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Research Article

Browder's Convergence for Uniformly Asymptotically Regular Nonexpansive Semigroups in Hilbert Spaces

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We give a sufficient and necessary condition concerning a Browder's convergence type theorem for uniformly asymptotically regular one-parameter nonexpansive semigroups in Hilbert spaces.

1. Introduction

Let C be a closed convex subset of a Hilbert space E. A mapping T on C is called a *nonexpansive* mapping if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Browder, see [1], proved that $F(T)$ is nonempty provided that C is, in addition, bounded. Kirk in a very celebrated paper, see [2], extended this result to the setting of reflexive Banach spaces with normal structure.

Browder [3] initiated the investigation of an implicit method for approximating fixed points of nonexpansive self-mappings defined on a Hilbert space. Fix $u \in C$, he studied the implicit iterative algorithm

$$z_t = tu + (1 - t)Tz_t. \tag{1.1}$$

Namely, z_t, $t \in (0, 1)$, is the unique fixed point of the contraction $x \mapsto tu + (1 - t)Tx$, $x \in C$. Browder proved that $\lim_{t \to +0} z_t = Pu$, where Pu is the element of $F(T)$ nearest to u. Extensions to the framework of Banach spaces of Browder's convergence results have been done by many authors, including Reich [4], Takahashi and Ueda [5], and O'Hara et al. [6].

A family of mappings $\{T(t) : t \geq 0\}$ is called a *one-parameter strongly continuous semigroup of nonexpansive mappings* (*nonexpansive semigroup*, for short) on C if the following are satisfied.

(NS1) For each $t \geq 0$, $T(t)$ is a nonexpansive mapping on C.

(NS2) $T(s+t) = T(s) \circ T(t)$ for all $s, t \geq 0$.

(NS3) For each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is strongly continuous.

There are many papers concerning the existence of common fixed points of $\{T(t) : t \geq 0\}$; see, for instance, [7–13]. As a matter of fact, Browder [8] proved that if C is bounded, then $\bigcap_{t\geq 0} F(T(t))$ is nonempty.

Browder's type convergence theorem for nonexpansive semigroups is proved in [11, 14–18] and others. For example, the following theorem is proved in [17].

Theorem 1.1 (see [17]). *Let C be a closed convex subset of a Hilbert space E. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C such that $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$ satisfying*

(C1) $0 < \alpha_n < 1$ *and* $0 \leq t_n$;

(C2) $\lim_n t_n = \lim_n \alpha_n / t_n = 0$, *where* $1/0 = \infty$.

Fix $u \in C$ and define a sequence $\{x_n\}$ in C by

$$x_n = \alpha_n u + (1 - \alpha_n) T(t_n) x_n. \tag{1.2}$$

Then $\{x_n\}$ converges strongly to the element of $\bigcap_{t\geq 0} F(T(t))$ nearest to u.

We note that (C1) is needed to define $\{x_n\}$.

A nonexpansive semigroup $\{T(t) : t \geq 0\}$ on C is said to be *uniformly asymptotically regular* (*u.a.r.*) if for every $t \geq 0$ and for every bounded subset K of C,

$$\lim_{s\to\infty} \sup_{x\in K} \|T(s+t)x - T(s)x\| = 0 \tag{1.3}$$

holds. The following is proved by Domínguez Benavides et al. [16]; see also [15].

Theorem 1.2 (see [16]). *Let E, C, and $\{T(t) : t \geq 0\}$ be as in Theorem 1.1. Assume that $\{T(t) : t \geq 0\}$ is u.a.r. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$ satisfying (C1) and*

(D2) $\lim_n \alpha_n = 0$ *and* $\lim_n t_n = \infty$.

Fix $u \in C$ and define a sequence $\{x_n\}$ in C by (1.2). *Then $\{x_n\}$ converges strongly to the element of $\bigcap_{t\geq 0} F(T(t))$ nearest to u.*

There is an interesting difference between Theorems 1.1 and 1.2, that is, $\{t_n\}$ in Theorem 1.1 converges to 0 and $\{t_n\}$ in Theorem 1.2 diverges to ∞. By the way, very recently, Akiyama and Suzuki [14] generalized Theorem 1.1. They replaced (C2) of Theorem 1.1 by the following:

(C2′) $\{t_n\}$ is bounded;

(C3′) $\lim_n \alpha_n/(t_n - \tau) = 0$ for all $\tau \in [0,\infty)$.

They also showed that the conjunction of (C2′) and (C3′) is best possible; see also [18].

In this paper, motivated by the previous considerations, we generalize Theorem 1.2 concerning $\{\alpha_n\}$ and $\{t_n\}$. Also, we will show that our new condition is best possible.

2. Main Results

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. For $t \in \mathbb{R}$, we denote by $[t]$ the maximum integer not exceeding t.

The following proposition plays an important role in this paper.

Proposition 2.1. *Let C be a set of a separated topological vector space E. Let $\{T(t) : t \geq 0\}$ be a family of mappings on C such that $T(s) \circ T(t) = T(s+t)$ for all $s, t \in [0,\infty)$. Assume that $\{T(t) : t \geq 0\}$ is asymptotic regular, that is,*

$$\lim_{s \to \infty} (T(t+s)x - T(s)x) = 0 \tag{2.1}$$

for all $t \in [0,\infty)$ and $x \in C$. Then

$$F(T(t)) = \bigcap_{s \geq 0} F(T(s)) \tag{2.2}$$

holds for all $t \in (0,\infty)$.

Proof. Fix $t \in (0,\infty)$. It is obvious that $F(T(t)) \supset \bigcap_s F(T(s))$ holds. Let $z \in C$ be a fixed point of $T(t)$. For every $h \in [0,\infty)$, we have

$$\begin{aligned} T(h)z - z &= \lim_{n \to \infty} \left(T(h) \circ T(t)^n z - T(t)^n z\right) \\ &= \lim_{n \to \infty} (T(h+nt)z - T(nt)z) \\ &= \lim_{s \to \infty} (T(h+s)z - T(s)z) \\ &= 0, \end{aligned} \tag{2.3}$$

and hence z is a common fixed point of $\{T(t) : t \geq 0\}$. □

It is well known that every Hilbert space has the Opial property.

Proposition 2.2 (Opial [19]). *Let E be a Hilbert space. Let $\{x_n\}$ be a sequence in E converging weakly to $z_0 \in H$. Then the inequality $\liminf_n \|x_n - z\| \leq \liminf_n \|x_n - z_0\|$ implies $z = z_0$.*

We generalize Theorem 1.2.

Theorem 2.3. *Let C be a closed convex subset of a Hilbert space E. Let $\{T(t) : t \geq 0\}$ be a u.a.r. nonexpansive semigroup on C such that $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$ satisfying (C1) and*

(D2′) $\lim_n \alpha_n = \lim_n \alpha_n / t_n = 0$.

Fix $u \in C$ and define a sequence $\{x_n\}$ in C by (1.2). *Then $\{x_n\}$ converges strongly to the element of $\bigcap_{t\geq 0} F(T(t))$ nearest to u.*

Proof. Put $F(\mathfrak{T}) = \bigcap_{t\geq 0} F(T(t))$. Let v be the element of $F(\mathfrak{T})$ nearest to u. Since

$$\begin{aligned}\|x_n - v\| &= \|(1-\alpha_n)T(t_n)x_n + \alpha_n u - v\| \\ &\leq (1-\alpha_n)\|T(t_n)x_n - v\| + \alpha_n\|u - v\| \\ &\leq (1-\alpha_n)\|x_n - v\| + \alpha_n\|u - v\|,\end{aligned} \tag{2.4}$$

we have $\|x_n - v\| \leq \|u - v\|$. Therefore $\{x_n\}$ is bounded. Hence $\{T(t)x_n : n \in \mathbb{N},\ t \geq 0\}$ is also bounded.

We put

$$M := \sup\{\|T(t)x_n - u\| : n \in \mathbb{N},\ t \geq 0\} < \infty. \tag{2.5}$$

Let $\{f(n)\}$ be an arbitrary subsequence of $\{n\}$. Then there exists a subsequence $\{g(n)\}$ of $\{n\}$ such that $\{x_{f\circ g(n)}\}$ converges weakly to x. We choose a subsequence $\{h(n)\}$ of $\{n\}$ such that

$$\tau := \lim_{n\to\infty} t_{f\circ g\circ h(n)} = \limsup_{n\to\infty} t_{f\circ g(n)}. \tag{2.6}$$

Put $y_j = x_{f\circ g\circ h(j)}$, $\beta_j = \alpha_{f\circ g\circ h(j)}$, and $s_j = t_{f\circ g\circ h(j)}$. We will show $x \in F(\mathfrak{T})$, dividing the following three cases:

(i) $\tau = \infty$,

(ii) $0 < \tau < \infty$,

(iii) $\tau = 0$.

In the first case, we fix $t \geq 0$. For sufficiently large $j \in \mathbb{N}$, we have

$$\begin{aligned}\|T(t)x - y_j\| &\leq \|T(t)x - T(t)y_j\| + \|T(t)y_j - y_j\| \\ &\leq \|x - y_j\| + \beta_j\|T(t)y_j - u\| + (1-\beta_j)\|T(t)y_j - T(s_j)y_j\| \\ &\leq \|x - y_j\| + \beta_j M + (1-\beta_j)\|T(s_j - t)y_j - y_j\| \\ &\leq \|x - y_j\| + \beta_j M + (1-\beta_j)\beta_j\|T(s_j - t)y_j - u\| + (1-\beta_j)^2\|T(s_j - t)y_j - T(s_j)y_j\| \\ &\leq \|x - y_j\| + \beta_j(2-\beta_j)M + (1-\beta_j)^2\|T(s_j - t + t)y_j - T(s_j - t)y_j\|,\end{aligned} \tag{2.7}$$

and hence

$$\liminf_{j\to\infty}\left\|T(t)x - y_j\right\| \le \liminf_{j\to\infty}\left\|x - y_j\right\|. \tag{2.8}$$

By the Opial property, we obtain $T(t)x = x$. Thus $x \in F(\mathcal{T})$.

In the second case, we have

$$\begin{aligned} \|T(\tau)x - y_j\| &\le \|T(\tau)x - T(s_j)x\| + \|T(s_j)x - T(s_j)y_j\| + \|T(s_j)y_j - y_j\| \\ &\le \|T(\tau)x - T(s_j)x\| + \|x - y_j\| + \beta_j\|T(s_j)y_j - u\| \\ &\le \|T(|\tau - s_j|)x - T(0)x\| + \|x - y_j\| + \beta_j M, \end{aligned} \tag{2.9}$$

and hence

$$\liminf_{j\to\infty}\left\|T(\tau)x - y_j\right\| \le \liminf_{j\to\infty}\left\|x - y_j\right\|. \tag{2.10}$$

By the Opial property, we obtain $T(\tau)x = x$. By Proposition 2.1, we obtain $x \in F(\mathcal{T})$.

In the third case, we fix $t \ge 0$. For sufficiently large $j \in \mathbb{N}$, we have

$$\begin{aligned} \|T(t)x - y_j\| &\le \|T(t)x - T([t/s_j]s_j)x\| + \|T([t/s_j]s_j)x - T([t/s_j]s_j)y_j\| \\ &\quad + \sum_{k=0}^{[t/s_j]-1}\|T(ks_j)y_j - T((k+1)s_j)y_j\| + \|T(0)y_j - y_j\| \\ &\le \|T(t - [t/s_j]s_j)x - T(0)x\| + \|x - y_j\| \\ &\quad + [t/s_j]\|T(s_j)y_j - y_j\| + \|T(0)y_j - T(s_j)y_j\| + \|T(s_j)y_j - y_j\| \\ &\le \|T(t - [t/s_j]s_j)x - T(0)x\| + \|x - y_j\| \\ &\quad + [t/s_j]\|T(s_j)y_j - y_j\| + \|y_j - T(s_j)y_j\| + \|T(s_j)y_j - y_j\| \\ &= \|T(t - [t/s_j]s_j)x - T(0)x\| + \|x - y_j\| + ([t/s_j] + 2)\|T(s_j)y_j - y_j\| \\ &= \|T(t - [t/s_j]s_j)x - T(0)x\| + \|x - y_j\| + ([t/s_j] + 2)\beta_j\|T(s_j)y_j - u\| \\ &\le \max\{\|T(s)x - T(0)x\| : 0 \le s \le s_j\} + \|x - y_j\| + (t\beta_j/s_j + 2\beta_j)M. \end{aligned} \tag{2.11}$$

Hence (2.8) holds. Thus we obtain $x \in F(\mathcal{T})$.

We next prove that $\{y_j\}$ converges strongly to v. Since

$$\begin{aligned} &\beta_j\left\|y_j - v\right\|^2 + (1 - \beta_j)\langle (y_j - T(s_j)y_j) - (v - T(s_j)v), y_j - v\rangle \\ &\qquad = \beta_j\langle u - v, y_j - v\rangle, \\ &\langle (y_j - T(s_j)y_j) - (v - T(s_j)v), y_j - v\rangle \\ &\qquad \ge \|y_j - v\|^2 - \|T(s_j)y_j - T(s_j)v\|\|y_j - v\| \ge 0, \end{aligned} \tag{2.12}$$

we obtain $\|y_j - v\|^2 \le \langle u - v, y_j - v\rangle$. Since $\langle u - v, x - v\rangle \le 0$, we have

$$\begin{aligned}\left\|y_j - v\right\|^2 &\le \langle u - v, y_j - v\rangle \\ &= \langle u - v, y_j - x\rangle + \langle u - v, x - v\rangle \\ &\le \langle u - v, y_j - x\rangle,\end{aligned} \tag{2.13}$$

and hence $\{y_j\}$ converges strongly to v. Since $\{x_{f(n)}\}$ is arbitrary, we obtain that $\{x_n\}$ converges strongly to v. □

Using [20, Theorem 7], we obtain the following Moudafi's type convergence theorem; see [21].

Corollary 2.4. *Let E, C, $\{T(t) : t \ge 0\}$, $\{\alpha_n\}$, and $\{t_n\}$ be as in Theorem 2.3. Let Φ be a contraction on C; that is, there exists $r \in [0,1)$ such that $\|\Phi x - \Phi y\| \le r\|x - y\|$ for $x, y \in C$. Define a sequence $\{x_n\}$ in C by*

$$x_n = \alpha_n \Phi x_n + (1 - \alpha_n) T(t_n) x_n. \tag{2.14}$$

Then $\{x_n\}$ converges strongly to the unique point $z \in C$ satisfying $P \circ \Phi z = z$, where P is the metric projection from C onto $\bigcap_{t\ge 0} F(T(t))$.

We will show that (D2′) is best possible.

Example 2.5. Put $E = \ell^2(\mathbb{N})$, that is, E is a Hilbert space consisting of all the functions x from $\mathbb{N}$ into $\mathbb{R}$ satisfying $\sum_{k\in\mathbb{N}} |x(k)|^2 < \infty$ with inner product $\langle x, y\rangle = \sum_{k\in\mathbb{N}} x(k)y(k)$. Define a bounded closed convex subset C of E by

$$C = \left\{x \in E : 0 \le x(k) \le p_k\right\}, \tag{2.15}$$

where $p_k = 2^{-k/2}$. Define a u.a.r. nonexpansive semigroup $\{T(t) : t \ge 0\}$ on C by

$$(T(t)x)(k) = \max\left\{x(k) - t{p_k}^2, 0\right\}. \tag{2.16}$$

Let $\{e_k\}$ be the canonical basis of E and put $u = \sum_{k=1}^{\infty} p_k e_k$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$ satisfying (C1) and define $\{x_n\}$ in C by (1.2). Then $\{x_n\}$ converges to a common fixed point of $\{T(t) : t \ge 0\}$ only if $\lim_n \alpha_n = \lim_n \alpha_n / t_n = 0$.

Proof. For $\alpha \in (0,1)$ and $t \ge 0$, we define $x(\alpha, t)$ by

$$x(\alpha, t) = \alpha u + (1 - \alpha) T(t) x(\alpha, t). \tag{2.17}$$

We note

$$x(\alpha,t)(k) = \begin{cases} \alpha p_k, & \text{if } \alpha \le t p_k, \\ \left(1 + t p_k - \dfrac{t p_k}{\alpha}\right) p_k, & \text{if } \alpha \ge t p_k. \end{cases} \tag{2.18}$$

So, $x(\alpha,t)(k) \ge \alpha p_k$. It is obvious that $\bigcap_{t\ge 0} F(T(t)) = \{0\}$. We assume $\lim_n x_n = \lim_n x(\alpha_n, t_n) = Pu = 0$. Then

$$0 = \lim_{n\to\infty} \frac{x_n(1)}{p_1} \ge \lim_{n\to\infty} \alpha_n. \tag{2.19}$$

Arguing by contradiction, we assume $\limsup_n \alpha_n / t_n > 0$. Then there exist $\kappa \in \mathbb{N}$ and a subsequence $\{f(n)\}$ of $\{n\}$ such that

$$\frac{\alpha_{f(n)}}{t_{f(n)}} \ge 2 p_\kappa. \tag{2.20}$$

Since $\lim_n x_{f(n)}(\kappa) = 0$, we have

$$\begin{aligned} 0 = \lim_{n\to\infty} \frac{x_{f(n)}(\kappa)}{p_\kappa} &= \lim_{n\to\infty} \left(1 + t_{f(n)} p_\kappa - \frac{t_{f(n)} p_\kappa}{\alpha_{f(n)}}\right) \\ &\ge \limsup_{n\to\infty} \left(1 - \frac{t_{f(n)} p_\kappa}{\alpha_{f(n)}}\right) \ge \frac{1}{2} > 0, \end{aligned} \tag{2.21}$$

which is a contradiction. Therefore we obtain $\lim_n \alpha_n / t_n = 0$. □

By Theorem 2.3 and Example 2.5, we obtain the following.

Theorem 2.6. *Let E be an infinite-dimensional Hilbert space. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$ satisfying (C1). Then the following are equivalent:*

(i) $\lim_n \alpha_n = \lim_n \alpha_n / t_n = 0$,

(ii) *if C is a bounded closed convex subset C of E, $\{T(t) : t \ge 0\}$ is a u.a.r. nonexpansive semigroup on C, $u \in C$, and $\{x_n\}$ is a sequence in C defined by* (1.2), *then $\{x_n\}$ converges strongly to the element of $\bigcap_{t\ge 0} F(T(t))$ nearest to u.*

Compare (D2′) with the conjunction of (C2′) and (C3′). We can tell that the difference between both conditions is u.a.r.

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