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A GENERALIZATION OF THE KALMAN RANK CONDITION FOR TIME–DEPENDENT COUPLED LINEAR PARABOLIC SYSTEMS

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Abstract. In this paper we present a generalization of the Kalman rank condition for linear ordinary differential systems to the case of systems of n coupled parabolic equations (posed in the time interval $(0,T)$ with $T>0$) where the coupling matrices A and B depend on the time variable t. To be precise, we will prove that the Kalman rank condition $\operatorname{rank}[A|B](t_0)=n$, with $t_0\in[0,T]$, is a sufficient condition (but not necessary) for obtaining the exact controllability to the trajectories of the considered parabolic system. In the case of analytic matrices A and B (and, in particular, constant matrices), we will see that the Kalman rank condition characterizes the controllability properties of the system. When the matrices A and B are constant and condition $\operatorname{rank}[A\,|\,B]=n$ holds, we will be able to state a Carleman inequality for the corresponding adjoint problem.

1. Statement of the problem. Main results

Let $\Omega\subset\mathbb{R}^N$ be a bounded connected open set with boundary $\partial\Omega$ of class C^2. Let $\omega\subset\Omega$ be a nonempty open subset and assume $T>0$. Let us consider the time-dependent second order elliptic operator

$$L(t)y(x,t)=-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y}{\partial x_j}(x,t)\right)+\sum_{i=1}^{N}b_i(x,t)\frac{\partial y}{\partial x_i}(x,t)$$
$$+c(x,t)y(x,t), \tag{1}$$

where

$$\begin{cases} a_{ij}\in W^{1,\infty}(Q), \quad b_i,c\in L^\infty(Q), \quad 1\leqslant i,j\leqslant N, \quad (Q=\Omega\times(0,T)),\\ a_{ij}(x,t)=a_{ji}(x,t)\ \text{a.e. in}\ Q, \end{cases} \tag{2}$$

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and the coefficients a_{ij} satisfy the uniform elliptic condition

$$\sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \geqslant a_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad \text{a.e. in } Q, \tag{3}$$

for a positive constant a_0.

For $n \geqslant 2$ and $m \geqslant 1$ given, we consider the linear parabolic system

$$\begin{cases} \partial_t y + L(t)y = A(t)y + B(t)v1_\omega \text{ in } Q = \Omega \times (0,T), \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T), \end{cases} \tag{4}$$

where

$$A(\cdot) \in C^{M-1}([0,T];\mathcal{L}(\mathbb{R}^n)) \text{ and } B(\cdot) \in C^M([0,T];\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)) \tag{5}$$

for an integer $M \geqslant n$. In (4), $y = (y_i)_{1\leqslant i\leqslant n}$ is the state, $v \in L^2(Q)^m$ is the control and 1_ω denotes the characteristic function of the open subset ω. Let us observe that, for every $y_0 \in L^2(\Omega)^n$ and $v \in L^2(Q)^m$, system (4) admits a unique solution $y \in L^2(0,T;H_0^1(\Omega)^n) \cap C^0([0,T];L^2(\Omega)^n)$ which satisfies $y(\cdot,0) = y_0$ in Ω.

The main goal of this paper is to analyze the controllability properties of system (4) when m distributed control forces are exerted on the system.

Let us fix $T_0, T_1 \in [0,T]$ with $T_0 < T_1$. It will be said that (4) is approximately controllable in $L^2(\Omega)^n$ on the time interval (T_0,T_1) if, for any $y_0, y_d \in L^2(\Omega)^n$ and any $\varepsilon > 0$, there exists a control function $v \in L^2(Q)^m$ such that the solution $y \in C^0([T_0,T_1];L^2(\Omega)^n)$ to (4) corresponding to the initial condition $y(\cdot,T_0) = y_0$ satisfies

$$||y(\cdot,T_1) - y_d||_{L^2} \leqslant \varepsilon.$$

On the other hand, it will be said that (4) is exactly controllable to the trajectories on (T_0,T_1) if for every trajectory $y^* \in C^0([0,T];L^2(\Omega)^n)$ of (4) (i.e., a solution to (4) corresponding to $v \equiv 0$) and $y_0 \in L^2(\Omega)^n$, there exists a control $v \in L^2(Q)^m$ such that the solution y to (4) corresponding to the initial condition $y(\cdot,T_0) = y_0$ satisfies

$$y(\cdot,T_1) = y^*(\cdot,T_1) \quad \text{in} \quad \Omega.$$

It will be said that system (4) is null controllable on the interval (T_0,T_1) when the previous condition is satisfied for the trajectory $y^* \equiv 0$. Let us remark that for linear systems as (4), the exact controllability to the trajectories and the null controllability are equivalent concepts.

There are few results on null controllability of system (4) when $n > 1$ and most of them are proved for $n = 2$ and $B = (1,0)^*$ $(m = 1)$. In [18], [5] and [13] the authors consider linear and nonlinear systems of two heat equations, one of them being forward and the other one backward in time, and show the null controllability of the considered systems. In [1] and [2], the authors give a null controllability result for a phase-field system and for reaction-diffusion systems (two nonlinear heat equations) when one distributed control force is exerted on the system. The results in [1] and [2] have been generalized in [11] in two directions: on the one hand, there are not restrictions on the dimension N, and on the other hand, the authors consider nonlinearities which depend on the gradient of the state. When $n>2$, we point out [12], [16], [3] and [4]. In [12] the authors provide a null controllability result for a general cascade parabolic system. In [16] the author gives a necessary and sufficient condition for the approximate controllability of a parabolic system with diagonalizable diffusion matrix. Finally, in [3] and [4] it is also proved a necessary and sufficient condition for the exact controllability to the trajectories of the autonomous system

$$\begin{cases} \partial_t y - DRy = Ay + Bv1_\omega \text{ in } Q = \Omega \times (0,T), \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T), \quad y(\cdot,0) = y_0 \text{ in } \Omega, \end{cases} \tag{6}$$

where R is a self adjoint elliptic second order operator given by

$$Ry(x) = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(r_{ij}(x) \frac{\partial y}{\partial x_j}(x) \right) + c(x)y(x),$$

with $r_{ij} \in W^{1,\infty}(\Omega)$, $\sum_{i,j} r_{ij}(x)\xi_i\xi_j \geqslant \widetilde{a}_0|\xi|^2$, for every $\xi \in \mathbb{R}^N$ ($\widetilde{a}_0 > 0$), and $r_{ij} = r_{ji}$ in Ω ($1 \leqslant i,j \leqslant N$), $c \in L^\infty(\Omega)$, $D = P^{-1}\mathrm{diag}\,(d_1,\dots,d_n)P$ (with $\det P \neq 0$ and $d_i > 0$, for every i, $1 \leqslant i \leqslant n$) and $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$ are constant matrices.

The controllability properties of ordinary differential systems are, nowadays, well known. To be precise, let us consider the system

$$x' = A(t)x + B(t)u \quad \text{on } (0,T), \tag{7}$$

where $A \in C^{n-2}([0,T];\mathcal{L}(\mathbb{R}^n))$ and $B \in C^{n-1}([0,T];\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n))$ are given and u is a control. Let us define

$$\begin{cases} B_0(t) = B(t), \\ B_i(t) = A(t)B_{i-1}(t) - \dfrac{d}{dt}B_{i-1}(t), \end{cases} \tag{8}$$

($1 \leqslant i \leqslant n-1$) and denote by $[A\,|\,B] \in C^0([0,T];\mathcal{L}(\mathbb{R}^{nm};\mathbb{R}^n))$ the matrix function given by:

$$[A\,|\,B](t) = (B_0(t)\,|\,B_1(t)\,|\,\cdots\,|\,B_{n-1}(t)).$$

Let us remark that when A and B are constant matrices, $[A\,|\,B] \in \mathcal{L}(\mathbb{R}^{nm},\mathbb{R}^n)$ is the matrix given by

$$[A\,|\,B] = \left(B\,|AB\,|A^2B\,|\cdots|A^{n-1}B \right). \tag{9}$$

With this notation, one has the following result.

THEOREM 1.1. (Silverman-Meadows [17]). *Under the previous assumptions, the following holds:*

1. *If there exists $t_0 \in [0,T]$ such that* rank $[A\,|\,B](t_0) = n$, *then system* (7) *is completely controllable on $(0,T)$, i.e., for every $x_0, x_d \in \mathbb{R}^N$ there exists a control $u \in L^2(0,T)^m$ such that the solution $x(\cdot)$ to* (7) *corresponding to the initial condition $x(0) = x_0$ satisfies $x(T) = x_d$.*

2. *System* (7) *is totally controllable on* $(0,T)$, *i.e., system* (7) *is completely controllable on every subinterval* (T_0,T_1) *of* $(0,T)$ $(0 \leqslant T_0 < T_1 \leqslant T)$ *if and only if there exists* E, *a dense subset of* $(0,T)$, *such that* $\operatorname{rank}[A\,|\,B](t) = n$ *for every* $t \in E$.

In the particular case in which A and B are constant matrices, the concepts of complete and total controllability on $(0,T)$ for system (7) coincide. Thus, the exact controllability of system (7) is equivalent to the well-known Kalman's rank condition (e.g., see [15])

$$\operatorname{rank}[A\,|\,B] = n. \tag{10}$$

The objective of the present paper is to extend the controllability results stated in Theorem 1.1 for ordinary differential systems to the case in which the parabolic system (4) is considered. On the other hand, all along this paper we will assume that the operator $L(t)$ is given by (1) and satisfies (2)-(3).

Under assumption (5), if $1 \leqslant p \leqslant M$, we can define

$$K_p(t) = (B_0(t)\,|\,B_1(t)\,|\cdots|\,B_{p-1}(t)) \in C^1([0,T];\mathcal{L}(\mathbb{R}^{mp};\mathbb{R}^n)), \tag{11}$$

where $B_i(t)$ is given by (8), for $0 \leqslant i \leqslant p-1$. With this notation, the first result of our work reads as follows.

THEOREM 1.2. *Assume that the matrices A and B satisfy* (5). *Then, the following holds:*

1. *If there exist* $t_0 \in [0,T]$ *and* $p \in \{1,...,M\}$ *such that*

$$\operatorname{rank} K_p(t_0) = n, \tag{12}$$

then system (4) *is exactly controllable to the trajectories on the time interval* $(0,T)$.

2. *System* (4) *is exactly controllable to the trajectories on every interval* (T_0,T_1) *with* $0 \leqslant T_0 < T_1 \leqslant T$ *if and only if there exists* E *a dense subset of* $(0,T)$ *such that* $\operatorname{rank}[A\,|\,B](t) = n$ *for every* $t \in E$, (*or equivalently,* $\operatorname{rank} K_p(t) = n$ *for all* $p \in \{n,...,M\}$ *and* $t \in E$).

Actually, we will show that, under assumption (12), if we write

$$B(t) = \left(b^1(t)\,|\,b^2(t)\,|\cdots|\,b^m(t)\right), \quad b^i(\cdot) \in C^M([0,T])^n, \quad 1 \leqslant i \leqslant m,$$

then, there exist $r \in \{1,...,m\}$ (the total number of *effective controls*), a set $\{b^{l_i} : 1 \leqslant i \leqslant r\}$ (the *effective controls*) and a non degenerate closed interval $[T_0,T_1] \subseteq [0,T]$ such that system (4) is exactly controllable to the trajectories on the time interval (T_0,T_1) (and therefore, on $(0,T)$) when we exert on the system the r control forces

$$\widetilde{B}(t) = (0\,|\cdots|\,0\,|\,b^{l_1}(t)\,|\,0\,|\cdots|\,0\,|\,b^{l_2}(t)\,|\,0\,|\cdots|\,0\,|\,b^{l_r}(t)\,|\,0\,|\cdots|\,0).$$

In fact, we will see that, on the time interval (T_0,T_1), system (4) is equivalent to a cascade parabolic system. We will deduce the null controllability properties of (4) from the corresponding results for cascade parabolic systems stated in [12].

REMARK 1.1. Assume that A and B satisfy (5) and let us fix an integer $p \in \{n,..,M\}$. At first sight it could seem that condition (12) is weaker that the following condition: there exists $t_1 \in [0,T]$ such that $\operatorname{rank} K_n(t_1) = n$. But, in fact, we will see that both conditions are equivalent. In Section 4 (see Corollary 4.2) it will be seen,

$$\max_{t\in[0,T]} \operatorname{rank} K_n(t) = \max_{t\in[0,T]} \operatorname{rank} K_p(t),$$

for every $p \geqslant n$.

A similar result has been proved by J.-M. Coron in [6] (see Proposition 1.19, p. 11): *"If $A(\cdot) \in C^\infty([0,T];\mathcal{L}(\mathbb{R}^n))$ and $B(\cdot) \in C^\infty([0,T];\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n))$ and there exists $t_0 \in [0,T]$ such that* (12) *holds, then there exists $\varepsilon > 0$ such that* $\operatorname{rank} K_n(t) = n$ *for every $t \in ([0,T] \cap (t_0-\varepsilon, t_0+\varepsilon)) \setminus \{t_0\}$ ".*

As in the case of finite-dimensional linear systems, it is interesting to point out that the existence of $t_0 \in [0,T]$ satisfying (12) is not a necessary condition to have the exact controllability to the trajectories on $(0,T)$ of system (4). Following [6], we will give an example of matrices $A \in C^\infty([0,T];\mathcal{L}(\mathbb{R}^n))$ and $B \in C^\infty([0,T];\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n))$ such that

$$\operatorname{rank} K_p(t) < n, \quad \forall p \geqslant 1 \text{ and } \forall t \in [0,T],$$

and system (4) is exactly controllable to the trajectories on $(0,T)$ (see Section 5). Nevertheless, when A and B are analytic functions on $[0,T]$, we will show that condition (12) is a necessary and sufficient condition for the exact controllability to trajectories of system (4). One has the following result.

THEOREM 1.3. *Let us suppose that A and B are analytic on $[0,T]$. Then, system* (4) *is exactly controllable to the trajectories on $(0,T)$ if and only if there exist $t_0 \in [0,T]$ and an integer p such that* (12) *holds.*

As in the case of system (7) and as a consequence of Theorem 1.2 (or Theorem 1.3), when A and B are constant matrices we obtain that the Kalman rank condition (10) is a necessary and sufficient condition for the exact controllability to the trajectories of system (4) on $(0,T)$. Thus, one has the following result.

THEOREM 1.4. *Assume $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$. Then, system* (4) *is exactly controllable to the trajectories on $(0,T)$ if and only if* (10) *holds.*

Let us remark that a similar result to Theorem 1.4 has been already proved in [3] and [4] for system (6). To be precise, in [3] and [4], a necessary and sufficient condition for the null controllability on the time interval $(0,T)$ of the system (6) is stated ($D = P^{-1}\operatorname{diag}(d_1,....,d_n)P$, $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$ are constant matrices and $\det P \neq 0$ and $d_i > 0$, for every i, $1 \leqslant i \leqslant n$). When $D \equiv Id$ we obtain system (4) for $L(t) \equiv -R$ and the condition given in [3] is equivalent to (10).

So as to state our next result, let us consider a trajectory of system (4) $y^* \in L^2(0,T;H_0^1(\Omega)^n) \cap C^0([0,T];L^2(\Omega)^n)$. Thus, it is possible to characterize the initial data y_0 that can be exactly driven to $y^*(\cdot,T)$ when A and B are constant matrices and condition (10) is not fulfilled. One has the following result.

THEOREM 1.5. *Assume* $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ *and* $\operatorname{rank}[A\,|\,B] = \ell < n$. *Let* $X \subset \mathbb{R}^n$ *be the linear space generated by the columns of* $[A\,|\,B]$. *Then, given* $y_0 \in L^2(\Omega)^n$ *and* $y^* \in L^2(0,T;H_0^1(\Omega)^n) \cap C^0([0,T];L^2(\Omega)^n)$, *a trajectory of* (4), *there exists a control* $v \in L^2(Q)^m$ *such that the solution to* (4) *corresponding to the initial condition* $y(\cdot,0) = y_0$ *in* Ω *satisfies*

$$y(\cdot,T) = y^*(\cdot,T) \text{ in } \Omega,$$

if and only if $y_0 - y^*(\cdot,0) \in L^2(\Omega;X)$.

In order to study the controllability properties of system (4), we will consider the corresponding adjoint problem

$$\begin{cases} -\partial_t \varphi + L^*(t)\varphi = A^*(t)\varphi + F_0 + \nabla \cdot F \text{ in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot,T) = \varphi_0 \text{ in } \Omega, \end{cases} \tag{13}$$

where $F_0 = (F_0^1,\dots,F_0^n)^* \in L^2(Q)^n$, $F = (F^1\,|\,F^2\,|\dots|\,F^n) \in L^2(Q;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^N))$, $\varphi_0 \in L^2(\Omega)^n$ and where, by means of $\nabla \cdot F$, we are denoting the column vector $\nabla \cdot F = (\nabla \cdot F^1, \nabla \cdot F^2, \dots, \nabla \cdot F^n)^*$. The operator $L^*(t)$, the adjoint operator to $L(t)$, is given by

$$\begin{aligned} L^*(t)y(x,t) = &-\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial y}{\partial x_j}(x,t)\Big) - \sum_{i=1}^N \frac{\partial (b_i y)}{\partial x_i}(x,t) \\ &+ c(x,t)y(x,t). \end{aligned}$$

It is by now well known that the exact controllability on $(0,T)$ to the trajectories of system (4) is equivalent to the existence of a positive constant C such that, for every $\varphi_0 \in L^2(\Omega)^n$, the solution $\varphi \in C^0([0,T];L^2(\Omega)^n)$ to the adjoint system (13) (corresponding to $F_0 = 0$ and $F = 0$) satisfies the *observability inequality*

$$||\varphi(\cdot,0)||^2_{L^2(\Omega)} \leqslant C \iint_{\omega\times(0,T)} |B^*(t)\varphi(x,t)|^2, \tag{14}$$

where we have omitted, as we do along all this paper when no confusion is possible, the Lebesgue measure $dxdt$. Under hypothesis (12), we will prove inequality (14) as a consequence of a global Carleman inequality for the solutions to (13) in the effective control interval (T_0,T_1). In the particular case in which A and B are constant matrices we will see that $(T_0,T_1) \equiv (0,T)$ and will obtain

THEOREM 1.6. *There exist a positive function* $\alpha_0 \in C^2(\overline{\Omega})$ (*only depending on* Ω *and* ω) *such that, if* $A \in \mathcal{L}(\mathbb{R}^n)$ *and* $B \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$ *satisfy* (10), *then there exist two positive constants* C_0 *and* σ_0 (*only depending on* Ω, ω, $(a_{ij})_{1\leqslant i,j\leqslant N}$, n, m, A *and* B) *and integers* $\ell \geqslant 3$, $\ell^1 \geqslant 0$ *and* $\ell^2 \geqslant 2$ (*only depending on* n, m, A *and* B) *such that, for every* $\varphi_0 \in L^2(Q)^n$, *the solution* φ *to* (13) *satisfies*

$$\begin{aligned} \mathcal{I}(3,\varphi) \leqslant \widetilde{C}_1 \Big(s^{\ell} &\iint_{\omega\times(0,T)} e^{-2s\alpha}\gamma(t)^{\ell}|B^*\varphi|^2 \\ &+ s^{\ell^1}\iint_Q e^{-2s\alpha}\gamma(t)^{\ell^1}|F_0|^2 + s^{\ell^2}\iint_Q e^{-2s\alpha}\gamma(t)^{\ell^2}|F|^2\Big), \end{aligned} \tag{15}$$

$\forall s \geqslant s_0 = \sigma_0 \left(T + T^2 + T^2 ||c||_\infty^{2/3} + T^2 ||b||_\infty^2 \right)$. *In inequality* (15), $\alpha(x,t)$, $\gamma(t)$ *and* $\mathfrak{I}(d,z)$ *are respectively given by:* $\alpha(x,t) \equiv \alpha_0(x)/t(T-t)$, $\gamma(t) \equiv (t(T-t))^{-1}$ *and*

$$\mathfrak{I}(d,z) \equiv s^{d-2} \iint_Q e^{-2s\alpha} \gamma(t)^{d-2} |\nabla z|^2 + s^d \iint_Q e^{-2s\alpha} \gamma(t)^d |z|^2.$$

We will prove Theorem 1.6 from the corresponding global Carleman inequality satisfied by the solutions to scalar parabolic equations with a right hand side in $L^2(0,T;H^{-1}(\Omega))$ (see e.g. [9] and [14]).

The rest of the work is organized as follows. In Section 2 we will recall some known results on controllability of cascade parabolic systems which will be used later. Section 3 will be devoted to proving Theorems 1.4, 1.5 and 1.6, i.e., we will study in this section the case of constant matrices $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. We will deal with the general case in Section 4. We will devote Section 5 to give some remarks and additional results. We will finalize the work with an appendix where we will give a sketch of the proofs of the results stated in Section 2.

2. Some known results

As said above, we will obtain the proofs of Theorems 1.2, 1.4 and 1.6 as a consequence of the results on controllability for cascade systems stated in [12]. For the sake of completeness, let us recall these results in the precise way they are going to be used.

Let us consider the controlled system

$$\begin{cases} \partial_t w + L(t) w = C(t) w + D u 1_\omega \text{ in } Q, \\ w = 0 \text{ on } \Sigma, \quad w(\cdot,0) = w_0 \text{ in } \Omega, \end{cases} \tag{16}$$

where $u = (u_1, u_2, \ldots, u_r)^*$, with $1 \leqslant r \leqslant n$, and the coupling and control matrices C and D satisfy: $C \in C^0([0,T]; \mathcal{L}(\mathbb{R}^n))$ is given by

$$C(t) = \begin{pmatrix} C_{11}(t) & C_{12}(t) & \cdots & C_{1r}(t) \\ 0 & C_{22}(t) & \cdots & C_{2r}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{rr}(t) \end{pmatrix}, \tag{17}$$

with

$$C_{ii}(t) = \begin{pmatrix} \alpha_{11}^i(t) & \alpha_{12}^i(t) & \alpha_{13}^i(t) & \ldots & \alpha_{1,s_i}^i(t) \\ 1 & \alpha_{22}(t) & \alpha_{23}^i(t) & \ldots & \alpha_{2,s_i}^i(t) \\ 0 & 1 & \alpha_{33}^i(t) & \ldots & \alpha_{3,s_i}^i(t) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & \alpha_{s_i,s_i}^i(t) \end{pmatrix},$$

$s_i \in \mathbb{N}$, $\sum_{i=1}^r s_i = n$, and $D \in \mathcal{L}(\mathbb{R}^r, \mathbb{R}^n)$ such that $D = (e_{S_1} | e_{S_2} | \cdots | e_{S_r})$ with $S_i = 1 + \sum_{j=1}^{i-1} s_j$, $1 \leqslant i \leqslant r$ (e_j is the j-th element of the canonical basis of $\mathbb{R}^n$). Observe that (16) is a cascade system and, by means of D, we are exerting r distributed controls. Therefore, we can apply the controllability results stated in [12] and obtain

THEOREM 2.1. *Under the previous assumptions, given* $w_0 \in L^2(\Omega)^n$*, there exists a control* $u \in L^2(Q)^r$ *such that the corresponding solution* w *to* (16) *satisfies* $w(x,T) = 0$ *in* Ω.

This controllability result is a consequence of an appropriate global Carleman inequality satisfied by the solutions to the adjoint problem to (16):

$$\begin{cases} -\partial_t \psi + L^*(t)\psi = C^*(t)\psi + G_0 + \nabla\cdot G \text{ in } Q, \\ \psi = 0 \text{ on } \Sigma, \quad \psi(\cdot,T) = \psi_0 \text{ in } \Omega, \end{cases} \tag{18}$$

where $G_0 = (G_0^1,\dots,G_0^n)^* \in L^2(Q)^n$, $G = (G^1\,|\,G^2\,|\dots|\,G^n) \in L^2(Q;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^N))$ and $\psi_0 \in L^2(\Omega)^n$. We recall that $\nabla\cdot G$ is the column vector given by $\nabla\cdot G = (\nabla\cdot G^1, \nabla\cdot G^2,\dots,\nabla\cdot G^n)^*$.

Under the previous hypotheses, one has the following result.

THEOREM 2.2. *There exist two positive constants* $\widetilde{C}_0$ *and* $\widetilde{\sigma}_0$ (*only depending on* Ω, ω, $(a_{ij})_{1\leqslant i,j\leqslant N}$, n, C *and* D) *and integers* $\ell_j \geqslant 3$, $\ell_k^1 \geqslant 0$ *and* $\ell_k^2 \geqslant 2$, *with* $1 \leqslant j \leqslant r$ *and* $1 \leqslant k \leqslant n$, (*only depending on* $(s_i)_{1\leqslant i\leqslant r}$, n *and* r) *such that, for every* $\psi_0 \in L^2(\Omega)^n$, *the solution* ψ *to* (18) *satisfies*

$$\sum_{i=1}^n \mathcal{I}(3,\psi_i) \leqslant \widetilde{C}_0\Bigg(\sum_{j=1}^r s^{\ell_j}\iint_{\omega\times(0,T)} e^{-2s\alpha}\gamma(t)^{\ell_j}|\psi_{S_j}|^2 \\ + \sum_{k=1}^n\left[s^{\ell_k^1}\iint_Q e^{-2s\alpha}\gamma(t)^{\ell_k^1}|G_0^k|^2 + s^{\ell_k^2}\iint_Q e^{-2s\alpha}\gamma(t)^{\ell_k^2}|G^k|^2\right]\Bigg), \tag{19}$$

for every $s \geqslant \widetilde{s}_0 = \widetilde{\sigma}_0\left(T + T^2 + T^2||c||_\infty^{2/3} + T^2||b||_\infty^2\right)$. *In the previous inequality,* $\alpha(x,t)$, $\gamma(t)$ *and* $\mathcal{I}(d,z)$ *are as in Theorem* 1.6.

Theorem 2.2 is proved in [12] but, for the sake of completeness, we will include a sketch of the proof in an appendix at the end of the paper.

3. The case in which A and B are constant

We will devote this section to prove Theorems 1.4, 1.5 and 1.6. As said above, we will show that, by means of an appropriate change of variables, system (4) is equivalent on $(0,T)$ to a cascade parabolic system as system (16). We will deduce the proofs of Theorems 1.4 and 1.6 from Theorems 2.1 and 2.2.

We first present the proofs when $B \in \mathbb{R}^n$ ($m = 1$, one control force) and then, the general case $B \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$.

3.1. One control force

All along this subsection, we will assume that $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathbb{R}^n$. Therefore, $[A\,|\,B] \in \mathcal{L}(\mathbb{R}^n)$.

Proof of Theorem 1.4. Firstly, observe that, as an easy consequence of Theorem 1.5, (10) is a necessary condition to obtain the exact controllability to trajectories of system (4) on the interval $(0,T)$.

On the other hand, let us assume that condition (10) holds. Then, we have that the matrix $P=[A\,|\,B]\in\mathcal{L}(\mathbb{R}^n)$ satisfies $\det P\neq 0$. Also, we can readily show the equalities

$$\begin{cases} AP=\big(AB|A^2B|\cdots|A^nB\big)=PC \\ Pe_1=B \end{cases} \tag{20}$$

where $e_1=(1,0,...,0)^*$, C is the matrix

$$C=\begin{pmatrix} 0 & 0 & 0 & \dots & -a_n \\ 1 & 0 & 0 & \dots & -a_{n-1} \\ 0 & 1 & 0 & \dots & -a_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & -a_1 \end{pmatrix} \tag{21}$$

and $a_i\in\mathbb{R}$, $1\leqslant i\leqslant n$, are the coefficients of the characteristic polynomial of A. Indeed, if $p(\lambda)=\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\cdots+a_{n-1}\lambda+a_n$ is the characteristic polynomial of A then, as a consequence of the Cayley-Hamilton theorem, we have $p(A)=0$, that is to say,

$$A^n=-a_1A^{n-1}-a_2A^{n-2}-\cdots-a_{n-1}A-a_nId,$$

hence

$$A^nB=-a_1A^{n-1}B-a_2A^{n-2}B-\cdots-a_{n-1}AB-a_nB.$$

Thus, $P^{-1}AP=C$ with C given by (21).

If y is the solution to (4) associated to the condition $y(\cdot,0)=y_0\in L^2(\Omega)^n$, then $w=P^{-1}y$ is the solution to

$$\begin{cases} \partial_t w+L(t)w=Cw+e_1v1_\omega \text{ in } Q, \\ w=0 \text{ on } \Sigma, \quad w(x,0)=P^{-1}y_0 \text{ in } \Omega, \end{cases} \tag{22}$$

with C given by (21). Clearly, system (4) is exactly controllable to the trajectories on the time interval $(0,T)$ if and only if system (4) is null controllable on $(0,T)$. Since $\det P\neq 0$, system (4) is null controllable on $(0,T)$ if and only if system (22) also is null controllable on $(0,T)$.

To finalize, the null controllability of system (22) can be obtained as a consequence of Theorem 2.1 applied to C, $D=e_1$, $m=1$, $r=1$, $S_1=1$ and $s_1=n$. $\square$

REMARK 3.1. Observe that condition (10) is independent of ω. Thus, if condition (10) is not satisfied, then system (4) is not exactly controllable to the trajectories on the time interval $(0,T)$ even if we take $\omega\equiv\Omega$.

On the other hand, we have deduced the exact controllability to the trajectories of system (4) showing that, under assumption (10), system (4) is equivalent to system (22) (C is given by (21)) on the time interval $(0,T)$. Observe that $[C\,|\,e_1]\equiv Id$. The pair (C,e_1) is the so-called Brunovsky's form of (A,B).

Proof of Theorem 1.5. By the Cayley-Hamilton theorem, we have that the space X is A-invariant, that is to say, we have $A(X) \subseteq X$ and, therefore, $\{B, AB, ..., A^{\ell-1}B\}$ is linearly independent and $X = \operatorname{span}\{B, AB, ..., A^{\ell-1}B\}$. In particular,

$$A^{\ell}B = \alpha_1 B + \alpha_2 AB + \cdots + \alpha_\ell A^{\ell-1}B.$$

Let $p_{\ell+1}, ..., p_n$ be $n-\ell$ vectors in $\mathbb{R}^n$ such that the set

$$\{B, AB, ..., A^{\ell-1}B, p_{\ell+1}, ..., p_n\}$$

is a basis of $\mathbb{R}^n$. If we set $P = \big(B\,|\,AB\,|\cdots|\,A^{\ell-1}B\,|\,p_{\ell+1}\,|\cdots|\,p_n\big) \in \mathcal{L}(\mathbb{R}^n)$, then $Pe_1 = B$ and

$$AP = P\begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix}, \quad \text{i.e.,} \quad P^{-1}AP = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix}$$

with $C_{12} \in \mathcal{L}(\mathbb{R}^{n-\ell}, \mathbb{R}^{\ell})$, $C_{22} \in \mathcal{L}(\mathbb{R}^{n-\ell})$ and $C_{11} \in \mathcal{L}(\mathbb{R}^{\ell})$ is given by

$$C_{11} = \begin{pmatrix} 0 & 0 & 0 & \ldots & \alpha_1 \\ 1 & 0 & 0 & \ldots & \alpha_2 \\ 0 & 1 & 0 & \ldots & \alpha_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & \alpha_\ell \end{pmatrix}.$$

Let us fix $y_0 \in L^2(\Omega)^n$ and a trajectory $y^* \in C^0([0,T]; L^2(\Omega))$ of system (4) and let y be the solution to (4) corresponding to $v \in L^2(Q)$ and the initial condition $y(\cdot,0) = y_0$. As in the proof of Theorem 1.4, if we set $w = P^{-1}(y - y^*)$, the function y satisfies $y(\cdot,T) = y^*(\cdot,T)$ in Ω if an only if the solution w to

$$\begin{cases} \partial_t w + L(t)w = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix} w + e_1 1_\omega v \text{ in } Q, \\ w = 0 \text{ on } \Sigma, \quad w(x,0) = P^{-1}\left(y_0 - y^*(\cdot,0)\right) \text{ in } \Omega \end{cases} \tag{23}$$

satisfies $w(\cdot,T) = 0$ in Ω.

On the other hand, given $u_0 \in L^2(\Omega)^n$, it is not difficult to see that $u_0 \in L^2(\Omega; X)$ if and only if there exists $w_0 \in L^2(\Omega)^{\ell}$ such that $u_0 = P\begin{pmatrix} w_0 \\ 0 \end{pmatrix}$.

If $y_0 - y^*(\cdot,0) \notin L^2(\Omega; X)$, then we have that $y_0 - y^*(\cdot,0) = P\begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$ with $w_0 \in L^2(\Omega)^{\ell}$, $w_1 \in L^2(\Omega)^{n-\ell}$ and $w_1 \not\equiv 0$ in Ω. Observe that, if $v \in L^2(Q)$, the corresponding solution to system (23) can be written as $w = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$ with $w^2 \in C^0([0,T]; L^2(\Omega)^{n-k})$ independent of v. Moreover, using the results on backward uniqueness for the parabolic system $\partial_t w^2 + L(t)w^2 = C_{22}w^2$ proved in [10], we can conclude that $w^2(\cdot,T) \not\equiv 0$ in Ω. So, systems (23) cannot be driven to zero at time T and (4) cannot be driven from y_0 at time 0 to $y^*(\cdot,T)$ at time T.

If $y_0 - y^*(\cdot,0) \in L^2(\Omega;X)$, then $y_0 - y^*(\cdot,0) = P\begin{pmatrix} w_0 \\ 0 \end{pmatrix}$ with $w_0 \in L^2(\Omega)^\ell$. Now, for a control $v \in L^2(Q)$ fixed, the solution to (4) which satisfies $y(\cdot,0) = y_0$ has the form $y = y^* + P\begin{pmatrix} w^1 \\ 0 \end{pmatrix}$ with $w^1 \in C^0([0,T];L^2(\Omega)^\ell)$ the solution to

$$\begin{cases} \partial_t w^1 + L(t)w^1 = C_{11}w^1 + e_1 1_\omega v \text{ in } Q, \\ w^1 = 0 \text{ on } \Sigma, \quad w^1(x,0) = w_0 \text{ in } \Omega. \end{cases}$$

We can readily verify that $[C_{11}\,|\,e_1]$ satisfies condition (10) and, therefore, the previous system is null controllable on $(0,T)$. We conclude that the solution to (4) corresponding to v and initial data y_0 satisfies $y(\cdot,T) = y^*(\cdot,T)$ in Ω. This finalizes the proof. $\square$

We will finish this section proving the Carleman inequality stated in Theorem 1.6.

Proof of Theorem 1.6. The proof is a consequence of Theorem 2.2. Let $\varphi \in C^0([0,T];L^2(\Omega)^n)$ be the solution to the adjoint system (13) corresponding to $F_0 = (F_0^1,\dots,F_0^n)^* \in L^2(Q)^n$, $F = (F^1\,|\,F^2\,|\dots|\,F^n) \in L^2(Q;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^N))$ and $\varphi_0 \in L^2(\Omega)^n$. Let us consider $P = [A\,|\,B]$ and $\psi = P^*\varphi$. Taking into account (20), we easily check that the function $\psi \in C^0([0,T];L^2(\Omega)^n)$ solves (18) with C given by (21)), $\psi_0 = P^*\varphi_0$, $G_0 = P^*F_0$ and $G = FP$.

It is easy to check that Theorem 2.2 can be applied to ψ with $D = e_1$, inferring the existence of a positive constant $\widetilde{C}_0$ (only depending on Ω, ω, $(a_{ij})_{1\leqslant i,j\leqslant N}$, n and C) and integers $\ell \geqslant 3$, $\ell_k^1 \geqslant 0$ and $\ell_k^2 \geqslant 2$, with $1 \leqslant k \leqslant n$, (only depending on n) such that (19) holds for every $s \geqslant \widetilde{s}_0$ with

$$\widetilde{s}_0 = \widetilde{\sigma}_0\left(T + T^2 + T^2||c||_\infty^{2/3} + T^2||b||_\infty^2\right).$$

If we set $\ell^1 = \max_{1\leqslant k\leqslant n}\ell_k^1$ and $\ell^2 = \max_{1\leqslant k\leqslant n}\ell_k^2$, we deduce

$$\begin{aligned} \mathcal{I}(3,\psi) \leqslant \widetilde{C}_1\Bigg(s^\ell\iint_{\omega\times(0,T)} & e^{-2s\alpha}\gamma(t)^\ell|e_1^*\cdot\psi|^2 \\ & + s^{\ell^1}\iint_Q e^{-2s\alpha}\gamma(t)^{\ell^1}|G_0|^2 + s^{\ell^2}\iint_Q e^{-2s\alpha}\gamma(t)^{\ell^2}|G|^2\Bigg), \end{aligned}$$

for every $s \geqslant \widetilde{s}_0$, with $\widetilde{C}_1$ a new positive constant which only depends on Ω, ω, $(a_{ij})_{1\leqslant i,j\leqslant N}$, n and C. In this last inequality we have used that, when $s \geqslant \widetilde{\sigma}_0 T^2$ and $\nu \leqslant \mu$, one has $(s\gamma(t))^\nu \leqslant C(s\gamma(t))^\mu$ in $(0,T)$, where C is a constant only depending on ν, μ and $\widetilde{\sigma}_0$.

From (10), one has $\det P \neq 0$, $e_1^* = B^*(P^*)^{-1}$ and $\varphi = (P^*)^{-1}\psi$. for a new positive constant $C_0 = C_0(\Omega,\omega,(a_{ij})_{1\leqslant i,j\leqslant N},n,A,B)$, from the previous inequality, we obtain that inequality (15) is satisfied for every $s \geqslant \widetilde{s}_0$. This ends the proof. $\square$

3.2. The constant case: m-control forces

In this section we will deal with the general problem of null controllability for system (4) in the autonomous case $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. To this aim, we will suppose that $m \geqslant 1$, $v \in L^2(Q)^m$ (m-control forces), and

$$B = \left(b^1 \,|\, b^2 \,|\, \cdots \,|\, b^m\right)$$

with $b^i \in \mathbb{R}^n$ $(1 \leqslant i \leqslant m)$. As in the previous section, $[A\,|\,B] \in \mathcal{L}(\mathbb{R}^{nm};\mathbb{R}^n)$ is given by (9).

Let X be the linear space generated by the columns of $[A\,|\,B]$. Then, $\dim X = \operatorname{rank}[A\,|\,B] = k \leqslant n$. In the next result, we will construct a special basis of X which allows us to prove Theorems 1.4, 1.5, and 1.6. One has:

LEMMA 3.1. *Assume that* $\operatorname{rank}[A\,|\,B] = k \leqslant n$. *Then, there exist* $r \in \{1,...,k\}$ *and sequences* $\{l_j\}_{1\leqslant j\leqslant r} \subset \{1,2,...,m\}$ *and* $\{s_j\}_{1\leqslant j\leqslant r} \subset \{1,2,...,n\}$ *with* $\sum_{j=1}^r s_j = k$, *such that*

$$\mathcal{B} = \bigcup_{j=1}^{r} \{b^{l_j}, Ab^{l_j}, \ldots, A^{s_j-1}b^{l_j}\}$$

is a basis of X. *Moreover, for every* j, *with* $1 \leqslant j \leqslant r$, *there exist* $\alpha^i_{k,s_j} \in \mathbb{R}$ $(1 \leqslant i \leqslant j,$ $1 \leqslant k \leqslant s_j)$ *such that*

$$A^{s_j}b^{l_j} = \sum_{i=1}^{j} \left(\alpha^i_{1,s_j} b^{l_i} + \alpha^i_{2,s_j} Ab^{l_i} + \cdots + \alpha^i_{s_i,s_j} A^{s_i-1} b^{l_i}\right). \tag{24}$$

Proof. In order to obtain the proof, we will give a constructive method which selects a basis of X that satisfies (24) from the columns of $[A\,|\,B]$. For every i with $1 \leqslant i \leqslant m$, let us set

$$X_1 = \operatorname{span}\{b^1, Ab^1, \ldots, A^{n-1}b^1\},$$
$$X_i = \operatorname{span}\{X_{i-1}, b^i, Ab^i, \ldots, A^{n-1}b^i\}, \quad 2 \leqslant i \leqslant m, \quad X_m = X.$$

Again, as a consequence of the Cayley-Hamilton theorem, we deduce $A(X_i) \subset X_i$. We are going to construct r_i and a basis of X_i for which (24) holds for every j, with $1 \leqslant j \leqslant r_i$.

Without loss of generality, we can assume that $b^i \neq 0$ for all i and thus, we set $l_1 = 1$ and $r_1 = 1$. If $s_1 = \dim X_1$, taking into account that X_1 is A-invariant, we infer that $\{b^{l_1}, Ab^{l_1}, ..., A^{s_{l_1}-1}b^{l_1}\}$ is a basis of X_1 and for $1 \leqslant j \leqslant r_1 \equiv 1$, (24) holds.

If $\dim X_2 > \dim X_1 (= s_1)$, then we set $r_2 = r_1 + 1 (= 2)$, $l_{r_2} = 2$ and $s_2 = \dim X_2 - \dim X_1$. If $\dim X_2 = \dim X_1$, then $r_2 = r_1 (= 1)$. In both cases and using again the fact that X_2 is A-invariant, one has that

$$\bigcup_{j=1}^{r_2} \{b^{l_j}, Ab^{l_j}, ..., A^{s_j-1}b^{l_j}\}$$

is a basis of X_2 and (24) is fulfilled for $1 \leqslant j \leqslant r_2$.

Finally, given i, with $1 \leqslant i \leqslant m-1$, and

$$\bigcup_{j=1}^{r_i} \{b^{l_j}, Ab^{l_j}, \ldots, A^{s_j-1}b^{l_j}\}$$

a basis of X_i such that (24) holds for every $1 \leqslant j \leqslant r_i$, we can construct a basis of X_{i+1} as follows: If $\dim X_{i+1} > \dim X_i$, we do $r_{i+1} = r_i + 1$, $l_{r_{i+1}} = i+1$ and $s_{i+1} = \dim X_{i+1} - \dim X_i$. If $\dim X_{i+1} = \dim X_i$, we set $r_{i+1} = r_i$. Again, using that X_{i+1} is A-invariant, we deduce that

$$\bigcup_{j=1}^{r_{i+1}} \{b^{l_j}, Ab^{l_j}, \ldots, A^{s_j-1}b^{l_j}\}$$

is a basis of X_{i+1} and, for $1 \leqslant j \leqslant r_{i+1}$, (24) is satisfied. If we take $r = r_m$ we obtain the proof of this lemma. This finalizes the proof. $\square$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Again the necessary condition can be deduced as an easy consequence of Theorem 1.5.

On the other hand, if condition (10) is fulfilled, then $\dim X = n$, X being the linear space defined in Lemma 3.1. Let $\mathcal{B}$ and P be, respectively, the basis of X provided by Lemma 3.1 and the matrix whose columns are the elements of $\mathcal{B}$, i.e.,

$$P = \left(b^{l_1} \,|\, Ab^{l_1} \,|\, \cdots \,|\, A^{s_1-1}b^{l_1} \,|\, \cdots \,|\, b^{l_r} \,|\, Ab^{l_r} \,|\, \cdots \,|\, A^{s_r-1}b^{l_r} \right). \tag{25}$$

Let us observe that the basis $\mathcal{B}$ has been constructed in such a way that (24) is satisfied. As a consequence of this equality we obtain

$$AP = PC \quad \text{and} \quad Pe_{S_i} = b^{l_i}, \quad 1 \leqslant i \leqslant r, \tag{26}$$

with $S_i = 1 + \sum_{j=1}^{i-1} s_j$, $1 \leqslant i \leqslant r$, and

$$C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ 0 & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{rr} \end{pmatrix}, \tag{27}$$

(e_{S_i} is the S_i–element of the canonical basis of $\mathbb{R}^n$). In (27), the matrices $C_{ii} \in \mathcal{L}(\mathbb{R}^{s_i})$ and $C_{ij} \in \mathcal{L}(\mathbb{R}^{s_j};\mathbb{R}^{s_i})$, $1 \leqslant i \leqslant j \leqslant r$, are given by

$$C_{ii} = \begin{pmatrix} 0 & 0 & 0 & \ldots & \alpha^i_{1,s_i} \\ 1 & 0 & 0 & \ldots & \alpha^i_{2,s_i} \\ 0 & 1 & 0 & \ldots & \alpha^i_{3,s_i} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & \alpha^i_{s_i,s_i} \end{pmatrix} \quad \text{and} \quad C_{ij} = \begin{pmatrix} 0 & 0 & \ldots & 0 & \alpha^i_{1,s_j} \\ 0 & 0 & \ldots & 0 & \alpha^i_{2,s_j} \\ 0 & 0 & \ldots & 0 & \alpha^i_{3,s_j} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \alpha^i_{s_i,s_j} \end{pmatrix}.$$

We will prove that, under condition (10), the system (4) is exactly controllable to trajectories on the time interval $(0,T)$ with r control forces. To be precise, we will prove that the system

$$\begin{cases} \partial_t y + L(t)y = Ay + \widetilde{B}v1_\omega \text{ in } Q = \Omega \times (0,T), \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T), \quad y(x,0) = y_0 \text{ in } \Omega, \end{cases} \tag{28}$$

with $\widetilde{B} = (0|\cdots|0|b^{l_1}|0|\cdots|0|b^{l_2}|0|\cdots|0|b^{l_r}|0|\cdots) \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$, is exactly controllable on $(0,T)$ to the trajectories of system (4) or, equivalently, we will see that this system is null controllable on $(0,T)$.

If we do $w = P^{-1}y$ and $v = (v_1,...,v_m)^* \in L^2(Q)^m$, the null controllability result on $(0,T)$ for system (28) with control v is equivalent to the null controllability on the interval $(0,T)$ of system (16) with $C(t) = C$, $D = (e_{S_1}|e_{S_2}|\cdots|e_{S_r})$ and control $u = (v_{l_1}, v_{l_2}, ..., v_{l_r})^*$.

Again, the null controllability of system (16) is deduced from Theorem 2.1. Let us observe that, from a technical point of view, system (16) behaves as r systems (with coupling matrices C_{ii}) controlled by one control force, the vectors

$$e_1^{s_i} = (1,0,...,0)^* \in \mathbb{R}^{s_i},$$

that satisfy the Kalman rank condition (10), i.e., $\det[C_{ii}\,|\,e_1^{s_i}] \neq 0$. $\square$

REMARK 3.2. As a consequence of the proof of Lemma 3.1 we deduce that system (4) can be controlled with r effective control forces. In fact, Lemma 3.1 provides us these effective controls ($\{b^{l_j} : 1 \leqslant j \leqslant r\}$) as well as a constructive method that selects them. On the other hand, again, if condition (10) is not satisfied, then system (4) is not exactly controllable to the trajectories on the interval $(0,T)$ even if $\omega \equiv \Omega$.

When condition (10) is not satisfied, let us prove the characterization of the initial data y_0 that can be exactly driven to a trajectory y^* of (4).

Proof of Theorem 1.5. Let $\mathcal{B} = \bigcup_{j=1}^r \{b^{l_j}, Ab^{l_j}, \ldots, A^{s_j-1}b^{l_j}\}$ be the basis of X provided by Lemma 3.1 with $k = \ell < n$. We complete $\mathcal{B}$ with the vectors $p_{\ell+1}, \ldots, p_n$ in order to have a basis $\widehat{\mathcal{B}}$ of $\mathbb{R}^n$. Let P be the matrix whose columns are the elements of $\widehat{\mathcal{B}}$. If we set $\widehat{B} = P^{-1}B$ and $C = P^{-1}AP$, then $\widehat{B} = \begin{pmatrix} \widehat{B}_1 \\ 0 \end{pmatrix}$ and

$$C = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix}$$

with $\widehat{B}_1 \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^\ell)$, $C_{12} \in \mathcal{L}(\mathbb{R}^{n-\ell},\mathbb{R}^\ell)$, $C_{22} \in \mathcal{L}(\mathbb{R}^{n-\ell})$ and $C_{11} \in \mathcal{L}(\mathbb{R}^\ell)$ are such that one has

$$\operatorname{rank}[C_{11}\,|\,\widehat{B}_1] = \ell.$$

With these considerations in mind, we can argue as in the case $m = 1$ and obtain the proof of the result. $\square$

Proof of Theorem 1.6. The proof is again a consequence of Theorem 2.2. We reason as in the case $m=1$. Let $\varphi \in C^0([0,T];L^2(\Omega)^n)$ be the solution to the adjoint system (13) corresponding to $F_0 \in L^2(Q)^n$, $F \in L^2(Q;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^N))$ and $\varphi_0 \in L^2(\Omega)^n$ and consider $P \in \mathcal{L}(\mathbb{R}^n)$ the matrix given by (25) and provided by Lemma 3.1. If we perform the change $\psi = P^*\varphi$, from (26), we deduce that $\psi \in C^0([0,T];L^2(\Omega)^n)$ is the solution to (18), with C given by (27),

$$D = (e_{S_1} \,|\, e_{S_2} \,|\, \cdots \,|\, e_{S_r}) \quad \Big(S_i = 1 + \sum_{j=1}^{i-1} s_j,\ 1 \leqslant i \leqslant r\Big),$$

$\psi_0 = P^*\varphi_0$, $G_0 = P^*F_0$ and $G = FP$. Now, from Theorem 2.2, we infer the existence of two positive constants $\widetilde{C}_0$ and $\widetilde{\sigma}_0$ (only depending on Ω, ω, $(a_{ij})_{1\leqslant i,j\leqslant N}$, n, C and D) and integers $\ell_j \geqslant 3$, $\ell_k^1 \geqslant 0$ and $\ell_k^2 \geqslant 2$, with $1 \leqslant j \leqslant r$ and $1 \leqslant k \leqslant n$, (only depending on s_i $(1 \leqslant i \leqslant r)$, n and r) such that the function ψ fulfills the inequality (19) for every $s \geqslant \widetilde{s}_0 = \widetilde{\sigma}_0 \left(T + T^2 + T^2 ||c||_\infty^{2/3} + T^2 ||b||_\infty^2\right)$. If we take $\ell = \max_{1\leqslant i\leqslant r} \ell_i$, $\ell^1 = \max_{1\leqslant k\leqslant n} \ell_k^1$ and $\ell^2 = \max_{1\leqslant k\leqslant n} \ell_k^2$ and we argue as in the case $m=1$, we get

$$\mathcal{I}(3,\psi) \leqslant \widetilde{C}_1 \Bigg(s^\ell \sum_{i=1}^{r} \iint_{\omega\times(0,T)} e^{-2s\alpha} \gamma(t)^\ell |e_{S_i}^* \cdot \psi|^2 + s^{\ell^1} \iint_Q e^{-2s\alpha} \gamma(t)^{\ell^1} |G_0|^2 + s^{\ell^2} \iint_Q e^{-2s\alpha} \gamma(t)^{\ell^2} |G|^2 \Bigg),$$

for every $s \geqslant \widetilde{s}_0$, with $\widetilde{C}_1$ a new positive constant which only depends on Ω, ω, $(a_{ij})_{1\leqslant i,j\leqslant N}$, n and C. Finally, from (26), we also have $e_{S_i}^* \psi = (b^{l_i})^* \varphi$ for every i, $1 \leqslant i \leqslant r$. Coming back to the last inequality and replacing $\varphi = (P^*)^{-1}\psi$ and taking into account that $|G_0(x,t)| \leqslant C|F_0(x,t)|$ and $|G(x,t)| \leqslant C|F(x,t)|$ a.e. in Q $(C>0$ only depends on $P)$, we deduce (15). This ends the proof. $\square$

4. The general case: Proofs of Theorems 1.2 and 1.3

We now consider the non-autonomous system of $n \in \mathbb{N}$ $(n \geqslant 2)$ coupled parabolic equations (4), with $v \in L^2(Q)^m$ and A and B satisfying (5), with $M \geqslant n$. Under this regularity assumption we have $K_p \in C^1([0,T];\mathcal{L}(\mathbb{R}^{mp},\mathbb{R}^n))$ for every $p : 1 \leqslant p \leqslant M$ (the matrix K_p is given by (11)). On the other hand, we write $B(t) = \big(b^1(t) \,|\, b^2(t) \,|\, \cdots \,|\, b^m(t)\big)$ and, for each j, $1 \leqslant j \leqslant m$, we also define

$$\begin{cases} b_0^j(t) = b^j(t), \\ b_i^j(t) = A(t) b_{i-1}^j(t) - \dfrac{d}{dt} b_{i-1}^j(t) \ \text{ for } i = 1, \ldots, p-1, \end{cases} \tag{29}$$

and $K_p^j(t) = (b_0^j(t) \,|\, b_1^j(t) \,|\, \cdots \,|\, b_{p-1}^j(t)) \in C^1([0,T];\mathcal{L}(\mathbb{R}^p;\mathbb{R}^n))$. It is not difficult to check that $b_i^j \in C^{M-i}([0,T];\mathbb{R}^n)$ for every i, j satisfying $0 \leqslant i \leqslant p-1$ and $1 \leqslant j \leqslant m$.

We will deduce the proof of Theorem 1.2 from the following result.

LEMMA 4.1. *Assume that* $\max\{\operatorname{rank} K_p(t) : t \in [0,T]\} = \ell \leqslant n$, *for an integer* $p \in \{n,...,M\}$. *Then, there exist* $T_0, T_1 \in [0,T]$, *with* $T_0 < T_1$, $r \in \{1,...,m\}$ *and sequences* $\{s_j\}_{1\leqslant j\leqslant r} \subset \{1,2,...,n\}$, *with* $\sum_{i=1}^r s_j = \ell$, *and* $\{l_j\}_{1\leqslant j\leqslant r} \subset \{1,2,...,m\}$ *such that, for every* $t \in [T_0,T_1]$, *the set*

$$\mathcal{B}(t) = \bigcup_{j=1}^{r} \{b_0^{l_j}(t), b_1^{l_j}(t), ..., b_{s_j-1}^{l_j}(t)\},$$

is linearly independent, generates the columns of $K_p(t)$ *and satisfies*

$$b_{s_j}^{l_j}(t) = \sum_{k=1}^{j} \left(\theta_{s_j,0}^{l_j,l_k}(t) b_0^{l_k}(t) + \theta_{s_j,1}^{l_j,l_k}(t) b_1^{l_k}(t) + \cdots + \theta_{s_j,s_k-1}^{l_j,l_k}(t) b_{s_k-1}^{l_k}(t) \right), \tag{30}$$

for every $t \in [T_0,T_1]$ *and* j, $1 \leqslant j \leqslant r$, *where* $\theta_{s_j,0}^{l_j,l_k}$, $\theta_{s_j,1}^{l_j,l_k}$, ..., $\theta_{s_j,s_k-1}^{l_j,l_k} \in C^1([T_0,T_1])$.

Proof. Since $K_p \in C^1([0,T]; \mathcal{L}(\mathbb{R}^{mp}; \mathbb{R}^n))$, there exists a non degenerate closed interval $I^0 \subseteq [0,T]$ such that

$$\operatorname{rank} K_p(t) = \ell, \quad \forall t \in I^0. \tag{31}$$

The proof will be made in the following steps.

Step 1. We define

$$l_1 = \min\{j \in \{1,...,m\} : \exists t_0 \in I^0 \text{ s.t. } b^j(t_0)(= b_0^j(t_0)) \neq 0\},$$

and $r_{l_1} = 1$. Observe that, if $1 \leqslant k \leqslant l_1 - 1$, $b^k(t) \equiv 0$ in I^0 and, therefore,

$$\left(K_p^1(t) \,|\, \cdots \,|\, K_p^{l_1-1}(t)\right) \equiv 0 \quad \text{in} \quad I^0.$$

Let $\widetilde{I^0} \subseteq I^0$ be a non degenerate closed interval such that $b_0^{l_1}(t) \neq 0$, for all $t \in \widetilde{I^0}$. We will show the statement of the lemma for l_1,

$$s_1 = \max_{t\in\widetilde{I^0}} \operatorname{rank} \left(K_p^1(t) \,|\, \cdots \,|\, K_p^{l_1}(t)\right) \equiv \max_{t\in\widetilde{I^0}} \operatorname{rank} K_p^{l_1}(t)$$

(which satisfies $1 \leqslant s_1 \leqslant n$), a non degenerate closed interval $I^{r_{l_1}}$ (to be determined) and r_{l_1} instead of m, ℓ, $[T_0,T_1]$ and r. Firstly, let us fix $I_0^1 \subseteq \widetilde{I^0}$ such that

$$\operatorname{rank} \left(K_p^1(t) \,|\, \cdots \,|\, K_p^{l_1}(t)\right) \equiv \operatorname{rank} K_p^{l_1}(t) = s_1, \quad \forall t \in I_0^1. \tag{32}$$

On the other hand, let us consider the set $\{b_0^{l_1}(t), b_1^{l_1}(t)\}$ with $t \in I_0^1$. Thus:

1. if $\{b_0^{l_1}(t), b_1^{l_1}(t)\}$ is linearly dependent for every $t \in I_0^1$, then, we deduce the existence of a function $\theta_{10}^{l_1} \in C^{M-1}(I_0^1)$ such that

$$b_1^{l_1}(t) = \theta_{10}^{l_1}(t) b_0^{l_1}(t), \quad \forall t \in I_0^1.$$

In this case, it is easy to show that, for every k, $1 \leqslant k \leqslant p-1$, one has

$$b_k^{l_1}(t) = \theta_{k0}^{l_1}(t) b_0^{l_1}(t), \quad \forall t \in I_0^1,$$

for a function $\theta_{k0}^{l_1} \in C^{M-k}(I_0^1)$. From (32), we deduce $s_1 \equiv 1$ and, therefore, for every $t \in I^{r_{l_1}} \equiv I_0^1$, we obtain that the set $\{b_0^{l_1}(t)\}$ is l.i., generates the columns of $\left(K_p^1(t) | \cdots | K_p^{l_1}(t)\right)$ and satisfies (30) for $j = r_{l_1} = 1$;

2. if $\{b_0^{l_1}(t_0^1), b_1^{l_1}(t_0^1)\}$, with $t_0^1 \in I_0^1$, is linearly independent, then, there exists a new non degenerate interval $I_1^1 \subseteq I_0^1$ such that $\{b_0^{l_1}(t), b_1^{l_1}(t)\}$ is linearly independent for every $t \in I_1^1$.

In this last case, we consider the set $\{b_0^{l_1}(t), b_1^{l_1}(t), b_2^{l_1}(t)\}$ with $t \in I_1^1$. If the set $\{b_0^{l_1}(t), b_1^{l_1}(t), b_2^{l_1}(t)\}$ is linearly dependent for every $t \in I_1^1$, there exist two functions $\theta_{20}^{l_1}, \theta_{21}^{l_1} \in C^{M-2}(I_1^1)$ such that

$$b_2^{l_1}(t) = \theta_{20}^{l_1}(t) b_0^{l_1}(t) + \theta_{21}^{l_1}(t) b_1^{l_1}(t), \quad \forall t \in I_1^1.$$

Again, we readily check that

$$b_k^{l_1}(t) = \theta_{k0}^{l_1}(t) b_0^{l_1}(t) + \theta_{k1}^{l_1}(t) b_1^{l_1}(t), \quad \forall t \in I_1^1, \quad \forall k : 2 \leqslant k \leqslant p-1,$$

with $\theta_{k0}^{l_1}, \theta_{k1}^{l_1} \in C^{M-k}(I_1^1)$. In this case, $s_1 \equiv 2$ and again we obtain the result for $I^{r_{l_1}} = I_1^1$.

On the other hand, if for $t_1^1 \in I_1^1$ the set $\{b_0^{l_1}(t_1^1), b_1^{l_1}(t_1^1), b_2^{l_1}(t_1^1)\}$ is linearly independent, we can continue the previous process until we find $\widetilde{s}_1 \in \{1, ..., p\}$ and a non degenerate interval $I^{r_{l_1}} \subseteq I^0$ such that for all $t \in I^{r_{l_1}}$ the set $\{b_0^{l_1}(t), b_1^{l_1}(t), ..., b_{\widetilde{s}_1 - 1}^{l_1}(t)\}$ is linearly independent and generates the columns of $K_p^{l_1}(t)$. From (32), it is clear that $\widetilde{s}_1 = s_1$ and, so, we have (30) for $j = 1$.

Step 2. Assume that, for $l_1 \leqslant i \leqslant m-1$, we have constructed $r_i \in \{1, ..., i\}$, a non degenerate closed interval $I^{r_i} \subseteq I^{r_{i-1}} \subseteq I^0$, $s_j \in \{1, ..., n\}$ and $l_j \in \{1, ..., m\}$ ($1 \leqslant j \leqslant r_i$) such that, if we define $\mathcal{B}_{l_j}(t) = \{b_0^{l_j}(t), b_1^{l_j}(t), ..., b_{s_j - 1}^{l_j}(t)\}$, one has that for every $t \in I^{r_i}$ the set

$$\bigcup_{j=1}^{r_i} \mathcal{B}_{l_j}(t)$$

is linearly independent, generates the columns of $\left(K_p^1(t) | \cdots | K_p^i(t)\right)$ and satisfies (30) for $1 \leqslant j \leqslant r_i$.

We argue as above and we consider the set

$$\mathcal{D}_{i+1} = \{q \in \{i+1, ..., m\} : \exists t_0 \in I^{r_i} \text{ s.t. } \{b^q(t_0)\} \cup \bigcup_{j=1}^{r_i} \mathcal{B}_{l_j}(t_0) \text{ is l.i.}\}.$$

1. If $\mathcal{D}_{i+1} = \emptyset$, then, for every q, $i+1 \leqslant q \leqslant m$, there exist coefficients $\theta_{00}^{q,l_j}(t)$, $\theta_{01}^{q,l_j}(t)$, ..., $\theta_{0,s_j-1}^{q,l_j}(t)$ $(1 \leqslant j \leqslant r_i)$ such that, for each $t \in I^{r_i}$, one has

$$b_0^q(t) = \sum_{j=1}^{r_i} \left(\theta_{00}^{q,l_j}(t) b_0^{l_j}(t) + \theta_{01}^{q,l_j}(t) b_1^{l_j}(t) + \cdots + \theta_{0,s_j-1}^{q,l_j}(t) b_{s_j-1}^{l_j}(t) \right). \tag{33}$$

It is not difficult to check that the previous coefficients satisfy

$$\theta_{00}^{q,l_j}, \theta_{01}^{q,l_j}, \ldots, \theta_{0,s_j-1}^{q,l_j} \in C^{M+1-\widetilde{S}_0}(I^{r_i}) \subset C^1(I^{r_i}),$$

where $\widetilde{S}_0 = \max_{1 \leqslant j \leqslant r_i} s_j$ $(1 \leqslant \widetilde{S}_0 \leqslant n)$.

From equality (33) and taking into account (29), we infer that the set

$$\{b_1^q(t)\} \cup \bigcup_{j=1}^{r_i} \mathcal{B}_{l_j}(t)$$

is linearly dependent for every $t \in I^{r_i}$. So, we deduce that $b_1^q(t)$ can be written as

$$b_1^q(t) = \sum_{j=1}^{r_i} \left(\theta_{10}^{q,l_j}(t) b_0^{l_j}(t) + \theta_{11}^{q,l_j}(t) b_1^{l_j}(t) + \cdots + \theta_{1,s_j-1}^{q,l_j}(t) b_{s_j-1}^{l_j}(t) \right),$$

for new coefficients $\theta_{10}^{q,l_j}, \theta_{11}^{q,l_j}, \ldots, \theta_{1,s_j-1}^{q,l_j} \in C^{M+1-\widetilde{S}_1}(I^{r_i}) \subset C^1(I^{r_i})$ $(1 \leqslant j \leqslant r_i)$, where $\widetilde{S}_1 = \max\{2, s_j : 1 \leqslant j \leqslant r_i\}$. By induction, we readily obtain that the set

$$\{b_l^q(t)\} \cup \bigcup_{j=1}^{r_i} \mathcal{B}_{l_j}(t) \tag{34}$$

is linearly dependent for every $l : 0 \leqslant l \leqslant p-1$ and $t \in I^{r_i}$ and $b_l^p(t)$ can be written as

$$b_l^q(t) = \sum_{j=1}^{r_i} \left(\theta_{l0}^{q,l_j}(t) b_0^{l_j}(t) + \theta_{l1}^{q,l_j}(t) b_1^{l_j}(t) + \cdots + \theta_{l,s_j-1}^{q,l_j}(t) b_{s_j-1}^{l_j}(t) \right),$$

for new coefficients $\theta_{l0}^{q,l_j}, \theta_{l1}^{q,l_j}, \ldots, \theta_{l,s_j-1}^{q,l_j} \in C^{M+1-\widetilde{S}_l}(I^{r_i})$ $(1 \leqslant j \leqslant r_i)$ with $\widetilde{S}_l = \max\{l+1, s_j : 1 \leqslant j \leqslant r_i\}$. In this case, we set $r_l = r_i$, for every $i+1 \leqslant l \leqslant m$ and the lemma is proved if we take $r = r_m$ and $[T_0, T_1] \equiv I^{r_m}$.

2. If $\mathcal{D}_{i+1} \neq \emptyset$, we set $\widetilde{l} = \min \mathcal{D}_{i+1}$, $r_j = r_i$ if $i+1 \leqslant j \leqslant \widetilde{l}-1$, $r_{\widetilde{l}} = r_i + 1$ and $l_{r_{\widetilde{l}}} = \widetilde{l}$. For $q : 1 \leqslant q \leqslant \widetilde{l}-1$, we can argue as above and obtain that for every $t \in I^{r_q} (= I^{r_i})$, the set (34) is linearly dependent for every $l : 0 \leqslant q \leqslant p-1$. So, for every $t \in I^{r_{\widetilde{l}-1}}$, the set $\cup_{j=1}^{r_{\widetilde{l}-1}} \mathcal{B}_{l_j}(t)$ is l.i., generates the columns of the matrix $\left(K_p^1(t) \,|\, \cdots \,|\, K_p^{\widetilde{l}-1}(t) \right)$ and (30) holds for $1 \leqslant j \leqslant r_{\widetilde{l}-1}$.

Now, we concentrate on the case $q = \widetilde{l}$. Our next goal is to construct a new non degenerate interval $I^{r_{\widetilde{l}}} \subseteq I^{r_i}$ and $s_{\widetilde{l}}$ such that, for every $t \in I^{r_{\widetilde{l}}}$, the set $\cup_{j=1}^{r_{\widetilde{l}}} \mathcal{B}_{l_j}(t)$ is l.i., generates the columns of the matrix

$$\left(K_p^1(t) \,|\, \cdots \,|\, K_p^{\widetilde{l}}(t) \right)$$

and (30) holds for $1 \leqslant j \leqslant r_{\widetilde{l}}$.

Taking into account the definition of $\widetilde{l}$, we deduce that there exists a new non degenerate interval $I_0^{r_{\widetilde{l}}} \subseteq I^{r_i}$ such that the set $\{b^{\widetilde{l}}(t)\} \cup \bigcup_{j=1}^{r_i} \mathcal{B}_{l_j}(t)$ is l.i. for every $t \in I_0^{r_{\widetilde{l}}}$. We now consider the set $\{b_0^{\widetilde{l}}(t), b_1^{\widetilde{l}}(t)\} \cup \bigcup_{j=1}^{r_i} \mathcal{B}_{l_j}(t)$ with $t \in I_0^{r_{\widetilde{l}}}$. If $\{b_0^{\widetilde{l}}(t), b_1^{\widetilde{l}}(t)\} \cup \bigcup_{j=1}^{r_i} \mathcal{B}_{l_j}(t)$ is l.d. for every $t \in I_0^{r_{\widetilde{l}}}$, there exist functions $\theta_{10}^{\widetilde{l},\widetilde{l}}, \theta_{10}^{\widetilde{l},l_j}, \theta_{11}^{\widetilde{l},l_j}, ..., \theta_{1,s_j-1}^{\widetilde{l},l_j} \in C^{M+1-\widetilde{S}_1}(I_0^{r_{\widetilde{l}}})$ $(1 \leqslant j \leqslant r_i)$, where $\widetilde{S}_1 = \max\{2, s_j : 1 \leqslant j \leqslant r_i\}$, such that, for each $t \in I_0^{r_{\widetilde{l}}}$, one has

$$b_1^{\widetilde{l}}(t) = \theta_{10}^{\widetilde{l},\widetilde{l}}(t) b_0^{\widetilde{l}}(t) + \sum_{j=1}^{r_i} \left(\theta_{10}^{\widetilde{l},l_j}(t) b_0^{l_j}(t) + \theta_{11}^{\widetilde{l},l_j}(t) b_1^{l_j}(t) + \cdots + \theta_{1,s_j-1}^{\widetilde{l},l_j}(t) b_{s_j-1}^{l_j}(t) \right).$$

From this equality and following the argument above, we infer that $b_l^{\widetilde{l}}(t)$, with $1 \leqslant l \leqslant p-1$ and $t \in I_0^{r_{\widetilde{l}}}$, can be written as a linear combination of $\{b_0^{\widetilde{l}}(t)\} \cup \bigcup_{j=1}^{r_i} \mathcal{B}_{l_j}(t)$. Thus, we obtain the previous property for the set $\cup_{j=1}^{r_{\widetilde{l}}} \mathcal{B}_{l_j}(t)$ by taking $s_{r_{\widetilde{l}}} = 1$ and $I^{r_{\widetilde{l}}} = I_0^{r_{\widetilde{l}}} \subseteq I^{r_i}$.

On the other hand, if there exists $t_0^{r_{\widetilde{l}}} \in I_0^{r_{\widetilde{l}}}$ such that the set $\{b_0^{\widetilde{l}}(t_0^{r_{\widetilde{l}}}), b_1^{\widetilde{l}}(t_0^{r_{\widetilde{l}}})\} \cup \bigcup_{j=1}^{r_i} \mathcal{B}_{l_j}(t_0^{r_{\widetilde{l}}})$ is l.i., then we can obtain a new non degenerate interval $I_1^{r_{\widetilde{l}}} \subset I_0^{r_{\widetilde{l}}}$ such that the previous set goes on being a l.i. set for every $t \in I_1^{\widetilde{l}}$. As in the previous step, we can continue the method until we obtain $s_{r_{\widetilde{l}}} \in \{1, ..., n\}$ and a non degenerate interval $I^{r_{\widetilde{l}}} \subset I^{r_i}$ such that the set $\bigcup_{j=1}^{r_{\widetilde{l}}} \mathcal{B}_{l_j}(t)$ is l.i. for every $t \in I^{r_{\widetilde{l}}}$, generates the columns of the matrix $\left(K_p^1(t)\,|\,K_p^2(t)\,|\cdots|\,K_p^{\widetilde{l}}(t)\right)$ and satisfies (30) for $1 \leqslant j \leqslant r_{\widetilde{l}}$.

Finally, from the reasoning by induction, we deduce the existence of $r(=r_m)$ and a non degenerate closed interval $[T_0, T_1] \subseteq I^0$ such that, for every $t \in [T_0, T_1]$, the set $\bigcup_{j=1}^{r} \mathcal{B}_{l_j}(t)$ is l.i., generates the columns of the matrix $\left(K_p^1(t)\,|\,K_p^2(t)\,|\cdots|\,K_p^m(t)\right)$ and satisfies (30) for $1 \leqslant j \leqslant r$. From (31), we also conclude that $\sum_{i=1}^{r} s_j = \ell$. This concludes the proof of Lemma 4.1. $\square$

As a consequence of Lemma 4.1 we deduce.

COROLLARY 4.2. *Under the assumptions of Lemma* 4.1*, let ℓ be a set defined by $\ell = \max\{\operatorname{rank} K_p(t) : t \in [0,T]\}$ with $p \in \{n, .., M\}$ an integer. Then,* $\max\{\operatorname{rank} K_n(t) : t \in [0,T]\} = \ell$ *and there exist $T_0, T_1 \in [0,T]$, with $T_0 < T_1$, such that*

$$\operatorname{rank} K_n(t) = \ell, \quad \forall t \in [T_0, T_1].$$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. All along this proof we will assume that A and B satisfy (5).
1. Let us suppose that condition (12) is satisfied for $t_0 \in [0,T]$ and $p \in \{1, ..., M\}$. If $p < n$, in particular $\operatorname{rank} K_n(t_0) = n$ and we can apply Lemma 4.1 with $\ell = n$. If

$p \in \{n, ..., M\}$ we can directly apply Lemma 4.1 also with $\ell = n$ and obtain in both cases the existence of an interval $[T_0, T_1] \subseteq [0, T]$ $(T_0 < T_1)$, $r \leqslant m$, $l_j \in \{1, 2, ..., m\}$ and $s_j \in \{1, 2, ..., n\}$, $s_j \leqslant p$, $(1 \leqslant j \leqslant r)$ with $\sum_{j=1}^r s_j = n$ such that if

$$P(t) = \left(b_0^{l_1} \,|\, b_1^{l_1} \,|\, \cdots \,|\, b_{s_1-1}^{l_1} \,|\, b_0^{l_2} \,|\, b_1^{l_2} \,|\, \cdots \,|\, b_{s_2-1}^{l_2} \,|\, \cdots \,|\, b_0^{l_r} \,|\, b_1^{l_r} \,|\, \cdots \,|\, b_{s_r-1}^{l_r} \right)(t),$$

then, $\det P(t) \neq 0$ for every $t \in [T_0, T_1]$. In addition, $P \in C^1([0, T]; \mathcal{L}(\mathbb{R}^n))$ and, thanks to the properties of the columns of $P(t)$ stated in Lemma 4.1, if $1 \leqslant j \leqslant r$ and $t \in [T_0, T_1]$ one has (30) for $\theta_{s_j,0}^{l_j,l_i}$, $\theta_{s_j,1}^{l_j,l_i}$, ..., $\theta_{s_j,s_i-1}^{l_j,l_i} \in C^1([T_0, T_1])$. Therefore,

$$A(t)P(t) - P'(t) = P(t)C(t) \quad \text{and} \quad P(t)e_{S_i} = b_0^{l_i}(t), \quad 1 \leqslant i \leqslant r,$$

with $S_i = 1 + \sum_{j=1}^{i-1} s_j$, $1 \leqslant i \leqslant r$, $C(t)$ given by (17) and $C_{ii} \in C^0([T_0, T_1]; \mathcal{L}(\mathbb{R}^{s_i}))$ and $C_{ij} \in C^0([T_0, T_1]; \mathcal{L}(\mathbb{R}^{s_j}; \mathbb{R}^{s_i}))$, $1 \leqslant i \leqslant j \leqslant r$, given by

$$C_{ii} = \begin{pmatrix} 0 & 0 & 0 & \dots & \theta_{s_i,0}^{l_i,l_i} \\ 1 & 0 & 0 & \dots & \theta_{s_i,1}^{l_i,l_i} \\ 0 & 1 & 0 & \dots & \theta_{s_i,2}^{l_i,l_i} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \theta_{s_i,s_i-1}^{l_i,l_i} \end{pmatrix}, \quad C_{ij} = \begin{pmatrix} 0 & 0 & \dots & 0 & \theta_{s_j,0}^{l_j,l_i} \\ 0 & 0 & \dots & 0 & \theta_{s_j,1}^{l_j,l_i} \\ 0 & 0 & \dots & 0 & \theta_{s_j,2}^{l_j,l_i} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \theta_{s_i,s_i-1}^{l_j,l_i} \end{pmatrix}.$$

As done in Section 3, we will prove that system (4) is in fact exactly controllable to trajectories on the interval $[T_0, T_1] \subseteq [0, T]$ when we exert r control forces. Obviously, this fact implies that system (4) is exactly controllable to the trajectories on the interval $(0, T)$. To this end, we consider the system

$$\begin{cases} \partial_t y + L(t)y = A(t)y + \widetilde{B}(t)v1_\omega \text{ in } \widetilde{Q} = \Omega \times (T_0, T_1), \\ y = 0 \text{ on } \widetilde{\Sigma} = \partial\Omega \times (T_0, T_1), \quad y(x, T_0) = \widetilde{y}_0 \text{ in } \Omega, \end{cases} \tag{35}$$

with $\widetilde{B}(t) = (0|\cdots|0|b_0^{l_1}(t)|0|\cdots|0|b_0^{l_2}(t)|0|\cdots|0|b_0^{l_r}(t)|0|\cdots)$ and $\widetilde{y}_0 \in L^2(\Omega)^n$. Let us see that system (35) is null controllable on $[T_0, T_1]$ (which clearly implies that system (4) is exactly controllable to the trajectories on the interval $(0, T)$).

If we do $w = P(t)^{-1}y$ and $v = (v_1, ..., v_m)^* \in L^2(\Omega)^m$, the null controllability result on the interval $[T_0, T_1]$ for system (35) is equivalent to the null controllability on the interval $[T_0, T_1]$ of system

$$\begin{cases} \partial_t w + L(t)w = C(t)w + \displaystyle\sum_{j=1}^{r} e_{S_j} v_{l_j} 1_\omega \text{ in } \widetilde{Q}, \\ w = 0 \text{ on } \widetilde{\Sigma}, \quad w(x, T_0) = P^{-1}(T_0)\widetilde{y}_0 \text{ in } \Omega, \end{cases}$$

where $C(t)$ is given by (17). Once again, the null controllability property of the previous system is deduced from Theorem 2.1.

2. Let us fix $p \in \{n,\dots,M\}$ and $T_0, T_1 \in [0,T]$ with $T_0 < T_1$. Let us assume that there exists a dense set E in $[0,T]$ such that $\operatorname{rank} K_p(t) = n$ for every $t \in E$. Then, one has $\max\{\operatorname{rank} K_p(t) : t \in [T_0,T_1]\} = n$. If we repeat the arguments developed in the previous point, we deduce that system (4) is exactly controllable to the trajectories on the interval $[T_0,T_1]$.

Now, let us assume that system (4) is exactly controllable to the trajectories on every non degenerate interval $[T_0,T_1]$ of $[0,T]$. We will prove the existence of a dense subset $E \subseteq [0,T]$ for which $\operatorname{rank} K_n(t) = n$ for every $t \in E$. Evidently, this fact implies that $\operatorname{rank} K_p(t) = n$ for every $p \in \{n,\dots,M\}$ and $t \in E$.

By contradiction, let us also assume that for a non degenerate interval $[\tau_0,\tau_1] \subseteq [0,T]$ one has $\max\{\operatorname{rank} K_n(t) : t \in [\tau_0,\tau_1]\} = \ell < n$. Thus, applying Lemma 4.1 in the interval $[\tau_0,\tau_1]$, we deduce the existence of a new non degenerate interval $[\widetilde{\tau}_0,\widetilde{\tau}_1] \subseteq [\tau_0,\tau_1]$, $r \in \{1,\dots,m\}$ and sequences $\{s_j\}_{1\leqslant j\leqslant r} \subset \{1,2,\dots,n\}$, with $\sum_{i=1}^r s_j = \ell$, and $\{l_j\}_{1\leqslant j\leqslant r} \subset \{1,2,\dots,m\}$ such that, for every $t \in [\widetilde{\tau}_0,\widetilde{\tau}_1]$, the set

$$\mathcal{B}(t) = \bigcup_{j=1}^{r} \{b_0^{l_j}(t), b_1^{l_j}(t), \dots, b_{s_j-1}^{l_j}(t)\},$$

is linearly independent, generates the columns of $K_n(t)$ and satisfies (30) in $[\widetilde{\tau}_0,\widetilde{\tau}_1]$. In particular, if we fix $\widetilde{\tau} \in (\widetilde{\tau}_0,\widetilde{\tau}_1)$, the set $\mathcal{B}(\widetilde{\tau})$ is linearly independent and generates the columns of $K_n(\widetilde{\tau})$. Let us now take a set $\{p_{\ell+1},\dots,p_n\}$ such that $\mathcal{B}(\widetilde{\tau}) \cup \{p_{\ell+1},\dots,p_n\}$ is a basis of $\mathbb{R}^n$. Using the continuity of $K_n(t)$ in $[0,T]$, we infer that the set $\mathcal{B}(t) \cup \{p_{\ell+1},\dots,p_n\}$ is a basis of $\mathbb{R}^n$ and $\mathcal{B}(t)$ generates the columns of $K_n(t)$ in a new non degenerate subinterval $[\widehat{\tau}_0,\widehat{\tau}_1]$ of $[\widetilde{\tau}_0,\widetilde{\tau}_1]$. For $t \in [\widehat{\tau}_0,\widehat{\tau}_1]$, let $P(t)$ be the matrix whose columns are the elements of $\mathcal{B}(t) \cup \{p_{\ell+1},\dots,p_n\}$. If we set

$$\widehat{B}(t) = P^{-1}(t)B(t) \quad \text{and} \quad C(t) = P^{-1}(t)A(t)P(t), \quad \forall t \in [\widehat{\tau}_0,\widehat{\tau}_1],$$

then, from the previous properties, we get $\widehat{B}(t) = \begin{pmatrix} \widehat{B}_1(t) \\ 0 \end{pmatrix}$ in $[\widehat{\tau}_0,\widehat{\tau}_1]$, and

$$C(t) = \begin{pmatrix} C_{11}(t) & C_{12}(t) \\ 0 & C_{22}(t) \end{pmatrix}, \quad t \in [\widehat{\tau}_0,\widehat{\tau}_1]$$

with

$$\widehat{B}_1 \in C^0([\widehat{\tau}_0,\widehat{\tau}_1];\mathcal{L}(\mathbb{R}^m;\mathbb{R}^\ell)),\ \ C_{11} \in C^0([\widehat{\tau}_0,\widehat{\tau}_1];\mathcal{L}(\mathbb{R}^\ell)),$$
$$C_{12} \in C^0([\widehat{\tau}_0,\widehat{\tau}_1];\mathcal{L}(\mathbb{R}^{n-\ell},\mathbb{R}^\ell)),\ \ C_{22} \in C^0([\widehat{\tau}_0,\widehat{\tau}_1];\mathcal{L}(\mathbb{R}^{n-\ell})),$$

and are such that one has

$$\operatorname{rank}[C_{11} \,|\, \widehat{B}_1] = \ell.$$

Again, reasoning as in the proof of Theorem 1.5, we obtain that system (4) is not null controllable on the time interval $[\widehat{\tau}_0,\widehat{\tau}_1]$ since on this time interval this system is equivalent to the decoupled system

$$\begin{cases} \partial_t w + L(t)w = \begin{pmatrix} C_{11}(t) & C_{12}(t) \\ 0 & C_{22}(t) \end{pmatrix} w + \begin{pmatrix} \widehat{B}_1(t) \\ 0 \end{pmatrix} 1_\omega v \text{ in } \Omega \times (\widehat{\tau}_0,\widehat{\tau}_1), \\ w = 0 \text{ on } \partial\Omega \times (\widehat{\tau}_0,\widehat{\tau}_1). \end{cases} \tag{36}$$

This contradicts our assumption and finalizes the proof of Theorem 1.2. □

REMARK 4.1. As said above, under condition (12), system (4) is exactly controllable to the trajectories on $(0,T)$ when we exert on the system $r \leqslant m$ effective control forces, the controls $\sum_{j=1}^{r} b^{l_j}(\cdot)v_{l_j}$. Observe that, in fact, theses control forces are exerted on the system in the effective time interval $[T_0,T_1]$.

As a consequence of Lemma 4.1, we can establish a Carleman estimate for the solutions of the adjoint problem (13) in the time interval (T_0,T_1). The following result holds true.

COROLLARY 4.3. *Let us assume that A and B satisfy hypothesis* (5). *Then, there exist a positive function $\alpha_0 \in C^2(\overline{\Omega})$ (only depending on Ω and ω) such that, if* (12) *is fulfilled, there exist a time interval $(T_0,T_1) \subseteq (0,T)$, two positive constants $C_0 = C_0(\Omega,\omega,(a_{ij})_{1\leqslant i,j\leqslant N},n,m,A(\cdot),B(\cdot))$ and $\sigma_0 = \sigma_0(\Omega,\omega,(a_{ij})_{1\leqslant i,j\leqslant N},n,m,A(\cdot),B(\cdot))$ and integers $\ell \geqslant 3$, $\ell^1 \geqslant 0$ and $\ell^2 \geqslant 2$ (only depending on n, m, $A(\cdot)$ and $B(\cdot)$) such that, for every $\varphi_0 \in L^2(Q)^n$, the solution φ to* (13) *satisfies*

$$\widetilde{\mathfrak{I}}(3,\varphi) \leqslant \widetilde{C}_1 \left(s^{\ell} \iint_{\omega\times(T_0,T_1)} e^{-2s\widetilde{\alpha}} \widetilde{\gamma}(t)^{\ell} |B^*\varphi|^2 \right.$$
$$\left. + s^{\ell^1} \iint_{\Omega\times(T_0,T_1)} e^{-2s\widetilde{\alpha}} \widetilde{\gamma}(t)^{\ell^1} |F_0|^2 + s^{\ell^2} \iint_{\Omega\times(T_0,T_1)} e^{-2s\widetilde{\alpha}} \widetilde{\gamma}(t)^{\ell^2} |F|^2 \right),$$

for every $s \geqslant s_0 = \sigma_0 \left(\widetilde{T} + \widetilde{T}^2 + \widetilde{T}^2 ||c||_{\infty}^{2/3} + \widetilde{T}^2 ||b||_{\infty}^2 \right)$ with $\widetilde{T} = T_1 - T_0$. In the previous inequality $\widetilde{\alpha}(x,t)$, $\widetilde{\gamma}(t)$ and $\widetilde{\mathfrak{I}}(d,z)$ are respectively given by:

$$\widetilde{\alpha}(x,t) \equiv \alpha_0(x)/(t-T_0)(T_1-t), \;\; \widetilde{\gamma}(t) \equiv ((t-T_0)(T_1-t))^{-1},$$

and

$$\widetilde{\mathfrak{I}}(d,z) \equiv s^{d-2} \iint_{\Omega\times(T_0,T_1)} e^{-2s\widetilde{\alpha}} \widetilde{\gamma}(t)^{d-2} |\nabla z|^2 + s^d \iint_{\Omega\times(T_0,T_1)} e^{-2s\widetilde{\alpha}} \widetilde{\gamma}(t)^{d} |z|^2.$$

Proof. Let us consider the adjoint problem (13) and let us assume that condition (12) holds. As in the proof of Theorem 1.2 1, we can apply Lemma 4.1 with $\ell = n$ and obtain an interval $[T_0,T_1] \subseteq [0,T]$ ($T_0 < T_1$) and a set $\mathcal{B}(t)$ such that $\mathcal{B}(t)$ is a basis of $\mathbb{R}^n$ for every $t \in (T_0,T_1)$ and satisfies (30). We now consider the matrix $P(t) \in C^1([T_0,T_1];\mathcal{L}(\mathbb{R}^n))$ whose columns are the elements of $\mathcal{B}(t)$. If φ is the solution to (13) corresponding to $\varphi_0 \in L^2(\Omega)^n$ and we perform the change $\psi = P(t)^*\varphi$ in (T_0,T_1), then, from the properties satisfied by the basis $\mathcal{B}(t)$ (see (30)), it is not difficult to check that ψ solves

$$\begin{cases} -\partial_t \psi + L(t)\psi = C^*(t)\psi + G_0 + \nabla\cdot G \text{ in } \Omega\times(T_0,T_1), \\ \psi = 0 \text{ on } \partial\Omega\times(T_0,T_1), \quad \psi(x,T_1) = P(T_1)^*\varphi(x,T_1) \text{ in } \Omega, \end{cases}$$

where $C(t)$ is the matrix given by (17), $G_0 = P^*(\cdot)F_0$ and $G = FP(\cdot)$. If we apply to ψ Theorem 2.2 in the interval (T_0,T_1) we deduce the existence of $\alpha_0 \in C^2(\overline{\Omega})$, only depending on Ω and ω, two positive constants $\widetilde{C}_0 = \widetilde{C}_0(\Omega,\omega,(a_{ij})_{1\leqslant i,j\leqslant N},n,m,A(\cdot),B(\cdot))$ and $\widetilde{\sigma}_0 = \widetilde{\sigma}_0(\Omega,\omega,(a_{ij})_{1\leqslant i,j\leqslant N},n,m,A(\cdot),B(\cdot))$ and integers $\ell_j \geqslant 3$, $\ell_k^1 \geqslant 0$ and $\ell_k^2 \geqslant 2$, with $1 \leqslant j \leqslant r$ and $1 \leqslant k \leqslant n$, (only depending on n, m, $A(\cdot)$ and $B(\cdot)$) in such a way that ψ satisfies the inequality

$$\sum_{i=1}^n \widetilde{\mathfrak{I}}(3,\psi_i) \leqslant \widetilde{C}_0 \Bigg(\sum_{j=1}^r s^{\ell_j} \iint_{\omega\times(T_0,T_1)} e^{-2s\widetilde{\alpha}} \widetilde{\gamma}(t)^{\ell_j} |\psi_{S_j}|^2 + \sum_{k=1}^n \left[s^{\ell_k^1} \iint_{\Omega\times(T_0,T_1)} e^{-2s\widetilde{\alpha}} \widetilde{\gamma}(t)^{\ell_k^1} |G_0^k|^2 + s^{\ell_k^2} \iint_{\Omega\times(T_0,T_1)} e^{-2s\widetilde{\alpha}} \widetilde{\gamma}(t)^{\ell_k^2} |G^k|^2 \right] \Bigg),$$

for every $s \geqslant \widetilde{s}_0 = \widetilde{\sigma}_0 \left(T + T^2 + T^2 ||c||_\infty^{2/3} + T^2 ||b||_\infty^2 \right)$, where r and s_i are provided by Theorem 2.2 and $S_i = 1 + \sum_{j=1}^{i-1} s_j$, $(1 \leqslant i \leqslant r)$.

Finally, the proof can be obtained if we take $\ell = \max_{1\leqslant j\leqslant r} \ell_j$, $\ell^1 = \max_{1\leqslant k\leqslant n} \ell_k^1$ and $\ell^2 = \max_{1\leqslant k\leqslant n} \ell_k^2$, we replace $\varphi = (P^*(t))^{-1}\psi$ in the previous inequality and we take into account the equality $e_{S_i}^*\psi_{S_j} = (b^{l_j})^*(t)\varphi$. This finalizes the proof. □

We will finish this section proving Theorem 1.3 when A and B are analytic in $[0,T]$.

Proof of Theorem 1.3. It is clear that if there exist $t_0 \in [0,T]$ and $p \geqslant 1$ such that condition (12) holds then system (4) is exactly controllable to the trajectories on the interval $[0,T]$.

Let us see the necessary part. By contradiction, assume that for every $t_0 \in [0,T]$ one has

$$\operatorname{rank} K_p(t_0) < n, \quad \forall p \geqslant 1.$$

If we fix $t_0 \in [0,T]$, the previous condition implies the existence of $\xi \in \mathbb{R}^n \setminus \{0\}$ such that

$$\xi^* B_i(t_0) = 0, \quad \forall i \geqslant 0. \tag{37}$$

Under this last condition we will see that system (4) is equivalent (on the time interval $[0,T]$) to an appropriate system which contains, at least, one equation which cannot be controlled. Therefore, we will deduce that system (4) is not null controllable on $[0,T]$.

Let us consider a fundamental matrix $F(t)$ associated to the ordinary differential system $x' = A(t)x$ on $[0,T]$. It is well known that $F(t)$ is nonsingular for all $t \in [0,T]$, is analytic on $[0,T]$ and satisfies

$$\frac{d}{dt}\left(F(t)^{-1}\right) = -F(t)^{-1}A(t), \quad \forall t \in [0,T].$$

It is not difficult to check the equality (see (8))

$$\frac{d^i}{dt^i}\left(F(t)^{-1}B(t)\right) = (-1)^i F(t)^{-1} B_i(t), \quad \forall t \in [0,T], \quad \forall i \geqslant 0.$$

We now take the function $g(t) = \xi^* F(t_0)F(t)^{-1}B(t)$ (analytic on $[0,T]$). The previous equality, together with condition (37), provides us the property

$$\frac{d^i g}{dt^i}(t_0) = 0, \quad \forall i \geqslant 0,$$

whence $g(t) = 0$, i.e., $\xi^* F(t_0)F(t)^{-1}B(t) = 0$, for every $t \in [0,T]$. Finally, we consider a nonsingular matrix $P \in \mathcal{L}(\mathbb{R}^n)$ whose first row is given by ξ^*. Thus, if system (4) is null controllable on $[0,T]$, performing the change $w = PF(t_0)F(t)^{-1}y$ and taking $\widetilde{B}(t) = PF(t_0)F(t)^{-1}B(t)$, the system

$$\begin{cases} \partial_t w + L(t)w = \widetilde{B}(t)v1_\omega \text{ in } Q, \\ w = 0 \text{ on } \Sigma, \end{cases} \tag{38}$$

is also null controllable on the time interval $[0,T]$. Nevertheless, the previous properties show

$$\widetilde{B}(t) = \begin{pmatrix} 0 \\ \widehat{B}(t) \end{pmatrix},$$

with $\widehat{B}(\cdot) \in C^\infty([0,T]; \mathcal{L}(\mathbb{R}^m; \mathbb{R}^{n-1}))$ (in fact, analytic on $[0,T]$). We readily deduce that system (38) is not null controllable on $[0,T]$ which contradicts the assumptions. This ends the proof. $\square$

REMARK 4.2. It is interesting to remark that, when A and B are analytic on $[0,T]$, the previous proof shows that a necessary and sufficient condition for the exact controllability to the trajectories of system (4) on $[0,T]$ is: *for every* $t \in [0,T]$ *there exists* $p \in \mathbb{N}$ *such that* $\operatorname{rank} K_p(t) = n$.

On the other hand, in the proof of Theorem 1.2 we have shown that if for $t_0 \in [0,T]$ and $p \geqslant 1$ we have $\operatorname{rank} K_p(t_0) = n$, then there exists a non degenerate interval $[T_0, T_1] \subseteq [0,T]$ such that $\operatorname{rank} K_n(t) = n$ for every $t \in [T_0, T_1]$. In particular, when A and B are analytic, we deduce the existence of a finite set $\mathcal{F}$ such that $\operatorname{rank} K_n(t) = n$ for every $t \in [0,T] \setminus \mathcal{F}$. Therefore, a necessary and sufficient condition for the exact controllability to the trajectories of system (4) on $[0,T]$ is the following one: *there exists a finite set* $\mathcal{F}$ *such that* $\operatorname{rank} K_n(t) = n$ *for every* $t \in [0,T] \setminus \mathcal{F}$.

5. Further results and comments

We will finalize this work doing some remarks and establishing some additional results.

1. All along this work we have assumed that A and B are matrix functions which satisfy (5) for $M \geqslant n$. It is not difficult to see that Theorem 1.2 is still valid under the following assumptions: $A \in W^{M-1,\infty}(0,T; \mathcal{L}(\mathbb{R}^n))$ and $B \in W^{M,\infty}(0,T; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$.
2. In the case of linear ordinary differential systems as (7) it is well known that the condition (10) is not necessary in order to obtain the null controllability result on the interval $[0,T]$. Let us show that for the system (4) condition (10) neither is a necessary

condition for the exact controllability to the trajectories on the interval $[0,T]$. Indeed, following [6], let us set $n=2$ and $m=1$ and let us take

$$A(t)\equiv 0 \quad \text{and} \quad B(t)=\begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}, \quad \forall t\in(0,T),$$

that is to say, let us take the system

$$\begin{cases} \partial_t y + L(t)y = \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} v 1_\omega \text{ in } Q, \\ y=0 \text{ in } \Sigma, \quad y(\cdot,0)=y_0 \text{ in } \Omega, \end{cases} \tag{39}$$

with $y_0\in L^2(\Omega)^2$, $b_1\in C_0^\infty(0,T/2)$ and $b_2\in C_0^\infty(T/2,T)$ such that for a positive constant β and nonempty time intervals $(\tau_0,\tau_1)\subset\subset(0,T/2)$ and $(\tau_0',\tau_1')\subset\subset(T/2,T)$ one has

$$|b_1(t)|\geqslant\beta>0, \quad \forall t\in(\tau_0,\tau_1) \quad \text{and} \quad |b_2(t)|\geqslant\beta>0, \quad \forall t\in(\tau_0',\tau_1').$$

Evidently, A and B satisfy (5) and $\operatorname{rank} K_p(t)<2$ for every $t\in[0,T]$ and $p\geqslant 1$. Let us show that system (39) is exactly controllable to the trajectories or equivalently, null controllable on the interval $[0,T]$: Let us denote $Y=(Y_1,Y_2)^*\in L^2(0,T;H_0^1(\Omega)^2)$ the solution to (39) corresponding to $y_0\in L^2(\Omega)^2$ and $v\equiv 0$. Firstly, thanks to the assumptions on the operator L and the function b_1, it is not difficult to prove that there exists $u_1\in L^2(\Omega\times(\tau_0,\tau_1))$ such that the solution $w_1\in L^2(\tau_0,\tau_1;H_0^1(\Omega))$ to the scalar equation

$$\begin{cases} \partial_t w_1 + L(t)w_1 = b_1(t)u_1 1_\omega \text{ in } \Omega\times(\tau_0,\tau_1) \\ w_1=0 \text{ on } \partial\Omega\times(\tau_0,\tau_1), \quad w_1(\cdot,\tau_0)=Y_1(\cdot,\tau_0) \end{cases}$$

satisfies $w_1(\cdot,\tau_1)=0$ in Ω. On the other hand, we can reason in a similar way and deduce the existence of $u_2\in L^2(\Omega\times(\tau_0',\tau_1'))$ such that the solution $w_2\in L^2(\tau_0',\tau_1';H_0^1(\Omega))$ to

$$\begin{cases} \partial_t w_2 + L(t)w_2 = b_2(t)u_2 1_\omega \text{ in } \Omega\times(\tau_0',\tau_1') \\ w_2=0 \text{ on } \partial\Omega\times(\tau_0',\tau_1'), \quad w_2(\cdot,\tau_0')=Y_2(\cdot,\tau_0') \end{cases}$$

satisfies $w_2(\cdot,\tau_1')=0$ in Ω. Finally, let us set

$$y_1(x,t)=\begin{cases} Y_1(x,t) & \text{if } t\in[0,\tau_0], \\ w_1(x,t) & \text{if } t\in[\tau_0,\tau_1], \\ 0 & \text{if } t\in[\tau_1,T], \end{cases} \qquad y_2(x,t)=\begin{cases} Y_2(x,t) & \text{if } t\in[0,\tau_0'], \\ w_2(x,t) & \text{if } t\in[\tau_0',\tau_1'], \\ 0 & \text{if } t\in[\tau_1',T], \end{cases}$$

$y=(y_1,y_2)^*$ and

$$v(x,t)=\begin{cases} v_1(x,t) \text{ if } t\in[0,T/2], \\ v_2(x,t) \text{ if } t\in[T/2,T], \end{cases}$$

where v_1 and v_2 are, respectively, the extensions by 0 of u_1 and u_2 to the whole interval $[0,T]$. Now, it is easy to check that y is the solution to system (39) corresponding to v and satisfies $y(\cdot,T)=0$ in Ω. In conclusion, condition (10) is not necessary to have on $[0,T]$ the exact controllability of system (4) to the trajectories.

3. *Approximate controllability.* As a consequence of the results stated in this work, we can obtain a result on approximate controllability of system (4) in $(0,T)$. To be precise, one has the following result.

THEOREM 5.1. *Assume that the matrices A and B satisfy* (5)*. Then, the following holds:*

(a) If there exist $t_0 \in [0,T]$ and $p \in \{1,...,M\}$ such that $\operatorname{rank} K_p(t_0) = n$*, then system* (4) *is approximately controllable on the interval $(0,T)$.*

(b) System (4) *is approximately controllable on every interval (T_0,T_1) with $0 \leqslant T_0 < T_1 \leqslant T$ if and only if there exists E a dense subset of $(0,T)$ such that* $\operatorname{rank}[A\,|\,B](t) = \operatorname{rank} K_n(t) = n$ *for every $t \in E$.*

Proof. (a) It is well known that the approximate controllability on $(0,T)$ of system (4) amounts to the following *unique continuation* property on $(0,T)$ for the adjoint system (13):

"*If $\varphi \in C^0([0,T];L^2(\Omega)^n)$ is a solution to* (13) *and $B^*(t)\varphi \equiv 0$ in $\omega \times (0,T)$, then $\varphi \equiv 0$ in Q.*"

Thus, assume that for $t_0 \in [0,T]$ and $1 \leqslant p \leqslant M$ we have $\operatorname{rank} K_p(t_0) = n$ and let φ be a solution to (13) such that $B^*(t)\varphi \equiv 0$. In particular, φ satisfies the Carleman inequality stated in Corollary 4.3 on an appropriate interval $(T_0,T_1) \subseteq (0,T)$. We can conclude that $\varphi \equiv 0$ in $\Omega \times (T_0,T_1)$, whence $\varphi \equiv 0$ in Q. This last fact can be inferred from the backward uniqueness result that fulfills the system satisfied by $\psi(x,t) = \varphi(x,T-t)$ (e.g., see [10]).

(b) If for a dense subset E of $(0,T)$ one has $\operatorname{rank}[A\,|\,B](t) = n$ in E then, the previous argument shows that system (4) is approximately controllable on every time interval $(T_0,T_1) \subseteq (0,T)$.

On the other hand, let us suppose that system (4) is approximately controllable on every non degenerate interval $[T_0,T_1]$ of $[0,T]$ and, again, by contradiction, let us assume that there exists a non degenerate interval $[\tau_0,\tau_1] \subseteq [0,T]$ with $\max\{\operatorname{rank}[A\,|\,B](t) : t \in [\tau_0,\tau_1]\} < n$. We can reason as in the proof of Theorem 1.2 2 and show that, for a new non degenerate interval $[\widehat{\tau}_0,\widehat{\tau}_1] \subseteq [0,T]$, system (4) is equivalent to the decoupled system (36) which, evidently, is not approximately controllable on $[\widehat{\tau}_0,\widehat{\tau}_1]$. This completes the proof. $\square$

4. The Carleman inequalities stated in Theorem 1.6 and Corollary 4.3 also permit to evaluate the cost of the exact controllability to the trajectories of system (4). Following the ideas of [8] it is possible to show the following result.

THEOREM 5.2. *Let us assume that $L(\cdot)$ is given by* (1) *and satisfies* (2) *and* (3)*. Let $y^* \in L^2(0,T;H_0^1(\Omega)^n) \cap C^0([0,T];L^2(\Omega)^n)$ be a trajectory of system* (4) *and let us fix $y_0 \in L^2(\Omega)^n$. Then, if* (12) *holds for $t_0 \in [0,T]$ and $p \in \{1,...,M\}$, there exists*

$v \in L^2(\Omega_T)^m$ such that the solution to (4) *satisfies $y(\cdot,T) = y^*(\cdot,T)$ in Ω. Moreover, there exist a time interval $(T_0,T_1) \subseteq (0,T)$ such that, for a positive constant C (which only depends on Ω, ω, $(a_{ij})_{1 \leqslant i,j \leqslant N}$, n, m, $A(\cdot)$ and $B(\cdot)$), one has*

$$||v||^2_{L^2(\Omega)^m} \leqslant \exp\left(C\left(1+\widetilde{T}+\frac{1}{\widetilde{T}}+T_1||c||_\infty+||c||_\infty^{2/3}+(1+T_1)||b||_\infty^2\right)\right)||y_0-y^*(\cdot,0)||^2,$$

with $\widetilde{T} = T_1 - T_0$ and $||.|| = ||.||_{L^2(\Omega)^n}$.

5. It is worthy of mention that Theorems 1.2, 1.3, 1.4 and 1.6 are still valid if in (4) we consider Neuman or Robin boundary conditions instead of Dirichlet boundary conditions.

6. *Boundary controls.* In view of known controllability results for a linear heat equation, it would be natural to wonder whether the controllability result for system (4) remains valid when one considers boundary controls exerted on a relative open subset γ of the boundary $\partial\Omega$. Nevertheless, there exist negative results for some 1-d cascade linear coupled parabolic systems with $n = 2$ which are null controllable in $(0,T)$ when we apply a distributed control $e_1 v 1_\omega$ and they are not if we take $y = e_1 v 1_\gamma$ on $\partial\Omega \times (0,T)$ as boundary control (cf. [7]). These counterexamples reveal the different nature of the controllability properties for a single heat equation and for coupled parabolic systems.

7. *Open problems.* As said above, in [3] and [4] the authors provide a necessary and sufficient condition for the null controllability of system (6). In this sense, it would be very interesting to generalize the results of [3] and [4] and give a characterization of the controllability properties on $(0,T)$ of the time-dependent system

$$\begin{cases} \partial_t y + DL(t)y = A(t)y + B(t)v1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \end{cases}$$

with $D = P^{-1}\mathrm{diag}\,(d_1,\dots,d_n)P$ ($d_i > 0$ for every $1 \leqslant i \leqslant n$ and $\det P \neq 0$), A and B as in (5), and $L(\cdot)$ satisfying (1), (2) and (3). Much more complicated is the case in which $D \in \mathcal{L}(\mathbb{R}^n)$ is a non diagonalizable symmetric definite positive matrix, even in the time-independent case $L(t) \equiv L$, $A(t) = A$ and $B(t) = B$ for every $t \in [0,T]$.

Finally, let us remark that there are few results about the controllability properties of general linear parabolic systems and, therefore, obtaining a general theory that characterizes the controllable parabolic systems is a widely open problem.

A. Appendix. Proof of Theorem 2.2

The starting point for proving Theorem 2.2 is a global Carleman estimate for the solutions to the scalar parabolic problem

$$\begin{cases} -\partial_t z - L^*(t)z = F_0 + \sum_{i=1}^N \dfrac{\partial F_i}{\partial x_i} \ \text{ in } Q, \\ z = 0 \ \text{ on } \Sigma, \quad z(x,T) = z^0(x) \ \text{ in } \Omega, \end{cases} \tag{40}$$

with $z^0 \in L^2(\Omega)$ and $F_i \in L^2(Q)$, $i = 0,1,\ldots,N$.

LEMMA A.1. *Let $\mathcal{B} \subset \Omega$ be a nonempty open subset and $d \in \mathbb{R}$. Then, there exist a function $\beta_0 \in C^2(\overline{\Omega})$ (only depending on Ω and $\mathcal{B}$) and two positive constants $\widehat{C}_0$ and $\widehat{\sigma}_0$ (which only depend on Ω, $\mathcal{B}$, $(a_{ij}(x,t))_{1\leqslant i,j\leqslant N}$ and d) such that, for every $z^0 \in L^2(\Omega)$, the solution z to (40) satisfies*

$$s^{d-2}\iint_Q e^{-2s\beta}\gamma(t)^{d-2}|\nabla z|^2 + s^d\iint_Q e^{-2s\beta}\gamma(t)^d|z|^2$$
$$\leqslant \widehat{C}_0\left(\mathcal{L}_{\mathcal{B}}(d,z) + s^{d-3}\iint_Q e^{-2s\beta}\gamma(t)^{d-3}|F_0|^2 + s^{d-1}\sum_{i=1}^N\iint_Q e^{-2s\beta}\gamma(t)^{d-1}|F_i|^2\right),$$

for all $s \geqslant \widehat{s}_0 = \widehat{\sigma}_0\left(T + T^2 + T^2||c||_\infty^{2/3} + T^2||b||_\infty^2\right)$. In (22), $\mathcal{L}_{\mathcal{B}}(d,z)$ and the functions β and γ are given by

$$\mathcal{L}_{\mathcal{B}}(d,z) \equiv s^d\iint_{\mathcal{B}\times(0,T)} e^{-2s\alpha}\gamma(t)^d|z|^2, \quad \beta(x,t) = \frac{\beta_0(x)}{t(T-t)}, \quad \forall (x,t) \in Q,$$

and $\gamma(t) = (t(T-t))^{-1}$, $t \in (0,T)$.

The proof of this result can be found in [14] although the authors do not specify the way the constant $\widehat{s}_0$ depends on T. This explicit dependence can be obtained arguing as in [8].

Proof of Theorem 2.2. Let us write $C(t) = (c_{ij}(t))_{1\leqslant i,j\leqslant n}$ with $c_{ij} \in C^0([0,T])$ $(1 \leqslant i,j \leqslant n)$. Observe that thanks to (17), for every $t \in [0,T]$ we have

$$\begin{cases} c_{ij}(t) = 0 & \text{if } 1 \leqslant j \leqslant n-2 \text{ and } j+2 \leqslant i \leqslant n, \\ c_{i,i-1}(t) = 1, & \text{if } 2 \leqslant i \leqslant n \text{ and } i \notin \{S_j : 1 \leqslant j \leqslant r\}, \\ c_{i,i-1}(t) = 0, & \text{if } 2 \leqslant i \leqslant n \text{ and } i \in \{S_j : 1 \leqslant j \leqslant r\}. \end{cases}$$

We reason as in [12] and we choose $\omega_0 \subset\subset \omega$. Let $\alpha_0 \in C^2(\overline{\Omega})$ be the function provided by Lemma A.1 and associated to Ω and $\mathcal{B} \equiv \omega_0$, and let $\alpha(x,t)$ the function given by $\alpha(x,t) = \alpha_0(x)/t(T-t)$. We will do the proof in two steps:

Step 1. Let $\psi = (\psi_i)_{1\leqslant i\leqslant n}$ be the solution to (18) associated to $\psi_0 \in L^2(\Omega)^n$, $G_0 = (G_0^i)_{1\leqslant i\leqslant n} \in L^2(Q)^n$ and $G = (G^1\,|\,G^2\,|\ldots|\,G^n) \in L^2(Q;\mathcal{L}(\mathbb{R}^n;\mathbb{R}^N))$. By Lemma A.1 for each function ψ_i $(1 \leqslant i \leqslant n)$ with $\mathcal{B} = \omega_0$, $d = 3$, $F_0 = \sum_{j=1}^{i+1} a_{ji}\psi_j + G_0^i$ and $G = G^i$, we get

$$\mathcal{I}(3,\psi_i) \leqslant \widehat{C}_1\left(\mathcal{L}_{\omega_0}(3,\psi_i) + M\sum_{j=1}^{i+1}\iint_Q e^{-2s\alpha}|\psi_j|^2\right.$$
$$\left. + \iint_Q e^{-2s\alpha}|G_0^i|^2 + s^2\iint_Q e^{-2s\alpha}\gamma(t)^2|G^i|^2\right)$$

for every $s \geqslant \widehat{s}_1 = \widehat{\sigma}_1 \left(T + T^2 + T^2 ||c||_\infty^{2/3} + T^2 ||b||_\infty^2\right)$. In this inequality $\widehat{\sigma}_1$ and $\widehat{C}_1$ are positive constants only depending on Ω, ω_0, $(a_{ij}(x,t))_{1\leqslant i,j\leqslant N}$ and n, and $M = \max_{1\leqslant i,j\leqslant n} ||c_{ij}||_\infty$. From this inequality and reasoning as in [12], we readily obtain the existence of two new positive constants C_1 and $\widetilde{\sigma}_0$ (only depending on Ω, ω_0, $(a_{ij}(x,t))_{1\leqslant i,j\leqslant N}$, n and M) for which

$$\sum_{j=1}^n \mathcal{I}(3,\psi_j) \leqslant C_1 \sum_{j=1}^n \left(\mathcal{L}_{\omega_0}(3,\psi_j) + \iint_Q e^{-2s\alpha}|G_0^j|^2 + s^2 \iint_Q e^{-2s\alpha}\gamma(t)^2 |G^j|^2 \right), \quad (41)$$

holds for every $s \geqslant \widetilde{s}_0 = \widetilde{\sigma}_0 \left(T + T^2 + T^2 ||c||_\infty^{2/3} + T^2 ||b||_\infty^2\right)$.

Step 2. In the previous inequality we can eliminate the local terms corresponding to ψ_k, with $2 \leqslant k \leqslant n$ and $k \notin \{S_i : 1 \leqslant i \leqslant r\}$, applying successively the following

LEMMA A.2. *Under assumptions of Theorem* 2.1 *and given* $l \in \mathbb{N}$, $\varepsilon > 0$, $2 \leqslant k \leqslant n$ *with* $k \notin \{S_i : 1 \leqslant i \leqslant r\}$, *and two open sets* $\mathcal{O}_0$ *and* $\mathcal{O}_1$ *such that* $\omega_0 \subset \mathcal{O}_0 \subset \subset \mathcal{O}_1 \subset \omega$, *there exists a positive constant* C_k (*only depending on* Ω, ω_0, $\mathcal{O}_0$, $\mathcal{O}_1$, $(a_{ij}(x,t))_{1\leqslant i,j\leqslant N}$, n *and* M), *such that, if* ψ *is the solution to* (18) *associated to* ψ_0, G_0 *and* G *and* $s \geqslant \widetilde{s}_0$, *one has*

$$\begin{aligned}\mathcal{L}_{\mathcal{O}_0}(l,\psi_k) \leqslant \varepsilon\,[\mathcal{I}(3,\psi_k) + \mathcal{I}(3,\psi_{k+1})] + C_k \Bigg(&\sum_{j=1}^{k-2} \mathcal{L}_{\mathcal{O}_1}(l_j,\psi_j) \\ &+ \left(1+\frac{1}{\varepsilon}\right)\mathcal{L}_{\mathcal{O}_1}(J,\psi_{k-1}) + \iint_Q e^{-2s\alpha}|G_0^k|^2 + s^2\iint_Q e^{-2s\alpha}\gamma(t)^2|G^k|^2 \\ &+ s^l \iint_Q e^{-2s\alpha}\gamma(t)^l |G_0^{k-1}|^2 + \left(1+\frac{1}{\varepsilon}\right) s^R \iint_Q e^{-2s\alpha}\gamma(t)^R |G^{k-1}|^2 \Bigg),\end{aligned}$$

where $J = \max\{l+4, 2l+1, 3l-2\}$, $R = \max\{l+1, 2l-1\}$ *and* $l_j = \max\{l,3\}$. (*In the previous inequality we have taken* $\varphi_{k+1} \equiv 0$ *when* $k = n$).

The proof of this lemma can be obtained if we reason as in [12].

In order to finalize the proof, we consider an open set $\widetilde{\mathcal{O}}_1$ such that $\omega_0 \subset\subset \widetilde{\mathcal{O}}_1 \subset\subset \omega$. Let us assume that $S_r < n$. Thus, we apply Lemma A.2 with $\mathcal{O}_0 = \omega_0$, $\mathcal{O}_1 = \widetilde{\mathcal{O}}_1$, $k = n$, $l = 3$ and $\varepsilon = 1/2C_1$ (with C_1 the constant appearing in (41)) and we deduce

$$\begin{aligned}\sum_{k=1}^n \mathcal{I}(3,\psi_k) \leqslant C_2 \Bigg(&\sum_{j=1}^{n-2} \mathcal{L}_{\widetilde{\mathcal{O}}_1}(3,\psi_j) + \mathcal{L}_{\widetilde{\mathcal{O}}_1}(7,\psi_{n-1}) \\ &+ \sum_{j=1}^n \left[\iint_Q e^{-2s\alpha}|G_0^j|^2 + s^2 \iint_Q e^{-2s\alpha}\gamma(t)^2|G^j|^2 \right] \\ &+ s^3 \iint_Q e^{-2s\alpha}\gamma(t)^3 |G_0^{n-1}|^2 + s^5 \iint_Q e^{-2s\alpha}\gamma(t)^5 |G^{n-1}|^2 \Bigg),\end{aligned}$$

for all $s \geqslant \widetilde{s}_0$, with C_2 a new positive constant only depending on $\widetilde{\mathcal{O}}_1$, $(a_{ij}(x,t))_{1 \leqslant i,j \leqslant N}$, Ω, ω_0, n and M. Observe that if $S_r = n$, we would reason as above with: $k = \max\{j : 2 \leqslant j \leqslant n \text{ and } j \notin \{S_i : 1 \leqslant i \leqslant r\}\}$ and $l = 3$.

Assume $S_r < n-1$. In the previous inequality we can then eliminate the local term corresponding to ψ_{n-1} reasoning as follows: we take a new open set $\widetilde{\mathcal{O}}_2$ such that $\widetilde{\mathcal{O}}_1 \subset\subset \widetilde{\mathcal{O}}_2 \subset\subset \omega$ and we again apply Lemma A.2 with $\mathcal{O}_0 = \widetilde{\mathcal{O}}_1$, $\mathcal{O}_1 = \widetilde{\mathcal{O}}_2$, $k = n-1$, $l = 7$ and $\varepsilon = 1/2C_2$. We get

$$\begin{aligned}\sum_{k=1}^{n} \mathcal{I}(3,\psi_k) \leqslant C_3 \Bigg(& \sum_{j=1}^{n-3} \mathcal{L}_{\widetilde{\mathcal{O}}_1}(7,\psi_j) + \mathcal{L}_{\widetilde{\mathcal{O}}_1}(19,\psi_{n-2}) \\ & + \sum_{j=1}^{n} \left[\iint_Q e^{-2s\alpha} |G_0^j|^2 + s^2 \iint_Q e^{-2s\alpha} \gamma(t)^2 |G^j|^2 \right] \\ & + s^7 \iint_Q e^{-2s\alpha} \gamma(t)^7 |G_0^{n-2}|^2 + s^{13} \iint_Q e^{-2s\alpha} \gamma(t)^{13} |G^{n-1}|^2 \\ & + s^3 \iint_Q e^{-2s\alpha} \gamma(t)^3 |G_0^{n-1}|^2 + s^5 \iint_Q e^{-2s\alpha} \gamma(t)^5 |G^{n-1}|^2 \Bigg),\end{aligned}$$

for all $s \geqslant \widetilde{s}_0$ (C_2 is a new positive constant only depending on Ω, $\widetilde{\mathcal{O}}_1$, $\widetilde{\mathcal{O}}_2$, n, M, and $(a_{ij}(x,t))_{1 \leqslant i,j \leqslant N}$). If we repeat this argument for each j, with $2 \leqslant j \leqslant n$ and $j \notin \{S_i : 1 \leqslant i \leqslant r\}$ we deduce the result. This ends the proof. $\square$

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