Sankhyā : The Indian Journal of Statistics
1997, Volume 59, Series A, Pt. 3, pp. 311-323

A CHARACTERIZATION OF DISTRIBUTIONS BASED ON LINEAR REGRESSION OF ORDER STATISTICS AND RECORD VALUES

By FERNANDO LÓPEZ BLÁQUEZ
and
J. LUIS MORENO REBOLLO
Universidad de Sevilla, Spain

SUMMARY. We obtain the family of distributions for which the regression of one order statistic on another, not necessarily adjacent, is linear. As a consequence, we present a characterization of uniform distributions on an interval. We also characterize the distributions that appear when we impose the condition of linearity of regression for record values.

1. **Introduction**

Ferguson (1967) characterized the distributions for which the regression of an order statistic on an adjacent one is linear. He pointed out that it is not known which distributions would be characterized if non-adjacent order statistics are considered. In the concluding remarks of his paper, Nagaraja (1988) affirms that the problem remains still unsolved. Another reference about this problem can be found in Arnold, Balakrishnan, and Nagaraja (1992) (p. 155).

Nagaraja (1977, 1988) obtains a characterization based on the linear regression of two adjacent record values. As in the case of order statistics, the problem is open if the condition of linearity of regression for nonadjacent record values is imposed.

After reviewing previous results in section 2, we give the characterization of the distributions which are characterized when the regression of two order statistics, not necessarily adjacent, is linear. In Section 4, we deal with a similar problem for record values.

Paper received. April 1995; revised November 1996.

AMS (1980) *subject clasification.* Primary 62E10; secondary: 62G30.

Key words and phrases. Characterization of distributions, order statistics, record values, linear regression.

2. **Previous Results**

Let us denote by $X_{(i:n)}$ the ith order statistic of a simple random sample (s.r.s.) of size n from a r.v. X, with c.d.f. F and density f. For any three fixed integers, i, k and n, such that $1 \leq i < i+k \leq n$, and a real number x satisfying $0 < F(x) < 1$, the conditional density of $X_{(i+k:n)}$ given $X_{(i:n)} = x$ is

$$\begin{aligned} f_{(i,k,n,x)}(y) &= \tfrac{C_{ikn}}{(1-F(x))^{n-i}}(F(y)-F(x))^{k-1}(1-F(y))^{n-i}f(y), \text{ if } y > x \\ &= 0, \text{ otherwise} \end{aligned}$$

with

$$C_{ikn} = \frac{(n-i)!}{(k-1)!(n-i-k)!}.$$

Denote by D_F the set of real numbers such that $0 < F(x) < 1$ and the conditional expectation $R_{ikn}(x,F) = E\left[X_{(i+k:n)}/X_{(i:n)} = x\right]$ exists. In that case, for any x in D_F,

$$R_{ikn}(x,F) = \frac{C_{ikn}}{(1-F(x))^{n-i}} \int_x^{\infty} y(F(y)-F(x))^{k-1}(1-F(y))^{n-i-k}f(y)dy.$$

Note also that, $X_{(i+k:n)}$ given $X_{(i:n)} = x$ is distributed as the k-th order statistic of a simple random sample of size $n-i$ from a left-truncated at x random variable, $Y^{(x)}$, with c.d.f.

$$F_x(y) = \frac{F(y)-F(x)}{1-F(x)}, \quad \text{if } y \geq x.$$

Then

$$R_{ikn}(x,F) = E[Y^{(x)}_{(k:n-i)}].$$

It will be useful to consider the quantile function of F, defined as

$$Q(u) = \inf\{x : F(x) \geq u\}, \quad u \in (0,1).$$

In the following lemma we quote without proof some properties of the quantile function (e.g. see Port (1994), p. 98).

LEMMA 2.1. *Let F be a distribution function and Q its quantile function, then*

(a) Q is non-decreasing.

(b) Q is continuous at any point of $(0,1)$ *save, perhaps, at a countable set of points at which Q is left-continuous.*

(c) If F is continuous, then Q is strictly increasing in $(0,1)$.

(d) If Q is continuous in $(0,1)$, then F is strictly increasing in $\{x : 0 < F(x) < 1\}$.
(e) If U is a $U(0,1)$ distribution, then the c.d.f. of $Q(U)$ is F.
(f) Moreover,

$$(X_{(1:n)}, \ldots, X_{(n:n)}) \stackrel{d}{\equiv} (Q(U_{(1:n)}), \ldots, Q(U_{(n:n)})).$$

(g) If F is absolutely continuous, the conditional expectation of $X_{(i+k:n)}$ given $X_{(i:n)} = x$, whenever it exists, is:

$$E\left[X_{(i+k:n)}|X_{(i:n)} = x\right] = \frac{C_{i,k,n}}{(1-u)^{n-i}} \int_u^1 Q(v)(v-u)^{k-1}(1-v)^{n-i-k}dv. \qquad \ldots(2.1)$$

with $u = F(x)$.

Lemma 2.2. *Let F be a continuous c.d.f., then*
(a) $R_{ikn}(., F)$ is continuous and non-decreasing.
(b) If F is strictly increasing, then $R_{ikn}(., F)$ also is.

Proof. (a) The continuity of $R_{ikn}(., F)$ follows from the continuity of F. To show that $R_{ikn}(., F)$ is non-decreasing, let us choose $x_1, x_2 \in D_F$, such that $x_1 \leq x_2$. Consider the r.v.'s

$$Y_{(k:n-i)}^{(x_m)} = Q(F(x_m) + (1 - F(x_m))U_{(k:n-i)}), \quad m = 1, 2, \qquad \ldots(2.2)$$

where $U_{(k:n-i)}$ is distributed as the kth order statistic of a s.r.s. of size $n-i$ from a $U(0,1)$ distribution. Then, the r.v.'s $Y_{(k:n-i)}^{(x_m)}$, $m = 1, 2$, are distributed as the kth order statistic of a s.r.s of size $n-i$ from the left-truncated distribution F_{x_m}, $m = 1, 2$. As Q is non-decreasing, from (2.2), we have

$$Y_{(k:n-i)}^{(x_1)} \leq Y_{(k:n-i)}^{(x_2)} \qquad \ldots(2.3)$$

and taking expected values in (2.3), it follows that

$$R_{ikn}(x_1) \leq R_{ikn}(x_2). \qquad \ldots(2.4)$$

(b) Note that, if F is strictly increasing the inequalities (2.3) and (2.4) are both strict. □

Lemma 2.3. *If F is an absolutely continuous c.d.f. such that*

$$R_{ikn}(x, F) = bx + a, \quad \textit{for any} \;\; x \in D_F, \qquad \ldots(2.5)$$

then

(a) F is strictly increasing in $\{x : 0 < F(x) < 1\}$.

(b) $b > 0$.

(c) If $b \neq 1$, the c.d.f. $F_\mu(x) = F(x - \mu)$, with $\mu = a/(1 - b)$, satisfies $R_{ikn}(x, F_\mu) = bx$

(d) If $b = 1$, then $a > 0$.

PROOF. (a) According to Lemma 2.1(d), it suffices to show that Q is continuous in $(0, 1)$. For that, define the function

$$S_{ikn}(u, Q) = \frac{C_{i,k,n}}{(1-u)^{n-i}} \int_u^1 Q(v)(v-u)^{k-1}(1-v)^{n-i-k} dv.$$

As Q is a quantile function, from Lemma 2.1(b), Q is continuous a.e. Thus S_{ikn} is continuous at any $u \in (0, 1)$, and from (2.5) we have,

$$S_{ikn}(u, Q) = bQ(u) + a, \quad \text{for any point of continuity of } Q, \qquad \ldots(2.6)$$

But as the LHS of (2.6) is continuous, the continuity of Q in $(0, 1)$ follows.

(b) As F is strictly increasing, from Lemma 2.2(b), we conclude $b > 0$.

(c) This is a consequence of the following relation

$$R_{ikn}(x, F_\mu) = R_{ikn}(x + \mu, F) - \mu.$$

(d) If X is a r.v. for which (2.5) holds with $b = 1$, we have

$$\begin{aligned} a &= E\left[R_{ikn}\left(X_{(i:n)}, F\right)\right] - E\left[X_{(i:n)}\right] \\ &= E\left\{E\left[X_{(i+k:n)} - X_{(i:n)} | X_{(i:n)}\right]\right\} = E\left[X_{(i+k:n)} - X_{(i:n)}\right] > 0. \end{aligned}$$

□

LEMMA 2.4. *Consider the polynomial equation*

$$P_k(z) = \frac{1}{b} P_k(n - i), \quad b > 0 \qquad \ldots(2.7)$$

with $P_k(z) = z(z-1)\cdots(z-k+1)$. Then,

(a) The real roots of (2.7) are at most double.

(b) If z_0 is a complex root of (2.7), with $Im(z_0) \neq 0$, then z_0 is simple.

(c) There is a unique simple real root of (2.7) in $(k - 1, +\infty)$. Moreover,

(c.1) *If $0 < b < 1$, there is a unique real root in $(n - i, +\infty)$.*

(c.2) *If $b > 1$, there is a unique root in $(k - 1, n - i)$.*

(c.3) *If $b = 1$, $z = n - i$ is the unique root in $(k - 1, +\infty)$.*

(d) $P_k(D)(t\exp(\alpha t)) = (P_k'(\alpha) + P_k(\alpha)t)\exp(\alpha t)$*, with* D *the derivative operator (with respect to the variable* t*), and* α *a real number.*

PROOF. (a), (b) Note that if $k > 2$, P_k is a polynomial of degree k, which has a local extremum (maximum or minimum) in each one of the open intervals $(j, j+1)$, $j = 0, \ldots, k-2$, in other words, P_k' is a polynomial of degree $k-1$ with $k-1$ simple real roots. From this fact (a) and (b) are immediate.

(c) As $P_k(z)$ is continuous, strictly increasing in $(k-1, +\infty)$, $P_k(k-1) = 0$, and $\lim_{z\to\infty} P_k(z) = +\infty$, then, for any $\alpha > 0$, there is a unique (simple) real root in $(k-1, +\infty)$ of the equation $P_k(z) = \alpha$. In particular, consider $\alpha = b^{-1}P_k(n-i) > 0$.

Note that, if $0 < b < 1$ then, $P_k(n-i) < \frac{1}{b}P_k(n-i)$, therefore the root is in the interval $(n-i, +\infty)$.

If $b > 1$, we have, $0 = P_k(k-1) < \frac{1}{b}P_k(n-i) < P_k(n-i)$, and therefore the root is in $(k-1, n-i)$.

If $b = 1$, as $k-1 < n-i$, then $0 = P_k(k-1) < P_k(n-i)$, thus the root is in $(k-1, +\infty)$.

(d) The proof follows easily by using induction. □

LEMMA 2.5. *Consider the functions,*

$$\begin{aligned}
Q_1(v) &= (1-v)^{r-(n-i)},\\
Q_2(v) &= (1-v)^{r-(n-i)}\log(1-v),\\
Q_3(v) &= (1-v)^{r-(n-i)}\log(1-v)\cos(s\log(1-v)), \quad s \neq 0,\\
Q_4(v) &= (1-v)^{r-(n-i)}\log(1-v)\sin(s\log(1-v)), \quad s \neq 0.
\end{aligned}$$

If $r \le k-1$*, then*

$$\int_u^1 Q_j(v)(v-u)^{k-1}(1-v)^{n-i-k}dv, \quad j = 1,2,3,4 \qquad \ldots (2.8)$$

is not convergent for any $u \in (0,1)$. □

PROOF. The proof is straightforward by using the classical criteria for the convergence of integrals.

3. Linear Regression of Order Statistics

In this section, we characterize the distributions for which the regression of two order statistics is linear. Our main result is presented in the following theorem.

THEOREM 3.1. *Let X be a r.v. with distribution function F which is k times differentiable in D_F, such that*

$$E\left[X_{(i+k:n)}|X_{(i:n)}\right] = bX_{(i:n)} + a. \qquad \ldots(3.1)$$

Then, except for location and scale parameters,

$$F(x) = 1- \mid x\mid^{\delta}, \quad \text{for} \quad x \in [-1, 0], \quad \text{if} \quad 0 < b < 1 \qquad \ldots(3.2)$$

$$F(x) = 1 - \exp(-x), \quad \text{for} \quad x \in [0, \infty), \quad \text{if} \quad b = 1 \qquad \ldots(3.3)$$

$$F(x) = 1 - x^{\delta}, \quad \text{for} \quad x \in [1, \infty), \quad \text{if} \quad b > 1 \qquad \ldots(3.4)$$

where, $\delta = (r - (n - i))^{-1}$ and r is the unique real root greater than $k - 1$ of the polynomial equation

$$P_k(x) = \frac{1}{b} P_k(n - i). \qquad \ldots(3.5)$$

PROOF. Let F be a c.d.f. for which (3.1) holds and Q its quantile function. Expression (3.1) can be rewritten as

$$C_{i,k,n} \int_u^1 Q(v)(v - u)^{k-1}(1 - v)^{n-i-k} dv = (bQ(u) + a)(1 - u)^{n-i}. \qquad \ldots(3.6)$$

From Lemma 2.3(a), F is strictly increasing in D_F, then $f(x) = F'(x) > 0$, for $x \in D_F$, and as F is k-times differentiable, it follows that $Q = F^{-1}$ is also k times differentiable in $(0, 1)$.

Differentiating k times both sides of (3.6), we obtain the ordinary differential equation,

$$(1 - u)^k H^{(k)}(u) = (-1)^k \frac{1}{b} P_k(n - i) \left\{H(u) - a(1 - u)^{n-i}\right\}, \qquad \ldots(3.7)$$

with,

$$H(u) = Q(u)(1 - u)^{n-i}.$$

The change of variables $t = \log(1 - u)$ transforms (3.7) into the linear differential equation

$$P_k(D)(G(t)) = \frac{1}{b} P_k(n - i) \{G(t) - a \exp(t(n - i))\}, \qquad \ldots(3.8)$$

where, D is the derivative operator and $G(t) = H(1 - e^t)$.

Let us distinguish two cases: $b \neq 1$ and $b = 1$.

Case A: $b \neq 1$. According to Lemma 2.3(c), we can assume, w.l.o.g., that $a = 0$. For obtaining the general solution of (3.8), we must solve the associated polynomial equation (3.5).

Let $r_1, \cdots, r_s$ and $r_{s+1}, \cdots, r_d$ be the simple and double real roots of (3.5) respectively, and $z_h = r_h + is_h$, $h > d$, ($s_h \neq 0$) the (simple) complex roots. Then, the complete set of solutions of the homogeneous linear differential equation (3.8) is the linear space generated by

$$\{\exp(r_h t),\ h = 1, \ldots, s; \quad t\exp(r_h t),\ h = s+1, \ldots, d;$$

$$t\exp(r_h t)\cos(s_h t), \quad t\exp(r_h t)\sin(s_h t), \quad h > d\}.$$

Hence, if Q is the quantile function of a r.v. X for which (3.1) holds, it must satisfy the following conditions:

(i) Q belongs to the linear space of functions generated by

$$(1-u)^{r_h-(n-i)}, \quad h = 1, \ldots, s; \qquad \ldots (3.9)$$

$$(1-u)^{r_h-(n-i)}\log(1-u), \quad h = s+1, \ldots, d; \qquad \ldots (3.10)$$

$$(1-u)^{r_h-(n-i)}\log(1-u)\cos(s_h\log(1-u)), \quad h > d; \qquad \ldots (3.11)$$

$$(1-u)^{r_h-(n-i)}\log(1-u)\sin(s_h\log(1-u)), \quad h > d. \qquad \ldots (3.12)$$

(ii) The conditional expectation $E[X_{(i+k:n)}|X_{(i:n)}]$ must exist, or equivalently, the integral

$$\int_u^1 Q(v)(v-u)^{k-1}(1-v)^{n-i-k}dv \qquad \ldots (3.13)$$

must exist, for any $u \in (0,1)$.

(iii) Q is monotone strictly increasing.

Let us analyze the functions in the basis of the linear space of solutions described in (i). The functions (3.11) and (3.12) change their signs infinitely many times in a neighborhood of $u = 1$, so that they are not monotone, in other words, they do not satisfy (iii). From Lemma 2.5, the integral (3.13) does not converge for the functions of the form (3.9) or (3.10) which have $r_h \leq k-1$. But, according to Lemma 2.4(c), there exist only one simple root of (3.5) greater than $k-1$, namely r, then the quantile functions which are solutions of our problem are of the form

$$Q(u) = A(1-u)^{\frac{1}{\delta}}, \qquad \ldots (3.14)$$

where, $\delta = (r-(n-i))^{-1}$ and A a real number chosen in such a way that (iii) holds, that is to say, $A/\delta < 0$.

We have two subcases. Firstly, if $b < 1$, Lemma 2.4(c.1) implies that $\delta > 0$, and w.l.o.g., it can be assumed $A = -1$. In this case, it follows easily that Q is the quantile function of (3.2). Secondly, if $b > 1$, Lemma 2.4(c.2) implies that $\delta < 0$, and choosing $A = 1$ we obtain the quantile of the c.d.f. (3.4).

Case B: $b = 1$. From Lemma 2.3(d), a is positive. We must solve the non-homogeneous linear differential equation (3.8). It can be shown, using Lemma 2.4(d), that a particular solution of (3.8) is

$$G_0(t) = -a\frac{P_k(n-i)}{P_k'(n-i)}t\exp((n-i)t)$$

The general solution of the homogeneous linear differential equation associated to (3.8) is obtained by solving the polynomial equation $P_k(z) = P_k(n-i)$. A similar argument to the one used in CASE A shows that the quantile functions which are solutions of our problem are of the form

$$Q(u) = -a\frac{P_k(n-i)}{P_k'(n-i)}\log(1-u) + A,$$

and, w.l.o.g., choosing $A = 0$ and $a = \frac{P_k'(n-i)}{P_k(n-i)}$, we obtain the quantile function of the c.d.f. given in (3.3). $\square$

We obtain the following dual result easily.

THEOREM 3.2 *Let X be a r.v. with distribution function F which is k-times differentiable in D_F, such that*

$$E\left[X_{(i:n)}|X_{(i+k:n)}\right] = cX_{(i+k:n)} + d, \qquad \ldots(3.15)$$

then, except for location and scale parameters,

$$F(x) = x^\theta, \text{ for } x \in [0,1], \text{ if } 0 < c < 1 \qquad \ldots(3.16)$$

$$F(x) = \exp(x), \text{ for } x \in (-\infty,0], \text{ if } c = 1 \qquad \ldots(3.17)$$

$$F(x) = \mid x \mid^\theta, \text{ for } x \in (-\infty,1], \text{ if } c > 1 \qquad \ldots(3.18)$$

where, $\theta = (r-(i+k-1))^{-1}$ and r is the unique real root greater than $k-1$ of the polynomial equation

$$P_k(x) = \frac{1}{c}P_k(i+k-1).$$

PROOF. Let X be a r.v. for which (3.15) holds, then the r.v. $Y = -X$ satisfies

$$E\left[Y_{(n-i+1:n)}|Y_{(n-i-k+1)} = y\right] = -(c(-y)+d) = cy - d,$$

thus the c.d.f. of Y belongs to one of the three families described in Theorem 3.1. $\square$

Note that, depending on the values of the slopes, we characterize in Theorem 3.2 three families of distributions which are just the same as those obtained by Ferguson (1967) in the adjacent case.

Combining the results stated in Theorems 3.1 and 3.2 gives the following characterizations of uniform distributions.

COROLLARY 3.1. *If F is k times differentiable, and for certain integers i, n, such that $1 \leq i < i+k \leq n$, the conditional expectations $E\left[X_{(i+k:n)}|X_{(i:n)}\right]$ and $E\left[X_{(i:n)}|X_{(i+k:n)}\right]$ are both linear, then F is the c.d.f. of a uniform distribution on a finite interval.*

COROLLARY 3.2. *Let F be a c.d.f. which is $(n-1)$-times differentiable. F is uniform on a finite interval iff $E\left[X_{(i:n)}|X_{(j:n)}\right]$ is linear for any i, j such that $1 \leq i, j \leq n$.*

4. **Linear Regression of Record Values**

Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. r.v.'s with common continuous distribution function F. The sequence of record times, $\{L(n)\}_{n\geq 0}$, is defined recursively as

$$\begin{aligned} &L(0) = 1, \\ &L(n) = \min\{j : j > L(n-1) \text{ and } X_j > X_{L(n-1)}\}, \quad n > 1, \end{aligned}$$

and the sequence of record values is defined as $\{X_{L(n)}\}_{n\geq 0}$.

If Q, the quantile function of F, is strictly increasing and W follows a standard negative exponential distribution then

$$(X_{L(0)}, \ldots, X_{L(j)}) \stackrel{d}{\equiv} (Q(1-\exp(-W_{L(0)})), \ldots, Q(1-\exp(-W_{L(j)}))),$$

and the joint density of $W_{L(i)}$ and $W_{L(i+k)}$, with $0 \leq i < i+k$ is

$$\begin{aligned} f_{i,k}(w,s) &= D_{i,k} w^i (s-w)^{k-1} \exp(-s) \quad &\text{if} \quad 0 < w < s < \infty \\ &= 0, &\text{otherwise} \end{aligned} \qquad \ldots(4.1)$$

with

$$D_{i,k} = \frac{1}{i!(k-1)!}$$

THEOREM 4.1. *Let F be a strictly increasing c.d.f. which is k-times differentiable, such that*

$$E\left[X_{L(i+k)}|X_{L(i)}\right] = bX_{L(i)} + a, \qquad \ldots(4.2)$$

holds for certain a and b. Then, except for location and scale parameters,

$$F(x) = 1- \mid x \mid^{\delta}, \quad \text{for} \quad x \in [-1,0], \qquad \text{if} \quad 0 < b < 1 \qquad \ldots (4.3)$$
$$F(x) = 1 - \exp(-x), \quad \text{for} \quad x \in [0,\infty), \qquad \text{if} \quad b = 1 \qquad \ldots (4.4)$$
$$F(x) = 1 - x^{\delta}, \quad \text{for} \quad x \in [1,\infty), \qquad \text{if} \quad b > 1 \qquad \ldots (4.5)$$

where, $\delta = (1-b^{1/k})^{-1}b^{1/k}$.

If $b \leq 0$, there is no c.d.f. for which (4.2) holds.

Proof. The proof of this result is similar to that of Theorem 3.1, so we omit some of the details.

It can be shown that, if $b \leq 0$ there is no c.d.f. for which (4.2) holds. So, we discuss the case $b > 0$.

The condition (4.2) can be rewritten as

$$\frac{1}{(k-1)!}\int_w^{\infty} Q(1-e^{-s})(s-w)^{(k-1)}e^{-s}\,ds = (bQ(1-e^{-w})+a)e^{-w}. \quad \ldots (4.6)$$

Differentiating k times both sides of (4.6), we obtain the following linear differential equation

$$bH^{(k)}(w) = (-1)^k\left(H(w) - a\exp(-w)\right) \qquad \ldots (4.7)$$

with,

$$H(w) = Q(1-\exp(-w))\exp(-w).$$

If $b \neq 1$, we can assume $a = 0$. In this case, the linear differential equation (4.7) is homogeneous, and the solutions are obtained by solving the associated polynomial equation $bz^k = (-1)^k$.

A detailed analysis of the solutions of (4.7) shows that the quantile functions satisfying (4.6) are of the form

$$Q(u) = A(1-u)^{1/\delta} \qquad \ldots (4.8)$$

with

$$\delta = b^{1/k}(1-b^{1/k})^{-1} \quad \text{and} \quad A/\delta < 0.$$

From (4.8), and depending on the values of b, we obtain the c.d.f.s (4.3) and (4.5).

If $b = 1$, then $a > 0$, and a particular solution of (4.7) is

$$H_0(w) = \frac{a}{k}w\exp(-w)$$

and the quantile functions which are solutions of (4.6), are of the form

$$Q(u) = A - \frac{a}{k}\log(1-u) \qquad \ldots (4.9)$$

which, except for location and scale parameters, is the quantile function corresponding to (4.4). $\square$

In particular, if in Theorem 4.1 we consider the case of adjacent record values, $k = 1$, we obtain the characterization given in Nagaraja (1977).

We can also give the following characterization.

THEOREM 4.2. *Let F be a strictly increasing c.d.f. which is k-times differentiable, such that*

$$E\left[X_{L(i)}|X_{L(i+k)}\right] = cX_{L(i+k)} + d \qquad \ldots (4.10)$$

holds for certain c and d, and $i \geq 0$. Then, except for location and scale parameters,

$$F(x) = 1 - \exp(-x^{\theta}), \quad \text{for} \ \ x \in [0,\infty)\,, \quad \text{if} \ \ 0 < c < 1 \qquad \ldots (4.11)$$

$$F(x) = 1 - \exp(-\exp(x)), \ \text{for} \ \ x \in (-\infty,\infty), \ \text{if} \ \ c = 1 \qquad \ldots (4.12)$$

$$F(x) = 1 - \exp(-\mid x \mid^{\theta}), \quad \text{for} \ \ x \in (-\infty, 0]\,, \quad \text{if} \ \ c > 1 \qquad \ldots (4.13)$$

with $\theta = (r - (i + k))^{-1}$, and r the unique real root greater than $k - 1$ of the equation

$$P_k(x) = \frac{P_k(i+k)}{c}. \qquad \ldots (4.14)$$

If $c \leq 0$, there is no c.d.f. for which (4.10) holds.

PROOF. Again, the proof is similar to that of Theorem 3.1. If $c \leq 0$, there is no c.d.f. such that (4.10) holds. From now on, assume $c > 0$.

The condition (4.10) can be expressed as

$$\frac{(i+k)!}{i!(k-1)!}\int_0^s Q(1-e^{-w})w^i(s-w)^{k-1}\,dw = (cQ(1-e^{-s})+d)s^{i+k} \quad \ldots (4.15)$$

Differentiating k times both sides of (4.15), we obtain the following differential equation

$$H^{(k)}(s) = \frac{P_k(i+k)}{c}\left(H(s)s^{-k} - ds^i\right) \qquad \ldots (4.16)$$

with

$$H(s) = Q(1-e^{-s})s^{i+k}$$

The change of variables $t = \log(s)$ transforms (4.16) into

$$P_k(D)\,(G(t)) = \frac{P_k(i+k)}{c}\,(G(t) - d\exp((i+k)t)) \qquad \ldots (4.17)$$

with

$$G(t) = H(e^t).$$

If $c \neq 1$, then, w.l.o.g., it can be considered $d = 0$, and (4.16) is a homogeneous differential linear equation which solutions are obtained by solving the associated polynomial equation (4.14).

The analysis of the solutions leads us to conclude that the quantile functions which are solutions of (4.15) are of the form

$$Q(u) = A\left(\log\frac{1}{1-u}\right)^{1/\theta} \qquad \ldots(4.18)$$

with $\theta = (r-(i+k))^{-1}$, and r the unique real root greater than k-1 of the equation (4.14).

It is easy to show that, with adequate choices of A, the quantile function (4.18) corresponds to the c.d.f.'s (4.11) and (4.13).

The solution in the case $c = 1$, except for location and scale parameters, is

$$Q(u) = \log\left(\log\frac{1}{1-u}\right) \qquad \ldots(4.19)$$

which is the quantile function of (4.12). □

It can be easily checked that in the adjacent case, $k = 1$, we obtain the characterization given in Nagaraja (1988). As a consequence of Theorems 4.1 and 4.2, we give the following characterization of exponential distributions.

Corollary 4.1. *If F is k times differentiable, and for certain nonnegative integer, i, the conditional expectations $E\left[X_{L(i+k)}|X_{L(i)}\right]$ and $E\left[X_{L(i)}|X_{L(i+k)}\right]$ are both linear, then, except for location and scale parameters, F is the c.d.f. of an exponential distribution.*

References

Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1992). *A First Course in Order Statistics.* Wiley & Sons. New York.

David, H.A. (1981). *Order Statistics*, (2nd ed.). Wiley, New York.

Ferguson, T.S. (1967). On characterizing distributions by properties of order statistics. *Sankhyā* A, **29**, 265-278.

López-Blázquez, F. (1990). Caracterización de Distribuciones mediante el Valor Esperado de Estadísticos Ordenados y Records. *Unpublished PhD. Thesis*, Universidad de Sevilla.

Nagaraja, H.N. (1977). On a characterization based on record values. *Austral. J. Statist.*, **19**(1), 70-73.

– – –– (1988). Some characterizations of continuos distributions based on regressions of adjacent order statistics and record values. *Sankhyā* A, **50**, Pt.1, 70-73.

Port, S.C. (1994). *Theoretical Probability for Applications.* Wiley, New York.

Dpto. de Estadística e I.O.
Facultad de Matemáticas.
Universidad de Sevilla.
Tarfia, s/n. 41012 Sevilla.
Spain
e-mail: puerto@cica.es