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# Wardowski conditions to the coincidence problem

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In this article we first discuss the existence and uniqueness of a solution for the coincidence problem: Find $p \in X$ such that $Tp = Sp$, where $X$ is a nonempty set, $Y$ is a complete metric space, and $T, S : X \to Y$ are two mappings satisfying a Wardowski type condition of contractivity. Later on, we will state the convergence of the Picard-Juncgk iteration process to the above coincidence problem as well as a rate of convergence for this iteration scheme. Finally, we shall apply our results to study the existence and uniqueness of a solution as well as the convergence of the Picard-Juncgk iteration process toward the solution of a second order differential equation.

**Keywords: coincidence points, iterative methods, rate of convergence, common fixed points**
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## 1. Introduction

Let $X, Y$ be two nonempty sets and let $T, S : X \to Y$ be two arbitrary mappings. The coincidence problem determined by the mappings $T$ and $S$ consists in

$$\text{Find } p \in X \text{ such that } Tp = Sp. \tag{1}$$

Quite often to solve problem (1), we have to assume that $Y$ is a complete metric space, and $T, S : X \to Y$ are two mappings satisfying some type of contractivity, for instance see [1–5]. Some nonlinear problems arising from many areas of applied sciences can be formulated, from a mathematical point of view, as a coincidence problem (see, [1–3, 6, 7] and references within).

Once the existence of a solution to problem (1) is known, a central question consists to study if there exists an approximating sequence $(x_n) \subseteq X$ generated by an iterative procedure $f(T, S, x_n)$ such that the sequence $(x_n)$ converges to the coincidence point of $T$ and $S$. Jungck [8] introduced the following iterative scheme: given $x_1 \in X$, there exists a sequence $(x_n)$ in $X$ such that $Tx_{n+1} = Sx_n$. This procedure becomes the Picard iteration when $X = Y$ and $T = I_d$, where $I_d$ is the identity map on $X$. In Jungck [8], the author proved that if $(X, d)$ and $(Y, \rho)$ are two complete metric spaces and $T$ and $S$ satisfy both that $S(X) \subseteq T(X)$ and that for every $x, y \in X$ the inequality $d(Sx, Sy) \leq \kappa\, d(Tx, Ty)$, with $0 \leq \kappa < 1$ holds, then $(x_n)$ converges to the unique coincidence point of $T$ and $S$. Later, this type of convergence results were generalized for more general classes of contractive type mappings, see [6, 7, 9, 10] (to see another type of iterative schemes we can quote [10, 11]).

Since it is well known that the existence of a solution to problem (1) is, under appropriate conditions, equivalent to the existence of a fixed point for a certain mapping. In this article, we will use the Wardowski fixed point theorem [12] in order to show that problem (1) has a unique solution and that the Picard-Jungck iterative scheme converges to the unique coincidence point, moreover a rate of convergence for this scheme will also be given. Finally, we will apply these results to a general second order differential equation.

## 2. Notations and Preliminaries

Throughout this article $\mathbb{R}_+$ and $\mathbb{N}$ will denote the set of all non-negative real numbers and the set of all positive integers respectively.

**Definition 2.1.** *Let X and Y be two nonempty sets and $T, S: X \to Y$ two mappings. If there exists $x \in X$ such that $Sx = Tx$ then x is said to be a coincidence point of S and T.*

**Definition 2.2.** *Let S and T be two self-mappings of a nonempty set X. The pair of mappings S and T is said to be weakly compatible if they commute at their coincidence points, that is, $TSx = STx$ whenever $Tx = Sx$.*

The following straightforward result states a relationship between coincidence points and common fixed points of two weakly compatible mappings, see Proposition 1.4 in Abbas and Jungck [9].

**Lemma 2.1.** *Let S and T be weakly compatible self-mappings of a nonempty set X. If S and T have a unique coincidence point x, then x is the unique common fixed point of S and T.*

Given $k \in (0, 1)$, by $\mathcal{F}_k$ denote the set of all strictly increasing real functions $f: (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

$(F_1)$ For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0 \iff \lim_{n\to\infty} f(\alpha_n) = -\infty$.

$(F_2)$ $\lim_{\alpha\to 0^+} \alpha^k f(\alpha) = 0$.

**Definition 2.3.** *Let $(X, d)$ be a complete metric space. A mapping $T: X \to X$ is said to be an F-contraction if there exist $\tau > 0$ and $f \in \mathcal{F}_k$ such that, for all $x, y \in X$,*

$$d(x, y) > 0 \Longrightarrow \tau + f(d(x, y)) \le f(d(Tx, Ty)). \tag{2}$$

The following result will be the key in the proof of our results. This result was proved by Wardowski [12].

**Theorem 2.1.** *[[12]] Let $(X, d)$ be a complete metric space and let $T: X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is convergent to $x^*$.*

## 3. Main results

### 3.1. Existence and Uniqueness

In this subsection we present a result which guarantees the existence and uniqueness of a solution to problem (1) when the mappings $T$ and $S$ satisfy a Wardowski's contractivity type condition.

**Theorem 3.1.** *Let X be a nonempty set and let $(Y, \rho)$ be a complete metric space. Assume that $T, S: X \to Y$ are two mappings satisfying the following conditions:*

*(i)* $T(X)$ *is closed;*
*(ii)* $S(X) \subseteq T(X)$*;*
*(iii)* *There exist $\tau > 0$ and $f \in \mathcal{F}_k$ such that, for all $x, y \in X$,*

$$\rho(Sx, Sy) > 0 \Longrightarrow \tau + f(\rho(Sx, Sy)) \le f(\rho(Tx, Ty)). \tag{3}$$

*Then, T and S have at least one coincidence point in X. If, moreover, T is one-to-one, then this coincidence point is unique.*

*Proof.* Consider $h: T(X) \to 2^{T(X)}$ given by $h(x) := S(T^{-1}x)$, where $T^{-1}x := \{\xi \in X : T(\xi) = x\}$. Notice that $h$ is single-valued. Indeed, if $u, v \in h(x)$ with $u \ne v$, then by definition we know that there exists $\xi_u, \xi_v \in T^{-1}x$ such that $u = S\xi_u$ and $v = S\xi_v$. Since $\rho(S\xi_u, S\xi_v) = \rho(u, v) > 0$, from (3), we have that

$$\tau + f(\rho(S\xi_u, S\xi_v)) \le f(\rho(T\xi_u, T\xi_v)) = f(\rho(x, x)) = f(0),$$

which is a contradiction, because $f$ is not defined at 0.

Therefore, $h: T(X) \to T(X)$ is a single valued map from $T(X)$ into itself. Furthermore, $h$ verifies Wardowski's contractive condition [12], since if $0 < \rho(h(x), h(y)) = \rho(S(T^{-1}x), S(T^{-1}y))$, then by (3) we have that

$$\tau + f(\rho(S(T^{-1}x), S(T^{-1}y))) \le f(\rho(T(T^{-1}x), T(T^{-1}y))),$$

that is, $\tau + f(\rho(h(x), h(y))) \le f(\rho(x, y))$.

Bearing in mind that $(Y, \rho)$ is complete and $T(X)$ is closed, Wardowski's Theorem states that $h$ has a unique fixed point $y^* \in T(X)$. Consider $x^* \in T^{-1}y^*$. Then, by definition, we have that $Sx^* = S(T^{-1}y^*) = h(y^*) = y^* = Tx^*$, that is, $x^*$ is a coincidence point of $T$ and $S$.

Now suppose that $T$ is injective. If there exist $x^*, x' \in X$ such that $Sx^* = Tx^*$, $Tx' = Sx'$ and $x^* \ne x'$, then $Sx^* = Tx^* \ne Tx' = Sx'$ because $T$ is injective. From (3), we obtain

$$\tau + f(\rho(Sx^*, Sx')) \le f(\rho(Tx^*, Tx')) = f(\rho(Sx^*, Sx')),$$

i.e., $\tau \le 0$ which is a contradiction. □

**Corollary 3.1.** *Let X be a nonempty set and $(Y, \rho)$ be a complete metric space. Assume that $T, S: X \to Y$ are two mappings such that:*

*(a)* $T(X)$ *is closed;*
*(b)* $S(X) \subseteq T(X)$*;*
*(c)* *There exist $\tau > 0$ such that, for all $x, y \in X$,*

$$\rho(Sx, Sy) > 0 \Longrightarrow \rho(Sx, Sy) \le \frac{\rho(Tx, Ty)}{(1 + \tau\sqrt{\rho(Tx, Ty)})^2}. \tag{4}$$

*Then, T and S have at least one coincidence point in X. If, moreover, T is one to one, then this coincidence point is unique.*

*Proof.* From (c) it follows that

$$\sqrt{\rho(Sx, Sy)} \leq \frac{\sqrt{\rho(Tx, Ty)}}{1 + \tau\sqrt{\rho(Tx, Ty)}},$$

that is,

$$\frac{1 + \tau\sqrt{\rho(Tx, Ty)}}{\sqrt{\rho(Tx, Ty)}} \leq \frac{1}{\sqrt{\rho(Sx, Sy)}},$$

therefore

$$\tau + \frac{1}{\sqrt{\rho(Tx, Ty)}} \leq \frac{1}{\sqrt{\rho(Sx, Sy)}}.$$

The above inequality can be written as

$$\tau - \frac{1}{\sqrt{\rho(Sx, Sy)}} \leq -\frac{1}{\sqrt{\rho(Tx, Ty)}}.$$

The last inequality means that $T$ and $S$ satisfy the conditions of Theorem 3.1 with respect to the function $f(t) = -\frac{1}{\sqrt{t}}$, which belongs to $\mathcal{F}_k$ for some $k \in (\frac{1}{2}, 1)$. □

## 3.2. Picard-Juncgk's Iteration Process

In this subsection we present the results on the convergence for the Picard-Jungck scheme when the conditions of Theorem 3.1 are satisfied. Before giving our convergence result, we state the following lemma proved implicitly in the proof of Wardowski's Theorem [12].

**Lemma 3.1.** *Let $\tau > 0$ and $f \in \mathcal{F}_k$ with $k \in (0, 1)$. If $\{\gamma_n\}_{n\in\mathbb{N}}$ is a sequence of real non-negative numbers satisfying $\tau + f(\gamma_{n+1}) \leq f(\gamma_n)$ for all $n \in \mathbb{N}$, then the series $\sum_{i=0}^{\infty} \gamma_i$ is convergent.*

**Theorem 3.2.** *Let $X$ be a nonempty set and $(Y, \rho)$ be a complete metric space. If $T, S : X \to Y$ satisfy the three conditions of Theorem 3.1 and $T$ is one-to-one, then given $x_1 \in X$ the iterative scheme $Tx_{n+1} = Sx_n$ satisfies that the sequences $\{Tx_n\}_{n\in\mathbb{N}}$ and $\{Sx_n\}_{n\in\mathbb{N}}$ converge to $Tp = Sp$, where $p \in X$ is the unique coincidence point of $T$ and $S$.*

*Proof.* Notice that under these assumptions, Theorem 3.1 guarantees the existence and uniqueness of a coincidence point of $T$ and $S$. Let $x_1 \in X$. It is worth pointing out that the sequence $\{x_n\}_{n\in\mathbb{N}}$, implicitly defined as

$$Tx_{n+1} = Sx_n \quad \text{for all } n \in \mathbb{N}, \tag{5}$$

is well-defined, since $S(X) \subseteq T(X)$. Furthermore, from the injectiveness of $T$, there exists $T^{-1} : T(X) \to X$ and, therefore, the sequence $\{x_n\}_{n\in\mathbb{N}}$ can be explicitly defined by $x_{n+1} = T^{-1}(Sx_n)$ for all $n \in \mathbb{N}$.

If there exists $n_0 \in \mathbb{N}$ such that $Sx_{n_0} = Sx_{n_0+1}$, then by (5) $x_{n_0+1}$ is a coincidence point of $T$ and $S$. But, in this case, we have that $Tx_{n_0+2} = Sx_{n_0+1} = Tx_{n_0+1}$, which implies $x_{n_0+2} = x_{n_0+1}$ because $T$ is injective. Again applying (5), we deduce that $Tx_{n_0+3} = Sx_{n_0+2} = Tx_{n_0+1}$. Bearing in mind the injectiveness of $T$, we get $x_{n_0+3} = x_{n_0+1}$. Hence, $\{x_n\}_{n>n_0}$ is a constant sequence.

Thus, we can assume that $Sx_n \neq Sx_{n+1}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we define $\gamma_n := \rho(Sx_n, Sx_{n+1})$. Thus, $\gamma_n > 0$ for all $n \in \mathbb{N}$. Moreover, from (3) and (5), $\tau + f(\gamma_{n+1}) \leq f(\gamma_n)$ for all $n \in \mathbb{N}$. By Lemma 3.1, the series $\sum_{i=0}^{\infty} \gamma_i$ is convergent. Then, $\{Sx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, since for $m \geq n$,

$$\rho(Sx_m, Sx_n) \leq \gamma_{m-1} + \gamma_{m-2} + \cdots + \gamma_n < \sum_{i=n}^{\infty} \gamma_i.$$

Since $T(X)$ is complete, there exists $q \in T(X)$ such that $Sx_n \to q$ as $n \to \infty$. By (5), we also deduce that $Tx_n \to q$ as $n \to \infty$. Since $q \in T(X)$, there exists $p \in X$ such that $q = Tp$. Let us see that $Tp = Sp$.

Notice that there exists $n_1 \in \mathbb{N}$ such that $Sx_n \neq Sp$ for all $n \geq n_1$. Otherwise there exists a subsequence $\{Sx_{n_k}\}_{n_k\in\mathbb{N}}$ such that $Sx_{n_k} = Sp$ for all $n_k \in \mathbb{N}$. In this case, $Sp = Tp$ since $Sx_n \to Tp$ as $n \to \infty$.

Therefore, we can assume that $Sx_n \neq Sp$ for all $n \geq n_1$. By the contractive condition (3), for each $n \geq n_1$,

$$\tau + f(\rho(Sx_n, Sp)) \leq f(\rho(Tx_n, Tp)).$$

Since $\tau > 0$ and $f$ is strictly increasing, we have that $\rho(Sx_n, Sp) < \rho(Tx_n, Tp)$ for all $n \geq n_1$. Taking limits and bearing in mind that $Tx_n \to Tp$ as $n \to \infty$, we infer that $Sx_n \to Sp$ as $n \to \infty$. Then, $Tp = Sp$. □

We now state the convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ to the unique coincidence point of $T$ and $S$.

**Theorem 3.3.** *Let $(X, d)$ and $(Y, \rho)$ be two metric spaces, with $Y$ being complete. Suppose that $T, S : X \to Y$ satisfy the three conditions of Theorem 3.1. If $T$ is injective and $T^{-1}$ is continuous, then the sequence $\{x_n\}_{n\in\mathbb{N}}$, defined by $x_{n+1} = T^{-1}Sx_n$ for each $n \in \mathbb{N}$, converges to the unique coincidence point of $T$ and $S$.*

*Proof.* Let $p$ be the unique coincidence point of $T$ and $S$, whose existence and uniqueness is guaranteed by Theorem 3.1. Fix $x_1 \in X$. By Theorem 3.2, we know that $\{Tx_n\}_{n\in\mathbb{N}}$ and $\{Sx_n\}_{n\in\mathbb{N}}$ converge to $Tp = Sp$. From the continuity of $T^{-1}$ we conclude that

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T^{-1}Sx_n = T^{-1}(Tp) = p.$$

□

Notice that it is not easy to check the continuity of $T^{-1}$. However, one can give some metric type condition for $T$ which implies the continuity of $T^{-1}$. In order to do this, we denote by $\mathcal{G}$ the set of functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any sequence $\{t_n\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty} g(t_n) = 0$ implies $\lim_{n\to\infty} t_n = 0$. On one hand, it is easily seen that if $g \in \mathcal{G}$ then $g(t) > 0$ for all $t > 0$. On the other hand, $\mathcal{G}$ contains a large number of functions, because $\mathcal{G}$ contains the set of all monotone nondecreasing real functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(t) = 0$ if and only if $t = 0$, see [10, Lemma 2.2].

**Corollary 3.2.** *Let $(X, d)$ and $(Y, \rho)$ be two metric spaces, with $Y$ being complete. Suppose that $T, S : X \to Y$ satisfy the three conditions of Theorem 3.1. If there exists $g \in \mathcal{G}$ such that*

$$g\big(d(x, y)\big) \leq \rho(Tx, Ty), \quad \textit{for all } x, y \in X, \tag{6}$$

*then the sequence $\{x_n\}_{n\in\mathbb{N}}$, defined by $x_{n+1} = T^{-1}Sx_n$ for each $n \in \mathbb{N}$, converges to the unique coincidence point of $T$ and $S$.*

*Proof.* It is sufficient to prove that $T$ is one to one and $T^{-1}$ is continuous. Notice first that (6) implies that $T$ is one to one. Indeed, if $Tx = Ty$ then $g(d(x, y)) = 0$ which implies that $d(x, y) = 0$, since $g \in \mathcal{G}$. Then, $T^{-1} : T(X) \to X$ is well-defined. We now see that $T^{-1}$ is continuous. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq T(X)$ be a sequence converging to $u \in T(X)$. From (6), we have $g\big(d(T^{-1}v, T^{-1}w)\big) \leq \rho(v, w)$ for all $v, w \in T(X)$. Then,

$$0 \leq \lim_{n\to\infty} g\big(d(T^{-1}u_n, T^{-1}u)\big) \leq \lim_{n\to\infty} \rho(u_n, u) = 0.$$

Since $g \in \mathcal{G}$, we deduce that $T^{-1}u_n \to T^{-1}u$ as $n \to \infty$. □

**Remark 3.1.** *It is worth pointing out that the continuity of $T^{-1}$ does not imply that (6) holds: Just take $T : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $Tx = \sqrt{x}$.*

**Remark 3.2.** *Since the identity mapping is weakly compatible with respect to any mapping, from Corollary 3.2, we recapture Theorem 2.1.*

**Corollary 3.3.** *Let $(X.d)$ and $(Y, \rho)$ be two metric spaces, with $Y$ being complete. Assume that $T, S : X \to Y$ are two mappings satisfying the conditions of Corollary 3.1 and, in addition, that there exists $g \in \mathcal{G}$ such that $T : X \to Y$ satisfies inequality (Equation 6). Then the sequence $\{x_n\}$, defined by $x_{n+1} = T^{-1}(Sx_n)$, converges to the unique coincidence point of $T$ and $S$.*

*Proof.* The proof of Corollary 3.1 shows that $T$ and $S$ satisfy the hypotheses of Corollary 3.2 with $f(t) = -\frac{1}{\sqrt{t}}$ and therefore we obtain the result. □

### 3.3. Rate of Convergence

The idea given in Kohlenbach [13] allows us to introduce the concept of modulus of uniqueness for the coincidence problem as follows.

**Definition 3.1.** *Let $(X, d)$ and $(Y, \rho)$ be two metric spaces and let $T, S : X \to Y$ be two mappings. A function $\psi : (0, \infty) \to (0, \infty)$ is said to be a modulus of uniqueness for the coincidence problem defined by $T$ and $S$ if, for any $\varepsilon > 0$, $\max\{\rho(Tx, Sx), \rho(Ty, Sy)\} < \psi(\varepsilon)$ implies that $d(x, y) < \varepsilon$.*

**Theorem 3.4.** *Let $(X, d)$ and $(Y, \rho)$ be two metric spaces. Suppose that $T, S : X \to Y$ satisfy the three conditions of Theorem 3.1 and also that there exists an increasing function $g : (0, +\infty) \to (0, +\infty)$ such that*

$$g\big(d(x, y)\big) \leq \rho(Tx, Ty), \quad \textit{for all } x, y \in X. \tag{7}$$

*If the function $\beta : (0, +\infty) \to (0, +\infty)$ defined by $\beta(t) := t - f^{-1}\big(f(t) - \tau\big)$ is increasing then $\psi := \frac{1}{2}\,\beta \circ g$ is a modulus of uniqueness for the coincidence problem defined by $T$ and $S$.*

*Proof.* Let $\varepsilon > 0$ and $x, y \in X$ such that $\max\{\rho(Tx, Sx), \rho(Ty, Sy)\} < \psi(\varepsilon)$. Notice that

$$\rho(Tx, Ty) \leq \rho(Tx, Sx) + \rho(Sx, Sy) + \rho(Sy, Ty) < 2\psi(\varepsilon) + f^{-1}\big(f(\rho(Tx, Ty)) - \tau\big).$$

Then, $\beta\big(\rho(Tx, Ty)\big) < 2\psi(\varepsilon) = \beta\big(g(\varepsilon)\big)$. Since $\beta$ is increasing, we get $\rho(Tx, Ty) < g(\varepsilon)$. From (7), we deduce that $d(x, y) < \varepsilon$ because $g$ is increasing. □

**Remark 3.3.** *As a direct consequence of the above theorem, we can get a new result on generalized Ulam-Hyers stability of the coincidence problem* (1).

Another consequence of Theorem 3.4 is the following result that states a rate of convergence for Picard-Juncgk's iteration process.

**Theorem 3.5.** *Under the hypotheses of Theorem 3.4. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence defined by $x_{n+1} = T^{-1}Sx_n$ for each $n \in \mathbb{N}$. Let $p \in X$ be some coincidence point of $T$ and $S$. Then, for all $n \geq \Phi(\varepsilon)$, we have that $d(x_n, p) < \varepsilon$, where $\Phi : (0, +\infty) \to \mathbb{N}$ is given as*

$$\Phi(\varepsilon) := \begin{cases} \left\lfloor \dfrac{f(\rho(Sx_1, Tx_1)) - f(\psi(\varepsilon))}{\tau} \right\rfloor + 2 & \textit{if } \psi(\varepsilon) \leq \rho(Sx_1, Tx_1), \\ 1 & \textit{if } \rho(Sx_1, Tx_1) < \psi(\varepsilon). \end{cases}$$

*Proof.* Fix $\varepsilon > 0$. By Theorem 3.4, if we prove that $\rho(Tx_n, Sx_n) < \psi(\varepsilon)$ for all $n \geq \Phi(\varepsilon)$, then we are done, since in this case it is enough to take $x = x_n$ and $y = p$.

Let us prove that $\rho(Tx_n, Sx_n) < \psi(\varepsilon)$ for all $n \geq \Phi(\varepsilon)$, i.e., $\rho(Sx_{n-1}, Sx_n) < \psi(\varepsilon)$ for all $n \geq \Phi(\varepsilon)$. From the proof of Theorem 3.2 we know that the sequence $\{\gamma_n\}_{n\in\mathbb{N}}$, defined by $\gamma_n := \rho(Sx_n, Sx_{n+1})$, satisfies

$$\tau + f(\gamma_{n+1}) \leq f(\gamma_n) \qquad \text{for all } n \in \mathbb{N}. \tag{8}$$

Since $f$ is increasing and $\tau > 0$, we have that $\{\gamma_n\}_{n\in\mathbb{N}}$ is strictly decreasing.

If $\gamma_1 := \rho(Sx_1, Tx_1) < \psi(\varepsilon)$, then $\gamma_n < \psi(\varepsilon)$ for all $n \geq 1 = \Phi(\varepsilon)$. Thus, we can assume that $\psi(\varepsilon) \leq \gamma_1$.

We claim that $\gamma_{\Phi(\varepsilon)} < \psi(\varepsilon)$. By contradiction, suppose that $\psi(\varepsilon) \leq \gamma_{\Phi(\varepsilon)}$. Using (8), we obtain that $\big(\Phi(\varepsilon) - 1\big)\,\tau + f(\gamma_{\Phi(\varepsilon)}) \leq f(\gamma_1)$. Bearing in mind that $f$ is increasing, we deduce that $\big(\Phi(\varepsilon) - 1\big)\,\tau + f\big(\psi(\varepsilon)\big) \leq f(\gamma_1)$, which contradicts the definition of $\Phi(\varepsilon)$. Therefore, $\gamma_{\Phi(\varepsilon)} < \psi(\varepsilon)$. Since $\{\gamma_n\}_{n\in\mathbb{N}}$ is decreasing, we conclude that $\gamma_n < \psi(\varepsilon)$ for all $n \geq \Phi(\varepsilon)$. □

**Corollary 3.4.** *Let $(X, d)$ and $(Y, \rho)$ be two metric spaces. If $T, S : X \to Y$ satisfy the condition of Corollary 3.3, then the function $\psi(\varepsilon) = \frac{1}{2}(\beta \circ g)(\varepsilon)$, where $\beta(t) = t - \frac{1}{(\tau + \frac{1}{\sqrt{t}})^2}$, is a modulus of uniqueness for the coincidence problem defined by $T$ and $S$.*

*Proof.* In this case, the proof of Corollary 3.1 shows that $f(t) = -\frac{1}{\sqrt{t}}$, and then it is clear that $f^{-1}(t) = \frac{1}{t^2}$. The above facts imply that $\beta(t) = t - \frac{1}{(\tau+\frac{1}{\sqrt{t}})^2}$ and then its derivative is $\beta'(t) = 1 - \frac{1}{(1+\tau\sqrt{t})^3} \geq 0$, which says that $\beta$ is an increasing function. Finally, by Theorem 3.4, we infer that $\psi(\epsilon) = \frac{1}{2}(\beta \circ g)(\epsilon)$ is a modulus of uniqueness. □

## 4. An Application to Differential Equations

We consider the following problem associated to a general differential equation of second order with homogeneous Dirichlet condition:

$$(P)\begin{cases} u''(t) = G\big(t, u(t), u'(t)\big), & \text{for } t \in [a,b] \\ u(a) = 0, \quad u(b) = 0, \end{cases}$$

where $G\colon [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is certain known function satisfying the following two general conditions:

$(H_1)$ $G$ is continuous in $[a,b]\times\mathbb{R}\times\mathbb{R}$;
$(H_2)$ there exist $\tau, \mu > 0$ and $f \in \mathcal{F}_k$, for some $k \in (0,1)$, such that

$$\left|G(t,x_1,x_2) - G(t,y_1,y_2)\right| \leq f^{-1}\Big(f\big(\mu \max_{i=1,2}\alpha_i \left|x_i - y_i\right|\big) - \tau\Big)$$

for all $t \in [0,1]$, $x_i, y_i \in \mathbb{R}$, with $i = 1, 2$; where,

$$0 \leq \alpha_1 \leq \frac{8}{\mu\,(b-a)^2} \qquad \text{and} \qquad 0 \leq \alpha_2 \leq \frac{2}{\mu\,(b-a)}.$$

Let $Y = (\mathcal{C}[a,b], \|\cdot\|_\infty)$ be the Banach space of the continuous functions $u : [a,b] \to \mathbb{R}$, with its norm $\|u\|_\infty := \max\{\left|u(t)\right| : a \leq t \leq b\}$. In the linear space $\mathcal{C}^2[a,b] := \{u : [a,b] \to \mathbb{R} : u'' \in \mathcal{C}[a,b]\}$ we consider the linear subspace $X := \{u \in \mathcal{C}^2[a,b] : u(a) = u(b) = 0\}$. Notice that $X$ endowed with the norm $\|u\|_* := \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$ is a Banach space.

In order to prove the existence and uniqueness of a solution of $(P)$ in $\mathcal{C}^2[a,b]$, we need the following result attributed to Tumura [14], see [15, p. 80].

**Lemma 4.1.** *For any $u \in X$ we have that $\|u\|_\infty \leq \frac{(b-a)^2}{8}\|u''\|_\infty$ and $\|u'\|_\infty \leq \frac{(b-a)}{2}\|u''\|_\infty$. Moreover, the above inequalities are sharp, since they become equalities for the function $u(t) = (t-a)(b-t)$.*

Now we are able to state the main result of this section on the existence and uniqueness of a solution of $(P)$.

**Theorem 4.1.** *With the previous notation, suppose that: $G : [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies conditions $(H_1)$ and $(H_2)$. Then, problem (P) has a unique solution $u_s \in \mathcal{C}^2[a,b]$.*

*Proof.* We define $T, S : X \to Y$ as $Tu(t) = u''(t)$ and $Su = G\big(t, u(t), u'(t)\big)$. In order to obtain the existence and uniqueness of the solution to the problem $(P)$, we will see that $T$ and $S$ satisfy the conditions of Theorem 3.1. Notice that $T$ is onto. Indeed, given $w \in Y$ it is enough to consider

$$u(t) := \int_a^t v(s)\,ds - \frac{t-a}{b-a}\int_a^b v(s)\,ds, \quad \text{where } v(s) := \int_a^s w(r)\,dr,$$

since in this case $u \in X$ and $Tu = w$. Thus, assumptions $(i)$ and $(ii)$ in Theorem 3.1 hold. Let us prove that $T$ and $S$ satisfy $(iii)$. Assume that $u, v \in X$ with $\|u - v\|_\infty \neq 0$. Then, there exists at least one $t \in [a,b]$ such that $u(t) \neq v(t)$. Hence, by $(H_2)$,

$$\begin{aligned}\left|Su(t) - Sv(t)\right| &= \left|G\big(t, u(t), u'(t)\big) - G\big(t, v(t), v'(t)\big)\right| \\ &\leq f^{-1}\Big(f\big(\mu \max\{\alpha_1\left|u(t) - v(t)\right|, \\ &\quad \alpha_2\left|u'(t) - v'(t)\right|\}\big) - \tau\Big) \\ &\leq f^{-1}\Big(f\big(\mu \max\{\alpha_1\left\|u(t) - v(t)\right\|_\infty, \\ &\quad \alpha_2\left\|u'(t) - v'(t)\right\|_\infty\}\big) - \tau\Big) \\ &\leq f^{-1}\Big(f\big(\left\|u'' - v''\right\|_\infty\big) - \tau\Big),\end{aligned}$$

the last inequality is obtained from Lemma 4.1 and because $f$ is increasing. Thus, $\|Su - Sv\|_\infty \leq f^{-1}\Big(f\big(\|Tu - Tv\|_\infty\big) - \tau\Big)$, that is, $T$ and $S$ satisfy $(iii)$. From Theorem 3.1, $T$ and $S$ have a unique coincidence point in $X$, i.e., problem $(P)$ has a unique solution $u_s \in \mathcal{C}^2[a,b]$. □

**Remark 4.1.** *Under the conditions of Theorem 4.1, applying Lemma 4.1 we obtain that $\|u\|_* \leq M\,\|u''\|_\infty$, where*

$$M := \max\left\{\frac{(b-a)^2}{8}, \frac{b-a}{2}, 1\right\}.$$

*Then*

*(a) If we define $g(t) = t/M$, it is clear that $T$ satisfies inequalities (Equations 6, 7). Therefore, by Corollary 3.2, we infer that for each $u_1 \in X$, the sequence $\{u_n\}_{n\in\mathbb{N}}$ defined by*

$$u_{n+1}(t) := \int_a^t \left(\int_a^s G_n(r)\,dr\right) ds - \frac{t-a}{b-a}\int_a^b \left(\int_a^s G_n(r)\,dr\right) ds, \tag{9}$$

*where $G_n(r) := G\big(r, u_n(r), u_n'(r)\big)$, converges to $u_s$,*

*(b) If the function $\beta : (0,\infty) \to \mathbb{R}$, defined by $\beta(t) := t - f^{-1}\big(f(t) - \tau\big)$, is increasing, Theorem 3.5 yields that for any $\varepsilon > 0$, $\|u_n - u_s\|_* < \varepsilon$ for all $n \geq \Phi(\varepsilon)$, where*

$$\Phi(\varepsilon) := \begin{cases} \left\lfloor \frac{f(\|u_2'' - u_1''\|_\infty) - f(\beta(g(\varepsilon))/2)}{\tau} \right\rfloor + 2, & \text{if } \beta(g(\varepsilon)) \leq 2\left\|u_2'' - u_1''\right\|_\infty, \\ 1, & \text{if } 2\left\|u_2'' - u_1''\right\|_\infty < \beta(g(\varepsilon)), \end{cases} \tag{10}$$

*which means that* $\Phi$ *given by* (10) *is a rate of convergence for* $\{u_n\}$ *to* $u_s$.

### 4.1. A Particular Case

Let $G : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that for every $t \in [0, 1]$, and for all $x, y \in \mathbb{R}$ the following inequality holds for some $\tau > 0$,

$$|G(t, x, y) - G(t, u, v)| \leq \frac{\max\{|x-u|, |y-v|\}}{(1+\tau\sqrt{\max\{|x-u|, |y-v|\}})^2}. \quad (11)$$

Let us check that $G$ satisfies condition $(H_2)$. Indeed, consider the function $f : (0, \infty) \to (-\infty, 0)$ defined by $f(t) = -\frac{1}{\sqrt{t}}$. It is clear that $f^{-1} : (-\infty, 0) \to (0, \infty)$ is given by $f^{-1}(s) = \frac{1}{s^2}$. Therefore, taking $\mu = \alpha_1 = \alpha_2 = 1$ we have:

$$\begin{aligned} &f^{-1}(f(\max\{|x-u|, |y-v|\}) - \tau)) \\ &= \frac{1}{(f(\max\{|x-u|, |y-v|\}) - \tau)^2} \\ &= \frac{\max\{|x-u|, |y-v|\}}{(1+\tau\sqrt{\max\{|x-u|, |y-v|\}})^2}, \end{aligned}$$

which means that $G$ satisfies condition $(H_2)$.

**Example 4.1.** *The second order differential equation with homogeneous Dirichlet condition*

$$\begin{cases} u''(t) = e^t + \frac{|u(t)|}{(\sqrt{2}+\sqrt{2|u(t)|})^2} + \frac{|u'(t)|}{(\sqrt{2}+\sqrt{2|u'(t)|})^2}, & \text{for } t \in [0, 1], \\ u(0) = 0, \quad u(1) = 0, \end{cases} \quad (12)$$

*has a unique classical solution.*

To see that Equation (12) has a unique classical solution it is enough to show that the conditions of Theorem 4.1 are satisfied. Since Equation (12) can be rewritten as

$$\begin{cases} u''(t) = G(t, u(t), u'(t)), & \text{for } t \in [0, 1], \\ u(0) = 0, \quad u(1) = 0, \end{cases} \quad (13)$$

where $G : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$G(t, x, y) = e^t + \frac{1}{2}\left[\frac{|x|}{(1+\sqrt{|x|})^2} + \frac{|y|}{(1+\sqrt{|y|})^2}\right],$$

we are going to prove that $G$ satisfies inequality (Equation 11). To do this, we notice first that the following elementary properties hold:

(1) the function $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(t) = \frac{t}{(1+\sqrt{t})^2}$, is increasing since $\varphi'(t) = \frac{1}{(1+\sqrt{t})^3} > 0$,
(2) $\varphi$ is concave since $\varphi''(t) = \frac{-3}{2\sqrt{t}(1+\sqrt{t})^4} < 0$,
(3) since $\varphi(0) = 0$ and $\varphi$ is concave, then it is sub-additive, that is $\varphi(t+s) \leq \varphi(t) + \varphi(s)$.

Since

$$\begin{aligned} |G(t, x, y) - G(t, u, v)| \leq &\frac{1}{2}\left|\frac{|x|}{(1+\sqrt{|x|})^2} - \frac{|u|}{(1+\sqrt{|u|})^2}\right| \\ &+ \frac{1}{2}\left|\frac{|y|}{(1+\sqrt{|y|})^2} - \frac{|v|}{(1+\sqrt{|v|})^2}\right| \end{aligned}$$

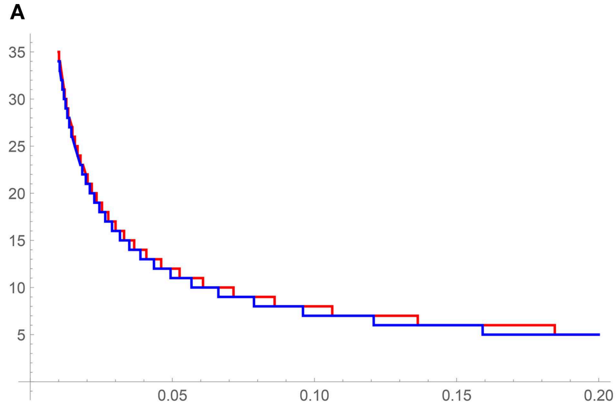
With the above three properties, the above inequality can be written as follows

$$\begin{aligned} |G(t, x, y) - G(t, u, v)| &\leq \frac{1}{2}\left|\varphi(|x|) - \varphi(|u|)\right| + \frac{1}{2}\left|\varphi(|y|) - \varphi(|u|)\right| \\ &\leq \frac{1}{2}\left|\varphi(|x| - |u|)\right| + \frac{1}{2}\left|\varphi(|y| - |v|)\right| \\ &\leq \frac{1}{2}\varphi(|x-u|) + \frac{1}{2}\varphi(|y-v|) \\ &\leq \varphi(\max\{|x-u|, |y-v|\}) \\ &= \frac{\max\{|x-u|, |y-v|\}}{(1+\sqrt{\max\{|x-u|, |y-v|\}})^2}, \end{aligned}$$

which means that the conditions of Theorem 4.1 are satisfied and therefore Equation (12) admits a unique classical solution.

Finally, let us give, by using expression (10), a rate of convergence for the iterative scheme given in

**A**




**B**

| Error $\varepsilon$ | $\Phi_0(\varepsilon)$ | $\Phi_1(\varepsilon)$ |
|---|---|---|
| $10^{-1}$ | 8 | 7 |
| $10^{-2}$ | 35 | 34 |
| $10^{-3}$ | 183 | 182 |
| $10^{-4}$ | 1008 | 1008 |
| $10^{-5}$ | 5638 | 5637 |
| $10^{-6}$ | 31647 | 31647 |

**FIGURE 1 | (A)** Red line is the ratio taking as a starting point $u_1 = 0$ and blue line is the ratio for the starting point $u_1 = t^2 - t$. **(B)** (The ratio of convergence $\Phi_0$ corresponds to $u_1 = 0$, while $\Phi_2$ corresponds to $u_1 = t^2 - t$).

Equation (9) concerning Equation (12). To apply Theorem 3.5, first we have to notice that the following facts hold:

1. $f : (0, +\infty) \to (-\infty, 0)$ is given by $f(t) = -\frac{1}{\sqrt{t}}$,
2. $f^{-1} : (-\infty, 0) \to (0, +\infty)$ is $f^{-1}(s) = \frac{1}{s^2}$.
3. $g : [0, \infty) \to [0, \infty)$ is given by $g(t) = t$,
4. $\tau = 1$,
5. $\beta(t) = t - \frac{t}{(1+\sqrt{t})^2}$,
6. $\psi(\epsilon) = \frac{1}{2}\beta(\epsilon)$.

In **Figure 1**, we use the above facts and expression (Equation 10) to compute the number of iterations that we have to do to obtain an error less than $\epsilon = 10^{-k}$ for $k = 1, \cdots, 6$ and taking as starting points $u_1 = 0$ and $u_1 = t^2 - t$ respectively.

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