varprojlim_p \{P_n; f_n\}$ where each P_n is a ray-extension of dimension $\leq n$. Moreover, the homeomorphism h extends to a homeomorphism $\widehat{h} : \widehat{X} \cong \varprojlim \{\widehat{P}_n; \widehat{f}_n\}$ for the corresponding Freudenthal compactifications.*

We do not know of any reference for this specific result in the literature. To ease the reading of the paper we give the proof of it in Appendix A. Besides Theorem 4.5, the two other ingredients in the proof of Proposition 4.3 are the two following results. The first one is an easy lemma.

Lemma 4.6. *Let $f : X_1 \to X_2$ be a proper map between graphs which are ray-extensions and such that $\widehat{f} : \widehat{X}_1 \to \widehat{X}_2$ is inessential. Then f is also inessential.*

Proof. By definition $X_i = G_i \cup T_i$ where G_i is a compact graph and T_i is a finite disjoint union of half-lines attached at G_i $(i = 1, 2)$. Then we simply observe that in the commutative diagram



the inclusions of the G_i's are homotopy equivalences and so are the inclusions $X_i \subset \widehat{X}_i$ for $i = 1, 2$. $\square$

The second result is essentially proved in [9, Theorem 11]; see also [13, Section 4]. We include the proof for the convenience of the reader.

Proposition 4.7. *Let X be a 1-dimensional generalized continuum which can be written as an inverse limit $X = \varprojlim_p \{X_n; f_n\}$ of graphs where the bonding proper maps are inessential. Then X is tree-like.*

Proof. Let $\pi_n : X \to X_n$ denote the canonical maps of the limit. For each n consider the universal covering space $p_n : \widetilde{X}_n \to X_n$ and choose a point $x^0 = (x^0_n)_{n\geq 1} \in X$. Since each map $\pi_n = f_n \circ \pi_{n+1}$ is inessential, given $\widetilde{x}^0_n \in \widetilde{X}_n$, there exists a lifting $\widetilde{\pi}_n : X \to \widetilde{X}_n$ with $\widetilde{\pi}_n(x) = \widetilde{x}^0_n$. Here we use [19, Theorem 2.2.3]. Similarly, the same homotopy covering property yields maps $\widetilde{f}_n : \widetilde{X}_{n+1} \to \widetilde{X}_n$ with $\widetilde{f}_n(\widetilde{x}^0_{n+1}) = \widetilde{x}^0_n$ such that

$$p_n \circ \widetilde{f}_n = f_n \circ p_{n+1}. \tag{4.1}$$

We next consider for each n the subtree $T_n = \widetilde{\pi}_n(X) \subset \widetilde{X}_n$ for which the restrictions $\overline{p}_n : T_n \to X_n$ and $\overline{\pi}_n : X \to T_n$ are readily checked to be proper. Moreover, one can easily show that $\widetilde{f}_n(T_{n+1}) = T_n$ and $\widetilde{f}_n : T_{n+1} \to T_n$ is proper. Let $Y = \varprojlim_p T_n$ be the inverse limit of the sequence with $\widetilde{f}_n$ as bonding maps and let $q_n : Y \to T_n$ denote the canonical maps.

Equality (4.1) yields that the restrictions $\overline{p}_n$ induce a map $\overline{p} : Y \to X$ for which the map $\phi : X \to Y$ defined by $\phi(x) = (\overline{\pi}_n(x))_{n\geq 1}$ is a section; that is, $\overline{p} \circ \phi = id_X$. Indeed, ϕ is well defined by the uniqueness of liftings [19, Theorem 2.2.2] since both $\widetilde{f}_n \circ \widetilde{\pi}_{n+1}$ and $\widetilde{\pi}_n$ are liftings of π_n at $\widetilde{x}^0_n$; moreover, $\overline{p}\phi(x) = (\overline{p}_n \overline{\pi}_n(x))_{n\geq 1} = (\pi_n(x))_{n\geq 1} = x$.

In addition, as $\phi^{-1}(A) \subset \overline{p}(A)$ for any set $A \subset Y$, it follows that ϕ is a proper embedding, and so $\phi(X)$ is a connected closed set of Y. Furthermore, $\phi(X) = \varprojlim_p q_n(\phi(X))$ by [10, Lemma 4.5], and so X is homeomorphic to the tree-like space $\phi(X)$ obtained as an inverse limit of the subtrees $q_n(\phi(X)) \subset T_n$. $\square$

We are now ready to prove Proposition 4.3

Proof of Proposition 4.3. Theorem 4.5 yields a homeomorphism $X \cong \varprojlim_p \{X_n; f_n\}$ extending to a homeomorphism $\widehat{X} \cong \varprojlim\{\widehat{X}_n; \widehat{f}_n\}$ where the X_n's are ray-extensions. By hypothesis, $\widehat{X}$ is a tree-like continuum and Lemma 3.3 allows us to assume without loss of generality that the bonding maps $\widehat{f}_n$ are inessential. Then the proper maps f_n are also inessential by Lemma 4.6. We conclude by applying Proposition 4.7. □

5. Some Consequences of Theorem 4.1

Let us start by observing that the maximality of the Freudenthal compactification and the ideas involved in the proof of Proposition 4.2 yield the following improvement of Theorem 4.1.

Theorem 5.1. *The following statements are equivalent:*

(1) *The generalized continuum X is tree-like.*
(2) *The Freudenthal compactification $\widehat{X}$ is tree-like.*
(3) *There is a tree-like compactification of X with* 0*-dimensional remainder.*

Proof. (1) ⇒ (2) is part of Theorem 4.1. Moreover, (2) ⇒ (3) is obvious.

In order to show (3) ⇒ (1), let Y be a tree-like compactification of X with 0-dimensional remainder $RY = Y - X$. As the Freudenthal compactification is maximal among these compactifications, there is a continuous extension of the identity of X, $\varphi : \widehat{X} \longrightarrow Y$.

Consider any map $f : \widehat{X} \to G$ to an arbitrary graph G. As done in the proof of Proposition 4.2, we can assume that $f(\mathcal{F}(X)) \subset \mathrm{Vert}(G)$ (Lemma 4.4) and so $f(\mathcal{F}(X)) \subset T_G$ where $T_G \subset G$ is a maximal tree. The induced map $\overline{f} : \widehat{X}/\mathcal{F}(X) \to G/T_G$ fits in the commutative diagram

$$\begin{array}{ccccc} G & \xleftarrow{\;f\;} & \widehat{X} & \xrightarrow{\;\varphi\;} & Y \\ {\scriptstyle\simeq}\big\downarrow{\scriptstyle\pi} & & {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ G/T_G & \xleftarrow{\;\overline{f}\;} & \widehat{X}/\mathcal{F}(X) & \xrightarrow[\cong]{\;\overline{\varphi}\;} & Y/RY \end{array}$$

where the homeomorphism $\overline{\varphi}$ is induced by φ and π is a well-known homotopy equivalence. Moreover, since Y is assumed to be tree-like, Theorem 3.1 yields that the composite $\rho = \overline{f}\overline{\varphi}^{-1}q : Y \to G/T_G$ is homotopically trivial, and so is $\pi \circ f = \rho \circ \varphi$ and hence f, since π is a homotopy equivalence. Therefore, $\widehat{X}$ is tree-like by again applying Theorem 3.1, whence X is tree-like by Theorem 4.1. □

Corollary 5.2. *The class of tree-like generalized continua is closed under inverse limits with proper bonding maps.*

Proof. Let $X = \varprojlim_p \{X_n, f_n\}$ be a generalized continuum where each X_n is a tree-like space. Hence, for every $n \geq 1$, the Freudenthal compactification $\widehat{X}_n$ is a tree-like continuum by Theorem 4.1 and so the continuum $L = \varprojlim\{\widehat{X}; \widehat{f}_n\}$ is also tree-like by [15, Lemma 2.5.15]. Moreover, by Lemma 2.4, $L = X \cup L_0$ where $L_0 = \varprojlim\{\mathcal{F}(X_n), f_{n*}\}$; therefore, L is a tree-like compactification of X with 0-dimensional remainder L_0. We conclude by Theorem 5.1 that X is tree-like. $\square$

We next use Theorem 4.1 to attain a proper analogue of the following celebrated theorem due to T. Bruce McLean.

Theorem 5.3 ([17])**.** *Let $f : X \to Y$ be a confluent map between continua. Assume that X is tree-like, then Y is also tree-like.*

Recall that, given two spaces X and Y, by a *confluent map* we mean a continuous surjection $f : X \to Y$ such that, for any subcontinuum $B \subset Y$, we have $f(A) = B$ for each connected component $A \subset f^{-1}(B)$. Namely, we prove the following theorem which answers affirmatively [9, Open Question 15].

Theorem 5.4. *Let $f : X \to Y$ be an end-faithful proper confluent surjection between generalized continua. If X is tree-like, then Y is tree-like as well.*

Recall that a proper map $f : X \to Y$ is said to be *end-faithful* if the induced map $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ is a bijection (or, equivalently, a homeomorphism).

Proof of Theorem 5.4. By [3, Theorem 7.5], the Freudenthal extension of f, $\widehat{f} : \widehat{X} \to \widehat{Y}$ is also confluent. Moreover, $\widehat{X}$ is a tree-like continuum by Theorem 4.1. Hence, $\widehat{Y}$ is also tree-like by Theorem 5.3; hence, Y is tree-like by applying Theorem 4.1 again. $\square$

Remark 5.5. As was observed in [9, Example 14], the end-faithfulness of the map f cannot be dropped in Theorem 5.4; that is, Theorem 5.3 does not hold with full generality in the proper category. We give the details for the sake of completeness.

Let $X \subset \mathbb{R}^2$ be the two-ended generalized continuum depicted in Figure 2(a) below.

The space X is tree-like since so is its Freudenthal compactification (that is, the space obtained by gluing the extremes of the sinoidal arcwise components of two copies of the $\sin 1/x$-curve). Here we use Theorem 4.1.

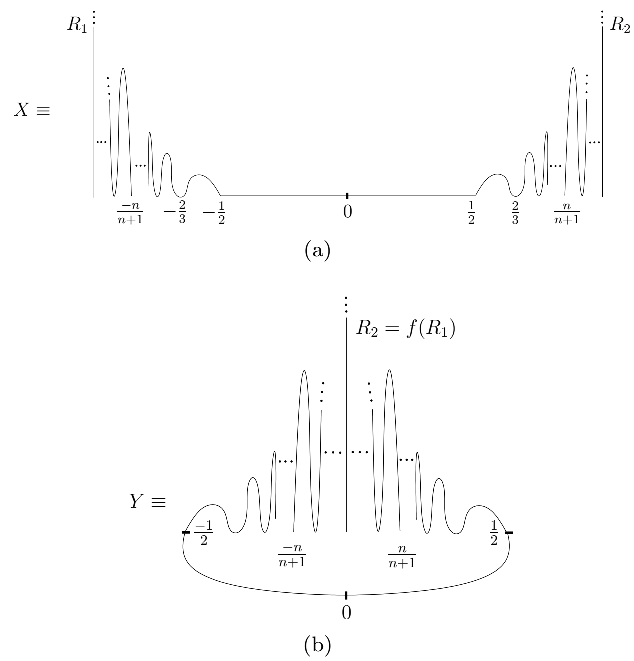


FIGURE 2

Let Y be the generalized continuum in Figure 2(b) obtained by identifying in X the two rays R_1 and R_2. Consider the quotient map $f : X \to Y$. It is obvious that f is proper; moreover, it is readily checked that f is confluent since any continuum $C \subset Y$ is an arc contained in either the ray $R_2 = f(R_1)$ or its complement $Y - R_2$.

However, Y fails to be tree-like. Indeed, let $\mathfrak{D}$ denote the decomposition of $\widehat{Y}$ whose single non-degenerate element is the complement of the lower arc in Figure 2(b) running from $-\frac{1}{2}$ to $\frac{1}{2}$. Then the quotient map $\pi : \widehat{Y} \to \widehat{Y}/\mathfrak{D} \cong S^1$ is a monotone (and hence, confluent) map onto the circle. As an immediate consequence of theorems 5.3 and 4.1, we get that Y is not tree-like.

Theorems 5.1 and 5.4 are now used to prove the following theorem.

Theorem 5.6. *Any connected locally compact subspace $U \subset X$ of a tree-like space X is also tree-like. In particular, tree-likeness is inherited by closed and by open subsets of tree-like spaces.*

Proof. Case 1: U is closed in X. Let $X = \varprojlim_p \{T_n, g_n\}$ where each T_n is a tree. Then $U = \varprojlim_p q_n(U)$ by [10, Lemma 4.5]. Moreover, as the bonding maps q_n are proper, the image $q_n(U)$ is a connected closed set of the tree T_n and hence a tree[1]. Therefore, U is also tree-like.

Case 2: U is open in X. Observe that U is still open in the Freudenthal compactification $\widehat{X}$ so that Theorem 4.1 allows us to replace X by $\widehat{X}$ so that we can assume without loss of generality that X is a compact tree-like space. Furthermore, the closure $\overline{U}$ is already tree-like by case 1; hence, we can also assume that U is dense in X. The complement $X - U$ is then a compact set of dimension ≤ 1. Here we use that tree-like spaces are 1-dimensional (this follows from connectedness and [6, Theorem 1.13.4]). If $\dim(X - U) = 0$, then X turns to be a compactification of U with compact 0-dimensional remainder $RX = X - U$, whence U is tree-like by Theorem 5.1.

Otherwise, if $\dim(X - U) = 1$, let $\mathfrak{D}$ be the decomposition of X whose non-degenerate elements are the 1-dimensional components of $X - U$. This decomposition is upper semicontinuous by [14, Theorem V.47.VI.6]. Hence, the quotient space $X/\mathfrak{D}$ is a continuum [14, Theorem IV.43.IV.1]. In addition, as the natural projection $\pi : X \to X/\mathfrak{D}$ is a monotone map, it follows from Theorem 5.4 that $X/\mathfrak{D}$ is tree-like.

Furthermore, the image $W = \pi(U)$ is a dense open set in $X/\mathfrak{D}$ homeomorphic to U since the restriction $\pi|U$ is an open map. This way the quotient space $X/\mathfrak{D}$ is a tree-like compactification of W with compact 0-dimensional reminder $X/\mathfrak{D} - W$. Then $U \cong W$ is a tree-like space by Theorem 5.1.

General Case: Let $U \subset X$ be any connected locally compact set in X. Let $P = \overline{U}$ be its closure in X. Then by case 1, P is tree-like. We claim that U is also open in P, and then case 2 yields that U is tree-like.

In order to show that U is open in P, let $p \in U$ be any point and let $\Omega \subset U$ be a compact neighborhood of p in U. Here we use the local compactness of U. Choose an open set V in P with $\operatorname{int}_U \Omega = V \cap U$. If $V - \Omega \neq \varnothing$, then $V - \Omega = V - U$ is a non-empty open set in P missing U; this contradicts the density of U in P. Therefore, $V - \Omega = \varnothing$, whence $V \subset \Omega \subset U$. This shows that p lies in the interior of U in P. □

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[1]If C is a closed connected subset of a tree T, then C reduces to an arc in an open edge of T if C does not contain vertices of T. Otherwise, C is the (locally finite) union of the subtree generated by the vertices in $\mathrm{Vert}(T) \cap C$ and (possibly) arcs in the open stars of the vertices in C

Appendix A. Proof of Theorem 4.5

This appendix contains a detailed proof of Theorem 4.5. We follow the ideas and notation of [6, Theorem 1.13.2] and [12]. Let us start with two preparatory lemmas, the first one being an immediate consequence of Lemma 2.2; see Remark 2.3.

Lemma A.1. *Any generalized continuum X admits an exhausting sequence $\{X_k\}_{k\geq 1}$ such that $X - X_k = \bigsqcup_{j\in J_k} W_j^k$ $(k \geq 1)$ is a finite disjoint union of closed and open sets in $X - X_k$ of non-compact closure. Moreover, for each $j \in J_k$ there is a unique $j' \in J_{k-1}$ with $W_j^k \subset W_{j'}^{k-1}$ and the Freudenthal ends of X are in $1-1$ correspondence with the sequences $(W^k_{j(k)})_{k\geq 1}$ with $W^k_{j(k)} \subset W^{k-1}_{j(k-1)}$ for all $k \geq 2$.*

Definition A.2. Given two exhausting sequences of X, $\mathcal{X} = \{X_k\}_{k\geq 1}$ and $\widetilde{\mathcal{X}} = \{\widetilde{X}_k\}_{k\geq 1}$, we say that $\mathcal{X}$ is a *shifting* of $\widetilde{\mathcal{X}}$ if $\widetilde{X}_k \subset \mathrm{int}X_k \subset X_k \subset \mathrm{int}\widetilde{X}_{k+1}$ $(k \geq 1)$.

Lemma A.3. *Given a generalized continuum X, let $\{X_k\}_{k\geq 1}$ be any exhausting sequence of X which is a shifting of $\{\widetilde{X}_k\}_{k\geq 1}$. Then there exists a sequence of proper maps $h_k : X - \mathrm{int}X_k \to [k, \infty)$ such that $h_k^{-1}(k) = \widetilde{X}_{k+1} - \mathrm{int}X_k$, while $h_k^{-1}(n) = \mathrm{Fr}X_n$ and $h_k^{-1}(n + \frac{1}{2}) = \mathrm{Fr}\widetilde{X}_{n+1}$ for all $n \geq k + 1$. Moreover, $h_{k+1} = h_k$ on $X - \mathrm{int}X_{k+2}$.*

Proof. For each $n \geq k$, write $D_n = D_n^1 \cup D_n^2$ where $D_n^1 = \widetilde{X}_{n+1} - \mathrm{int}X_n$ and $D_n^2 = X_{n+1} - \mathrm{int}\widetilde{X}_{n+1}$. As metric spaces are perfectly normal, there is a continuous map $g_k : D_k \to [k, k + 1]$ with $g_k^{-1}(k + 1) = \mathrm{Fr}X_{k+1}$ and $g_k^{-1}(k) = D_k^1$; see [5, Theorem 1.5.19]. Similarly, for $n \geq k + 1$, we find continuous maps $g_{n,1} : D_n^1 \to [n, n + \frac{1}{2}]$ and $g_{n,2} : D_n^2 \to [n + \frac{1}{2}, n + 1]$ such that $g_{n,1}^{-1}(n) = \mathrm{Fr}X_n$, $g_{n,1}^{-1}(n + \frac{1}{2}) = g_{n,2}^{-1}(n + \frac{1}{2}) = \mathrm{Fr}\widetilde{X}_{n+1}$, and $g_{n,2}^{-1}(n + 1) = \mathrm{Fr}X_{n+1}$. Then the map h_k is defined as the union of maps $h_k = g_k \bigcup_{n\geq k+1}(g_{n,1} \cup g_{n,2})$. □

The basic ideas for the proof of Theorem 4.5 are well known (see the proof of [6, Theorem 1.13.2]). The variations are due to the special kind of open covers needed to produce ray-extensions. A prototype of these covers is constructed as follows. We start with an exhausting sequence $\mathcal{X} = \{X_k\}_{k\geq 1}$ as in Lemma A.1, which is also a shifting of some exhausting sequence $\widetilde{\mathcal{X}} = \{\widetilde{X}_k\}_{k\geq 1}$ (for instance, $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ can be chosen to be the families of sets of even and odd indexes, respectively, of the exhausting sequence in Lemma A.1). In particular, for each $k \geq 1$, $X - X_k = \bigsqcup_{j\in J_k} W_j^k$ is the disjoint union of a finite family of open and closed sets in $X - X_n$ of non-compact closure $\mathcal{W}^k = \{W_j^k\}_{j\in J_k}$. Then we consider open covers of X of the form

$$\mathcal{U} = \mathcal{U}^0 \cup \mathcal{W}^k \tag{A.1}$$

where $\mathcal{U}^0$ is a finite cover of X_k with open sets in $\mathrm{int}\widetilde{X}_{k+1}$. As $\dim X \leq n$, then $\dim U^0 \leq n$ for the open set $U^0 = \bigcup\{U; U \in \mathcal{U}^0\} \subset \mathrm{int}\widetilde{X}_{k+1}$. Therefore, any open cover of U^0 admits a refinement of order[2] $\leq n+1$ (see [6, Proposition 3.2.2]). Hence, the compact set X_k is covered by finitely many open sets of that refinement, and these sets, together with the given family $\mathcal{W}^k$, can replace the original cover $\mathcal{U}$. Thus, we can assume without loss of generality that the subcover $\mathcal{U}^0$ in (A.1) has order $\leq n+1$.

If the cardinality of the open cover $\mathcal{U}$ is s, its nerve $N(\mathcal{U})$ will be regarded as a subcomplex of dimension $\leq n$ of the canonical simplex $\Delta^{s-1} \subset \mathbb{R}^s$ after identifying the sets in $\mathcal{U}$ with the vertices $p_i = (0, \ldots, 1, \ldots, 0)$ of Δ^{s-1}. If this simplex is metrized by $d(x, y) = \|x - y\|$ where $\|z\| = \sum_{i=1}^{s} |\lambda_i|$ for $z = (\lambda_1, \ldots, \lambda_s)$, it is obvious that diam $\sigma \leq$ diam $\Delta^{s-1} \leq 2$ for any simplex $\sigma \in N(\mathcal{U})$.

Now we attach rays to the compact nerve $N(\mathcal{U})$ to get a ray-extension $S(\mathcal{U}) \subset \mathbb{R}^s$ as follows. For each $j \in J_k$, let $[k, \infty)_j$ denote a copy of the ray $[k, \infty)$. Then the polyhedron

$$S(\mathcal{U}) = N(\mathcal{U}) \bigcup_{j \in J_k} [k, \infty)_j \subset \mathbb{R}^{s_k} \tag{A.2}$$

is obtained after identifying k_j with the vertex of $N(\mathcal{U})$ corresponding to the set $W_j^k \in \mathcal{W}^k$. Notice that the mesh[3] of $S(\mathcal{U})$ is ≤ 2. Note also that the connectedness of X implies that, for any $U, U' \in \mathcal{U}$, there exists a finite sequence $U_1, \ldots, U_m \in \mathcal{U}$ with $U_1 = U$, $U_m = U'$ and $U_i \cap U_{i+1} \neq \varnothing$ $(1 \leq i \leq m-1)$. Thus, $N(\mathcal{U})$, and hence $S(\mathcal{U})$, is connected. However, $N(\mathcal{U}^0)$ needs not be connected.

Next, we proceed to define a proper map $f : X \to S(\mathcal{U})$. We start by using the canonical barycentric map $\alpha : X \to N(\mathcal{U})$ given by $\alpha(x) = \sum_{U \in \mathcal{U}} \alpha_U(x) U$ where $\alpha_U(x) = \frac{d(x, X-U)}{\sum_{U \in \mathcal{U}} d(x, X-U)}$. In particular, we have $\alpha(x) = 1$ for all $x \in \mathrm{Fr}\widetilde{X}_{k+1}$. Note also $\alpha(X_k) \subset N(\mathcal{U}^0)$. Therefore, the restriction $\alpha_{k+1} = \alpha | \widetilde{X}_{k+1} : \widetilde{X}_{k+1} \to N(\mathcal{U})$ extends to a proper map $f : X \to S(\mathcal{U})$ by setting $f(x) = h_k(x)_j$ for all $x \in W_j^k - \mathrm{int}\widetilde{X}_{k+1}$, where $h_k : X - \mathrm{int}X_k \to [k, \infty)$ is the function in Lemma A.3. Here, $h_k(x)_j$ denotes the value $h_k(x)$ placed on the copy $[k, \infty)_j$. The map f will be

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[2] The *order* of a cover $\mathcal{U}$ is the largest integer m (if it exists) such that $\mathcal{U}$ does not contain $m+1$ sets with non-empty intersection.

[3] The *mesh* of a family of sets is the supremum of the diameters of its sets.

termed a *relative barycentric map* associated to the cover $\mathcal{U}$. Note

$$\text{(A.3)} \qquad \begin{array}{l} \text{(a) } f(x) = W_j^k = k_j \text{ if } x \in W_j^k \cap \mathrm{Fr}\widetilde{X}_{k+1}; \text{ and} \\ \text{(b) } f(X_k) \subset N(\mathcal{U}^0). \end{array}$$

We are now ready to prove Theorem 4.5.

Proof of Theorem 4.5. Consider a sequence $\mathcal{U}_1, \dots, \mathcal{U}_k, \dots,$ of finite open covers $\mathcal{U}_k = \mathcal{U}_k^0 \cup \mathcal{W}^k$ as in (A.1) and relative barycentric maps $f_k : X \to S(\mathcal{U}_k)$ with properties (i) and (ii) below. Let $\mathrm{Cov}(f_{k-1}, S(\mathcal{U}_{k-1}))$ denote the open cover of X_k consisting of the counterimages by f_{k-1} of the open stars $st^\circ(v; S(\mathcal{U}_{k-1})^{(1)})$ with v ranging over the set of vertices in $N(\mathcal{U}_{k-1})^{(1)} \bigcup_{j\in J_k} [k-1,k]_j^{(1)}$. Here the upper index "(1)" stands for the first barycentric subdivision. Notice that $f_{k-1}^{-1}(st^\circ(v; S(\mathcal{U}_{k-1})^{(1)}) \subset \mathrm{int}\widetilde{X}_{k+1}$ for every vertex v.

(i) $\mathcal{U}_k^0$ is a refinement of $\mathrm{Cov}(f_{k-1}, S(\mathcal{U}_{k-1}))$.

(ii) The mesh of $\mathcal{U}_k^0$ is $< \frac{1}{2^k}$ and the order of each $\mathcal{U}_k^0$ is $\leq n+1$.

Condition (i) allows us to define a map π_{k-1} between the sets of vertices of $S(\mathcal{U}_k)$ and $S(\mathcal{U}_{k-1})$ as follows. Given a vertex $U \neq W_j^k$ in $N(\mathcal{U}_k)$ corresponding to an open set $U \in \mathcal{U}_k^0$, we set $\pi_{k-1}(U) = v$ where $v \in N(\mathcal{U}_{k-1})^{(1)} \bigcup_{j\in J_{k-1}} [k-1,k]_j^{(1)}$ is a vertex for which $f_{k-1}(U) \subset st^\circ(v; S(\mathcal{U}_{k-1})^{(1)})$. Otherwise, if $U = t_j$ is a vertex in $[k,\infty)_j$ (in particular, $U = W_j^k = k_j$), then we define $\pi_{k-1}(t_j) = t_{j'}$ where $j' \in J_{k-1}$ is the unique index for which $W_j^k \subset W_{j'}^{k-1}$.

Actually, if $S(\mathcal{U}_{k-1})^{\#} = N(\mathcal{U}_{k-1})^{(1)} \bigcup_{j\in J_{k-1}} ([k-1,k]_j^{(1)} \cup [k,\infty)_j)$, then π_{k-1} determines a proper simplicial map

$$\text{(A.4)} \qquad \pi_{k-1} : S(\mathcal{U}_k) \to S(\mathcal{U}_{k-1})^{\#}.$$

Indeed, there is nothing to be checked on $[k,\infty)_j$ ($j \in J_k$). Let $\sigma = \langle U_1, \dots, U_s \rangle \in N(\mathcal{U}_k)$. If $U_i \neq W_j^k$ for all i, then $f_{k-1}(x) \in \bigcap_{i=1}^s st^\circ(\pi_{k-1}(U_i); S(\mathcal{U}_{k-1})^{(1)})$ and $\pi_{k-1}(U_1), \dots, \pi_{k-1}(U_s)$ lie in a simplex of $N(\mathcal{U}_{k-1})^{(1)} \bigcup_{j\in J_{k-1}} [k-1,k]_j^{(1)}$. Otherwise, if some W_j^k appears as vertex of σ, it must be unique, say $U_1 = W_j^k$, and let $x \in \bigcap_{i=1}^s U_i$. The definition of π_{k-1} yields $f_{k-1}(x) \in st^\circ(\pi_{k-1}(U_i); S(\mathcal{U}_{k-1})^{(1)})$ for each $i \neq 1$. Moreover, as $x \in W_j^k$, $f_{k-1}(x) \in [k,k+1]_{j'}$ for some $j' \in J_{k-1}$. Hence, $\pi_{k-1}(U_i) = k_{j'}$ for all i. Finally, the properness of π_{k-1} is immediate since it is readily checked that $\pi_{k-1}^{-1}(v)$ is a finite set for every vertex $v \in S(\mathcal{U}_{k-1})^{\#}$.

Remark A.4. Notice that $\dim \pi_{k-1}(S(\mathcal{U}_k)) \leq n$ since the only possible simplices $\tau \in S(\mathcal{U}_k)$ with $\dim \tau \geq n+1$ must contain a vertex of the form W_j^k and then $\pi_{k-1}(\tau) = k_{j'}$ for some $j' \in J_{k-1}$.

Although f_{k-1} needs not agree with $\pi_{k-1} f_k$, the following properties hold:

$$\begin{aligned} &\pi_{k-1}(\sigma_k(x)) \subset \sigma_{k-1}(x) \text{ for all } x \in X, \\ &\pi_{k-1}(\sigma_k(x)) = \sigma^{\#}_{k-1}(x) \text{ if } x \in X_m \text{ and } k > m. \end{aligned} \tag{A.5}$$

Here, $\sigma_k(x)$ and $\sigma^{\#}_k(x)$ denote the support[4] of $f_k(x)$ in $S(\mathcal{U}_k)$ and $S(\mathcal{U}_k)^{\#}$, respectively.

Indeed, there is nothing to be checked if $x \in X - \mathrm{int}\widetilde{X}_{k+1}$ since $\pi_{k-1} f_k = f_{k-1}$ on this difference; moreover, if $x \in \mathrm{int}\widetilde{X}_{k+1}$ and $\sigma_k(x) = \langle U_1, \dots, U_m \rangle \in N(\mathcal{U}_k)$, then the support of $\pi_{k-1} f_k(x)$ is a simplex η with vertices (possibly repeated) $\pi_{k-1}(U_1), \dots, \pi_{k-1}(U_m)$. If $U_i \neq W_j^k$ for all i (for instance, if $x \in X_m$ with $m < k$), the definition of π_{k-1} yields $x \in U_i \subset f^{-1}_{k-1}(st^\circ(\pi_{k-1}(U_i); S(\mathcal{U}_{k-1})^{\#})$ $(1 \leq i \leq m)$, whence $f_{k-1}(x)$ lies in the interior of η and so $\eta = \sigma^{\#}_{k-1}(x) \subset \sigma_{k-1}(x)$.

It remains to check the case when some W_j^k appears among the vertices of $\sigma_k(x)$, say $U_1 = W_j^k$. In this instance, π_{k-1} is the constant map $k_{j'}$ on $\sigma_k(x)$ for the unique index $j' \in J_{k-1}$ with $W_j^k \subset W_{j'}^{k-1}$. Therefore, $\eta = \langle k_{j'} \rangle$ reduces to a vertex. On the other hand, as $x \in W_j^k \subset W_{j'}^{k-1} - X_k \subset \mathrm{int}\widetilde{X}_{k+1} - X_k$, we have $k_{j'} < f_{k-1}(x) < (k+1)_{j'}$; that is, $\sigma_{k-1}(x) = \sigma^{\#}_{k-1}(x) = [k_{j'}, (k+1)_{j'}]$ is the support of $f_{k-1}(x)$ in $S(\mathcal{U}_{k-1})$. This shows (A.5).

The proof will be accomplished by showing a homeomorphism $h : X \cong S$ where $S = \varprojlim_p \{S(\mathcal{U}_k), \pi_k\}_{k\geq 1}$ is the limit of the inverse sequence

$$S(\mathcal{U}_1) \xleftarrow{\pi_1} S(\mathcal{U}_2) \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_{k-1}} S(\mathcal{U}_{k-1}) \xleftarrow{\pi_{k-1}} S(\mathcal{U}_k) \xleftarrow{\pi_k} \cdots$$

since $S = \varprojlim_p \{\pi_{k-1} S(\mathcal{U}_k), \pi_k\}_{k\geq 2}$ and $\dim \pi_{k-1}(S(\mathcal{U}_k)) \leq n$ for all $k \geq 2$; see Remark A.4.

Let $q_k : S \to S(\mathcal{U}_k)$ denote the canonical projection $(k \geq 1)$. It is not hard to show that the projections q_k inherit the properness of the bonding maps π_k; see [10, Lemma 3.1]. The following statement is the crucial fact in the construction of the homeomorphism h. For $r > k$, let π_k^r denote the composite $\pi_k^r = \pi_k \circ \ldots \circ \pi_r : S(\mathcal{U}_{r+1}) \to S(\mathcal{U}_{k-1})$. Given a sequence of points $y_k \in S(\mathcal{U}_k)$, assume that we have an inverse sequence

$$F_1 \xleftarrow{\pi_1} F_2 \xleftarrow{\pi_2} \cdots \tag{A.6}$$

[4]The *support* of a point x in a simplicial complex K is the unique simplex $\sigma \in K$ which contains x in its interior.

where $F_k \subset Y_k = \bigcup\{\sigma \in S(\mathcal{U}_k); y_k \in \sigma\}$ is a non-empty compact set for each $k \geq 1$. Then there exists $m \geq 1$ such that $F_k \subset N(\mathcal{U}_k^0)$ for all $k > m$ and

$$\text{(A.7)} \qquad \operatorname{diam} \pi_k^r(F_{r+1}) \leq 4(\frac{n}{n+1})^{r-k} \text{ for all } r > k > m.$$

Indeed, choose m such that $F_1 \subset S_0 = N(\mathcal{U}_1^0) \bigcup_{j \in J_1} [1, m]_j \subset S(\mathcal{U}_1)$. Then the definition of the maps π_k shows that $F_{k+1} \subset \pi_1^{k^{-1}}(S_0) \subset N(\mathcal{U}_{k+1}^0)$ whenever $k > m$. Thus, all simplices $\sigma \subset Y_{k+1}$ which meet F_{k+1} do not contain any vertex W_j^k $(j \in J_k)$ and so $\dim \sigma \leq n$; see Remark A.4. Moreover, (A.6) and the definition of π_k imply that $\pi_k(\sigma)$ is part of the barycentric subdivision of a simplex $\rho \in S(\mathcal{U}_{k-1})$ which is a face of some $\sigma' \subset Y_{k-1}$, and now [6, Lemma 1.13.1] yields $d(\pi_k(y), \pi_k(y')) \leq \frac{n}{n+1} d(y, y')$ for all $y, y' \in \sigma$. Assumption (A.6) allows us to iterate this inequality starting with any $x, x' \in \mu \subset Y_{r+1}$ with $\mu \cap F_{k+1} \neq \varnothing$ to obtain the inequality $d(\pi_k^r(x), \pi_k^r(x')) \leq (\frac{n}{n+1})^{r-k} d(x, x') \leq 2(\frac{n}{n+1})^{r-k}$. Here, we use that the mesh of any ray-extension in (A.2) is chosen to be ≤ 2. Therefore, we derive $d(z, z') \leq d(z, y_{k+1}) + d(y_{k+1}, z') \leq 4(\frac{n}{n+1})^{r-k}$ if $z, z' \in F_{r+1}$.

Let L stand for the inverse limit of the sequence in (A.6). Since $q_k(L) \subset \bigcap_{r=k+1}^{\infty} \pi_k^r(F_{r+1})$ for all $k > m$, we derive from (A.7) that the inverse limit $L = \{*\}$ is a singleton.

Any $x \in X$ induces the sequence $f_k(x) \in S(\mathcal{U}_k)$, and by (A.5) the inverse sequence

$$\text{(A.8)} \qquad \sigma_1(x) \stackrel{\pi_1}{\leftarrow} \sigma_2(x) \stackrel{\pi_2}{\leftarrow} \dots$$

is well defined. Then the previous observations show that the inverse limit of this sequence, termed $L(x)$, reduces to a point. This way we have a well-defined map $h : X \to S$ by setting $h(x) = L(x)$. Furthermore, the proof of (A.7) shows that, for the sequence in (A.8), the following inequality holds.

$$\text{(A.9)} \quad \operatorname{diam} \pi_k^r(\sigma_{r+1}(x)) \leq 2(\frac{n}{n+1})^{r-k} \text{ for all } r > k > m \text{ and } x \in X_m.$$

Here, we use (A.3) to get $f_k(X_m) \subset f_k(X_k) \subset N(\mathcal{U}_k^0)$, whence $\sigma_k(x)$ does not contain any vertex W_j^k $(j \in J_k)$ for $k > m$.

We will check that h is a homeomorphism by showing that h is a continuous injection, as well as a proper surjection.

Clearly, h is injective since by condition (ii), given two points $x \neq x'$ of X, if $x, x' \in X_m$ we can find a natural number $k > m$ such that no open set in $\mathcal{U}_k$ containing x contains x'. Therefore, the vertex sets of the supports $\sigma_k(x)$ and $\sigma_k(x')$ in $N(\mathcal{U}_k)$ must be disjoint (recall that x belongs to the open sets which are the vertices of $\sigma_k(x)$) and the definition of h yields

$h(x) = L(x) \neq L(x') = h(x')$. In order to prove that h is continuous, it will suffice to check that the composite $h_k = q_k h$ is continuous for all $k \geq 1$. Given a point $x_0 \in X$, let m be the first natural number for which $x \in X_m$. Since for any $r > k$ we have $h_k = \pi_k^r h_{r+1}$, it will be enough to check the continuity of h_k with $k > m$. For this, given $\epsilon > 0$, we choose a natural number r such that $r > k$ and $(\frac{n}{n+1})^{r-k} < \frac{\epsilon}{2}$. The intersection $U = \bigcap\{U; U \in \mathcal{U}_r; x_0 \in U\}$ is an open neighborhood of x_0 such that for any $x \in U$ we have

$$\text{(A.10)} \qquad \sigma_{r+1}(x_0) \subset \sigma_{r+1}(x);$$

indeed, as $\sigma_{r+1}(x_0) = \langle U_0, \ldots, U_p\rangle$ contains $f_{r+1}(x_0)$ in its interior, the definition of $f_{r+1} : X \to S(\mathcal{U}_{r+1})$ yields that $x_0 \in \bigcap_{i=1}^{p} U_i$, and hence $x \in U_i$ for all i. Therefore, $\alpha_{U_i}(x) > 0$ for all i and $\sigma_{r+1}(x)$ contains the vertices of $\sigma_{r+1}(x_0)$.

The inclusion (A.10) yields $h_k(x_0) \in \pi_k^r(\sigma_{r+1}(x_0)) \subset \pi_k^r(\sigma_{r+1}(x))$ for $x \in U$. Then, by (A.9), diam $(\pi_k^r(\sigma_{r+1}(x)) \leq 2(\frac{n}{n+1})^{r-k} < \epsilon$ and since $h_k(x) \in \pi_k^r(\sigma_{r+1}(x))$, then $d(h_k(x_0), h_k(x)) < \epsilon$ for $x \in U$; that is, h_k is continuous. This proves that h is a continuous injection.

To verify the properness of h, let $K \subset S$ be any compact set. Then each projection $q_k(K)$ is compact and so is the union $A_k = \bigcup\{\sigma \in S(\mathcal{U}_k); \sigma \cap q_k(K) \neq \varnothing\}$ for each k. Furthermore, we have

$$\text{(A.11)} \qquad \pi_k(A_{k+1}) \subset A_k;$$

in fact, for any $x \in \sigma$ such that there is $y \in K$ with $q_{k+1}(y) \in \sigma$, we have by (A.4) that $\pi_k(\sigma)$ is a simplex in $S(\mathcal{U}_k)^{\#}$ and so there is a simplex $\eta \in S(\mathcal{U}_k)$ with $\pi_k(\sigma) \subset \eta$. Hence, both $\pi_k(x)$ and $q_k(y) = \pi_k q_{k+1}(y)$ lie in η, whence $\pi_k(x) \in \eta \subset A_k$. In particular, the inverse limit $A = \varprojlim\{A_k, \pi_k\}_{k\geq 1}$ is a compact set. We will show $h^{-1}(K) \subset A$ and so the closed set $h^{-1}(K)$ is compact. The inclusion is obvious if $h^{-1}(K)$ is empty; otherwise, given $z \in h^{-1}(K)$, we have that $h(z) = L(z) \in K = \varprojlim\{q_k(K); \pi_k\}_{k\geq 1}$ (the last equality holds by [5, Corollary 2.5.7]). Then, necessarily, $\sigma_k(z) \cap q_k(K) \neq \varnothing$ and $\sigma_k(z) \subset A_k$ $(k \geq 1)$, whence $h(z) = L(z) \in A$ as claimed.

Finally, we will check that h is onto; that is, $h^{-1}(y) \neq \varnothing$ for any $y \in S$. This will show that h is a proper surjection and the proof will be complete.

Note that for $K = \{y\}$, the set A_k in (A.11) coincides with Y_k in (A.6) for each $k \geq 1$. We claim that each counterimage $B_k = h_k^{-1}(A_k)$ is not empty. Indeed, if $q_k(y) \in \sigma$ for some $\sigma \in S(\mathcal{U}_k)$, let $\sigma_0 = \langle U_1, \ldots, U_s\rangle$ be a maximal simplex in $S(\mathcal{U}_k)$ which contains $q_k(y)$, and $x \in \bigcap_{i=1}^{s} U_i$. Then, necessarily, $\sigma_k(x) = \sigma_0 \subset A_k$, and so $h_k(x) = q_k(L(x)) \in \sigma_k(x) \subset A_k$; that is, $x_k \in B_k$. Furthermore, as h and q_k are proper maps so is h_k, whence all sets B_k are compact. In addition, (A.11) implies

$$B_{k+1} = h_{k+1}^{-1}(A_{k+1}) \subset h_{k+1}^{-1}(\pi_k(A_k)) = h_k^{-1}(A_k) = B_k \ (k \geq 1)$$

and so, by compactness of B_1, we have $B = \bigcap_{i=k}^{\infty} B_k \neq \varnothing$. Given any $x \in B$, we have that $h_k(x) = q_k h(x)$ lies in the intersection $Z_k = A_k \cap \sigma_k(x)$ for every $k \geq 1$. Moreover, (A.11) and (A.5) yield $\pi_k(Z_{k+1}) \subset Z_k$. Hence, the non-empty inverse limit $\varprojlim\{Z_k, \pi_k\}_{k\geq 1}$ is contained in the singletons $h(x) = L(x) = \varprojlim\{\sigma_k(x), \pi_k\}_{k\geq 1}$ and $A = \varprojlim\{A_k; \pi_k\} = \{y\}$ (see (A.6)). Thus, $h(x) = y$, and we are done.

Finally, if $Q = \varprojlim\{\widehat{S(\mathcal{U}_k)}, \widehat{\pi}_k\}_{k\geq 1}$ is the inverse limit of the Freudenthal compactifications, it is readily checked that $S \subset Q$ and the homeomorphism h extends to a homeomorphism $\widehat{h} : \widehat{X} \to Q$ by setting $\widehat{h}(\varepsilon) = (\varepsilon^k_{j(k)})_{k\geq 1} \in Q$ where $\varepsilon \in \mathcal{F}(X)$ is the end determined by the sequence $(W^k_{j(k)})_{k\geq 1}$, and $\varepsilon^k_{j(k)} \in \mathcal{F}(S(\mathcal{U}_k))$ is the end of the ray $[k, \infty)_{j(k)} \subset S(\mathcal{U}_k)$. □

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