*Séminaires & Congrès*
**8**, 2004, p. 89–128

# EXPLICIT CALCULATIONS IN RINGS OF DIFFERENTIAL OPERATORS

*by*

Francisco J. Castro-Jiménez & Michel Granger

***Abstract.*** — We use the notion of a standard basis to study algebras of linear differential operators and finite type modules over these algebras. We consider the polynomial and the holomorphic cases as well as the formal case.
Our aim is to demonstrate how to calculate classical invariants of germs of coherent (left) modules over the sheaf $\mathscr{D}$ of linear differential operators over $\mathbb{C}^n$. The main invariants we deal with are: the characteristic variety, its dimension and the multiplicity of this variety at a point of the cotangent space.
In the final chapter we shall study more refined invariants of $\mathscr{D}$-modules linked to the question of irregularity: The slopes of a $\mathscr{D}$-module along a smooth hypersurface of the base space.

***Résumé* (Calculs explicites dans l'anneau des opérateurs différentiels).** — Dans ce cours on développe la notion de base standard, en vue d'étudier les algèbres d'opérateurs différentiels linéaires et les modules de type fini sur ces algèbres. On considère le cas des coefficients polynomiaux, des coefficients holomorphes ainsi que le cas des algèbres d'opérateurs à coefficients formels.
Notre but est de montrer comment les bases standards permettent de calculer certains invariants classiques des germes de modules (à gauche) cohérents sur le faisceaux $\mathscr{D}$ des opérateurs différentiels linéaires sur $\mathbb{C}^n$. Les principaux invariants que nous examinons sont : la variété caractéristique, sa dimension et sa multiplicité en un point du fibré cotangent.
Dans le dernier chapitre nous étudions des invariants plus fins des $\mathscr{D}$-modules qui sont reliés aux questions d'irrégularité : les pentes d'un $\mathscr{D}$-module, le long d'une hypersurface lisse.

**2000 *Mathematics Subject Classification.*** — 13N10, 13P10, 16S32.
***Key words and phrases.*** — D-modules, Gröbner basis, slopes.

F.C.: Partially supported by DGESIC-PB97-0723; BFM2001-3164 and FQM-218.
Both authors partially supported by Picasso-HF2000-0044.

## Introduction

The purpose of these notes is to make an account of explicit methods, using the notion of a standard basis, which could be used in studying algebras of linear differential operators and finite type modules over these algebras. We consider in parallel each of the following cases: coefficients in a ring of polynomials $\boldsymbol{k}[x_1, \ldots, x_n]$ for the Weyl algebra $A_n(\boldsymbol{k})$, in the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$ for $\mathscr{D}_n$, or in the ring of formal power series for $\widehat{\mathscr{D}_n}$. We denote $\mathscr{R}$ any of these rings of operators and $\mathscr{B}$ the corresponding commutative ring of coefficients.

Our aim is to demonstrate how to calculate classical invariants of germs of coherent (left) modules over the sheaf $\mathscr{D}$ of linear differential operators over $\mathbb{C}^n$. In practice we shall look at finite type modules over $\mathscr{D}_n$ or $\widehat{\mathscr{D}_n}$. The main invariants we are dealing with are: the characteristic variety, and the multiplicity of this variety at a point of the cotangent space. See [**25**] and [**19**] for an introduction to the theory of $\mathscr{D}$-modules and for the definition of the characteristic variety, of its dimension and and of its multiplicity. In the last chapter we shall study more refined invariants of $\mathscr{R}$-modules linked to the question of irregularity: The slopes of a $\mathscr{D}_n$-module or an $A_n(\boldsymbol{k})$-module along a smooth hypersurface of the base space. In these notes we deal mainly with the case of monogenic modules $\mathscr{M} = \mathscr{R}/I$ with $I$ a (left) ideal of $\mathscr{R}$. We provide an algorithm to build standard bases of $I$ and in the context of chapter II these bases yield a special kind of system of generators for which the module of relations is easy to describe. There is a straightforward generalisation for the case $\mathscr{M} = \mathscr{R}^p/\mathscr{N}$ involving a submodule $\mathscr{N}$ of $\mathscr{R}^p$. Then continuing the process of building standard bases for submodules we can thus obtain a (locally) free resolution of $\mathscr{M}$. The techniques used are the notion of privileged exponents with respect to an ordering and a theorem of division. They were introduced by H. Hironaka (*cf.* [**26**] or [**1**]). In the polynomial case the notion of a standard basis was developed by Buchberger under the name of a Gröbner basis in [**13**] where he also gives an algorithm for its calculation.

The commutative case is treated in chapter I, where we recall the notions of a privileged exponent of a polynomial or a power series with respect to a convenient ordering, the definition of a standard basis and the algorithm for calculating it, which is the Buchberger's algorithm in the polynomial case. We also draw attention to the elegant proof in the convergent case taken from Hauser and Muller (*cf.* [**20**].) We finish by giving some applications in commutative algebra such as calculating multiplicities, syzygies, and the intersections of ideals.

In chapter II, we consider division processes in algebras of operators which are compatible with a filtration which may either be the filtration by the order of operators or in the particular case of $A_n(\boldsymbol{k})$, the Bernstein filtration by the total order. At the same time, for the sake of completeness we treat a weighted homogeneous version of these filtrations. Using a compatible ordering on monomials we again develop a division algorithm and an algorithm for the construction of a standard basis. These algorithms are very similar to those developed in chapter I, since in fact a division by a family of operators $\{P_1, \ldots, P_r\}$, or by a standard basis of an ideal $I$ induces the same object via the principal symbols in the commutative associated graded rings. The references for these results are [**11**] and [**14**]. Let us also notice that it is only in the case of $\boldsymbol{k}[x_1, \ldots, x_r]$ or $A_n(\boldsymbol{k})$ that the suitable orderings used in chapters I and II are well orderings and therefore that the algorithms are effective. In the power series case they depend on formal or convergent processes in the local rings of series.

In chapter III we give an algorithm for the calculation of the slopes of a coherent $\mathscr{R}$-module along a smooth hypersurface $Y$ of $\boldsymbol{k}^n$ or $\mathbb{C}^n$ in the neighbourhood of a point of $Y$. The material is essentially taken from our work with A.Assi [**2**] where however only the case of $A_n(\boldsymbol{k})$ is considered.

The notion of a slope of a coherent $\mathscr{D}$-module $\mathscr{M}$ was introduced by Y. Laurent under the name of a critical index. He considers, in the more general context of microdifferential operators a family of filtrations $L_r = pF + qV$ (with $r$ a rational number such that $0 \leqslant r = p/q \leqslant +\infty$), which is an interpolation between the filtration by the order $F$ and the $V$-filtration of Malgrange and Kashiwara (*cf.* [**22**]). The critical indices are those for which the $L_r$-characteristic variety of $\mathscr{M}$ is not bihomogeneous with respect to $F$ and $V$. Laurent proved in loc. cit. the finiteness of the number of slopes and then C. Sabbah and F. Castro proved the same result in [**30**] by using a local flattener. In [**28**] Z. Mebkhout introduced the notion of a transcendental slope of a holonomic $\mathscr{D}$-module $\mathscr{M}$, as being a jump in the Gevrey filtration $\mathrm{Irr}_Y^{(r)}(\mathscr{M})$ of the irregularity sheaf $\mathrm{Irr}_Y(\mathscr{M})$. The irregularity sheaf is the complex of solutions of $\mathscr{M}$ with values in the quotient of the formal completion along $Y$ of the structural sheaf $\mathscr{O}$, by $\mathscr{O}$ itself. By the main result of [**28**], it is a perverse sheaf, and $\mathrm{Irr}_Y^{(r)}(\mathscr{M})$ is the sub-perverse sheaf of solutions in formal series of Gevrey type $r$ along $Y$. In [**23**] Laurent and Z. Mebkhout proved that the transcendental slopes of an holonomic $\mathscr{D}$-module are equal to the slopes in the sense of Laurent called algebraic slopes. The analogue in dimension one is Malgrange's paper [**27**] for the perversity of the irregularity sheaf and Ramis's paper [**29**] for the theorem of the comparison of slopes.

In chapter III, we recall the principle of the algorithm of calculation of the algebraic slopes of an $\mathscr{R}$-module that we developed in [**2**] and we give some supplementary information. Here the additional difficulty is that the linear form $L_r$ which yields the similarly called filtration now possesses a negative coefficient in the variable $x_1$. Although we can still speak of privileged exponents and standard bases, the standard bases are no longer systems of generators of the ideal $I$ which we consider but only induce a standard basis of the graded associated ideal. A more serious consequence of non-positivity, is that the straightforward division algorithm does not work inside *finite order* operators. The way to solve this problem is to homogenize the operators in $\mathscr{R}[t]$ with respect to the order filtration or, in the case of $A_n(\boldsymbol{k})$, with respect to the Bernstein filtration. We notice in chapter III, following a remark made by L. Narváez

[**16**] that we can simplify the original proof in [**2**] by considering on $A_n[t]$ a different structure as a Rees ring. Another improvement to [**2**] lies in the distinction between the slopes in the sense of Laurent and the values of $r$ for which the ideal $I$ gives a non-bihomogeneous graded ideal $\mathrm{gr}_{L_r}(I)$. We call those $r$, the idealistic slopes of $I$. In [**2**] we considered only this set of slopes and proved its finiteness; this paper however already contains the hard part of the algorithm of the calculation of algebraic slopes. Let us end this introduction by pointing out two other extensions of the original material of our paper [**2**]. First we make the same algorithm work for the rings of operators $\mathscr{D}_n$, or $\widehat{\mathscr{D}_n}$. Secondly we give some significant examples of the calculations of slopes: the slopes of the direct image of $\mathscr{D}_{\mathbb{C}}e^{1/x^k}$ by an immersion in $\mathbb{C}^2$, with respect to a smooth curve $Y$ tangent to the support. This example contains idealistic slopes which end up not being algebraic slopes. Finally, we calculate the slopes of $\mathscr{D}_{\mathbb{C}^2}e^{1/(y^p-x^q)}$ along any line through the origin.

*Added on March 21, 2003.* — This paper was written in September 1996, as material for a six hour course given in the CIMPA summer school "Differential Systems" (Sevilla, September 1996). Consequently, the bibliography is outdated. Since then, many papers have been published about the computational aspects in $\mathscr{D}$-modules theory. We have therefore decided to add, after the references, a complementary list of recent publications on the subject.

## 1. Division theorems in polynomial rings and in power series rings

**1.1.** Let $\boldsymbol{k}$ be a field, with an arbitrary characteristic unless otherwise stated. Let $n$ be a positive integer . We denote by:

- $\boldsymbol{k}[\boldsymbol{X}] = \boldsymbol{k}[X_1, \ldots, X_n]$ the ring of polynomials with coefficients in $\boldsymbol{k}$ and variables $X_1, \ldots, X_n$.
- $\boldsymbol{k}[[\boldsymbol{X}]] = \boldsymbol{k}[[X_1, \ldots, X_n]]$ the ring of formal power series with coefficients in $\boldsymbol{k}$ and variables $X_1, \ldots, X_n$.
- $\boldsymbol{k}\{\boldsymbol{X}\} = \boldsymbol{k}\{X_1, \ldots, X_n\}$ the ring of convergent power series with coefficients in $\boldsymbol{k}$ and variables $X_1, \ldots, X_n$, if $\boldsymbol{k} = \mathbb{R}$ or $\mathbb{C}$. [(1)]

If $f \in \boldsymbol{k}[[\boldsymbol{X}]]$, $f \neq 0$, we write $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \boldsymbol{X}^\alpha$ where $f_\alpha \in \boldsymbol{k}$. If $f \in \boldsymbol{k}[\boldsymbol{X}]$ $f \neq 0$, then this sum is finite. The set $\mathscr{N}(f) = \{\alpha \in \mathbb{N}^n \mid f_\alpha \neq 0\}$ is called the Newton diagram of the power series or of the polynomial $f$.

**1.2. $L$-degree and $L$-valuation.** — Let $L : \mathbb{Q}^n \to \mathbb{Q}$ be a linear form with non negative coefficients.

***Definition 1.2.1***. — Let $0 \neq f \in \boldsymbol{k}[\boldsymbol{X}]$. We define the $L$-degree of $f$ (and we denote it by $\deg_L(f)$) as being $\max\{L(\alpha) \mid f_\alpha \neq 0\}$. We set $\deg_L(0) = -\infty$.

(1) Or, more generally, a complete valued field.

***Definition 1.2.2***. — Let $0 \neq f \in \boldsymbol{k}[[\boldsymbol{X}]]$. We define the $L$-valuation of $f$ (which we denote by $\mathrm{val}_L(f)$) as being $\min\{L(\alpha) \mid f_\alpha \neq 0\}$. We set $\mathrm{val}_L(0) = +\infty$.

We have $\deg_L(fg) = \deg_L(f) + \deg_L(g)$ if $f, g \in \boldsymbol{k}[\boldsymbol{X}]$ and $\mathrm{val}_L(fg) = \mathrm{val}_L(f) + \mathrm{val}_L(g)$ if $f, g \in \boldsymbol{k}[[\boldsymbol{X}]]$.

***Definition 1.2.3***. — Let $0 \neq f \in \boldsymbol{k}[[\boldsymbol{X}]]$. We call the sum $\mathrm{in}_L(f) = \sum_{L(\alpha)=\mathrm{val}_L(f)} f_\alpha \boldsymbol{X}^\alpha$ the $L$- initial form of the power series $f$ [(2)]. Let $I$ be an ideal of $\boldsymbol{k}[[\boldsymbol{X}]]$. We call the ideal of $\boldsymbol{k}[[\boldsymbol{X}]]$ generated by $\{\mathrm{in}_L(f) \mid f \in I\}$, the initial ideal of $I$ . We denote it by $\mathrm{In}_L(I)$ (or simply $\mathrm{In}(I)$)

***Notation***. — The following notation will be useful. If $f = \sum_\alpha f_\alpha \boldsymbol{X}^\alpha$ is a power series, we set $\mathrm{in}_{L,\nu}(f) = \sum_{L(\alpha)=\nu} f_\alpha \boldsymbol{X}^\alpha$. When no confusion can occur, we write $\mathrm{in}_\nu(f)$ instead of $\mathrm{in}_{L,\nu}(f)$. We have: $f = \sum_\nu \mathrm{in}_\nu(f)$.

***Definition 1.2.4***. — Let $0 \neq f \in \boldsymbol{k}[\boldsymbol{X}]$. We call the sum $\mathrm{fin}_L(f) = \sum_{L(\alpha)=\deg_L(f)} f_\alpha \boldsymbol{X}^\alpha$ the $L$-final form of the polynomial $f$. Let $I$ be an ideal of $\boldsymbol{k}[\boldsymbol{X}]$. We call the ideal of $\boldsymbol{k}[\boldsymbol{X}]$ generated by $\{\mathrm{fin}_L(f) \mid f \in I\}$ the final ideal of $I$. We denote it by $\mathrm{Fin}_L(I)$ (or simply by $\mathrm{Fin}(I)$).

**1.3. Orderings in $\mathbb{N}^n$.** — Let $<$ be a total well ordering on $\mathbb{N}^n$ compatible with sums (*i.e.* if $\alpha, \beta \in \mathbb{N}^n$ and $\alpha < \beta$ then we have $\alpha + \gamma < \beta + \gamma$ for any $\gamma \in \mathbb{N}^n$). Let $L : \mathbb{Q}^n \to \mathbb{Q}$ be a linear form with non negative coefficients . The relation $<_L$, defined by:

$$\alpha <_L \beta \text{ if and only if } \begin{cases} L(\alpha) < L(\beta) \\ \text{or } L(\alpha) = L(\beta) \text{ and } \alpha < \beta \end{cases}$$

is a total well ordering on $\mathbb{N}^n$ compatible with sums.

**1.4. The privileged exponent of a polynomial or of a power series.** — The notion of the privileged exponent of a power series is due to H. Hironaka. It was introduced in [**26**] (see also [**1**], [**10**]). We fix, once and for all, a total well ordering $<$, compatible with sums, in $\mathbb{N}^n$. Let $L : \mathbb{Q}^n \to \mathbb{Q}$ be a linear form as above.

***Definition 1.4.1***. — Let $f = \sum_\alpha f_\alpha \boldsymbol{X}^\alpha \in \boldsymbol{k}[\boldsymbol{X}], f \neq 0$. We call:

- The $n$-uple $\exp_L(f) = \max_{<_L}\{\alpha \mid f_\alpha \neq 0\}$, the $L$-privileged exponent of $f$
- The monomial $\mathrm{mp}_L = f_{\exp_L(f)} \boldsymbol{X}^{\exp_L(f)}$, the $L$-privileged monomial of $f$

Let $f = \sum_\alpha f_\alpha \boldsymbol{X}^\alpha \in \boldsymbol{k}[[\boldsymbol{X}]], f \neq 0$. We call:

- The $n$-uple $\exp_L(f) = \min_{<_L}\{\alpha \mid f_\alpha \neq 0\}$, the $L$-privileged exponent of $f$.
- The monomial $\mathrm{mp}_L = f_{\exp_L(f)} \boldsymbol{X}^{\exp_L(f)}$, the $L$-privileged monomial of $f$.

(2) If all the coefficients of $L$ are positive, then the initial form of a power series is a polynomial.

When it becomes necessary, we shall use the more precise notation, $\exp_{<_L}(f) = \exp_L(f)$ and $\mathrm{mp}_{<_L}(f) = \mathrm{mp}_L(f)$. In all the cases, when no confusion can result, we shall write $\exp(f)$ instead of $\exp_L(f)$ and $\mathrm{mp}(f)$ instead of $\mathrm{mp}_L(f)$.

***Note 1.4.2***. — When $f \in \boldsymbol{k}[\boldsymbol{X}], f \neq 0$, we shall take care not to confuse the privileged exponent of the polynomial $f$ with the privileged exponent of the power series $f$, in spite of the notation. If necessary, we shall use the notation $\mathrm{expp}(f)$ for the privileged exponent of the polynomial $f$ and $\mathrm{exps}(f)$ for the privileged exponent of the power series $f$.

***Proposition 1.4.3***. — *Let $f, g \in \boldsymbol{k}[\boldsymbol{X}]$ (resp. $f, g \in \boldsymbol{k}[[\boldsymbol{X}]]$) be non zero elements. We have:*

(1) $\exp(fg) = \exp(f) + \exp(g)$.
(2) $\mathrm{mp}(fg) = \mathrm{mp}(f)\,\mathrm{mp}(g)$.
(3) *If* $\exp(f) \neq \exp(g)$ *then*

$$\exp(f+g) = \max_{<_L}\{\exp(f), \exp(g)\} \quad (\textit{resp.}\ \exp(f+g) = \min_{<_L}\{\exp(f), \exp(g)\}).$$

Let $I$ be a non zero ideal of $\boldsymbol{k}[\boldsymbol{X}]$ (resp. $\boldsymbol{k}[[\boldsymbol{X}]]$). We denote

$$\mathrm{E}_{<_L}(I) = \{\exp_L(f) \mid f \in I \smallsetminus \{0\}\}.$$

When no confusion can result, we write $\mathrm{E}(I)$ instead of $\mathrm{E}_{<_L}(I)$. Because of 1.4.3, we have $\mathrm{E}(I) + \mathbb{N}^n = \mathrm{E}(I)$. We denote by $\mathrm{mp}(I)$, the ideal of $\boldsymbol{k}[\boldsymbol{X}]$ generated by the family of monomials $\{\mathrm{mp}(f) \mid f \in I\}$[(3)].

***Proposition 1.4.4***. — *Let $I$ be a non zero ideal of $\boldsymbol{k}[\boldsymbol{X}]$ (resp. $\boldsymbol{k}[[\boldsymbol{X}]]$). Then we have:*

$$\mathrm{E}(I) = \mathrm{E}(\mathrm{mp}(I)) = \mathrm{E}(\mathrm{Fin}(I)) \quad (\textit{resp.}\ \mathrm{E}(I) = \mathrm{E}(\mathrm{mp}(I)) = \mathrm{E}(\mathrm{In}(I))).$$

*Proof*. — By definition, for every non zero polynomial $f$, we have

$$\exp(f) = \exp(\mathrm{fin}(f)) \quad \text{and} \quad \exp(f) = \exp(\mathrm{mp}(f))$$

(see 1.4.1). If $f$ is a non zero power series, then we have: $\exp(f) = \exp(\mathrm{in}(f))$ and $\exp(f) = \exp(\mathrm{mp}(f))$ (see 1.4.1). □

***Note 1.4.5***. — With the notations of 1.4.2, if $f$ is a power series such that $\mathrm{in}(f)$ is a polynomial, (this condition is verified if every coefficient in the linear form $L$ is positive) then we have, in general, $\exp(f) \neq \mathrm{expp}(\mathrm{in}(f))$.

Assume that every coefficient in the linear form $L$ is positive (we then just say that $L$ is a positive linear form). Consider the ordering $\lhd_L$ defined on $\mathbb{N}^n$ by the formula:

$$\alpha \lhd_L \beta \text{ if and only if } \begin{cases} L(\alpha) < L(\beta) \\ \text{or } L(\alpha) = L(\beta) \text{ and } \beta < \alpha \end{cases}$$

(3) This is a monomial ideal, which means that a polynomial $f$ is an element of the ideal if and only if every monomial of $f$ is in the ideal.

This is a total well ordering [4] on $\mathbb{N}^n$ compatible with the sum.

If $f$ is a power series, then we have: $\exp_{<_L}(f) = \mathrm{exps}_{<_L}(\mathrm{in}_L(f)) = \mathrm{expp}_{\lhd_L}(\mathrm{in}_L(f))$.

***Proposition 1.4.6***. — *Let $E \subset \mathbb{N}^n$ such that $E + \mathbb{N}^n = E$. Then $E$ contains a finite family of generators; In other words, there exists a finite family $F \subset E$ such that $E = \cup_{\alpha\in F}(\alpha + \mathbb{N}^n)$.*

*Proof*. — This is a version of Dickson's lemma. The proof is by induction on $n$. For $n = 1$ a (finite) family of generators is given by the smallest element of $E$ (for the usual ordering in $\mathbb{N}$). Assume that $n > 1$ and that the result is true for $n - 1$. Let $E \subset \mathbb{N}^n$ be such that $E + \mathbb{N}^n = \mathbb{N}^n$. We can assume that $E$ is non empty. Let $\alpha \in E$. For any $i = 1, \dots, n$ and $j = 0, \dots, \alpha_i$ we consider the bijective mapping

$$\begin{aligned} \phi_{i,j} : \mathbb{N}^{i-1} \times \{j\} \times \mathbb{N}^{n-i} &\longrightarrow \mathbb{N}^{n-1} \\ (\beta_1, \dots, \beta_{i-1}, j, \gamma_{i+1}, \dots, \gamma_n) &\longmapsto (\beta_1, \dots, \beta_{i-1}, \gamma_{i+1}, \dots, \gamma_n) \end{aligned}$$

and we denote $E_{i,j} = \phi_{i,j}(E \cap (\mathbb{N}^{i-1} \times \{j\} \times \mathbb{N}^{n-i}))$. It is clear that $E_{i,j} + \mathbb{N}^{n-1} = E_{i,j}$ and by the induction hypothesis there is a finite subset $F_{i,j} \subset E_{i,j}$ generating $E_{i,j}$. The family $F = \{\alpha\} \cup \big(\cup_{i,j}(\phi_{i,j})^{-1}(F_{i,j})\big)$ generates $E$. The proof above is taken from [**18**]. $\square$

***Remark***. — The previous proposition can be rephrased as follows: *Any monomial ideal in $\boldsymbol{k}[\boldsymbol{X}]$ is finitely generated.* This is a particular case of the Hilbert basis theorem. In the same way we can see that any increasing sequence $E_k$ of subsets of $\mathbb{N}^n$, stable under the action of $\mathbb{N}^n$, is stationary. We shall often use this property called the *Noetherian property for* $\mathbb{N}^n$.

We can adapt the proof above to show that, given $E \subset \mathbb{N}^n$ as in the proposition, we can find in any set of generators, a finite subset of generators of $E$ . This proves in particular that in any system of generators made of monomials of a monomial ideal of $\boldsymbol{k}[\boldsymbol{X}]$, we can find a finite subset of generators. This is Dickson's lemma.

***Definition 1.4.7***. — Let $I$ be a non zero ideal of $\boldsymbol{k}[\boldsymbol{X}]$ (resp. $\boldsymbol{k}[[\boldsymbol{X}]]$). A standard basis[5] of $I$, relative to $L$ (or $L$-standard basis of $I$) is any family $f_1, \dots, f_m$ of elements in $I$ such that $\mathrm{E}(I) = \cup_{i=1}^m(\exp_L(f_i) + \mathbb{N}^n)$.

***Remark***. — There always exist a standard basis for $I$, because of the definition of $\mathrm{E}(I)$ and 1.4.6.

---

(4) If the form $L$ has at least one non positive coefficient the previous formula defines a total ordering over $\mathbb{N}^n$, but not a well ordering.

(5) The notion of a standard basis, introduced by H. Hironaka in [**21**], is similar to the notion of a Gröbner basis, introduced by Buchberger in [**13**]. We shall come back to this analogy later.

**1.5. Here are the divisions.** — We shall prove here that a standard basis of an ideal $I$ is a system of generators of this ideal.

With any $m$-uple $(\alpha^1,\dots,\alpha^m)$ of elements of $\mathbb{N}^n$ we shall associate a partition[(6)] $\Delta_1,\dots,\Delta_m,\overline{\Delta}$ of $\mathbb{N}^n$ in the following way. We set:

$$\Delta^1=\alpha^1+\mathbb{N}^n,\quad \Delta^{i+1}=(\alpha^{i+1}+\mathbb{N}^n)\smallsetminus(\Delta^1\cup\cdots\cup\Delta^i) \text{ if } i\geqslant 1,$$
$$\overline{\Delta}=\mathbb{N}^n\smallsetminus(\cup_{i=1}^m\Delta^i)$$

***Theorem 1.5.1***. — *Let $(f_1,\dots,f_m)$ be an $m$-uple of non zero elements of $\boldsymbol{k}[[\boldsymbol{X}]]$ (resp. of $\boldsymbol{k}[\boldsymbol{X}]$). We denote by $\Delta^1,\dots,\Delta^m,\overline{\Delta}$ the partition of $\mathbb{N}^n$ associated with $(\exp(f_1),\dots,\exp(f_m))$. Then, for any $f$ in $\boldsymbol{k}[[\boldsymbol{X}]]$ (resp. in $\boldsymbol{k}[\boldsymbol{X}]$) there exists a unique $(m+1)$-uple $(q_1,\dots,q_m,r)$ of elements of $\boldsymbol{k}[[\boldsymbol{X}]]$ (resp. of $\boldsymbol{k}[\boldsymbol{X}]$) such that:*

1) $f=q_1f_1+\cdots+q_mf_m+r$,
2) $\exp(f_i)+\mathscr{N}(q_i)\subset\Delta^i$, $i=1,\dots,m$,
3) $\mathscr{N}(r)\subset\overline{\Delta}$.

*If $\boldsymbol{k}$ is either $\mathbb{R}$ or $\mathbb{C}$ and if the $f_i$ are convergent power series, then for any convergent power series $f$ the series $q_i$ and $r$ are convergent.*

***Remark***. — The element $q_i$ in the theorem is called the $i$-th quotient and $r$ is called the remainder of the division of $f$ by $(f_1,\dots,f_m)$. We shall denote the remainder by $r(f;f_1,\dots,f_m)$. Of course, the quotients as well as the remainder depend on the well ordering $<_L$.

*Proof of theorem 1.5.1*. — Assume that two $(m+1)$-uples, $(q_1,\dots,q_m,r)$ and $(q'_1,\dots,q'_m,r')$, satisfy the conditions of the theorem. We have:

$$\sum_{i=1}^m(q_i-q'_i)f_i+r-r'=0 \tag{1}$$

If $q_i\neq q'_i$ then $\exp((q_i-q'_i)f_i)\in\Delta^i$. If $r\neq r'$ then $\exp(r-r')\in\overline{\Delta}$. Since $\Delta^1,\dots,\Delta^m,\overline{\Delta}$ is a partition of $\mathbb{N}^n$, the equality (1) is only possible if $q_i=q'_i$ for any $i$ and if $r=r'$. This proves the uniqueness in the theorem. We shall now prove the existence. Let us first consider the polynomial case. Since the set $\mathbb{N}^n$ is well ordered with respect to $<_L$, we use an induction on unitary monomials of $\boldsymbol{k}[\boldsymbol{X}]$. If $\boldsymbol{X}^\alpha=1$ (*i.e.* if $\alpha=(0,\dots,0)$), then either $\exp(f_i)\neq(0,\dots,0)$ for any $i$ and in this case it is enough to write $1=\sum_{i=1}^m 0f_i+1$, or there exists an integer $j$ such that $\exp(f_j)=(0,\dots,0)$. In this case $f_j$ is a non zero constant.[(7)] Assume that $j$ is minimal. We write $1=\sum_{i\neq j}0\cdot f_i+(1/f_j)f_j+0$. This proves the result at the first step of the induction. Assume that the result is proved for any $\beta$ such that $\beta<_L\alpha$. Let $j$ be such that $\alpha\in\Delta^j$. If there is no such $j$ we write $\boldsymbol{X}^\alpha=\sum_{i=1}^m 0f_i+\boldsymbol{X}^\alpha$. If

[(6)] We use the word partition in a broad sense, which means that an element of the family may be empty.
[(7)] We use here the fact that for the well ordering $<_L$, $(0,\dots,0)$ is the first element of $\mathbb{N}^n$.

$j$ exists, let $\gamma \in \mathbb{N}^n$ be such that $\alpha = \exp(f_j) + \gamma$. We can write, $\boldsymbol{X}^\alpha = \frac{1}{c_j}\boldsymbol{X}^\gamma f_j + g_j$ where $c_j$ is the coefficient of the privileged monomial of $f_j$ and all the monomials in $g_j$ are smaller (with respect to $<_L$) than $\alpha$. By the induction hypothesis there exists $(q'_1, \ldots, q'_m, r')$ satisfying the conditions of the theorem for $f = g_j$. In particular we have:

$$\boldsymbol{X}^\alpha = \sum_{i \neq j} q'_i f_i + \Big(\frac{1}{c_j}\boldsymbol{X}^\gamma + q'_j\Big) f_j + r'.$$

This proves the result for $\alpha$. Thus, existence is proved for the polynomials.

We say that a polynomial $g$ is $L$-homogeneous if all its monomials have the same $L$-degree.

It is clear in the proof above that if $f$ is $L$-homogeneous of $L$-degree $d \in \mathbb{Q}$ and if $f_i$ is $L$-homogeneous of $L$-degree $d_i \in \mathbb{Q}$ (for any $i$) then the quotient $q_i$, if it is non zero is $L$-homogeneous of $L$-degree $d - d_i$, and the remainder $r$, if it is non zero is $L$-homogeneous of $L$-degree $d$.

Assume now that $f$ is a power series. Let us now see the existence in that case, first assuming that $L$ is a positive linear form (see 1.4.5). Any non zero power series $f = \sum_\alpha f_\alpha \boldsymbol{X}^\alpha$ can be represented, in a unique way, as a sum $f = \sum_{\nu \in L(\mathbb{N}^2)} f_\nu$ where $f_\nu = \sum_{L(\alpha)=\nu} f_\alpha \boldsymbol{X}^\alpha$ is a $L$-homogeneous polynomial. By definition (see 1.2.2) we have: $\mathrm{val}_L(f) = \min\{\nu \mid f_\nu \neq 0\}$.

Because of 1.4.5 we have, for any $i$: $\exp(f_i) = \mathrm{expp}_{\lhd_L}(\mathrm{in}(f_i))$ and we can apply the division, in the polynomial case, of $\mathrm{in}(f)$ by $(\mathrm{in}(f_1), \ldots, \mathrm{in}(f_m))$. There exists a (unique) $(m+1)$-uple $(\sigma_1, \ldots, \sigma_m, \rho)$ such that

$$\mathrm{in}(f) = \sum_{i=1}^m \sigma_i \,\mathrm{in}(f_i) + \rho$$

and satisfying the conditions similar to 2) and 3) in the theorem. The following notations will be useful: $\sigma_i(f) = \sigma_i$, $\rho(f) = \rho$ and for any power series $g$, $\widehat{g} = g - \mathrm{in}(g)$. We have:

$$f = \mathrm{in}(f) + \widehat{f} = \sum_{i=1}^m \sigma_i(f) f_i + \rho(f) + \widehat{f} - \sum_{i=1}^m \sigma_i(f)\widehat{f_i}$$

We introduce the following notation:

$$s^0(f) = f, \quad s(f) = s^1(f) = \widehat{f} - \sum_{i=1}^m \sigma_i(f)\widehat{f}_i, \quad s^j(f) = s(s^{j-1}(f)).$$

We have:

- $\mathrm{val}_L(s^{j+1}(f)) > \mathrm{val}_L(s^j(f))$ for any $j$.
- $\deg_L(\sigma_i(s^{j+1}(f))) > \deg_L(\sigma_i(s^j(f)))$ for any $i$ and any $j$.
- $\deg_L(\rho(s^{j+1}(f))) > \deg_L(\rho(s^j(f)))$ for any $i$ and any $j$.
- For any $i$, the series

$$\sum_{j \geqslant 0} \sigma_i(s^j(f))$$

is convergent in the $(\boldsymbol{X})$-adic topology of $\boldsymbol{k}[[\boldsymbol{X}]]$.

- The series

$$\sum_{j\geqslant 0}\rho(s^j(f))$$

is convergent in the $(\boldsymbol{X})$-adic topology of $\boldsymbol{k}[[\boldsymbol{X}]]$.

- $f=\sum_{i=1}^m(\sum_{j\geqslant 0}\sigma_i(s^j(f)))f_i+(\sum_{j\geqslant 0}\rho(s^j(f)))$

If we write $q_i=\sum_{j\geqslant 0}\sigma_i(s^j(f))$ and $r=\sum_{j\geqslant 0}\rho(s^j(f))$, the $(m+1)$-uple $(q_1,\dots,q_m,r)$ satisfies the conditions 1), 2) et 3) in the theorem, relative to $f$.

Now, we prove the convergent case. We will follow the proof of H. Hauser and G. Müller [**20**]. Set

$$\boldsymbol{k}\{\boldsymbol{X}\}^{\overline{\Delta}}=\{r\in\boldsymbol{k}\{\boldsymbol{X}\}\mid\mathscr{N}(r)\subset\overline{\Delta}\}$$

and

$$\boldsymbol{k}\{\boldsymbol{X}\}^{m,\Delta}=\{(q_1,\dots,q_m)\in\boldsymbol{k}\{\boldsymbol{X}\}^m\mid\exp(f_i)+\mathscr{N}(q_i)\subset\Delta^i\text{ for all }i\}.$$

Let $L':\mathbb{Q}^n\to\mathbb{Q}$ be a positive linear form such that $\exp_{L'}(f_i)=\exp_L(f_i)$, for all $i$ and the Newton diagram of each $\mathrm{in}_{L'}(f_i)$ is reduced to a point.

For $s\in\mathbb{R}, s>0$ consider

(1) the pseudo-norm defined on $\boldsymbol{k}\{\boldsymbol{X}\}$ by $|g|_s=\sum_\alpha|g_\alpha|s^{L'(\alpha)}$,

(2) the pseudo-norm defined on $\boldsymbol{k}\{\boldsymbol{X}\}^m$ by $|(g_1,\dots,g_m)|_s=\sum_i|g_i|_s$.

We define $\boldsymbol{k}\{\boldsymbol{X}\}_s=\{g\in\boldsymbol{k}\{\boldsymbol{X}\}\mid |g|_s<\infty\}$ and

$$\boldsymbol{k}\{\boldsymbol{X}\}_s^{\overline{\Delta}}=\boldsymbol{k}\{\boldsymbol{X}\}^{\overline{\Delta}}\cap\boldsymbol{k}\{\boldsymbol{X}\}_s,$$

which are Banach spaces with norm $||_s$. Similarly we define

$$\boldsymbol{k}\{\boldsymbol{X}\}_s^{m,\Delta}=\{\boldsymbol{q}\in\boldsymbol{k}\{\boldsymbol{X}\}^{m,\Delta}\mid|\boldsymbol{q}|_s<\infty\}$$

which is a Banach space with norm $||_s$. There are constants $c>0$ and $\epsilon>0$ such that $|m_i|_s\leqslant|f_i|_s\leqslant c|m_i|_s$ and $|f_i-m_i|_s\leqslant s^\epsilon|m_i|_s$ for all $i$ and all sufficiently small $s$, where $m_i$ is the monomial of $f_i$ corresponding to $\exp(f_i)$. For such an $s$ we consider the continuous linear map

$$u_s:\boldsymbol{k}\{\boldsymbol{X}\}_s^{m,\Delta}\oplus\boldsymbol{k}\{\boldsymbol{X}\}_s^{\overline{\Delta}}\longrightarrow\boldsymbol{k}\{\boldsymbol{X}\}_s$$

defined by $u_s(\boldsymbol{q},r)=\sum_i q_if_i+r$. We will show that $u_s$ is onto for small $s$.

We define on $\boldsymbol{k}\{\boldsymbol{X}\}_s^{m,\Delta}\oplus\boldsymbol{k}\{\boldsymbol{X}\}_s^{\overline{\Delta}}$ the norm $||(\boldsymbol{q},r)||_s=\sum_i|q_i|_s|m_i|_s+|r|_s$. With this norm this space becomes a Banach space. The linear map

$$v_s:\boldsymbol{k}\{\boldsymbol{X}\}_s^{m,\Delta}\oplus\boldsymbol{k}\{\boldsymbol{X}\}_s^{\overline{\Delta}}\longrightarrow\boldsymbol{k}\{\boldsymbol{X}\}_s$$

defined by $v_s(\boldsymbol{q},r)=\sum_i q_im_i+r$ is bijective and bicontinuous of norm 1. Its inverse $v_s^{-1}$ has norm 1. Let $w_s$ denote the continuous linear map $u_s-v_s$. We have $w_s(\boldsymbol{q},r)=\sum_i q_i(f_i-m_i)$.

There exits $s_0>0$ such that, for $s<s_0$, we have $||w_s||\leqslant s^\epsilon$ and $||w_sv_s^{-1}||\leqslant s^\epsilon<1$. We have $u_sv_s^{-1}=Id+w_sv_s^{-1}$ and so $u_sv_s^{-1}$ is invertible. Then $u_s$ is invertible.

Assume now that $L$ is general (with non negative coefficients). We remark that there is a form $L''$, with positive coefficients, such that $\exp_L(f_i) = \exp_{L''}(f_i)$ for any $i$ (see *e.g.* [**6**]). We perform the division of the series $f$ by $(f_1, \ldots, f_m)$ relative to the form $L''$. Because of the note 1.5.2 below this division is also a division relative to $L$. □

***Remark***. — It follows from the proof that for any division $f = q_1 f_1 + \cdots + q_m f_m + r$ as in the polynomial case of the theorem we have $\max\{\max_i\{\exp_L(q_i f_i)\}, \exp_L(r)\} = \exp_L(f)$ and as a consequence: If $f \in \boldsymbol{k}[\boldsymbol{X}]$ then $\max\{\max_i\{\deg_L(q_i f_i)\}, \deg_L(r)\} = \deg_L(f)$. In the power series case the same is true with max and deg replaced respectively by min and val.

***Note 1.5.2***. — We must remark that if $L$ and $L'$ are two linear forms (with non negative coefficients) such that $\exp_L(f_i) = \exp_{L'}(f_i)$ then the quotients and the remainders of a division of $f$ by $(f_1, \ldots, f_m)$ relative to $<_{L'}$ also give a division relative to $<_L$.

***Corollary 1.5.3***. — *Let $I$ be a non zero ideal of the ring $\mathscr{B}$ ($\mathscr{B} = \boldsymbol{k}[\boldsymbol{X}]$, $\boldsymbol{k}[[\boldsymbol{X}]]$ or $\boldsymbol{k}\{\boldsymbol{X}\}$). Let $E = \mathrm{E}(I)$ with respect to an arbitrary linear form L and $\mathscr{B}^E = \{f = \sum_\alpha f_\alpha \boldsymbol{X}^\alpha \in \mathscr{B} \mid \mathscr{N}(f) \cap E = \varnothing\}$. Then, the natural mapping*

$$\varpi : \mathscr{B}^E \longrightarrow \mathscr{B}/I$$

*is an isomorphism of $\boldsymbol{k}$-vector spaces.*

*Proof*. — The mapping $\varpi$ is defined as the composition of the l'inclusion $\mathscr{B}^E \subset \mathscr{B}$ and the projection $\mathscr{B} \to \mathscr{B}/I$. Thus $\varpi$ is a homomorphism of vector spaces. Let us prove that it is onto. Let $f \in \mathscr{B}$. Let $\{f_1, \ldots, f_m\}$ be a standard basis of $I$ with respect to $L$ and let $r = r(f; f_1, \ldots, f_m)$ be the remainder of the division of $f$ by the standard basis. Because of 1.4.7 and 1.5.1 $r$ is an element of $\mathscr{B}^E$ and $\varpi(r) = r + I = f + I$. Let us now see the injectivity of $\varpi$. Let $b \in \mathscr{B}^E$. If $\varpi(b) = 0 + I$ then $b \in I$. If $b \neq 0$ this would imply $\exp(b) \in E$ which contradicts the fact that $\mathscr{N}(b) \cap E = \varnothing$. □

***Corollary 1.5.4***. — *Let $I$ be a non zero ideal of $\mathscr{B}$ and let $f_1, \ldots, f_m$ be a family of elements of $I$. The following conditions are equivalent:*

1) *$f_1, \ldots, f_m$ is a standard basis of $I$.*
2) *For any $f$ in $\mathscr{B}$ we have: $f \in I$ if and only if $r(f; f_1, \ldots, f_m) = 0$.*

***Corollary 1.5.5***. — *Let $I$ be a non zero ideal of $\mathscr{B}$ and let $f_1, \ldots, f_m$ be a standard basis of $I$. Then $f_1, \ldots, f_m$ is a system of generators of $I$.*

*Proof*. — Because of 1.5.4, if $f$ is in $I$, we obtain by division $f = \sum q_i f_i$. □

***Note 1.5.6***. — Let $L, L'$ be two linear forms with non negative coefficients. Let $\{f_1, \ldots, f_m\}$ be a standard basis of an ideal $I$ (in $\mathscr{B}$) relative to $L$. Assume that we have $\exp_{L'}(f_i) = \exp_L(f_i)$ for any $i$. Then $\{f_1, \ldots, f_m\}$ is a standard basis relative to $L'$. This result will be very useful in the calculations that follows. It is a direct application of the note 1.5.2 and of 1.5.4.

### 1.6. Semisyzygies and the explicit calculation of a standard basis

***Definition 1.6.1***. — Let $g_1,\ g_2$ be elements of $\mathscr{B}$. The *semisyzygy* relative to $(g_1,\ g_2)$, is the polynomial (resp. the power series) (defined up to a factor $c \in \boldsymbol{k}^*$)

$$S(g_1,\ g_2) = m_1 g_1 - m_2 g_2$$

characterized by the following conditions:

(1) $m_i$ is a monomial.
(2) $\exp(m_1 g_1) = \exp(m_2 g_2) = \mu$
(3) Any pair of monomial $(m_1', m_2')$ such that $\exp(m_1' g_1) = \exp(m_2' g_2)$ satisfies $\exp(m_1' g_1) = \exp(m_2' g_2) \in (\mu + \mathbb{N}^n)$
(4) $\exp(S(g_1,\ g_2)) <_L \mu$ (resp. $\mu <_L \exp(S(g_1,\ g_2))$

In other words $\mu$ is the *gcd* —in the sense of $\mathbb{N}^n$— of $\exp(g_1)$, $\exp(g_2)$.

***Proposition 1.6.2***. — *Let $\mathscr{F} = \{p_1, \dots, p_r\}$ be a system of generators of the ideal $I$ of $\mathscr{B}$ such that for any $(i,j)$, the remainder of the division of $S(p_i, p_j)$ by $(p_1, \dots, p_r)$ is zero. Then $\mathscr{F}$ is an $L$-standard basis of $I$.*

*Proof*. — For the polynomial case we refer to [**13**] and [**24**]. Remember that if $f = \sum_{i=1}^r b_i f_i$ the problem is to reduce to the case when the exponent $\alpha = \max\{\exp(b_i f_i)\}$ is equal to $\exp(f)$. This can be done by induction on $\alpha$. For that purpose we change the above decomposition of $f$ by using the division of the semisyzygies. Consider now the power series case, assuming first that $L$ is positive. The proof in this case is the same but by descending induction on $\alpha = \min\{\exp(b_i f_i)\}$. In the case of a general $L$ (with non negative coefficients), let $L'$ be a form with positive coefficients and such that $\exp_L(f_i) = \exp_{L'}(f_i)$ (see *e.g.* [**6**]). The result follows now from the notes 1.5.2 and 1.5.6. □

***Note 1.6.3***. — This proposition gives an algorithmic process in the case of polynomials, in order to calculate a standard basis starting from a system of generators, by a finite number of divisions. This is Buchberger's algorithm for polynomials [**13**]. The version for differential operators is given in detail in the next chapter.

***Corollary-definition 1.6.4***. — *Let $\mathscr{F} = \{p_1, \dots, p_r\}$ be a system of generators of the ideal $I$ of $\mathscr{B}$ such that for any $(i,j)$, the remainder of the division of $S(p_i, p_j)$ by $(p_1, \dots, p_r)$ is zero. Let $\boldsymbol{r}_{i,j}$ be the relation obtained by this division. Then, the module of the relations $\mathscr{S}$ between the $p_i$ is generated by the relations $\boldsymbol{r}_{i,j}$. Each of these relations is called an* elementary relation *between the $p_i$.*

*Proof.* — The module $\mathscr{S}$ is the set of $r$-uples $s = (a_1, \dots, a_r) \in \mathscr{B}^r$ such that $a_1 p_1 + \cdots + a_r p_r = 0$. Let $\mathscr{S}'$ be the module generated by the family $\boldsymbol{r}_{i,j}$. By the same principle as in the previous proof, if $\boldsymbol{b} = (b_1, \dots, b_r)$ is in $\mathscr{S}$, and if $\delta = \max\{\exp(b_i p_i)\}$ (resp. $\delta = \min\{\exp(b_i p_i)\}$) we can, modulo $\mathscr{S}'$, replace $\boldsymbol{b}$ by a relation $\boldsymbol{b'}$ with $\max\{\exp(b'_i p_i)\} < \delta$ (resp. $\min\{\exp(b'_i p_i)\} > \delta$). In the polynomial case we end the proof by induction. In the case of power series, we first reduce the proof to the case when $L$ is positive. By iterating the process above we find, for any positive integer $N$, $\mathscr{S} \subset \mathscr{S}' + (X_1, \dots, X_n)^N \boldsymbol{k}[[\boldsymbol{X}]]^r$ (or a similar formula in the case of $\boldsymbol{k}\{\boldsymbol{X}\}$). This allows us to conclude by applying the intersection theorem of Krull. □

Thus we have a way to calculate the first step of a free resolution of the module $M = \mathscr{B}/I$. Indeed, we have an exact sequence:

$$\mathscr{B}^r \xrightarrow{\phi_1} \mathscr{B} \xrightarrow{\phi_0} M \longrightarrow 0$$

where $\phi_1(b_1, \dots, b_r) = \sum_i b_i p_i$ and $\phi_0$ is the natural morphism. The kernel of $\phi_1$ is the module of relations between the $p_i$ (denoted above by $\mathscr{S}$). By 1.6.4 this module is generated by the elementary relations between the $p_i$. Let $s$ be the number of these relations. We then have a natural morphism $\phi_2 : \mathscr{B}^s \to \mathscr{B}^r$ which sends each element $\boldsymbol{e}_{i,j}$ of the canonical basis of the free module $\mathscr{B}^s$ to the relation $\boldsymbol{r}_{i,j}$. We deduce from this an exact sequence:

$$\mathscr{B}^s \xrightarrow{\phi_2} \mathscr{B}^r \xrightarrow{\phi_1} \mathscr{B} \xrightarrow{\phi_0} M \longrightarrow 0.$$

**1.7. Application 1. Elimination of variables.** — Let $I$ be an ideal of $\boldsymbol{k}[\boldsymbol{X}]$ and $k$ be an integer $0 \leqslant k \leqslant n-1$. We define $I_k = I \cap \boldsymbol{k}[X_{k+1}, \dots, X_n]$. $I_k$ is the set of polynomials in $I$ which depend only on the variables $X_{k+1}, \dots, X_n$. We write $I_n = \boldsymbol{k} \cap I$. The set $I_k$ is, for any $k$, an ideal of the ring $\boldsymbol{k}[X_{k+1}, \dots, X_n]$. The ideal $I_k$ is called the $k$-th elimination ideal of $I$. We shall return later on this definition.

***Note 1.7.1.*** — The lexicographic ordering in $\mathbb{N}^n$ is by definition the total well ordering $<_{\text{lex}}$ defined by:

$$\alpha_{<_{\text{lex}}}\beta \text{ if and only if } \begin{cases} \text{in the vector } \alpha - \beta \\ \text{the first non zero component} \\ \text{is negative} \end{cases}$$

The lexicographic ordering is compatible with the sum in $\mathbb{N}^n$.

***Lemma 1.7.2.*** — *Let $f$ be a polynomial in $\boldsymbol{k}[\boldsymbol{X}]$. Then $\mathrm{mp}_{<_{\text{lex}}}(f)$ is in $\boldsymbol{k}[X_{k+1}, \dots, X_n]$ if and only if $f$ is in $\boldsymbol{k}[X_{k+1}, \dots, X_n]$.*

*Proof.* — Let $\boldsymbol{X}^\alpha$ be the unitary monomial corresponding to $\mathrm{mp}_{<_{\text{lex}}}(f)$. Let $\boldsymbol{X}^\beta$ be another monomial of $f$. We have: $\beta <_{\text{lex}} \alpha$ and thus, by 1.7.1, the first non zero component of the vector $\beta - \alpha$ is negative. But $\alpha_1 = \cdots = \alpha_k = 0$ and thus $\beta_1 = \cdots = \beta_k = 0$, which proves the lemma. $\square$

For any integer $k$ such that $0 \leqslant k \leqslant n-1$, we identify $\mathbb{N}^{n-k}$ with a subset of $\mathbb{N}^n$ by the injective mapping $\varphi_k : \mathbb{N}^{n-k} \to \mathbb{N}^n$ defined by $\varphi_k(\beta_{k+1}, \ldots, \beta_n) = (0, \ldots, 0, \beta_{k+1}, \ldots, \beta_n)$. The lexicographic ordering on $\mathbb{N}^n$ induces on $\mathbb{N}^{n-k}$ the lexicographic ordering of $\mathbb{N}^{n-k}$

***Theorem 1.7.3***. — *Let $I$ be an ideal of $\boldsymbol{k}[\boldsymbol{X}]$ and $k$ an integer such that $0 \leqslant k \leqslant n$. Let $\mathscr{G}$ be a standard basis of the ideal I relative to the lexicographic ordering. Let $\mathscr{G}_k = \mathscr{G} \cap \boldsymbol{k}[X_{k+1}, \ldots, X_n]$. Then we have:*

(1) *If $\mathscr{G}_k = \varnothing$ then $I_k = (0)$.*

(2) *If $\mathscr{G}_k \neq \varnothing$ then $\mathscr{G}_k$ is a standard basis of the ideal $I_k$ relative to the lexicographic ordering.*

*Proof.* — Assume that there a non zero $f$ in $I_k$. Since $f \in I$ and since $\mathscr{G}$ is a standard basis of $I$, there is $g \in \mathscr{G}$ and $\beta \in \mathbb{N}^n$ such that $\mathrm{mp}(f) = \boldsymbol{X}^\beta \, \mathrm{mp}(g)$ (see 1.4.7). Therefore $\mathrm{mp}(g) \in \boldsymbol{k}[X_{k+1}, \ldots, X_n]$ and , by lemma 1.7.2 $g \in \mathscr{G}_k$. In particular $\mathscr{G}_k$ is non empty.

Assume that $\mathscr{G}_k$ is non empty. Let $f$ be a non zero polynomial in $I_k$. In the proof just above we also get $\boldsymbol{X}^\beta \in \boldsymbol{k}[X_{k+1}, \ldots, X_n]$ so that we have:

$$\mathrm{E}(I_k) \subset \left(\bigcup_{g \in \mathscr{G}_k} \exp_{<_{\mathrm{lex}}}(g) + \mathbb{N}^{n-k}\right)$$

The other inclusion being obvious, $\mathscr{G}_k$ is a standard basis of $I_k$. $\square$

## 1.8. Application 2. Some useful calculations on ideals of $\boldsymbol{k}[\boldsymbol{X}]$

*1.8.1. Intersections of ideals.*— Let $I, J$ be two ideals of $\boldsymbol{k}[\boldsymbol{X}]$. Let $y$ be a new indeterminate. We denote by $I^e$ (resp. $J^e$) the extension of the ideal $I$ (resp. $J$) to the ring $\boldsymbol{k}[\boldsymbol{X}, y]$. If $h$ is a polynomial in $\boldsymbol{k}[y]$ we denote by $hI^e$ (resp. $hJ^e$) the product of the ideals[(8)] $(h)$ and $I^e$ (resp. $(h)$ and $J^e$). With these notations we have:

***Theorem 1.8.2***. — *Let $I, J$ be two ideals of $\boldsymbol{k}[\boldsymbol{X}]$. Then $I \cap J = (yI^e + (1-y)J^e) \cap \boldsymbol{k}[\boldsymbol{X}]$.*

*Proof.* — If $f \in I \cap J$ then $yf \in yI^e$ and $(1-y)f \in (1-y)J^e$. Therefore $f = yf + (1-y)f \in (yI^e + (1-y)J^e) \cap \boldsymbol{k}[\boldsymbol{X}]$. Conversely, let $f \in (yI^e + (1-y)J^e) \cap \boldsymbol{k}[\boldsymbol{X}]$. We can write:

$$f = yG + (1-y)H, \tag{1}$$

with $G = G(\boldsymbol{X}, y) \in I^e$ and $H = H(\boldsymbol{X}, y) \in J^e$. We set $y = 0$ in the equation (1) and we get $f = H(\boldsymbol{X}, 0)$ and it is clear that $H(\boldsymbol{X}, 0) \in J$. On the other side, if we set $y = 1$ in the equation (1), we get $f = G(\boldsymbol{X}, 1)$ and it is clear that $G(\boldsymbol{X}, 1) \in I$. $\square$

(8) these are ideals of the ring $\boldsymbol{k}[\boldsymbol{X}, y]$

This theorem gives a way to find a standard basis of $I \cap J$ by eliminating the variable $y$.

*1.8.3. The radical of an ideal.*— Let $I$ be an ideal of $\boldsymbol{k}[\boldsymbol{X}]$. We recall that the radical of the ideal $I$ is the set of polynomials $f \in \boldsymbol{k}[\boldsymbol{X}]$ such that $f^j \in I$ for some integer $j$. The radical of $I$ is denoted by $\sqrt{I}$. It is an ideal of $\boldsymbol{k}[\boldsymbol{X}]$.

Let us consider the problem of deciding whether an element of the ring belongs to the ideal $\sqrt{I}$. Let $y$ be a new variable. Let $f \in \boldsymbol{k}[\boldsymbol{X}]$.

**Theorem 1.8.4**. — *With the notations above, we have $f \in \sqrt{I}$ if and only if the ideal $I^e + (1 - yf)$ of the ring $\boldsymbol{k}[\boldsymbol{X}, y]$ is the whole ring.*

*Proof.* — Exercise. See e.g. [**17**]. □

## 1.9. Application 3. The calculation of the dimension and the multiplicity of a local algebra $\boldsymbol{k}[[\boldsymbol{X}]]/I$

*1.9.1. The Hilbert-Samuel function.*— Recall that if $(A, \mathfrak{m})$ is a Noetherian local ring, the *Hilbert-Samuel function* of $A$ is the mapping:

$$\begin{aligned} FHS_A : \mathbb{N} &\longrightarrow \mathbb{N} \\ k &\longmapsto \dim_{A/\mathfrak{m}}(A/\mathfrak{m}^{k+1}) \end{aligned}$$

Recall also that there is a polynomial $PHS_A(t) \in \mathbb{Q}[t]$ –called *the Hilbert-Samuel polynomial* of $A$- such that, for $k \gg 0$, we have $FHS_A(k) = PHS_A(k)$.

**Theorem 1.9.2 (The dimension theorem)**. — *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then the Krull dimension of $A$ (denoted by $\dim(A)$) is equal to the degree of the Hilbert-Samuel polynomial of $A$.*

*Proof.* — See [**5**] chapter 11 or [**8**]. □

The highest degree monomial of $PHS_A(t)$ can be written $\frac{e(A)}{\dim(A)!}t^{\dim(A)}$, where $e(A)$ is a positive integer called the multiplicity of $A$.

This applies in particular, to the case when $A = \boldsymbol{k}[[\boldsymbol{X}]]/I$ (or $A = \boldsymbol{k}\{\boldsymbol{X}\}/I$) where $I$ is an ideal of $\boldsymbol{k}[[\boldsymbol{X}]]$ (or $\boldsymbol{k}\{\boldsymbol{X}\}$). If we denote by $\mathfrak{m}$ the maximal ideal $(X_1, \ldots, X_n)$ then we have:

$$FHS_A(k) = \dim_{\boldsymbol{k}}\left(\frac{\boldsymbol{k}[[\boldsymbol{X}]]}{I + \mathfrak{m}^{k+1}}\right)$$

The aim in this section is to compute the dimension and the multiplicity of the $\boldsymbol{k}$-local algebra $\boldsymbol{k}[[\boldsymbol{X}]]/I$ (or $\boldsymbol{k}\{\boldsymbol{X}\}/I$), in terms of $\mathrm{E}(I)$ for a well chosen ordering in $\mathbb{N}^n$.

**Proposition 1.9.3**. — *Let $L$ be the linear form on $\mathbb{Q}^n$ defined by $L(\alpha) = \alpha_1 + \cdots + \alpha_n = |\alpha|$. Let $I$ be an ideal of $\boldsymbol{k}[[\boldsymbol{X}]]$ (or $\boldsymbol{k}\{\boldsymbol{X}\}$) and $E(I) = \mathrm{E}_{<_L}(I)$. Then, for any $k \in \mathbb{N}$*

*we have:*

$$FHS_A(k) = \dim_{\boldsymbol{k}} \left( \frac{A}{A\mathfrak{m}^{k+1}} \right) = \#\{\alpha \in (\mathbb{N}^n \smallsetminus E(I)) \mid |\alpha| \leqslant k\}.$$

*Proof.* — Let us consider the formal power series case, the convergent case being similar. We have a natural isomorphism of vector spaces $A/A\mathfrak{m}^{k+1} \simeq \boldsymbol{k}[[\boldsymbol{X}]]/(I + \mathfrak{m}^{k+1})$. For the ordering $<_L$ we have the equality $\mathrm{E}(I + \mathfrak{m}^{k+1}) = \mathrm{E}(I) \cup \mathrm{E}(\mathfrak{m}^{k+1})$. Indeed, it is enough to prove the inclusion $\mathrm{E}(I + \mathfrak{m}^{k+1}) \subset \mathrm{E}(I) \cup \mathrm{E}(\mathfrak{m}^{k+1})$, the other being obvious. Let $f \in I$ and $g \in \mathfrak{m}^{k+1}$. If $\mathrm{val}(f) < \mathrm{val}(g)$ then $\mathrm{in}(f + g) = \mathrm{in}(f)$ and thus $\exp(f+g) = \exp(f) \in \mathrm{E}(I)$. If $\mathrm{val}(f) \geqslant \mathrm{val}(g)$ then $\mathrm{val}(f+g) \geqslant \min\{\mathrm{val}(f), \mathrm{val}(g)\} \geqslant \mathrm{val}(g) \geqslant k + 1$. Whence $f + g \in \mathfrak{m}^{k+1}$.

We end the proof of the proposition by applying 1.5.3. □

Let us denote by $\wp$ the set of the subsets $\{1, \ldots, n\}$. We introduce the following notations:

- For each $\sigma \in \wp$ we write:
  - $S(\sigma) = \{\alpha \in \mathbb{N}^n \mid \alpha_i = 0 \text{ if } i \in \sigma\}$
  - $T(\sigma) = S(\{1, \ldots, n\} \smallsetminus \sigma)$
  - $\#\sigma$ = cardinal of $\sigma$
- For each non-empty subset $E \subset \mathbb{N}^n$ such that $E + \mathbb{N}^n = E$:
  - $cd(E) = \min\{\#\sigma \mid S(\sigma) \cap E = \varnothing\}$
  - $d(E) = n - cd(E)$

***Proposition 1.9.4***. — *Let $\varnothing \neq E \subset \mathbb{N}^n$ be such that $E + \mathbb{N}^n = E$. Let $\sigma \in \wp$ be such that $\#\sigma = cd(E)$. Then the set*

$$\{\alpha \in T(\sigma) \mid (\alpha + S(\sigma)) \cap E = \varnothing\}$$

*is finite.*

*Proof.* — We remark that the set defined in the proposition is the complement of $p(E)$ in $T(\sigma)$, $p$ being the natural projection of $\mathbb{N}^n$ onto $T(\sigma)$. Since $p(E)$ is stable by addition in $T(\sigma)$, this complement could only be infinite if it contained a coordinate axis in $T(\sigma)$, which would contradict the minimality of the cardinal of $\sigma$. □

Let us denote by $e_\sigma(E)$ the cardinal of the set defined in the previous proposition and by $e(E)$ the sum

$$e(E) = \sum_{\#\sigma = cd(E)} e_\sigma(E)$$

***Theorem 1.9.5***. — *With the notations above we have:*

(1) $d(E(I)) = \dim(A)$
(2) $e(E(I)) = e(A)$.

*Proof.* — See [**15**], [**7**]. □

## 2. Division theorems in the rings of differential operators

**2.1.** The aim of this section is to adapt the division theorems proved in chapter I to the case of the rings of differential operators and to give some applications: The calculation of free resolutions, of characteristic varieties and of multiplicities. The references are [**11**] and [**14**].

Let $\boldsymbol{k}$ be a field of characteristic zero. We denote:

- $A_n(\boldsymbol{k}) = \boldsymbol{k}[\boldsymbol{X}, \boldsymbol{\partial}] = \boldsymbol{k}[X_1, \ldots, X_n; \partial_1, \ldots, \partial_n]$ the Weyl algebra, *i.e.* the ring of linear differential operators with polynomial coefficients in $n$ variables.
- $\widehat{\mathscr{D}}_n(\boldsymbol{k}) = \boldsymbol{k}[[\boldsymbol{X}]][\boldsymbol{\partial}] = \boldsymbol{k}[[X_1, \ldots, X_n]][\partial_1, \ldots, \partial_n]$ the ring of linear differential operators with formal power series in $n$ variables as coefficients.
- $\mathscr{D}_n(\boldsymbol{k}) = \boldsymbol{k}\{\boldsymbol{X}\}[\boldsymbol{\partial}] = \boldsymbol{k}\{X_1, \ldots, X_n\}[\partial_1, \ldots, \partial_n]$ the ring of linear differential operators with convergent power series in $n$ variables as coefficients, if $\boldsymbol{k} = \mathbb{R}$ or $\mathbb{C}$ or, more generally, a complete valued field of characteristic zero.

For the sake of brevity we shall write when no confusion is possible: $A_n$, $\widehat{\mathscr{D}_n}$, $\mathscr{D}_n$. We denote by $\mathscr{R}$ any of these three rings.

If $P$ is an operator we develop it in the following way:

$$P = \sum_{(\alpha,\beta)\in\mathbb{N}^{2n}} a_{(\alpha,\beta)} \boldsymbol{X}^\alpha \boldsymbol{\partial}^\beta = \sum_{\beta\in\mathbb{N}^n} f_\beta \boldsymbol{\partial}^\beta$$

where $a_{(\alpha,\beta)} \in \boldsymbol{k},\ f_\beta \in \boldsymbol{k}[\boldsymbol{X}], \boldsymbol{k}[[\boldsymbol{X}]]$ or $\boldsymbol{k}\{\boldsymbol{X}\}$.

We call the following subset of $\mathbb{N}^{2n}$, denoted by $\mathscr{N}(P)$, the *Newton's diagram* of $P$:

$$\mathscr{N}(P) = \{(\alpha,\beta) \in \mathbb{N}^{2n} \mid a_{(\alpha,\beta)} \neq 0\}$$

**2.2. The order of an operator.** — We fix a linear form $L$ on $\mathbb{Q}^{2n}$ with non negative coefficients, whose restriction $L_2$ to $\{0\}\times\mathbb{Q}^n$ has strictly positive coefficients. This condition is only necessary in the case of power series coefficients.

***Definition 2.2.1.*** — Let $0 \neq P \in \mathscr{R} = A_n, \widehat{\mathscr{D}_n}$ or $\mathscr{D}_n$. We define the $L_2$-order of $P$ (and we denote it by $\mathrm{ord}_{L_2}(P)$) as being $\max\{L_2(\beta) \mid f_\beta \neq 0\}$. We set $\mathrm{ord}_{L_2}(0) = -\infty$.

We have $\mathrm{ord}_{L_2}(PQ) = \mathrm{ord}_{L_2}(P) + \mathrm{ord}_{L_2}(Q)$ for any operators $P$ and $Q$.

For each $k \in L_2(\mathbb{Q}^n)$, we write

$$F_k^{L_2}(\mathscr{R}) = \{P \in \mathscr{R} \mid \mathrm{ord}_{L_2}(P) \leqslant k\}.$$

The family $F_\bullet^{L_2}(\mathscr{R})$ is an increasing filtration of the ring $\mathscr{R}$. Let $\mathrm{gr}_k^{L_2}(\mathscr{R})$ (or, more briefly, $\mathrm{gr}_k(\mathscr{R})$) denote the quotient $F_k^{L_2}(\mathscr{R})/F_{<k}^{L_2}(\mathscr{R})$. We call the mapping $\sigma_k^{L_2}$ : $F_k(\mathscr{R}) \to \mathrm{gr}_k(\mathscr{R})$ the *symbol function of order $k$*.

***Definition 2.2.2.*** — Let $P \in F_k(\mathscr{R}) \smallsetminus F_{<k}(\mathscr{R})$. We call $\sigma_k^{L_2}(P)$ the $L_2$-principal symbol of $P$. We denote the $L_2$-principal symbol of $\partial_i$ by $\xi_i$. Thus, $\sigma_k^{L_2}(P) = \sum_{L_2(\beta)=k} f_\beta \boldsymbol{\xi}^\beta$. We shall write it simply $\sigma^{L_2}(P)$.

The ring $\mathrm{gr}^{L_2}(\mathscr{R}) = \bigoplus_k \mathrm{gr}_k^{L_2}(\mathscr{R})$ is commutative and isomorphic to the ring $\mathscr{B}[\xi_1,\ldots,\xi_n]$ where as the case may be $\mathscr{B} = \boldsymbol{k}[\boldsymbol{X}]$, $\boldsymbol{k}[[\boldsymbol{X}]]$, or $\boldsymbol{k}\{\boldsymbol{X}\}$.

***Definition 2.2.3.*** — Let $I$ be an ideal [(9)] of $\mathscr{R}$. We call the ideal of $\mathrm{gr}^{L_2}(\mathscr{R})$, denoted by $\mathrm{gr}^{L_2}(I)$, generated by $\{\sigma^{L_2}(P) \mid P \in I\}$ the $L_2$-graded ideal associated with $I$.

***Definition 2.2.4.*** — Let $I$ be an ideal of $\mathscr{R}$. We call the set

$$\{(\boldsymbol{x},\boldsymbol{\xi}) \in \boldsymbol{k}^{2n} \mid \sigma^{L_2}(P)(\boldsymbol{x},\boldsymbol{\xi}) = 0 \text{ for all } P \in I\},$$

denoted by $\mathrm{Char}_{L_2}(\mathscr{R}/I)$, the $L_2$-characteristic variety of the $\mathscr{R}$-module $\mathscr{R}/I$.

When $\mathscr{R} = A_n$ we also have the possibility of mixing the variables $\boldsymbol{X}$ and $\boldsymbol{\partial}$:

***Definition 2.2.5* (The $L$-Bernstein filtration).** — Let $P \in A_n(\boldsymbol{k})$. We call the integer

$$\max\{L(\alpha,\beta) \mid a_{(\alpha,\beta)} \neq 0\}$$

the $L$-order of $P$ (and we denote it by $\mathrm{ord}_L(P)$). The $L$-principal symbol of $P$ is the sum $\sigma^L(P) = \sum_{L((\alpha,\beta))=\mathrm{ord}_L(P)} a_{(\alpha,\beta)}\boldsymbol{X}^\alpha\boldsymbol{\xi}^\beta$.

We have once again the notion of graded ideal associated with an ideal $I$ of $A_n$ and the notion of $L$-characteristic variety of $A_n/I$, for the $L$-Bernstein filtration.

On the other hand when $L_2(\beta) = \beta_1 + \cdots + \beta_n$, the filtration induced by $L_2$ is the usual filtration by the order of operators with respect to derivation variables.

**2.3. Orderings in $\mathbb{N}^{2n}$ and the privileged exponent of an operator.** — Let $<$ be a total well ordering on $\mathbb{N}^{2n}$ compatible with sums. We define an ordering denoted by $<_L$, on $\mathbb{N}^{2n}$, in a different way according to whether we are in $A_n$ or with power series coefficients.

- In $A_n$:

$$(\alpha,\beta) <_L (\alpha',\beta') \text{ if and only if } \begin{cases} L_2(\beta) < L_2(\beta') \\ \text{or } L_2(\beta) = L_2(\beta') \text{ and } L(\alpha,\beta) < L(\alpha',\beta') \\ \text{or } \begin{cases} L_2(\beta) = L_2(\beta'),\ L(\alpha,\beta) = L(\alpha',\beta') \\ \text{and } (\alpha,\beta) < (\alpha',\beta') \end{cases} \end{cases}$$

This is a total well ordering compatible with sums.

- In $\widehat{\mathscr{D}_n}$ or $\mathscr{D}_n$:

$$(\alpha,\beta) <_L (\alpha',\beta') \text{ if and only if } \begin{cases} L_2(\beta) < L_2(\beta') \\ \text{or } L_2(\beta) = L_2(\beta') \text{ and } L(\alpha,\beta) > L(\alpha',\beta') \\ \text{or } \begin{cases} L_2(\beta) = L_2(\beta'),\ L(\alpha,\beta) = L(\alpha',\beta') \\ \text{and } (\alpha,\beta) > (\alpha',\beta') \end{cases} \end{cases}$$

***Definition 2.3.1.*** — Let $P \in A_n$, $\widehat{\mathscr{D}_n}$ or $\mathscr{D}_n$. We call the $2n$-uple $\exp_L(P) = \max_{<_L}\{(\alpha,\beta) \mid a_{(\alpha,\beta)} \neq 0\}$, the $L$-privileged exponent of $P$.

(9) All the ideals under consideration are left ideals.

***Remark***. — We have in every case the formula $\exp_L(P) = \exp_L(\sigma^{L_2}(P))$ with $\sigma^{L_2}(P) \in \boldsymbol{k}[\boldsymbol{X},\boldsymbol{\xi}]$, $\boldsymbol{k}[[\boldsymbol{X}]][\boldsymbol{\xi}]$ or $\boldsymbol{k}\{\boldsymbol{X}\}[\boldsymbol{\xi}]$ respectively, the two last rings being seen as subrings of $\boldsymbol{k}[[\boldsymbol{X},\ \boldsymbol{\xi}]]$ or of $\boldsymbol{k}\{\boldsymbol{X},\ \boldsymbol{\xi}\}$ and the privileged exponents being taken in the sense of the first chapter.

Then we can state the following propositions which can be proved exactly as in the first chapter:

***Proposition 2.3.2***. — *Let $P, Q \in \mathscr{R}$ . We have:*

1) $\exp(PQ) = \exp(P) + \exp(Q)$.
2) *If* $\exp(P) \neq \exp(Q)$ *then* $\exp(P+Q) = \max_{<_L}\{\exp(P), \exp(Q)\}$.

For each non zero ideal $I$ of $\mathscr{R}$ let $\mathrm{E}_{<_L}(I)$ denote the set $\{\exp_L(P) \mid P \in I \smallsetminus \{0\}\}$. If no confusion is possible we write $\mathrm{E}(I)$ instead of $\mathrm{E}_{<_L}(I)$. We have, by 2.3.2, $\mathrm{E}(I) + \mathbb{N}^{2n} = \mathrm{E}(I)$ and as we prove in 1.4.6 we have:

***Proposition 2.3.3***. — *Let $E \subset \mathbb{N}^{2n}$ be such that $E + \mathbb{N}^{2n} = E$. Then there is a finite subset $F \subset E$ such that $E = \cup_{(\alpha,\beta)\in F}((\alpha,\beta) + \mathbb{N}^{2n})$.*

***Definition 2.3.4***. — Let $I$ be a non zero ideal of $\mathscr{R}$. We call any family $P_1, \ldots, P_m$ of elements in $I$ such that $\mathrm{E}(I) = \cup_{i=1}^m(\exp_L(P_i) + \mathbb{N}^{2n})$, a *standard basis* of $I$, relative to $L$ (or an $L$-standard basis of $I$)

***Remarks***

1) There always exists a standard basis of $I$ by definition of $\mathrm{E}(I)$ and 2.3.3.

2) In the case of $A_n$ we can also consider the $L$-Bernstein filtration, and the following ordering similar to the one given in the preceding chapter up to the change of $n$ into $2n$:

$$(\alpha,\beta) <_L (\alpha',\beta') \text{ if and only if } \begin{cases} L(\alpha,\beta) < L(\alpha',\beta') \\ \text{or } L(\alpha,\beta) = L(\alpha',\beta') \text{ and } (\alpha,\beta) < (\alpha',\beta') \end{cases}$$

**2.4. More divisions.** — The statements below narrowly follow those in the preceding chapter and we shall only give the proofs of the points specific to the case of the operators.

With each $m$-uple $((\alpha^1,\beta^1),\ldots,(\alpha^m,\beta^m))$ of elements of $\mathbb{N}^{2n}$, we associate a partition $\Delta_1,\ldots,\Delta_m,\overline{\Delta}$ of $\mathbb{N}^{2n}$ in the same way as in chapter I. We set:

$$\Delta^1 = (\alpha^1,\beta^1) + \mathbb{N}^{2n}, \quad \Delta^{i+1} = ((\alpha^{i+1},\beta^{i+1}) + \mathbb{N}^{2n}) \smallsetminus (\Delta^1 \cup \cdots \cup \Delta^i) \text{ if } i \geqslant 1,$$
$$\overline{\Delta} = \mathbb{N}^{2n} \smallsetminus (\cup_{i=1}^m \Delta^i).$$

***Theorem 2.4.1***. — *Let $(P_1,\ldots,P_m)$ be an $m$-uple of non zero elements of $\mathscr{R}$ and let $\Delta_1,\ldots,\Delta_m,\overline{\Delta}$ be the partition of $\mathbb{N}^{2n}$ associated with $(\exp(P_1),\ldots,\exp(P_m))$. Then, for any $P$ in $\mathscr{R}$, there is a unique $(m+1)$-uple $(Q_1,\ldots,Q_m,R)$ of elements in $\mathscr{R}$, such that:*

(1) $P = Q_1P_1 + \cdots + Q_mP_m + R$.

(2) $\exp(P_i) + \mathscr{N}(Q_i) \subset \Delta^i$, $i = 1, \ldots, m$.
(3) $\mathscr{N}(R) \subset \overline{\Delta}$.

*Proof*. — Uniqueness can be proved as in the commutative case. For existence, we consider $\sigma^{L_2}(P)$ and $\sigma^{L_2}(P_i)$ as elements of $\boldsymbol{k}[\boldsymbol{X}, \boldsymbol{\xi}]$ (resp. $\boldsymbol{k}[[\boldsymbol{X}, \boldsymbol{\xi}]]$, or $\boldsymbol{k}\{\boldsymbol{X}, \boldsymbol{\xi}\}$), which are $L_2$-homogeneous with respect to the variable $\boldsymbol{\xi}$. Let us write the division in the sense of chapter I, in any of the three cases:

$$\sigma^{L_2}(P) = \sum_{i=1}^{m} q_i \sigma^{L_2}(P_i) + r,$$

the $q_i(\boldsymbol{X}, \boldsymbol{\xi})$ and $r(\boldsymbol{X}, \boldsymbol{\xi})$ being polynomials and $L_2$-homogeneous with respect to variables $\boldsymbol{\xi}$ (since the coefficients of $L_2$ are strictly positive). Suppose that $d = \mathrm{ord}_{L_2}(P)$ and that $d_i = \mathrm{ord}_{L_2}(P_i)$. Then the degrees of the quotients and of the remainder are given by the relations:

$$\mathrm{ord}_{L_2}(q_i) = d - d_i, \text{ or } q_i = 0, \ \mathrm{ord}_{L_2}(r) = d \text{ or } r = 0.$$

Let then $Q_i$ and $R$ be the obvious operators such that $q_i = \sigma^{L_2}(Q_i)$ and $r = \sigma^{L_2}(R)$ (for example if $q_i = \sum_{L_2(\beta)=d-d_i} a_{(\alpha,\beta)} \boldsymbol{X}^\alpha \boldsymbol{\xi}^\beta$, $Q_i = \sum_{L_2(\beta)=d-d_i} a_{(\alpha,\beta)} \boldsymbol{X}^\alpha \boldsymbol{\partial}^\beta$).

Then the operator $P' = P - \sum_{i=1}^m Q_i P_i - R$ is of $L_2$-order strictly smaller than $d$. We remark that the $Q_i$ and $R$ have the properties 2) and 3) above since $q_i$ and $r$ have the corresponding properties and $\exp(P_i) = \exp(\sigma^{L_2}(P_i))$.

We end the proof by an induction (finite since the coefficients of $L_2$ are $> 0$) on the $L_2$-order. □

***Remark***. — The element $Q_i$ in the theorem is called the $i$-th quotient and $R$ is called the remainder of the division of $P$ by $(P_1, \ldots, P_m)$. The remainder will be denoted by $R(P; P_1, \ldots, P_m)$.

***Remark***. — It follows from the proof that for any division $P = Q_1P_1 + \cdots + Q_mP_m + R$ as in the theorem we have $\max\{\max_i\{\exp_L(Q_iP_i)\}, \exp_L(R)\} = \exp_L(P)$ and as a consequence $\max\{\max_i\{\mathrm{ord}_{L_2}(Q_iP_i)\}, \mathrm{ord}_{L_2}(R)\} = \mathrm{ord}_{L_2}(P)$.

***Remark***. — We have a similar (and simpler to prove) division theorem in the ring $\mathrm{gr}^{L_2}(\mathscr{R}) = \mathscr{B}[\xi]$. We let the reader state (and prove) a division theorem in $A_n$, relative to the $L$-Bernstein filtration. See [**14**].

***Corollary 2.4.2***. — *Let $I$ be a non zero ideal of $\mathscr{R}$ (or $\mathrm{gr}^{L_2}(\mathscr{R})$) and let $P_1, \ldots, P_m$ be a family of elements of $I$. The following conditions are equivalents:*

1) *$P_1, \ldots, P_m$ is a standard basis of $I$.*
2) *For any $P$ in $\mathscr{R}$, we have: $P \in I$ if and only if $R(P; P_1, \ldots, P_m) = 0$.*

***Corollary 2.4.3***. — *Let $I$ be a non zero ideal of $\mathscr{R}$ (or $\mathrm{gr}^{L_2}(\mathscr{R})$) and let $P_1, \ldots, P_m$ be a standard basis of $I$. Then $P_1, \ldots, P_m$ is a system of generators of $I$.*

These two statements can be proved exactly as in the commutative case.

***Remark***. — Let $I$ be an ideal of $\mathscr{R}$. Then $\{P_1, \ldots, P_m\}$ is a standard basis of $I$ if and only if $\{\sigma(P_1), \ldots, \sigma(P_m)\}$ is a standard basis of $\mathrm{gr}^{L_2}(I)$.

**2.5. The calculation of a standard basis and its applications.** — Let $P_1, P_2$ be two operators with privileged exponents $(\alpha^1, \beta^1), (\alpha^2, \beta^2)$. As in chapter I, we call the semisyzygy of $P_1, P_2$ the operator $M_1P_1 - M_2P_2 = S(P_1, P_2)$ where $M_1, M_2$ are two monomials whose exponents $\nu^1, \nu^2$ are such that $\nu^1 + (\alpha^1, \beta^1) = \nu^2 + (\alpha^2, \beta^2)$ and minimal for this property and furthermore such that the leading coefficients satisfy $c(M_1)c(P_1) = c(M_2)c(P_2)$ so that we get $\exp_L(S(P_1, P_2)) <_L \exp_L(M_1P_1) = \exp_L(M_2P_2)$. We have again:

***Proposition 2.5.1***. — *Let $P_1, \ldots, P_r$ be a system of generators of the ideal $I$ of $\mathscr{R}$ such that for any $(i,j)$ the remainder of the division of $S(P_i, P_j)$ by $(P_1, \ldots, P_r)$ is zero. Then, $\{P_1, \ldots, P_r\}$ is a standard basis of the ideal $I$.*

*Proof*. — We deduce the proof from the result in the commutative case by considering the $\sigma(P_i) \in \mathscr{B}[\xi_1, \ldots, \xi_n]$, and by using the fact that $P_i$ and $\sigma(P_i)$ have the same privileged exponent. If $M_iP_i - M_jP_j = S(P_i, P_j) = A_1P_1 + \cdots + A_rP_r$ is a division, we have $\mathrm{ord}_{L_2}(A_kP_k) \leqslant \mathrm{ord}_{L_2}(M_iP_i) = \mathrm{ord}_{L_2}(M_jP_j)$.

We set $m_i = \sigma(M_i)$, $a_k = \sigma_{\nu_k}(A_k)$ where $\nu_k = \mathrm{ord}_{L_2}(M_iP_i) - \mathrm{ord}_{L_2}(A_k)$, and then we get the relation:

$$m_i\sigma(P_i) - m_j\sigma(P_j) = a_1\sigma(P_1) + \cdots + a_r\sigma(P_r).$$

This is a division in $\boldsymbol{k}[\boldsymbol{X}, \boldsymbol{\xi}]$, $\boldsymbol{k}[[\boldsymbol{X}, \boldsymbol{\xi}]]$ or $\boldsymbol{k}\{\boldsymbol{X}, \boldsymbol{\xi}\}$ as the case may be. Furthermore, it is $L_2$-homogeneous, hence in $\mathscr{B}[\boldsymbol{\xi}]$.

Thus, $\{\sigma(P_1), \ldots, \sigma(P_r)\}$ gives a standard basis of the ideal which they generate in the above rings hence also in $\mathscr{B}[\boldsymbol{\xi}]$. It remains to prove that the $\sigma(P_i)$'s generate $\mathrm{gr}(I)$. We consider $P \in I$ and we write:

$$P = A_1P_1 + \cdots + A_rP_r \qquad (*)$$

If $\mathrm{ord}_{L_2}(P) < \delta = \max(\mathrm{ord}_{L_2}(A_kP_k))$, we have $a_1\sigma(P_1) + \cdots + a_r\sigma(P_r) = 0$, where $a_k = \sigma_{\delta - \mathrm{ord}_{L_2}(P_k)}(A_k)$.

We deduce from 1.6.4 the fact that in $\mathscr{B}[\boldsymbol{\xi}]$, $L_2$-homogeneous relations between the $\sigma(P_k)$ are generated by those which come from the divisions of semisyzygies. This allows us to change the relation $(*)$ in order to lower $\delta$.

We finally obtain a decomposition $(*)$ for which $\delta = \mathrm{ord}_{L_2}(P)$ in which case we have $\sigma(P) = a_1P_1 + \cdots + a_rP_r \in \mathrm{gr}(I)$. $\square$

Let $I \subset \mathscr{R}$ be an ideal given by a system of generators $P_1, \ldots, P_s$. The process that we are going to describe enables us to build a standard basis $(P_1, \ldots, P_s, P_{s+1}, \ldots, P_{s+t})$ by a finite sequence of divisions. This algorithm is the analogue for algebraic differential operators of Buchberger's [**13**] (see 1.6.3).

• Assume that $(P_1,\dots,P_s,P_{s+1},\dots,P_{s+q})$ are already built and define $E_q = \bigcup_{k=1}^{s+q}(\exp(P_k)+\mathbb{N}^{2n})$.

• If there is $(i,j)$ such that the remainder of the division of $S(P_i,P_j)$ by $(P_1,\dots,P_{s+q})$ is non zero, let us choose the first of these $(i,j)$ (for the lexicographic ordering) and denote by $P_{s+q+1}$ the remainder thus obtained. Thus we have $E_q \subset E_{q+1}$ and $E_q \neq E_{q+1} \subset \mathrm{E}(I)$.

• By a Noetherian argument, this process stops and there exists an integer $t$ such that $E_{s+t} = \mathrm{E}(I)$. This can be detected by the lack of a non zero remainder since then $(P_1,\dots,P_s,\dots,P_{s+t})$ is a standard basis.

• We can eliminate (one by one) the $P_k$ whose privileged exponents are contained in the $\mathbb{N}^{2n}$-subset generated by the remaining exponents.

*Application 1. The calculation of the characteristic variety of a $\mathscr{R}$-module of type $\mathscr{R}/I$*

**Proposition 2.5.2**. — *Let $(P_1,\dots,P_r)$ be a L-standard basis of the ideal $I$ of $\mathscr{R}$. Then the equations of the $L_2$-characteristic variety of $\mathscr{R}/I$ are:*

$$\sigma(P_1)(\boldsymbol{X},\boldsymbol{\xi}) = \cdots = \sigma(P_r)(\boldsymbol{X},\boldsymbol{\xi}) = 0$$

Indeed the equations $\sigma(P)(\boldsymbol{X},\boldsymbol{\xi}) = 0$ for all $P\in I$ are linear combinations of these equations.

*Application 2. Free resolutions of an $\mathscr{R}$-module of type $\mathscr{R}/I$.* — Let $(P_1,\dots,P_r)$ be a standard basis of the ideal $I$ of $\mathscr{R}$. Let $\mathscr{S}$ be the module of relations between the operators $P_k$. This module is the set of $r$-uples $R=(A_1,\dots,A_r)\in\mathscr{R}^r$ such that $A_1P_1+\cdots+A_rP_r=0$. We say that $R$ is of order $k$ if $k=\max(\mathrm{ord}_{L_2}(A_iP_i))$ and we set: $\sigma_k(R) = (\sigma_{k-d_1}(A_1),\dots,\sigma_{k-d_r}(A_r))$.

Let us denote the relations following from the division of semisyzygies by $R_{i,j}$ and $r_{i,j}=\sigma(R_{i,j})$.

**Proposition 2.5.3**. — *We have an exact sequence: $\mathscr{D}^{r(r+1)/2} \xrightarrow{\varphi} \mathscr{D}^r \xrightarrow{\psi} \mathscr{D} \to \mathscr{D}/I$ with:*

$$\psi(Q_1,\dots,Q_r) = Q_1P_1+\cdots+Q_rP_r, \quad \varphi((A_{i,j})) = \sum A_{i,j}R_{i,j}.$$

*Proof*. — This is equivalent to stating that the relations between the $P_\ell$ are generated by the relations $R_{i,j}$. If $R$ is such a relation, $\sigma(R)=r$ is a homogeneous relation between the $\sigma(P_\ell)$, of degree $k=\mathrm{ord}_{L_2}(R)$. By the commutative analogue (see 1.6.4), we can write $r=\sum\lambda_{i,j}r_{i,j}$ with $\mathrm{ord}(\lambda_{i,j})+k_{i,j}\leqslant k$ where $k_{i,j}=\mathrm{ord}(R_{i,j})$. We choose $\Lambda_{i,j}\in\mathscr{R}$ such that $\sigma(\Lambda_{i,j})=\lambda_{i,j}$.

Then, $R' = R-\sum\Lambda_{i,j}R_{i,j}$ is a relation between the operators $P_\ell$ of $L_2$-order $<k$. We conclude by an induction on the $L_2$-order. □

*Application 3. Elimination of variables in $A_n$ and intersection of ideals.* — Rename the vector $(x_1,\dots,x_n,\partial_1,\dots,\partial_n)$ as $(y_1,\dots,y_n,y_{n+1},\dots,y_{2n})$ and consider new variables $z_1,\dots,z_n,z_{n+1},\dots,z_{2n}$. Let $\tau$ be a permutation of $2n$ symbols and denote

$z_i = y_{\tau(i)}$. Denote by $\rho$ the inverse of $\tau$. Then $A_n$ is isomorphic to the $\boldsymbol{k}$-algebra generated by $z_1, \dots, z_n, z_{n+1}, \dots, z_{2n}$ with relations $[z_{\rho(i)}, z_{\rho(j)}] = 0$ $(i \leqslant j)$ except for $j = i + n$ in which case $[z_{\rho(i)}, z_{\rho(j)}] = -1$.

Let $I$ be a left ideal of $A_n$ and $k$ be an integer $0 \leqslant k \leqslant 2n-1$. We denote by $A_{n,k}$ the subalgebra of $A_n$ generated by $z_{k+1}, \dots, z_{2n}$. We define $I_k = I \cap A_{n,k}$. The (left) ideal $I_k$ of $A_{n,k}$ is the set of operators in $I$ which depend only on $z_{k+1}, \dots, z_{2n}$. We write $I_{2n} = \boldsymbol{k} \cap I$. The ideal $I_k$ is called the $k$-th elimination ideal of $I$. We shall return later to this definition.

Using the lexicographic ordering on $\mathbb{N}^{2n}$ (as in 1.7.1) we can prove the following results which are similar as well as the proofs to those in 1.7 and 1.8.1.

***Lemma 2.5.4***. — *Let $P$ be an element of $A_n$. Then $\mathrm{mp}_{<_{\mathrm{lex}}}(P)$ is in $A_{n,k}$ if and only if $P$ is in $A_{n,k}$.*

***Theorem 2.5.5***. — *Let $I$ be a left ideal of $A_n$ and $k$ an integer such that $0 \leqslant k \leqslant 2n$. Let $\mathscr{G}$ be a standard basis of the ideal $I$ relative to the lexicographic ordering. Let $\mathscr{G}_k = \mathscr{G} \cap A_{n,k}$. Then we have:*

(1) *If $\mathscr{G}_k = \varnothing$ then $I_k = (0)$.*

(2) *If $\mathscr{G}_k \neq \varnothing$ then $\mathscr{G}_k$ is a standard basis of the ideal $I_k$ relative to the lexicographic ordering.*

Let $I, J$ be two left ideals of $A_n$. Let $\theta$ be a new indeterminate. We denote by $I^e$ (resp. $J^e$) the extension of the ideal $I$ (resp. $J$) to the ring $A_n[\theta]$ (here $\theta$ is a central element). If $h$ is an element of $\boldsymbol{k}[\theta]$ we denote by $hI^e$ (resp. $hJ^e$) the product of the ideals[10] $(h)$ and $I^e$ (resp. $(h)$ and $J^e$). With these notations we have:

***Theorem 2.5.6***. — *Let $I, J$ be two left ideals of $A_n$. Then $I \cap J = (\theta I^e + (1-\theta) J^e) \cap A_n$.*

***Remark***. — The theory of standard bases can be easily generalized to the case of sub–modules of $\mathscr{R}^N$, see [**14**]. For that purpose we only have to adapt the notions of ordering and of privileged exponents to exponents in $\mathbb{N}^{2n} \times \{1, \dots, N\}$. By applying this to the calculation of a standard basis of $ker(\varphi)$ and then of the successive kernels, we build a free resolution of any $\mathscr{R}$-module $\mathscr{M}$ of finite presentation, whence for example a realization of the complex of solutions and of the De Rham complex $\mathbb{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{M}, \mathscr{O})$ and $\Omega^n \overset{\mathbf{L}}{\otimes} \mathscr{M}$. This is algorithmic in the algebraic case.

**2.6. An example: The characteristic cycle of $\mathscr{O}[1/f]$ for a quasihomogeneous $f$ in two variables.**— In this example we are dealing with the form $L_2(i,j) = i + j$. In this case and more generally in the case of the diagonal form $L_2$ on $\mathbb{Q}^n$, we refer to [**25, 19**] for the definition of the multiplicity at a point of the cotangent space. The characteristic cycle of a coherent $\mathscr{D}$-module is the linear

(10) these are ideals of the ring $A_n[\theta]$

combination of the irreducible components of the characteristic variety, each counted with its multiplicity at a generic point.

Let $f \in \mathbb{C}[x, y]$ be a quasi-homogeneous polynomial. We denote by $w_1$ and $w_2$ the weights of variables and by $\chi$ the Euler vector field:

$$\chi = w_1 x\partial_x + w_2 y\partial_y$$

We have $\chi(f) = f$. We verify that $\mathscr{O}[1/f] = \mathscr{D} \cdot \frac{1}{f}$, because the Bernstein polynomial of $f$ has no $\leqslant -2$ integer root (see [**31**]). It is easier to deal with the quotient $\mathscr{O}[1/f]/\mathscr{O}$ and we find that the annihilator ideal of its generator $c\ell(\frac{1}{f})$ is the ideal generated by the following three operators:

- $P_1 = f'_y\partial_x - f'_x\partial_y$
- $P_2 = w_1 x\partial_x + w_2 y\partial_y + 1 (= \chi + 1)$
- $P_3 = f$

Let us first consider the case

$$f = y^p + c_1 x^{q_1} y^{p-p_1} + \cdots + c_k x^{kq_1} y^{p-kp_1} + \cdots$$

with $q_1 > p_1 \geqslant 1$, $p = 0$ or $1 (\operatorname{mod} p_1)$ and $w_2 = 1/p$, $q_1 w_1 = p_1 w_2$.

In this situation we verify by computing the semisyzygies that $\{P_1, P_2, P_3\}$ is a standard basis for the ordering (of series type) associated with $L(j, i, \beta, \alpha) = j + i + \alpha + \beta$ the monomial with the same $L$-order being further ordered by $y > x > \partial_y > \partial_x$.

The privileged exponents are respectively: $(p-1, 0, 0, 1)$, $(0, 1, 0, 1)$, $(p, 0, 0, 0)$.

By applying 1.9 we can compute the multiplicity at the origin of $\mathscr{O}[1/f]/\mathscr{O}$ which is therefore $(p-1)+0+p+0+0+0 = 2p-1$. The characteristic cycle has the following form: $sT^*_0(\mathbb{C}^2) + 1.T^*_{f^{-1}(0)}(\mathbb{C}^2)$ for some integer $s$. The multiplicity of $f^{-1}(0)$ at the origin being $p$ we get from this $2p - 1 = s + 1.p$, or: $s = p - 1$.

For the case $f = x \cdot g$, where $g$ is a polynomial as in the previous case we refer to [**9**].

## 3. Generalized division theorems. The calculation of slopes

The reference for this chapter is [**2**] for the case of the Weyl algebra. We denote by $\mathscr{R}$ any of the rings $A_n$, $\mathscr{D}_n$ or $\widehat{\mathscr{D}_n}$.

### 3.1. Orders and filtrations with respect to a smooth hypersurface.

Let $Y$ be a hypersurface of $\mathbb{C}^n$ defined by $x_1 = 0$. Given a linear form $L(a, b) = pa + qb$ on $\mathbb{Q}^2$ (with non negative and relatively prime integer coefficients $p, q$), we define the $L$-order along $Y$ of $P = P(x, \partial)$ in $\mathscr{R}$ denoted by $\operatorname{ord}_L(P)$, as the maximum of $L(|\beta|, \beta_1 - \alpha_1)$ for $(\alpha, \beta)$ in the Newton diagram of $P$. To shorten we write here $x$ instead of $\boldsymbol{X}$ and $\partial$ instead of $\boldsymbol{\partial}$.

Notice that here $L$ is a linear form on $\mathbb{Q}^2$ whereas in the previous chapters this letter was used to denote a linear form on $\mathbb{Q}^{2n}$ whose part is now taken by:

$$\widetilde{L}(\alpha,\beta) = L(|\beta|, \beta_1 - \alpha_1) = (p+q)\,|\beta| - q(\beta_2 + \cdots + \beta_n + \alpha_1).$$

Let $F_{L,\bullet}(\mathscr{R})$ be the filtration induced by the $L$-order on $\mathscr{R}$ *i.e.* $F_{L,k}$ is the set of operators $P$ such that $\mathrm{ord}_L(P) \leqslant k$. Let $F$ (resp. $V$) denote the filtration associated with the linear form $L(a,b) = a$ (resp. $L(a,b) = b$). By extension we also write $F$ (resp. $V$) for the corresponding linear forms. If $L \neq F, V$ the graded ring associated with this filtration

$$\mathrm{gr}^L(\mathscr{R}) = \bigoplus_{k\in\mathbb{Z}} F_{L,k}(\mathscr{R})/F_{L,k-1}(\mathscr{R})$$

is isomorphic to one of the graded commutative rings $\mathbb{C}[x,\xi] = \mathbb{C}[x_1,\ldots,x_n,\xi_1,\ldots,\xi_n]$ or $\mathbb{C}\{x_2,\ldots,x_n\}[x_1,\xi_1,\ldots,\xi_n]$ or $\mathbb{C}[[x_2,\ldots,x_n]][x_1,\xi_1,\ldots,\xi_n]$ where the degree of the monomial $x^\alpha\xi^\beta$ is $L(|\beta|,\beta_1-\alpha_1)$. If $L = F$, the filtration $F_{L,\bullet}$ is the filtration by the order of operators. The graded ring $\mathrm{gr}^V(\mathscr{R})$ is isomorphic to one of the rings $A_n$, $\mathbb{C}\{x_2,\ldots,x_n\}[x_1,\partial_1,\ldots,\partial_n]$ or $\mathbb{C}[[x_2,\ldots,x_n]][x_1,\partial_1,\ldots,\partial_n]$ where the degree of the monomial $x^\alpha\partial^\beta$ is $\beta_1-\alpha_1$.

Given an ideal $I$ of $\mathscr{R}$ let $\mathrm{gr}^L(I)$ be the graded ideal associated with the filtration induced by $F_{L,\bullet}$ on $I$. The ideal $\mathrm{gr}^L(I)$ is generated by the set $\{\sigma^L(P) \mid P \in I\}$ where $\sigma^L(P)$ is the principal symbol of $P$ with respect to $L$. By definition, if $L \neq V$,

$$\sigma^L(P) = \sum_{L(|\beta|,\beta_1-\alpha_1)=\mathrm{ord}_L(P)} p_{\alpha,\beta}x^\alpha\xi^\beta.$$

If $L$ is the form $V$, the symbol of $P$ with respect to $V$ is the differential operator

$$\sigma^V(P) = \sum_{\beta_1-\alpha_1=\mathrm{ord}_V(P)} p_{\alpha,\beta}x^\alpha\partial^\beta.$$

Notice that for $L \neq V$, $(\alpha,\beta) \to L(|\beta|,\beta_1-\alpha_1)$ is a linear form whose coefficients on the $\beta_i$ are all strictly positive. What follows works in the same way for any family of linear forms of this type for which the variables $\alpha_i$ having non-positive coefficients are fixed and for which $\mathrm{ord}_L([P,Q]) < \mathrm{ord}_L(P) + \mathrm{ord}_L(Q)$ whence $\mathrm{gr}^L(\mathscr{R})$ is commutative. We shall not write this generalization. In the case of an ideal of $A_n$, the following lemma shows how to deal with the ideal generated by $I$ in $\mathscr{D}_n$ (or in $\widehat{\mathscr{D}_n}$) and conversely:

***Lemma 3.1.1***. — *Let $I$ be an ideal in $A_n$. Then $\mathrm{gr}^L(\mathscr{D}_nI) = \mathrm{gr}^L(\mathscr{D}_n)\,\mathrm{gr}^L(I)$. More precisely, if $\mathscr{F} = \{P_1,\ldots,P_r\}$ is a system of generators of $I$ such that $\mathscr{G} = \{\sigma^L(P_i)\}_{i=1}^r$ generates $\mathrm{gr}^L(I)$, then $\mathscr{G}$ generates $\mathrm{gr}^L(\mathscr{D}_nI)$ over $\mathrm{gr}^L(\mathscr{D}_n)$.*

***Remark***. — We shall see later that such a family $\mathscr{F}$ can be calculated effectively starting from a system of generators of the ideal $I$.

*Proof*. — See [**2**]. The same result is valid in $\widehat{\mathscr{D}_n}$. □

**3.2. Privileged exponents.** — As in 2.3 let $<$ be a well ordering, compatible with the sum, in $\mathbb{N}^{2n}$ (*i.e.* a well ordering such that $(\alpha+\alpha'',\beta+\beta'') < (\alpha'+\alpha'',\beta'+\beta'')$ if and only if $(\alpha,\beta) < (\alpha',\beta')$) and let $\Lambda$ be a linear form on $\mathbb{Q}^{2n}$ with non negative coefficients and such that $\Lambda_2 := \Lambda_{|\{0\}\times\mathbb{Q}^n}$ have strictly positive coefficients. We shall use the ordering $<_\Lambda$ (2.3) in the following definition.

***Definition 3.2.1***. — Let $L$ be a linear form on $\mathbb{Q}^2$, with non negative integer coefficients. We define on $\mathbb{N}^{2n}$ the total ordering (denoted $<_L$) by:

$$(\alpha,\beta) <_L (\alpha',\beta') \Longleftrightarrow \begin{cases} L(|\beta|,\beta_1-\alpha_1) < L(|\beta'|,\beta'_1-\alpha'_1) \\ \text{or } \begin{cases} L(|\beta|,\beta_1-\alpha_1) = L(|\beta'|,\beta'_1-\alpha'_1) \\ \text{and } (\alpha,\beta) <_\Lambda (\alpha',\beta') \end{cases} \end{cases}$$

***Remark***. — Because of $<_\Lambda$ this definition must be read in a different way for $A_n$ and for $\mathscr{D}_n$ or $\widehat{\mathscr{D}_n}$.

For any $d\in\mathbb{Z}$, the restriction of $<_L$ to the set $\{(\alpha,\beta) \mid L(|\beta|,\beta_1-\alpha_1)=d\}$ is a well ordering in the case of $A_n$ and this is not the case for $\mathscr{D}_n$ or $\widehat{\mathscr{D}_n}$.

***Definition 3.2.2***. — Let $L$ be a linear form on $\mathbb{Q}^2$ with non-negative coefficients, and $P\in\mathscr{R}$. The element of $\mathbb{N}^{2n}$ $\max_{<_L}\{\mathscr{N}(P)\}$ (where $\mathscr{N}(P)$ is the Newton diagram of $P$) is called the *L-privileged exponent* of $P\in\mathscr{R}$ (and we denote it by $\exp_L(P)$). We write $\exp(P)$ when no confusion is possible.

The privileged exponent of $P\in A_n$ seen as an element of $\mathscr{D}_n$ (or of $\widehat{\mathscr{D}_n}$) is different from the one first defined. This second definition gives quotients and remainders which are elements of $\mathscr{D}_n$ even when starting from elements of $A_n$ in the division theorems that we are going to state.

We can also remark that in any case $\exp_L(P)=\exp_L(\sigma^L(P))$, where the second privileged exponent is taken in the sense of polynomials or of power series in the variables $(x,\xi)$ for a convenient ordering.

We have again:

***Lemma 3.2.3***. — *Let $Q,P$ be elements of $\mathscr{R}$. We have:*

(1) $\exp(Q\cdot P)=\exp(Q)+\exp(P)$.

(2) *If* $\exp(Q)\neq\exp(P)$ *then* $\exp(P+Q)=\max_{<_L}\{\exp(P),\exp(Q)\}$.

(3) *If* $\exp(Q)=\exp(P)$ *and if* $c(P)+c(Q)\neq 0$ *then* $\exp(P+Q)=\exp(P)$ *and* $c(P+Q)=c(P)+c(Q)$.

(4) *If* $\exp(Q)=\exp(P)$ *and if* $c(P)+c(Q)=0$ *then* $\exp(P+Q) <_L \exp(P)$.

Let $I$ be an ideal of $\mathscr{R}$. We denote the set $\{\exp_L(P)\mid P\in I\}$ by $\mathrm{E}_L(I)$ (or simply by $\mathrm{E}(I)$ when no confusion is possible). By 3.2.3, $\mathrm{E}_L(I)+\mathbb{N}^{2n}=\mathrm{E}_L(I)$.

***Definition 3.2.4***. — Let $I$ be an ideal of $\mathscr{R}$. A family $\{P_1, \ldots, P_r\}$ of elements of $I$ is called a *standard basis* (relative to the order $<_L$, or an $L$-standard basis) of $I$ if

$$\mathrm{E}_L(I) = \bigcup_{i=1}^{r} (\exp_L(P_i) + \mathbb{N}^{2n}).$$

Unlike what happens in Chapters I and II, a standard basis of an ideal $I$ of $\mathscr{R}$ is not necessarily a system of generators of $I$. However we have the following result:

***Lemma 3.2.5***. — *Let $\mathscr{F} = \{P_1, \ldots, P_r\}$ be a system of generators of an ideal $I$ of $\mathscr{R}$. If $\mathscr{F}$ is an $L$-standard basis of $I$ then :*

(1) *$\{\sigma^L(P_1), \ldots, \sigma^L(P_r)\}$ is a system of generators of $\mathrm{gr}^L(I)$.*

(2) *If, furthermore $\mathrm{E}_V(\mathrm{gr}^L(I)) = \cup_{i=1}^r (\exp_V(\sigma^L(P_i)) + \mathbb{N}^{2n})$ (resp. $\mathrm{E}_F(\mathrm{gr}^L(I)) = \cup_{i=1}^r (\exp_F(\sigma^L(P_i)) + \mathbb{N}^{2n})$), then the family $\{\sigma^V(\sigma^L(P_i))\}_{i=1}^r$ (resp. $\{\sigma^F(\sigma^L(P_i))\}_{i=1}^r$) generates $\mathrm{gr}^V(\mathrm{gr}^L(I))$ (resp. $\mathrm{gr}^F(\mathrm{gr}^L(I))$).*

*Proof*. — By definition we have

$$\mathrm{E}_L(I) = \mathrm{E}_L(\mathrm{gr}^L(I)) = \mathrm{E}_{<_\Lambda}(\mathrm{gr}^L(I))$$

because, for any $P \in R$ we have in fact $\exp_L(P) = \exp_L(\sigma^L(P)) = \exp_{<_\Lambda}(\sigma^L(P))$. Thus the first part of the lemma follows from the commutative case of 2.4.3 applied to $<_\Lambda$. □

### 3.3. Homogenisation. Orderings in $\mathbb{N}^{2n+1}$ and division theorems in $\mathscr{R}[t]$

Following the idea of the preceding lemma we can write a division step $P' = P - \sum_{i=1}^r Q_i P_i - R$ such that the $L$-order of the operator $P'$ is strictly smaller than the order of $P$. But the degree of $P'$ with respect to the derivatives can increase. Thus if we continue this process we have not only to face the question of the convergence of power series but also the fact that we cannot stay in the frame of finite order operators. To avoid this problem we introduce a division process on appropriate homogeneized operators which allows the construction of an $L$-standard basis for an ideal $I$ in $\mathscr{R}$.

We set $\mathscr{R}[t] = \mathscr{R} \otimes_{\mathbb{C}} \mathbb{C}[t]$. If $P = \sum_{\alpha,\beta} p_{\alpha,\beta} x^\alpha \partial^\beta$ (resp. $P = \sum_\beta f_\beta \partial^\beta$) is an element of $A_n$, (resp. $\mathscr{D}_n$ or $\widehat{\mathscr{D}_n}$) we call the integer $\max\{|\alpha| + |\beta| \mid p_{\alpha,\beta} \neq 0\}$ (resp. $\max\{|\beta| \mid f_\beta \neq 0\}$) the *total order* of $P$ (resp. *order* of $P$), and we denote it by $\mathrm{ord}^T(P)$ (resp. $\mathrm{ord}(P)$) .

In the same way as for $\mathscr{R}$ we define the notion of the Newton diagram of an operator in $\mathscr{R}[t]$.

***Definition 3.3.1***. — Let $P = \sum_{\alpha,\beta} p_{\alpha,\beta} x^\alpha \partial^\beta \in A_n$. Then the differential operator

$$h(P) = \sum_{\alpha,\beta} p_{\alpha,\beta} t^{ord^T(P) - |\alpha| - |\beta|} x^\alpha \partial^\beta \in A_n[t],$$

is called the homogenisation of $P$.

***Definition 3.3.2***. — Let $P = \sum_\beta f_\beta \partial^\beta \in \mathscr{D}_n$ or $\widehat{\mathscr{D}_n}$. Then the differential operator

$$h(P) = \sum_\beta f_\beta t^{\mathrm{ord}(P)-|\beta|}\partial^\beta \in \mathscr{D}_n[t] \text{ (or } \widehat{\mathscr{D}_n}[t])$$

is called the homogenisation of $P$.

We define an ordering on $\mathbb{N}^{2n+1}$ denoted by $\prec_L$, in the following way:

- In the case of $A_n[t]$

$$(k,\alpha,\beta)\prec_L(k',\alpha',\beta') \iff \begin{cases} k+|\alpha|+|\beta| < k'+|\alpha'|+|\beta'| \\ \text{or} \begin{cases} k+|\alpha|+|\beta| = k'+|\alpha'|+|\beta'| \text{ and} \\ (\alpha,\beta) <_L (\alpha',\beta') \end{cases} \end{cases}$$

This ordering on $\mathbb{N}^{2n+1}$ is a well ordering compatible with sums.

- In the case of $\mathscr{D}_n[t]$ or $\widehat{\mathscr{D}_n}[t]$

$$(k,\alpha,\beta)\prec_L(k',\alpha',\beta') \iff \begin{cases} k+|\beta| < k'+|\beta'| \\ \text{or} \begin{cases} k+|\beta| = k'+|\beta'| \text{ and} \\ (\alpha,\beta) <_L (\alpha',\beta') \end{cases} \end{cases}$$

We can again define the notion of a privileged exponent and the usual properties hold.

***Definition 3.3.3***. — Let $H = \sum_{k,\alpha,\beta} h_{k,\alpha,\beta} t^k x^\alpha \partial^\beta$ be an element of $\mathscr{R}[t]$. Then the greatest element, with respect to the total order $\prec_L$, of the Newton diagram of $H$ is called the *privileged exponent* of $H$ relative to $\prec_L$ and is denoted by $\exp_{\prec_L}(H)$. The monomial of $H$ whose exponent is equal to the privileged exponent is called the *initial monomial* of $H$ and is denoted by $\mathrm{In}_{\prec_L}(H)$. The coefficient of the initial monomial of $H$ is called the *initial coefficient* of $H$ and is denoted by $c_{\prec_L}(H)$. We write $\exp(H), In(H)$ and $c(H)$ when no confusion is possible.

***Lemma 3.3.4***. — *For all $H_i$ in $\mathscr{R}[t]$, $P$ and $Q$ in $\mathscr{R}$, the following relations hold (where* exp *denotes the exponent either for $\prec_L$ or for $<_L$):*

(1) $\exp(H_1H_2) = \exp(H_1) + \exp(H_2)$.

(2) *If* $\exp(H_1) \neq \exp(H_2)$ *then* $\exp(H_1+H_2) = \max_{\prec_L}\{\exp(H_1), \exp(H_2)\}$.

(3) *If* $\exp(H_1) = \exp(H_2)$ *and if* $c(H_1)+c(H_2) \neq 0$ *then* $\exp(H_1+H_2) = \exp(H_1)$ *and* $c(H_1+H_2) = c(H_1)+c(H_2)$.

(4) *If* $\exp(H_1) = \exp(H_2)$ *and if* $c(H_1)+c(H_2) = 0$ *then* $\exp(H_1+H_2)\prec_L \exp(H_1)$.

(5) $\exp(h(QP)) = \exp(h(Q)h(P))$.

(6) $\pi(\exp(h(P))) = \exp(P)$, *where* $\pi : \mathbb{N}^{2n+1} = \mathbb{N}^{2n}\times\mathbb{N} \to \mathbb{N}^{2n}$ *is the natural projection.*

Given an $r$-uple $(P_1,\dots,P_r)$ of $\mathscr{R}^r$ we define as usual a partition of $\mathbb{N}^{2n+1}$:

$$\Delta_1 = (\exp(h(P_1)) + \mathbb{N}^{2n+1}, \quad \Delta_i = ((\exp(h(P_i)) + \mathbb{N}^{2n+1}) \smallsetminus (\textstyle\bigcup_{j=1}^{i-1}\Delta_j) \text{ if } 2 \leqslant i \leqslant r$$

$$\overline{\Delta} = \mathbb{N}^{2n+1} \smallsetminus (\textstyle\bigcup_{j=1}^{r}\Delta_j).$$

***Theorem 3.3.5***. — *Let $(P_1, \ldots, P_r)$ be in $\mathscr{R}^r$. Let $\{\Delta_1, \ldots, \Delta_r, \overline{\Delta}\}$ be the partition of $\mathbb{N}^{2n+1}$, associated with $(\exp(h(P_1)), \ldots, \exp(h(P_r)))$. Then, for any $H \in \mathscr{R}[t]$ there exists a unique element $(Q_1, \ldots, Q_r, R)$ in $\mathscr{R}[t]^{r+1}$ such that:*

(1) $H = Q_1 h(P_1) + \cdots + Q_r h(P_r) + R$.
(2) $\exp(h(P_i)) + \mathscr{N}(Q_i) \subset \Delta_i$ *and* $\exp(Q_i h(P_i)) \preceq_L \exp(H)$ *for* $1 \leqslant i \leqslant r$.
(3) $\mathscr{N}(R) \subset \overline{\Delta}$ *and* $\exp(R) \preceq_L \exp(H)$.

*Proof*. — In the case of $A_n[t]$ the proof is standard since $\prec_L$ is then a well ordering.

In the case of $\mathscr{D}_n[t]$ or $\widehat{\mathscr{D}_n}[t]$, we notice that $\exp(H)$ is equal to the privileged exponent of the total symbol $\sigma^T(H)$ of $H$, associated with the filtration by the order $T = k + |\beta|$. The graded ring associated with this filtration is $\mathbb{C}\{x\}[\xi][t]$ or $\mathbb{C}[[x]][\xi][t]$. In this ring we can carry out the division of $\sigma^T(H)$ by the $\sigma^T(h(P_i)) = h(P_i)$. In fact, this is a division in the ring of power series $\mathbb{C}\{x, \xi, t\}$ or $\mathbb{C}[[x, \xi, t]]$, but it is $T$-homogeneous and so polynomial over the variables $(\xi, t)$. We can lift the quotients and the remainders to operators $Q_i^{(1)}$, $R^{(1)}$ having the required properties with respect to the partition of $\mathbb{N}^{2n+1}$ so that we find $H' = H - \sum Q_i^{(1)} h(P_i) - R^{(1)}$ of total order smaller than the one of $H$. We obtain the existence of the division by an induction on the total order. Uniqueness is treated as before. $\square$

### 3.4. Semisyzygies and a criterion for standard bases

***Definition 3.4.1***. — Let $G_1, G_2$ be elements of $\mathscr{R}[t]$. Let $S(G_1, G_2) = M_1 G_1 - M_2 G_2$ be the operator, defined up to a multiplicative constant $c \in \mathbb{C}^*$ and characterized by the following conditions:

(1) $M_i$ is a monomial with exponent $\nu_i$
(2) $\mu = \nu_1 + \exp_{\prec_L}(G_1) = \nu_2 + \exp_{\prec_L}(G_2)$
(3) Any $\mu'$ having the two properties above is in $\mu + \mathbb{N}^{2n+1}$
(4) $\exp_{\prec_L}(S(G_1, G_2)) \prec_L \mu$.

We call it the *semisyzygy* relative to $(G_1, G_2)$.

***Theorem 3.4.2***. — *Let $\mathscr{F} = \{P_1, \ldots, P_r\}$ be a system of generators of the ideal $I$ of $\mathscr{R}$ such that, for any $(i, j)$, the remainder of the division of $S(h(P_i), h(P_j))$ by $(h(P_1), \ldots, h(P_r))$ is equal to zero, modulo $(t-1)\mathscr{R}[t]$. Then $\mathscr{F}$ is an $L$-standard basis of $I$.*

*Proof*. — We shall only sketch the proof by referring to [**2**] where the case of $A_n$ is treated and explain what happens for $\mathscr{D}_n$ or $\widehat{\mathscr{D}_n}$. Let $\Delta = \bigcup_{i=1}^r (\exp(P_i) + \mathbb{N}^{2n})$. We have to prove that $\mathrm{E}_L(I) \subset \Delta$. Let $P \in I$. We write it $P = \sum_{i=1}^r Q_i P_i$. Let us set:

- $d_i = \mathrm{ord}^T(Q_i P_i)$, $d = \max_{i=1,\ldots,r}\{d_i\}$, $\delta = \mathrm{ord}^T(P) \leqslant d$,
- $\mu^i = \exp_{\prec_L}(h(P_i))$, $\nu^i = \exp_{\prec_L}(h(Q_i) h(P_i))$,
- $(\alpha^0, \beta^0) = \max_{i=1,\ldots,r}\{\exp_L(Q_i P_i)\}$.

Let $\{i_0,\dots,i_s\}$ be the indices $i$ at which the maximum $(\alpha^0,\beta^0)$ is reached. If $s=0$ then $\exp(P)=\exp(Q_{i_0}P_{i_0})\in\Delta$. Therefore we can assume $s\geqslant 1$. By writing

$$t^{d-\delta}h(P)=\sum_{i=1}^{r}t^{d-d_i}h(Q_i)h(P_i)\ \mathrm{mod.}(t-1)\mathscr{R}[t],$$

and by using the division of $S(h(P_{i_0}),h(P_{i_1}))$, without a remainder modulo $(t-1)$, we find a new decomposition of $P$: $P=\sum_{i=1}^{r}Q'_iP_i$, where the $Q'_i$ have the following properties:

(1) $\max_{<_L}\{\exp_L(Q'_iP_i)\mid i=1,\dots,r\}\leqslant_L(\alpha_0,\beta_0)$
(2) $\exp_L(Q'_{i_0}P_{i_0})<_L(\alpha^0,\beta^0)$

The proof goes then by an induction, after remarking that $|\alpha|+|\beta|$ in the case of $A_n[t]$ or $|\beta|$ in the case of $\mathscr{D}_n[t]$ (or $\widehat{\mathscr{D}_n}[t]$) is bounded for $(\alpha,\beta)$ in the Newton diagrams of the successive $Q_i$. The case of $A_n[t]$, treated in [**2**], is then immediate.

The case of $\mathscr{D}_n[t]$ (or $\widehat{\mathscr{D}_n}[t]$) is slightly more complicated: Since $|\beta|$ is bounded, say by $K$, $\alpha_1$ cannot increase indefinitely with a constant $L(|\beta|,\beta_1-\alpha_1)$, in the transformation of the decomposition of $P$ described above. Therefore when we repeat the process a sufficient number of times we can obtain for any $N\in\mathbb{N}$ a decomposition of $P$ where the coefficients of $Q_i$ of $L$-order equal to $L(\left|\beta^0\right|,\beta_1^0-\alpha_1^0)-\mathrm{ord}_L(P_i)$, are in the ideal $(x_2,\dots,x_n)^N$ of $\mathbb{C}\{x_2,\dots,x_n\}$. By applying the theorem of Krull (see *e.g.* [**5**, Corollary 10.18]) to the $\mathbb{C}\{x_2,\dots,x_n\}$-module generated by the operators $x_1^{\alpha_1}\boldsymbol{\partial}^{\beta}P_i$, for $i=1,\dots,r$, $|\beta|\leqslant K$ and $L(|\beta|,\beta_1-\alpha_1)=L(\left|\beta^0\right|,\beta_1^0-\alpha_1^0)-\mathrm{ord}_L(P_i)$, we obtain a new decomposition of $P$ where the $L$-order has strictly decreased, in the case where $\mathrm{ord}_L(P)>L(\left|\beta^0\right|,\beta_1^0-\alpha_1^0)$. When we have the equality $\mathrm{ord}_L(P)=L(\left|\beta^0\right|,\beta_1^0-\alpha_1^0)$, the same argument allows us to end at the case where $\exp_L(P)=(\alpha^0,\beta^0)$. □

According to the theorem 3.4.2 one might try to build a standard basis in the following way: If $P_1,\dots,P_r$ are generators of $I$ and if $H_i=h(P_i)$, assume that there is a $S(H_i,H_j)$ which gives, in a division by $H_1,\dots,H_r$, a remainder $R\neq 0$ (mod $(t-1)\mathscr{R}[t]$). One might expect by adding the new element $R_{|t=1}=P$ of $I$ to the list of generators, to start a process of constructing a standard basis.

The trouble is that $R$ is not homogeneous in general and therefore is different from $h(P)$. Possibly $h(P)$ does not create a new exponent in $\mathbb{N}^{2n+1}$. We solved this problem in [**2**] by using a technical trick. We divide again $h(P)$ by $H_1,\dots,H_r$ and by iterating this we get at the end either a new exponent in $\mathbb{N}^{2n+1}$ or a remainder in $(t-1)\mathscr{R}[t]$. This gives an algorithm for constructing an $L$-standard basis by an appropriate modification of the theorem 3.4.2.

As L. Narváez pointed out to us, there is another way to get an algorithm. We put on $\mathscr{R}[t]$ another structure given by $[\partial_i,x_i]=t^2$ for $A_n[t]$ and $[\partial_i,a(x)]=(\partial a/\partial x_i)t$ for $\mathscr{D}_n[t]$ which is the Rees ring for the Bernstein or for the order filtration. We denote this ring by $\widetilde{\mathscr{R}[t]}$. There exists a non commutative graded division on this ring [**3**] and [**4**]. We get a standard basis of $I$ directly from a standard basis of $\sum\widetilde{\mathscr{R}[t]}h(P_i)$.

***Remark***. — In some cases (see in particular the examples below), it may happen that the semisyzygies in $\mathscr{R}$, give a finite division process without having to homogenize. In this case if the remainders of all the divisions are zero we can again apply the same result and assert that we have an $L$-standard basis in the following situation: It is enough to verify that for a convenient choice of an order $<$, each privileged exponent of an element of the basis is, among the monomial of maximal $L$-order, one of those which have the highest degree $|\alpha|+|\beta|$ or $|\beta|$ according to the case under consideration.

**3.5. The determination of slopes.** — As explained in the introduction, the slopes of a $\mathscr{D}$-module, defined algebraically by Y. Laurent in [**22**] (see also below) are closely related to the irregularity sheaf of this module. The slopes are, according to Laurent and Mebkhout [**23**], the degrees of the non zero graded parts of this filtration. We may also remark that these slopes control the growth rates of the exponential parts of the solutions. The calculation of the slopes is therefore an important aspect of an effective approach to $\mathscr{D}$-module theory. This was the aim of the paper [**2**] whose results we recall below, at the same time as we develop some examples in the subsequent section.

The ring $\operatorname{gr}^L(\mathscr{R})$ if $L \neq F, V$ has a graduation with respect to $F$ and another with respect to $V$.

***Definition 3.5.1* (After Y. Laurent [22])**. — Let $I$ be an ideal of $\mathscr{R}$ and let $L \neq F, V$ be a linear form as before. We say that $L$ is a slope of $\mathscr{D}_n/I$ if the ideal $\sqrt{\operatorname{gr}^L(I)}$ is not $(F,V)$-bihomogeneous. We call the variety defined by the ideal $\operatorname{gr}^L(I)$ the $L$-characteristic variety of $\mathscr{R}/I$ and we denote it by $\operatorname{Char}^L(\mathscr{R}/I)$.

***Definition 3.5.2***. — With $I$ as above, if $\operatorname{gr}^L(I)$ is not $(F,V)$-bihomogeneous we say that $L$ is an *idealistic* slope of $I$.

***Remark***. — There are idealistic slopes which are not slopes in the sense of Laurent: see example 1 below. In [**2**] only the idealistic slopes are treated which is the hard part of the algorithm that follows.

The determination of slopes can be done in the same way in $A_n$ $\mathscr{D}_n$ or $\widehat{\mathscr{D}}_n$. We refer to the paper [**2**], for the details, and simply recall here some points.

***Lemma 3.5.3***. — *Let $I$ be an ideal of $\mathscr{R}$, and $L \neq V$ (resp. $L \neq F$). Then, there exists a linear form $L^{(1)}$ (resp. $L^{(2)}$), such that for any linear form $L'$ having an intermediate slope, i.e.* $\operatorname{slope}(L) < \operatorname{slope}(L') < \operatorname{slope}(L^{(1)})$ *(resp.* $\operatorname{slope}(L) > \operatorname{slope}(L') > \operatorname{slope}(L^{(2)})$*) we have* $\operatorname{gr}^{L'}(I) = \operatorname{gr}^V(\operatorname{gr}^L(I))$ *(resp.* $\operatorname{gr}^{L'}(I) = \operatorname{gr}^V(\operatorname{gr}^L(I))$*).*

By a compactness argument this implies the finiteness of the number of idealistic slopes. An $L$-standard basis of $I$ still remains to be determined in order to write the equations $\sigma^L(P_i)(\boldsymbol{x},\boldsymbol{\xi}) = 0$ of the $L$-characteristic variety and then equations of the reduced variety. This allows us to determine among these $L$ those which are slopes in the sense of Y. Laurent.

Recall, to end this chapter, the process which allows us to determine these idealistic slopes. Of course this is an algorithm only in the case of $A_n$.

- We determine an $L$-standard basis $\{P_1, \ldots, P_r\}$ of $I$ where $L$ is $F$ or a previously determined slope. We make sure that it is also a $V$-standard basis of $\mathrm{gr}^L(I)$.
- We determine the form $L^{(1)}$ with minimal slope $>$ slope$(L)$ such that one of the $\sigma^{L^{(1)}}(P_i)$ is not bihomogeneous. Precisely $L^{(1)}$ is the linear form with smallest slope greater than slope$(L)$ appearing in the $(F, V)$-Newton diagram of the operators $P_i$.
- By a finite division process we can decide whether one of the bihomogeneous components of one of the $\sigma^{L^{(1)}}(P_i)$ is not an element of $\mathrm{gr}^{L^{(1)}}(\mathscr{R})$. In this case $L^{(1)}$ is a new idealistic slope. In the other case we can modify $P_i$ in order to eliminate $L^{(1)}$, and obtain a basis which is standard for $L$ and for $L^{(1)}$. We prove in [**2**] that this type of cancellation can happen only a finite number of times before we come upon a new slope or upon $V$.

### 3.6. Examples of calculations of slopes

*Example 1.* — In this example we consider the direct image of the $\mathscr{D}_{\mathbb{C}}$-module $\mathscr{D}_{\mathbb{C}} e^{1/v^k}$, by an immersion in $\mathbb{C}^2$ and the slopes relative to a hypersurface tangent to the support. The advantage of this example is that we can carry out all the calculations in many cases and that it shows idealistic slopes which are not slopes. For $k \in \mathbb{N}$ we write:

$$\mathscr{M} = \mathscr{D}_{\mathbb{C}} e^{1/v^k} \simeq \frac{\mathscr{D}_{\mathbb{C}}}{\mathscr{D}_{\mathbb{C}}(v^{k+1}\partial_v + 1)}, \quad \mathscr{N} = i_+\mathscr{M} \simeq \frac{\mathscr{D}_{\mathbb{C}^2}}{\mathscr{D}_{\mathbb{C}^2}(v^{k+1}\partial_v + 1) + \mathscr{D}_{\mathbb{C}^2} u}$$

where $i$ is the immersion $\mathbb{C} \to \mathbb{C}^2$ given by $i(v) = (0, v)$. We want to calculate the slopes of $\mathscr{N}$ along the curve $v^m + u = 0$.

We carry out the change of variables: $u = x - y^m,\ v = y$. We have: $\partial_u = \partial_x,\ \partial_v = \partial_y + my^{m-1}\partial_x$. We then find that $\mathscr{N}$ is the quotient of $\mathscr{D}_{\mathbb{C}^2}$ by the ideal $I$ generated by the following operators:

- $P'_1 = y^{k+1}\partial_y + my^{k+m}\partial_x + 1$
- $P_2 = y^m - x$

We then have to look at the slopes along $x = 0$. In what follows we say that the slope is $-p/q$ if $L(a, b) = pa + qb$.

*Subexample 1.1: $m = 1$.* — We find the slope $-k$. We are in the same situation as for the calculation of the slope of $\mathscr{N}$ along $v = 0$. This is also a particular case of the following.

*Subexample 1.2: $k = mp$.* — We then find the slope $-p$. We have:

$$\begin{aligned} P'_1 = y^{mp+1}\partial_y + my^{m(p+1)}\partial_x + 1 &= (y\partial_y - mp)y^{mp} + m\partial_x x^{p+1} + 1 \\ &= (y\partial_y - mp)x^p + mx^{p+1}\partial_x + m(p+1)x^p + 1 \pmod{\mathscr{D}P_2} \end{aligned}$$

This gives the presentation of $\mathscr{N}$:

- $P_1 = mx^{p+1}\partial_x + x^p y\partial_y + mx^p + 1$
- $P_2 = y^m - x$

We choose the ordering $<_F$ for which the variables are ordered as follows: $\partial_x > \partial_y > y > x$. We then find that $(P_1, P_2)$ is an $F$-standard basis. Indeed the remainder of the division of:

$$S(P_1, P_2) = y^m P_1 - mx^{p+1}\partial_x$$

by $\{P_1, P_2\}$ is zero. In this standard basis the privileged exponent of $P_i$ is also the privileged exponent of $\sigma^V(\sigma^F(P_i))$, so that by looking at $P_1$ we can say that the $-p$ is an idealistic slope. For that we verify that $\sigma^L(P_1) = mx^{p+1}\xi + x^p y\eta + mx^p + 1$ is not bihomogeneous [(11)]. This last result comes for example from the fact that $1 \notin \mathrm{gr}^V(\mathrm{gr}^F(I))$ ($\mathscr{N} \neq 0$).

There is no other slope because for any $L'$ of slope $> -p$, $\mathrm{gr}^{L'}(I) = (1)$.

*Subexample 1.3: $m = 2$ and $k = 2n-1$.* — By changing $P_1' = y^{2n}\partial_y + 2y^{2n+1}\partial_x + 1$ modulo $P_2 = y^2 - x$, as in the preceding subexample, we find $I = \mathscr{D}P_1 + \mathscr{D}P_2$, with: $P_1 = 2x^n y\partial_x + x^n\partial_y + 1$.

The first semisyzygy $S(P_1, P_2) = yP_1 - 2x^n\partial_x P_2$ divided by $\{P_1, P_2\}$ gives a remainder $P_3$ whence the following generators for $I$:

- $P_1 = 2x^n y\partial_x + x^n\partial_y + 1$
- $P_2 = y^2 - x$
- $P_3 = 2x^{n+1}\partial_x + x^n y\partial_y + y + 2x^n$

We find that the remainders of the divisions of $S(P_1, P_3) = xP_1 - yP_3$ and $S(P_2, P_3) = -2x^{n+1}\partial_x P_2 + y^2 P_3$ by $\{P_1, P_2, P_3\}$ are zero. Thus this is an $F$-standard basis which is also as in the preceding subexample a $V$-basis of $\mathrm{gr}^F(I)$. Let $L$ be the linear form corresponding to the first eventual slope, the slope $-n$.

We have $\sigma^L(P_3) = 2x^{n+1}\xi + x^n y\eta + y$. It is impossible that $y \in \mathrm{gr}^V(\mathrm{gr}^F(I)$ because $I$ would contain the two elements of order 0, $y^2 - x$ and $y + x\phi(x, y)$ and $\mathscr{N}$ would be supported by the origin.

Thus we have pointed out an idealistic slope of the ideal $I$. It is not a slope of $\mathscr{D}_{\mathbb{C}^2}/I$ because the equations of the $L$-characteristic variety are:

- $\sigma^L(P_1) = 2x^n y\xi = 0$
- $\sigma^L(P_2) = y^2 = 0$
- $\sigma^L(P_3) = 2x^{n+1}\xi + x^n y\eta + y = 0$

and the associated reduced variety is bihomogeneous with equations $y = x\xi = 0$. Let us set $F_i = \sigma^L(P_i)$ and look for a $V$-standard basis of $\mathrm{gr}^L(I)$.

Let us remark that the privileged exponent of $F_3$ has changed and is now the monomial $y$. The semisyzygy $S(F_1, F_3) = -F_1 + 2x^n\xi F_3$ gives the remainder: $F_4 = 4x^{2n+1}\xi^2 + 2x^{2n}y\xi\eta$, and we find that all the other remainders are zero, so that

[(11)] Here $L$ is the linear form of slope $-p$

$\{F_1, F_2, F_3, F_4\}$ is a $V$-standard basis of $\mathrm{gr}^L(I)$. We can lift the preceding semisyzygy in $\mathscr{D}$, as $-P_1 + 2x^n\partial_x P_3$, and this gives the operator:

$$P_4 = 4x^{2n+1}\partial_x^2 + 2x^{2n}y\partial_x\partial_y + 4(n+2)x^{2n}\partial_x + 2nx^{2n-1}y\partial_y - x^n\partial_y + 4nx^{2n-1} - 1.$$

The family $\{P_1, P_2, P_3, P_4\}$ is therefore a system of generators of $I$ which gives a standard basis of $\mathrm{gr}^V(\mathrm{gr}^L(I))$ and by $P_4$ we point out the linear form $L'$ of slope $-(n-1/2)$. By the algorithm above, it is enough to verify that $\sigma^{L'}(P_4) = 4x^{2n+1}\xi^2 + 2x^{2n}y\xi\eta - 1$ is not in $\mathrm{gr}^V(\mathrm{gr}^L(I))$, which amounts to find out that $1 \notin \mathrm{gr}^V(\mathrm{gr}^L(I)) = (2x^n y\xi, y^2, y, 4x^{2n+1}\xi^2 + 2x^{2n}y\xi\eta)$.

Finally, there is no other slope because for any form $L''$ of slope $> -(n-1/2)$ we have $1 = -\sigma^{L''}(P_4) \in \mathrm{gr}^{L''}(I)$.

*Example 2: $\mathscr{D}e^{1/(y^p-x^q)}$ (with G. Brevet).* — To apply our algorithm to the determination of the slopes of the $\mathscr{D}$-module generated by $e^{1/(y^p-x^q)}$, it is necessary to know the annihilator in $\mathscr{D}$ of the function $e^{1/(y^p-x^q)}$. The answer to this last question was given by J. Briançon and Ph. Maisonobe in [**12**] in the more general case where $f$ is quasi-homogeneous with an isolated singularity. The annihilator is:

$$\mathscr{D}(f\chi + 1) + \mathscr{D}(\frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x})$$

where $\chi$ is a vector field such that $\chi(f) = f$. Thus we have:

$$I = \mathrm{Ann}_{\mathscr{D}}(e^{1/(y^p-x^q)}) = \mathscr{D}(P_1, P_2)$$

with

$$P_1 = py^{p-1}\partial_x + qx^{q-1}\partial_y, \quad P_2 = qy^{p+1}\partial_y - px^{q+1}\partial_x - 2qx^q y\partial_y + pq$$

Furthermore, we take as an $L$-ordering (where $L$ is a linear form on $\mathbb{Q}^2$ with rational positive coefficients) the ordering on $\mathbb{N}^4$ defined as follows:

$$(i,j,\alpha,\beta) <_L (i',j',\alpha',\beta') \iff \begin{cases} L(\alpha+\beta, \alpha-i) < L(\alpha'+\beta', \alpha'-i') \\ \text{or } L(\alpha+\beta, \alpha-i) = L(\alpha'+\beta', \alpha'-i') \text{ and} \\ i+j > i'+j' \\ \text{or } \begin{cases} L(\alpha+\beta, \alpha-i) = L(\alpha'+\beta', \alpha'-i'), \\ \hfill i+j = i'+j' \\ \text{and } (\alpha,\beta,j,i) <_{\mathrm{lex}} (\alpha',\beta',j',i') \end{cases} \end{cases}$$

*1. The calculation of a standard basis of $I$ for the form $L = F$.* — If $q > p$, we have: $\mathrm{mp}_F(P_1) = py^{p-1}\partial_x$ and $\mathrm{mp}_F(P_2) = qy^{p+1}\partial_y$. Let us set then $\Delta_1 = (0, p-1, 1, 0) + \mathbb{N}^4$ and $\Delta_2 = ((0, p+1, 0, 1) + \mathbb{N}^4) \smallsetminus \Delta_1$. The syzygy relative to $P_1$ and $P_2$ is equal to: $S(P_1, P_2) = qy^2\partial_y P_1 - p\partial_x P_2$. The remainder of the division gives a third operator

$$P_3 = p^2x^{q+1}{\partial_x}^2 + 2pqx^q y\partial_x\partial_y + q^2x^{q-1}y^2{\partial_y}^2 \\ + p^2(q+1)x^q\partial_x + q^2(p+1)x^{q-1}y\partial_y - p^2q\partial_x.$$

**Proposition 3.6.1.** — *For $2 \leqslant p < q$, $\{P_1, P_2, P_3\}$ is an $F$-standard basis of $I$.*

*Proof.* — We have $\mathrm{mp}_F(P_3) = p^2x^{q+1}{\partial_x}^2$ and $\Delta_3 = ((q+1,0,2,0)+\mathbb{N}^4)\smallsetminus(\Delta_1\cup\Delta_2)$. We then have to prove that the remainders of the divisions by $(P_1,P_2,P_3)$ of the semisyzygies $S(P_1,P_3)$ and $S(P_2,P_3)$ are zero.

We find first:

$$\begin{aligned}
S(P_1,P_3) &= px^{q+1}\partial_x P_1 - y^{p-1}P_3\\
&= (p^2x^{q+1}y^{p-1}{\partial_x}^2 + pqx^{2q}\partial_x\partial_y + pq(q-1)x^{2q-1}\partial_y)\\
&\quad - \big(p^2x^{q+1}y^{p-1}{\partial_x}^2 + 2pqx^qy^p\partial_x\partial_y + q^2x^{q-1}y^{p+1}{\partial_y}^2 + p^2(q+1)x^qy^{p-1}\partial_x\\
&\qquad\qquad + q^2(p+1)x^{q-1}y^p\partial_y - p^2qy^{p-1}\partial_x\big)\\
&= \cdots = ((pq-2q-p)x^q - 2qx^qy\partial_y + pq)P_1 - qx^{q-1}\partial_yP_2.
\end{aligned}$$

Let us denote by $Q_1 = (pq-2q-p)x^q - 2qx^qy\partial_y + pq$ the quotient relative to $P_1$ in this division.

We must now deal with the semisyzygy $S(P_2,P_3) = p^2x^{q+1}{\partial_x}^2P_2 - qy^{p+1}\partial_yP_3$. Instead of directly applying the division algorithm we are going to use the above equalities:

$$\begin{aligned}
y^{p-1}P_3 &= Q_1P_1 + qx^{q-1}\partial_yP_2\\
p\partial_xP_2 &= (qy^2\partial_y - (p-1)qy)P_1 - P_3
\end{aligned}$$

and we denote by $Q_1' = qy^2\partial_y - (p-1)qy$ the quotient relative to $P_1$.

Thus we have on one hand:

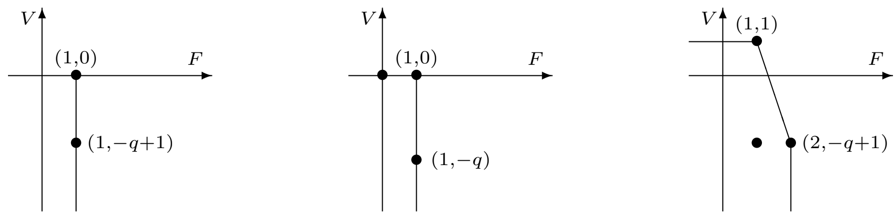
$$\begin{aligned}
qy^2(\partial_yy^{p-1} - (p-1)y^{p-2})P_3 &= (qy^2\partial_y - (p-1)qy)y^{p-1}P_3\\
&= (qy^2\partial_y - (p-1)qy)(Q_1P_1 + qx^{q-1}\partial_yP_2)
\end{aligned}$$

and the obtained quotients for $P_1$ and $P_2$ are allowed for the division. We have, on the other hand:

$$p^2x^{q+1}\partial_x^2P_2 = px^{q+1}\partial_x(Q_1'P_1 - P_3)$$

and the obtained quotients are allowed for the division. This shows that $S(P_2,P_3)$ has by division a zero remainder. □

*2. The calculation of the slopes.* — Let us draw first the Newton polygons associated with the operators $P_1,P_2,P_3$:




$\mathscr{N}(P_1)$ $\mathscr{N}(P_2)$ $\mathscr{N}(P_3)$

Since $\{P_1, P_2, P_3\}$ is an $F$-standard basis of $I$ and a system of generators of $I$, the ideal $\mathrm{gr}^F(I)$ is generated by the principal symbols

$$\begin{aligned}
\sigma^F(P_1) &= py^{p-1}\xi + qx^{q-1}\eta\\
\sigma^F(P_2) &= qy^{p+1}\eta - px^{q+1}\xi - 2qx^q y\eta\\
\sigma^F(P_3) &= p^2x^{q+1}\xi^2 + 2pqx^q y\xi\eta + q^2x^{q-1}y^2\eta^2
\end{aligned}$$

***Proposition 3.6.2***. — *$(\sigma^F(P_i))_{1\leqslant i\leqslant 3}$ is a $V$-standard basis of $\mathrm{gr}^F(I)$.*

*Proof*. — We have $\mathrm{mp}_V(\sigma^F(P_1)) = py^{p-1}\xi$, $\mathrm{mp}_V(\sigma^F(P_2)) = qy^{p+1}\eta$ and $\mathrm{mp}_V(\sigma^F(P_3)) = p^2x^{q+1}\xi^2$. The divisions by $\sigma^F(P_1)$, $\sigma^F(P_2)$, $\sigma^F(P_3)$ give:

$$\begin{aligned}
S(\sigma^F(P_1), \sigma^F(P_2)) &= qy^2\eta\sigma^F(P_1) - p\xi\sigma^F(P_2) = \sigma^F(P_3) \equiv 0,\\
S(\sigma^F(P_1), \sigma^F(P_3)) &= px^{q+1}\xi\sigma^F(P_1) - y^{p-1}\sigma^F(P_3)\\
&= -2q^2x^q y\eta\sigma^F(P_1) - qx^{q-1}\eta\sigma^F(P_2) \equiv 0,\\
S(\sigma^F(P_2), \sigma^F(P_3)) &= p^2x^{q+1}\xi^2\sigma^F(P_2) - qy^{p+1}\eta\sigma^F(P_3))\\
&= -2q^2x^q y^3\eta^2\sigma^F(P_1) - q^2x^{q-1}y^2\eta^2\sigma^F(P_2) - px^{q+1}\xi\sigma^F(P_3) \equiv 0.
\end{aligned}$$

This proves the proposition. □

We now have to consider the linear form $L (L < F)$ with the greatest possible slope such that one of the principal symbols of one of the $P_i$ is not bihomogeneous. We have: $L(a, b) = qa + b$ (slope equal to $-q$).

***Proposition 3.6.3***. — *The $\mathscr{D}$-module $\mathscr{D}e^{1/(y^p-x^q)}$ has only the slope $-q$ along the hypersurface $x = 0$.*

*Proof*

*First step: $-q$ is a slope.* We know (see [**2**]) that if $L < \Lambda < F$, then $\mathrm{gr}^\Lambda(I) = \mathrm{gr}^V(\mathrm{gr}^F(I))$. If $\mathrm{gr}^L(I)$ was bihomogeneous, then it would also be equal to $\mathrm{gr}^V(\mathrm{gr}^F(I))$. But by the previous proposition, we have:

$$\begin{aligned}
\mathrm{gr}^V(\mathrm{gr}^F(I)) &= (\sigma^V(\sigma^F(P_1)), \sigma^V(\sigma^F(P_2)), \sigma^V(\sigma^F(P_3)))\mathbb{C}\{y\}[x, \xi, \eta]\\
&= (py^{p-1}\xi, qy^{p+1}\eta, p^2x^{q+1}\xi^2 + 2pqx^q y\xi\eta + q^2x^{q-1}y^2\eta^2)\mathbb{C}\{y\}[x, \xi, \eta]
\end{aligned}$$

Since $\sigma^L(P_3) = \sigma^V(\sigma^F(P_3)) - p^2q\xi$ and $\sigma^V(\sigma^L(P_3)) = -p^2q\xi$ it is therefore enough to prove that $\xi \notin \mathrm{gr}^V(\mathrm{gr}^F(I))$: this can be seen by writing $\xi$ as a linear combination of $\sigma^V(\sigma^F(P_i))$ then by evaluating at $x = y = 0$ (we find then $\xi \equiv 0$!).

*Second step: there is no other slope.* Take $V < L' < L$. Let us show that $L'$ is not a slope. We have:

$$(\sigma^{L'}(P_1), \sigma^{L'}(P_2), \sigma^{L'}(P_3))\mathbb{C}\{y, x\}[\xi, \eta] \subset \mathrm{gr}^{L'}(I)$$

that is $(y^{p-1}\xi, y^{p+1}\eta, \xi) \subset \mathrm{gr}^{L'}(I)$ and so $\mathrm{Char}^{L'}(\mathscr{D}e^{1/(y^p-x^q)}) \subset \{y=\xi=0\} \cup \{\eta = \xi = 0\}$. Thus this characteristic variety is of dimension 2 and:

$$\sqrt{\mathrm{gr}^{L'}(I)} = (\xi, y) \quad \text{or} \quad (\xi, \eta) \quad \text{or} \quad (\xi, y\eta).$$

$\sqrt{\mathrm{gr}^{L'}(I)}$ is therefore bi-homogeneous and $L'$ is not a slope. □

***Remark***. — The arguments given do not allow one to deal directly with the case $p = q$ because the $F$-standard basis of $I$ which we build does not then give a $V$-standard basis of $\mathrm{gr}^V(\mathrm{gr}^F(I))$. For $p > q$ it works in a similar way, with a suitable order.

## References

[1] J. Aroca, H. Hironaka & J. Vicente – *The theory of maximal contact*, Memorias, vol. 29, Instituto "Jorge Juan", Madrid, 1975.

[2] A. Assi, F. Castro-Jiménez & J.-M. Granger – How to calculate the slopes of a $\mathscr{D}$-module, *Compositio Math.* **104** (1996), p. 107–123.

[3] ______, The Gröbner fan of a $A_n$-module, *J. Pure Appl. Algebra* **150** (2000), p. 27–39.

[4] ______, The standard fan of an analytic $\mathscr{D}$-module, *J. Pure Appl. Algebra* **164** (2001), p. 3–21, Effective methods in algebraic geometry (Bath, 2000).

[5] M.F. Atiyah & I.G. MacDonald – *Introduction to commutative algebra*, Addison-Wesley, Reading MA, 1969.

[6] D. Bayer – The division algorithm and the Hilbert scheme, Ph.D. Thesis, Harvard University, 1982.

[7] T. Becker & V. Weispfenning – *Gröbner bases, a computational approach to commutative algebra*, G.T.M., vol. 141, Springer-Verlag, New York, 1993.

[8] N. Bourbaki – *Algèbre commutative,* Chap. 8 et 9, Masson, Paris, 1983.

[9] G. Brevet – Notes manuscrites, 1996.

[10] J. Briançon – Weierstrass préparé à la Hironaka, Astérisque, vol. 7, Société Mathématique de France, 1973.

[11] J. Briançon & Ph. Maisonobe – Idéaux de germes d'opérateurs différentiels à une variable, *Enseign. Math.* **30** (1984), p. 7–38.

[12] ______, Remarques sur la variété caractéristique de $\mathscr{D}$-modules irréguliers le long de $f = 0$, prépub. Université de Nice, juillet 1994.

[13] B. Buchberger – Ein algorithmisches Kriterium fur die Lösbarkeit eines algebraischen Gleichungssystems, *Aequationes Math.* **4** (1970), p. 374–383, phd. thesis.

[14] F. Castro – Théorème de division pour les opérateurs différentiels et calcul des multiplicités, Ph.D. Thesis, Université de Paris VII, 1984.

[15] ______, Calculs effectifs pour les idéaux d'opérateurs différentiels, in *Actas de la II Conferencia Internacional de Geometría Algebraica. La Rábida.*, Travaux en Cours, vol. 24, Hermann, Paris, 1987.

[16] F. Castro & L. Narváez – Homogenising differential operators, Prepublicaciones de la Facultad de Matemáticas de la Universidad de Sevilla, n° 36, June 1997.

[17] J. L. D. Cox & D. O'Shea – *Ideals, Varieties and Algorithms*, U.T.M., Springer-Verlag, New York, 1992.

[18] A. Galligo – Théorème de division et stabilité en Géométrie Analytique locale, *Ann. Inst. Fourier (Grenoble)* **29** (1979), p. 107–184.

[19] M. Granger & Ph. Maisonobe – A basic course on differential modules, in *Éléments de la théorie des systèmes différentiels*, Les cours du CIMPA, Travaux en cours, vol. 45, Hermann, Paris, 1993, p. 103–168.

[20] H. Hauser & G. Muller – A rank theorem for analytic maps between power series spaces, *Publ. Math. Inst. Hautes Études Sci.* **80** (1994), p. 95–115.

[21] H. Hironaka – Resolution of singularities of an algebraic variety over a field of characteristic 0, *Ann. of Math.* **79** (1964), p. 109–326.

[22] Y. Laurent – Polygone de Newton et $b$-fonctions pour les modules microdifférentiels, *Ann. scient. Éc. Norm. Sup.* $4^{\text{e}}$ *série* **20** (1987), p. 391–441.

[23] Y. Laurent & Z. Mebkhout – Pentes algébriques et pentes analytiques d'un $\mathscr{D}$-module, *Ann. scient. Éc. Norm. Sup.* $4^{\text{e}}$ *série* **32** (1999), p. 39–69.

[24] M. Lejeune-Jalabert – Effectivité des calculs polynomiaux, Cours de DEA, Université de Grenoble I, 1985.

[25] M. Lejeune-Jalabert & B. Malgrange (eds.) – *Séminaire sur les opérateurs différentiels*, Institut Fourier, Grenoble, 1975.

[26] M. Lejeune-Jalabert & B. Teissier – Quelques calculs utiles pour la résolution des singularités, Prépub. Centre de Mathématiques, École polytechnique, 1971.

[27] B. Malgrange – Sur les points singuliers des équations différentielles, *Enseign. Math.* **20** (1974), p. 147–176.

[28] Z. Mebkhout – Le théorème de positivité de l'irrégularité pour les $\mathscr{D}_X$-modules, in *The Grothendieck Festschrift*, Progress in Math., vol. 88, no. 3, Birkhäuser, Basel, Boston, 1990, p. 83–132.

[29] J.-P. Ramis – *Théorèmes d'indice Gevrey pour les équations différentielles ordinaires*, Mem. Amer. Math. Soc., vol. 48, no. 296, American Mathematical Society, Providence RI, 1984.

[30] C. Sabbah – Proximité évanescente, I. La structure polaire d'un $\mathscr{D}$-module, Appendice en collaboration avec F. Castro, *Compositio Math.* **62** (1987), p. 283–328.

[31] A.N. Varchenko – Asymptotic Hodge structure on the cohomology of the Milnor fiber, *Math. USSR Izv.* **18** (1982), p. 469–512.

## A complementary list of recent publications

[1] R. Bahloul – Algorithm for computing Bernstein-Sato ideals associated with a polynomial mapping, *J. Symb. Comput.* **32** (2001), p. 643–662, Effective methods in rings of differential operators.

[2] F. Castro-Jiménez & N. Takayama – Slopes of a Hypergeometric System Associated with a Monomial Curve, *Trans. Amer. Math. Soc.*, **355** (2003), no. 9, p. 3761–3775.

[3] F. Castro-Jiménez & J. Ucha Enríquez – A duality formula in $\mathscr{D}$-modules, in *Proceedings of the Meeting of Andalusian Mathematicians, Vol. II (Spanish) (Sevilla, 2000)*, Colecc. Abierta, vol. 52, Univ. Sevilla Secr. Publ., Sevilla, 2001, p. 283–288.

[4] ———, Explicit comparison theorems for $\mathscr{D}$-modules, *J. Symb. Comput.* **32** (2001), p. 677–685, Effective methods in rings of differential operators.

[5] M.J. Gago-Vargas – Bases for Projective $A_n(k)$-modules, *J. Symb. Comput.*, vol. 36 (2003), no. 6, p. 845–853.

[6] M.I. Hartillo Hermoso – About an algorithm of T. Oaku, in *Ring theory and algebraic geometry (León, 1999)*, Lecture Notes in Pure and Appl. Math., vol. 221, Dekker, New York, 2001, p. 241–250.

[7] ———, Slopes in Gel'fand-Kapranov-Zelevinsky systems of hypergeometric equations, in *Proceedings of the Meeting of Andalusian Mathematicians, Vol. II (Spanish) (Sevilla, 2000)*, Colecc. Abierta, vol. 52, Univ. Sevilla Secr. Publ., Sevilla, 2001, p. 565–568.

[8] ———, Métodos efectivos en anillos de operadores diferenciales y en sistemas hipergeométricos, Ph.D. Thesis, University of Sevilla, 2002.

[9] ———, Slopes for hypergeometric systems in codimension 1, *Revista Matemática Iberoamericana*, vol. 19 (2003), no. 2, p. 456–466.

[10] A. Leykin – Constructibility of the set of polynomials with a fixed Bernstein-Sato polynomial: an algorithmic approach, *J. Symb. Comput.* **32** (2001), p. 663–675, Effective methods in rings of differential operators.

[11] ———, Computing local cohomology in Macaulay 2, in *Local cohomology and its applications (Guanajuato, 1999)*, Lecture Notes in Pure and Appl. Math., vol. 226, Dekker, New York, 2002, p. 195–205.

[12] ———, $D$-modules for Macaulay 2, in *Mathematical software (Beijing, 2002)*, World Sci. Publishing, River Edge, NJ, 2002, p. 169–179.

[13] M.A. Moreno – Métodos computacionales en los sistemas de ecuaciones en derivadas parciales, Ph.D. Thesis, University of Sevilla, 2000.

[14] M. Mustaţă, G.G. Smith, H. Tsai & U. Walther – $\mathscr{D}$-modules on smooth toric varieties, *J. Algebra* **240** (2001), p. 744–770.

[15] M. Noro – Gröbner bases—theory, more efficient calculation and applications, *Sūrikaisekikenkyūsho Kōkyūroku* **1138** (2000), p. 127–171, Research on the theory and applications of computer algebra (Japanese) (Kyoto, 1999).

[16] ———, Gröbner basis calculation on Weyl algebra in Risa/Asir and their applications, *Sūrikaisekikenkyūsho Kōkyūroku* **1199** (2001), p. 43–50, Theory and application in computer algebra (Japanese) (Kyoto, 2000).

[17] ———, An efficient modular algorithm for computing the global $b$-function, in *Mathematical software (Beijing, 2002)*, World Sci. Publishing, River Edge, NJ, 2002, p. 147–157.

[18] M. Noro & al. – RISA/ASIR, `ftp://endeavor.fujitsu.co.jp/pub/isis/asir`.

[19] T. Oaku – Algorithms for $b$-functions, restrictions, and algebraic local cohomology groups of $D$-modules, *Adv. in Appl. Math.* **19** (1997), p. 61–105.

[20] ———, Algorithms for the $b$-function and $D$-modules associated with a polynomial, *J. Pure Appl. Algebra* **117/118** (1997), p. 495–518, Algorithms for algebra (Eindhoven, 1996).

[21] ———, An algorithm of computing $b$-functions, *Duke Math. J.* **87** (1997), p. 115–132.

[22] T. Oaku & N. Takayama – Algorithms for $D$-modules—restriction, tensor product, localization, and local cohomology groups, *J. Pure Appl. Algebra* **156** (2001), p. 267–308.

[23] ———, Minimal free resolutions of homogenized $D$-modules, *J. Symb. Comput.* **32** (2001), p. 575–595, Effective methods in rings of differential operators.

[24] T. Oaku, N. Takayama & H. Tsai – Polynomial and rational solutions of holonomic systems, *J. Pure Appl. Algebra* **164** (2001), p. 199–220, Effective methods in algebraic geometry (Bath, 2000).

[25] T. Oaku, N. Takayama & U. Walther – A localization algorithm for $D$-modules, *J. Symb. Comput.* **29** (2000), p. 721–728, Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998).

[26] C.J. Rust, G.J. Reid & A.D. Wittkopf – Existence and uniqueness theorems for formal power series solutions of analytic differential systems, in *Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation (Vancouver, BC)*, 1999, electronic, p. 105–112.

[27] M. Saito – Isomorphism classes of $A$-hypergeometric systems, *Compositio Math.* **128** (2001), p. 323–338.

[28] ———, Logarithm-free $A$-hypergeometric series, *Duke Math. J.* **115** (2002), p. 53–73.

[29] M. Saito, B. Sturmfels & N. Takayama – Gröbner deformations of regular holonomic systems, *Proc. Japan Acad. Ser. A Math. Sci.* **74** (1998), p. 111–113.

[30] ———, Hypergeometric polynomials and integer programming, *Compositio Math.* **115** (1999), p. 185–204.

[31] ———, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics, vol. 6, Springer-Verlag, Berlin, 2000.

[32] B. Sturmfels & N. Takayama – Gröbner bases and hypergeometric functions, in *Gröbner bases and applications (Linz, 1998)*, London Math. Soc. Lecture Note Ser., vol. 251, Cambridge Univ. Press, Cambridge, 1998, p. 246–258.

[33] N. Takayama – Kan: A system for computation in algebraic analysis, Source code available at `http://www.math.kobe-u.ac.jp/KAN/`, version 1 (1991), version 2 (1994).

[34] H. Tsai & U. Walther – Computing homomorphisms between holonomic $D$-modules, *J. Symb. Comput.* **32** (2001), p. 597–617, Effective methods in rings of differential operators.

[35] J.M. Ucha-Enríquez – Métodos constructivos en álgebras de operadores diferenciales, Ph.D. Thesis, University of Sevilla, 1999.

[36] ———, Computation of the slopes of a $\mathscr{D}$-module of type $\mathscr{D}^r/N$, in *Ring theory and algebraic geometry (León, 1999)*, Lecture Notes in Pure and Appl. Math., vol. 221, Dekker, New York, 2001, p. 311–324.

[37] U. Walther – Algorithmic computation of local cohomology modules and the local cohomological dimension of algebraic varieties, *J. Pure Appl. Algebra* **139** (1999), p. 303–321, Effective methods in algebraic geometry (Saint-Malo, 1998).

[38] ———, Algorithmic computation of de Rham cohomology of complements of complex affine varieties, *J. Symb. Comput.* **29** (2000), p. 795–839, Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998).

[39] ———, Algorithmic determination of the rational cohomology of complex varieties via differential forms, in *Symbolic computation: solving equations in algebra, geometry, and engineering (South Hadley, MA, 2000)*, Contemp. Math., vol. 286, Amer. Math. Soc., Providence, RI, 2001, p. 185–206.

[40] ———, Computing the cup product structure for complements of complex affine varieties, *J. Pure Appl. Algebra* **164** (2001), p. 247–273, Effective methods in algebraic geometry (Bath, 2000).

[41] ———, $D$-modules and cohomology of varieties, in *Computations in algebraic geometry with Macaulay 2*, Algorithms Comput. Math., vol. 8, Springer, Berlin, 2002, p. 281–323.

---

F.J. Castro-Jiménez, Departamento de Álgebra, Facultad de Matemáticas, Universidad de Sevilla, E-41012 Sevilla, Spain • *E-mail :* `castro@us.es`
*Url :* `http://thales.cica.es/usr/castro/`

M. Granger, Département de mathématiques, Université d'Angers, 2 Boulevard Lavoisier, F-49045 Angers cedex 01, France • *E-mail :* `michel.granger@univ-angers.fr`