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$L^1 \to L^q$ POINCARÉ INEQUALITIES FOR $0 < q < 1$ IMPLY REPRESENTATION FORMULAS

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Dedicated to Dick Wheeden on the occasion of his 60th birthday with appreciation and admiration

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ABSTRACT. Given two doubling measures μ and ν in a metric space $(\mathcal{S}, \rho)$ of homogeneous type and let $B_0 \subset \mathcal{S}$ be a given ball. It has been a well-known result by now (see [FLW], [FW], [LW1], [LW2]) that the validity of an $L^1 \to L^1$ Poincaré inequality of the following form:

$$\fint_B |f - f_B| d\nu \leq c r(B) \fint_B g d\mu$$

for all metric balls $B \subset B_0 \subset \mathcal{S}$ implies a variant of representation formula of fractional intergral type: for ν-a.e. $x \in B_0$,

$$|f(x) - f_{B_0}| \leq C \int_{B_0} g(y) \frac{\rho(x,y)}{\mu(B(x, \rho(x,y)))} \, d\mu(y) + C \frac{r(B_0)}{\mu(B_0)} \int_{B_0} g(y) \, d\mu(y).$$

One of the main results of this paper shows that an L^1 to L^q Poincaré inequality for some $0 < q < 1$, i.e.,

$$\left(\fint_B |f - f_B|^q d\nu \right)^{1/q} \leq c r(B) \fint_B g d\mu$$

for all metric balls $B \subset B_0$ will suffice to imply the above representation formula. As an immediate corollary, we can show that the weak type condition

$$\sup_{\lambda > 0} \frac{\lambda \nu \left(\{ x \in B : |f(x) - f_B| > \lambda \} \right)}{\nu(B)} \leq C r(B) \fint_B g d\mu$$

also implies the same formula.

Analogous theorems related to high order Poincaré inequalities and Sobolev spaces in metric spaces are also proved.

§1. Introduction

It is known that $L^1 \to L^1$ Poincaré inequalities are equivalent to the fractional integral estimates in general metric spaces of homogeneous type (see [FLW2], [FW], [LW1-2]). A natural question thus arises: Is the $L^1 \to L^1$ Poincaré inequality the least we need to start in order to derive such representation formulas? In this paper, we study this issue and weaken the hypothesis that an $L^1 \to L^1$ Poincaré inequality has to hold to obtain any kind of representation formulas of fractional type. More precisely, we will show that an $L^1 \to L^q$ Poincaré inequality for some $0 < q < 1$ will suffice to derive the pointwise fractional estimates. On the other hand it is well known that the following Kolmogorovs inequality holds for $0 < q < 1$, any nonnegative function g and arbitrary measurable set E with finite measure

$$\left(\frac{1}{\mu(E)} \int_E g(x)^q \, d\mu \right)^{1/q} \leq c_q \, |g|_{L^{1,\infty}(E,\mu)}. \tag{1.1}$$

See [GCRdF] p. 485 for instance. We will be using the following notation for the local average Marcinkiewicz quasi-norm

$$|g|_{L^{1,\infty}(E,\mu)} = \sup_{\lambda > 0} \lambda \frac{\mu \left(\{ x \in E : |f(x)| > \lambda \} \right)}{\mu(E)}.$$

Hence as an interesting corollary of our main result, we prove that weak type $L^{1,\infty} \to L^1$ Poincaré inequality is sufficient to imply fractional representation formulas.

More precisely, given two doubling measures μ and ν in a metric space $(\mathcal{S}, \rho)$ of homogeneous type and let $B_0 \subset \mathcal{S}$ be a given ball. It has been a well-known result by now (see [FLW2], [FW], [LW1], [LW2]) that the validity of an $L^1 \to L^1$ Poincaré inequality of the following form:

$$\fint_B |f - f_B| d\nu \le cr(B) \fint_B g\, d\mu \tag{1.2}$$

for all metric balls $B \subset B_0 \subset \mathcal{S}$ implies a variant of representation formula of fractional integral type: for ν-a.e. $x \in B_0$,

$$|f(x) - f_{B_0}| \le C \int_{B_0} g(y) \frac{\rho(x,y)}{\mu(B(x, \rho(x,y)))}\, d\mu(y) + C \frac{r(B_0)}{\mu(B_0)} \int_{B_0} g(y)\, d\mu(y).$$

As usual we use the following notation for the average of f over a ball B, $f_B = \frac{1}{\mu(B)} \int_B f\, d\mu$

Our first main result of this paper demonstrates that an L^1 to L^q Poincaré inequality for some $0 < q < 1$, i.e.,

$$\left(\fint_B |f - f_B|^q d\nu \right)^{1/q} \le cr(B) \fint_B g\, d\mu$$

for all metric balls $B \subset B_0$ will suffice to imply the above representation formula. As a by-product of this, we derive our second main result that the weak type condition

$$|f - f_B|_{L^{1,\infty}(B,\nu)} \le C\, r(B) \fint_B g\, d\mu$$

also implies the same pointwise estimates.

We note that, similar to what was first shown in [LW1] and then in [LW2], the integrals on the right hand side is on the same ball B_0, rather than on the enlarged ball (see [FLW2], [FW]). We make emphasis on the fact that the only assumption we make on the measure μ is the doubling property. Indeed, it has recently been shown in [LW2] that there is no need to require additional assumptions of reverse doubling of order $1+\epsilon$ or 1 (see [FLW2], [FW]). However, we need to add the second term on the right hand side, which is not harmful at all as far as the Poincaré type estimates concerned. If we also assume that the measure μ is of reverse doubling order 1, then this second term can be dropped (see also [FW] and [LW2]). We mention that the authors in [HK2] derived independently from [LW2] a formula without the second term without the assumption that μ is doubling, but with f_{B_0} replaced by $f_{\frac{1}{2}B_0}$. It seems that the passage from $f_{\frac{1}{2}B_0}$ to f_{B_0} would also result in the second term in the formula.

As applications, we provide weaker, but equivalent, definitions of Sobolev spaces of first order in metric spaces than those defined in [H], and further exploited in [FLW2], [FHK] and [LW2]. The implications of $L^1 \to L^q$ Poincaré inequalities of high order to representation formulas also hold and improve those in [LW2]. These also provide us with weaker, and also equivalent, definitions of high order Sobolev spaces in metric spaces defined in [LLW1].

The methods used in this paper are extensions of several techniques adapted from [FLW2], [FW], [LW1], [LW2] and [LLW1]. In particular, we will use similar ideas from [LW2]. However, our case is concerned with the situation $q < 1$, and there are some subtleties we have to overcome. Some inequalities which hold for $q \geq 1$ fail to be true for $q < 1$. Thus, we have to proceed with caution.

We remark in passing that there has been extensive research of proving $L^p \to L^q$ Poincaré inequalities, if a certain type of $L^p \to L^p$ Poincaré inequality is already known to exist in the given setting, see [SC], [HK1-2], [BM], [MSC], [GN], [BCSC], [FPW], [MP1-2], [OP]. This is the so-called self-improving property, which can be used to prove Poincaré inequalities without using representation formula. Thus, by combining with Jerison's result for Poincaré inequalities with $q = p$ for Hörmander vector fields [J], this argument will recapture the sharp Poincaré inequalities for Hörmander vector fields first proved in [L2] (for $p > 1$) and [FLW1] (for $p = 1$) by using representation formulas. Direct proofs of representation formulas for Hörmander vector fields or Grushin vector fields have been given in [F], [FL], [FSe], [L1], [FLW1], [FGW], [CDG], [LM].

Furthermore, it is shown in the papers [FPW], [MP1-2] and [OP] that the Sobolev-Poincaré inequalities are special cases of a more general theory that includes, for instance, the classical theorem of John-Nirenberg as well as the Trudinger inequality. The idea there is to replace the expression on the right hand side of (1.2) by a more general "functional" $a(B)$ and to use the Calderon-Zygmund theory, under a certain mild geometric condition on a (see [P] for a survey).

We mention that the self-improving property by assuming the initial inequality

$$\left(\fint_B |f - f_B|^q d\nu \right)^{1/q} \leq a(B)$$

for some $0 < q < 1$ and some quantity $a(B)$ to hold has also been established recently in [FLPW].

To make our paper self-contained, and for the sake of clarity of our presentation, we have decided to treat the case of first order Poincaré inequalities separately from the ones of high order. The plan of the paper is as follows. In section 2, we prove our results for the first order in general metric spaces. Section 3 contains new definitions of Sobolev spaces of first order in metric spaces. Section 4 deals with the implication of high order Poincaré inequalities to representation formulas in metric spaces and provides with new definitions of Sobolev spaces of any high order in metric spaces.

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§2 Representation formulas of first order in metric spaces

We begin with the definition of "weak Boman chain domain" defined in [LW2]. Boman chain domains in Euclidean spaces were introduced by Boman in his unpublished work [Bom] and used to prove Poincaré inequalities on such domains (see [Boj], [Ch], [IN]). Such a notion in metric space seems to be first used in [FGW] and [L2] by slightly modifying the definition in Euclidean spaces.

Definition 2.1 [LW2]. *A domain (i.e., an open connected set) Ω in $\mathcal{S}$ is said to satisfy the Boman chain condition of type σ, M, or to be a member of $\mathcal{F}(\sigma, M)$, if there exist constants $\sigma > 1$, $M > 0$, and a family $\mathcal{F}$ of metric balls $B \subset \Omega$ such that*

(1) $\Omega = \bigcup_{B \in \mathcal{F}} B$
(2) $\sum_{B \in \mathcal{F}} \chi_{\sigma B}(x) \le M\, \chi_\Omega(x)$ *for all* $x \in \mathcal{S}$
(3) *There is a "central ball" $B_0 \in \mathcal{F}$ such that for each ball $B \in \mathcal{F}$, there is a positive integer $k = k(B)$ and a chain of balls $\{B_j\}_{j=0}^k$ for which $B_k = B$ and each $B_j \bigcap B_{j+1}$ contains a ball D_j with $B_j \bigcup B_{j+1} \subset M D_j$.*
(4) $B \subset M B_j$ *for all* $j = 0, \ldots, k(B)$

If we replace the hypothesis that $\sigma > 1$ by $\sigma = 1$, we say that Ω satisfies the **weak Boman chain condition***.*

We do not know if this weaker definition can actually be equivalent to the "Boman chain domain", where τ has to be taken bigger than 1. It will also be interesting to know if the class of weaker Boman chain domains is strictly larger than the Boman chain domains. We mention that Boman domain is equivalent to John domain as shown independently in [BKL] and [GN].

We now state the following four hypotheses that are modifications of those given in [LW1] and [LW2]. The crucial difference is that we have replaced (H1) there by our $L^1 \to L^q$ Poincaré inequality for some $0 < q < 1$, rather than the $L^1 \to L^1$ inequality. We note that not all four hypotheses are needed in every theorem. As always, $(\mathcal{S}, \rho)$ is a metric space. Let μ and ν be doubling measures with respect to metric balls, and let Ω be a domain in $\mathcal{S}$.

(H1) f is a function satisfying L^1 to L^q Poincaré inequality for some $0 < q < 1$, i.e.,

$$\left(\fint_B |f - f_B|^q d\nu \right)^{1/q} \le c r(B) \fint_B g d\mu$$

for metric balls $B \subset \Omega$.

(H2) The measure μ in (H1) satisfies a reverse doubling condition of order 1, i.e., there is a constant $C > 0$ such that if B and $\tilde{B}$ are balls with centers in Ω and with $B \subset \tilde{B}$, then

$$\mu(\tilde{B}) \ge C \left(\frac{r(\tilde{B})}{r(B)} \right) \mu(B).$$

(H3) $(\mathcal{S}, \rho)$ has the *segment (or geodesic) property* that for each pair of points $x, y \in \mathcal{S}$, there is a continuous curve γ connecting x and y such that $\rho(\gamma(t), \gamma(s)) = |t - s|$.

(H4) Ω is a weak Boman chain domain.

The main results of this section are improvements of those in [LW2] where $L^1 \to L^1$ Poincaré inequalities have to be assumed.

Remark. Since weak L^1 implies locally strong L^1 for $0 < q < 1$ as mentioned in the introduction, thus our theorems below still remain to be true if we replace (H1) above by

(WH1) f is a function satisfying weak L^1 to L^1 Poincaré inequality, i.e.,

$$|f - f_B|_{L^{1,\infty}(B,\nu)} \leq C\, r(B) \fint_B g\, d\mu$$

for metric balls $B \subset \Omega$.

From Kolmogorov's inequality (1.1) we see that (WH1) implies (H1) for all $0 < q < 1$.

Theorem 2.2. *Let ν, μ be doubling measures on a metric space $(\mathcal{S}, \rho)$. Let B_0 be a ball and suppose that (H1) and (H3) hold with $\Omega = B_0$ and $f_B = \fint_B f(y) d\nu(y)$. Then for ν-a.e. $x \in B_0$,*

$$\begin{aligned}&|f(x) - f_{B_0}|\\ \leq& C \int_{B_0} g(y)\, \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\, d\mu(y) + C \frac{r(B_0)}{\mu(B_0)} \int_{B_0} g(y)\, d\mu(y),\end{aligned}$$

where C depends only on ν, μ and the constants in (H1).

If in addition we impose the reverse doubling condition (H2) in Theorem 2.2, then we have

Theorem 2.3. *Let ν, μ be doubling measures on a metric space $(\mathcal{S}, \rho)$. Let B_0 be a ball and suppose that (H1), (H2) and (H3) hold with $\Omega = B_0$. Then for ν-a.e. $x \in B_0$,*

$$|f(x) - f_{B_0}| \leq C \int_{B_0} g(y) \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\, d\mu(y),$$

where C depends only on ν, μ and the constants in (H1), (H2).

The next theorem is a generalization of Theorem 2.3 to any weak Boman chain domain Ω.

Theorem 2.4. *Suppose that ν and μ are doubling measures on a metric space $(\mathcal{S}, \rho)$ and that hypotheses (H1)–(H4) hold for a domain $\Omega \subset \mathcal{S}$. Then for ν-a.e. $x \in \Omega$,*

$$|f(x) - f_{B_0}| \leq C \int_{\Omega} g(y) \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\, d\mu(y),$$

where B_0 is the central ball in Ω, f, g, ν and μ in (H1), and C depends only on ν, μ and the constants in (H1), (H2) and (H4).

As is well-known, under the segment hypothesis (H3), any metric ball is a Boman chain domain (see [FGW], [L2]), and thus Theorem 2.3 is a special case of Theorem 2.4 .

The proof of Theorem 2.2 relies on the construction of the following chain of metric balls given in [LW2], assuming the segment hypothesis (H3). A similar construction was given in [FW], but the following one enables us to select all balls in the chain lying inside entirely the given ball B_0. The chain of balls will allow us to prove the representation formulas on the same ball on both sides directly (see [LW2]), rather than using the formula on the enlarged ball to get the corresponding one on the same ball (see [LW1]). A somewhat different chain of finite length is given independently in [HK2].

Theorem 2.5 [LW2]. *Let $(\mathcal{S}, \rho)$ be a metric space in which the segment property (H3) holds. Let B_0 be a ball in $\mathcal{S}$. Given $x \in B_0$, there exists a chain $\{B_k\}_{k\geq 1}$ of balls with the following properties:*

(1) *$B_k \subset B_0$ and $\rho(B_k, x) \to 0$ as $k \to \infty$.*
(2) *$r(B_1) \approx r(B_0)$ and $r(B_k) \to 0$ as $k \to \infty$.*
(3) *If $y \in B_k$, then $\rho(y, x) \approx r(B_k)$.*
(4) *$B_k \bigcap B_{k-1}$ contains a ball S_k with $r(S_k) \approx r(B_k) \approx r(B_{k+1}) \approx 2^{-k} r(B_0)$.*
(5) *If $j < k$, then $B_k \subset cB_j$.*
(6) *$\{B_k\}_{k\geq 1}$ has bounded overlaps, i.e., $\sum_k \chi_{B_k}(y) \leq c$ for all y.*

The constants of equivalence in (2), (3) and (4) and the constants c in (5) and (6) are independent of x, k, j and B_0, but the chain $\{B_k\}$ depends on x.

The following remark is in order. The argument given here is similar to the proof of Theorem A in [LW2]. However, since our case is for $q < 1$ and then the Minkowski's inequality fails. Thus, our situation becomes more delicate than the case for $q = 1$. In particular, we will use the inequality

$$\left(\int_E (f+g)^q d\nu\right)^{1/q} \leq 2^q \left[\left(\int_E f^q\right)^{1/q} + \left(\int_E g^q\right)^{1/q}\right]. \tag{2.6}$$

However, this inequality does not hold when we have infinitely many terms in the integrand unlike the case for $q \geq 1$, namely, we do not have

$$\left(\int_E \left(\sum_{i=1}^{\infty} f_i\right)^q d\nu\right)^{1/q} \leq C(q) \sum_{i=1}^{\infty} \left(\int_E f_i^q d\nu\right)^{1/q}.$$

Therefore we have to proceed with caution, see the estimate for I_2 below.

Proof of Theorem 2.2. We will use Theorem 2.5 to prove Theorem 2.2. Let B_0 be a ball in $\mathcal{S}$ and suppose that (H1) and the segment property (H3) hold for B_0. Given $x \in B_0$, let $\{B_k\}_{k\geq 1}$ be a sequence of balls with the properties guaranteed by Theorem 2.5. Then

$$|f(x) - f_{B_0}| \leq |f(x) - f_{B_1}| + |f_{B_1} - f_{B_0}|. \tag{2.7}$$

For the second term on the right in (2.7), we get for $\nu-$a.e. $x \in B_0$ that

$$\begin{aligned}
|f_{B_1} - f_{B_0}| &= \left(\fint_{B_1} |f_{B_1} - f_{B_0}|^q d\nu\right)^{1/q} \\
&\leq C\left(\fint_{B_1} |f(y) - f_{B_1}|^q\, d\nu(y)\right)^{1/q} + C\left(\fint_{B_1} |f(y) - f_{B_0}|^q\, d\nu(y)\right)^{1/q} \\
&\leq C\left(\fint_{B_1} |f(y) - f_{B_1}|^q\, d\nu(y)\right)^{1/q} + C\left(\fint_{B_0} |f(y) - f_{B_0}|^q\, d\nu(y)\right)^{1/q} \\
&\quad \text{since } \nu(B_1) \approx \nu(B_0) \text{ and } \nu \text{ is doubling} \\
&\leq C\frac{r(B_1)}{\mu(B_1)} \int_{B_1} g\, d\mu + C\frac{r(B_0)}{\mu(B_0)} \int_{B_0} g\, d\mu \,\text{by the Poincaré inequality (H1)} \\
&\leq C\frac{r(B_0)}{\mu(B_0)} \int_{B_0} g\, d\mu
\end{aligned}$$

since $B_1 \subset B_0$, $r(B_1) \approx r(B_0)$ and $\mu(B_1) \approx \mu(B_0)$.

Assuming as we may that x is a Lebesgue point for both $|f - f_{B_1}|^q$ and g with respect to ν and using properties (1)–(3) from Theorem 2.5 and the inequality (2.6), we have for the first term on the right in (2.7) that

$$\begin{aligned}|f(x) - f_{B_1}| =& \lim_{k\to\infty} \left(\fint_{B_k} |f(y) - f_{B_1}|^q d\nu(y) \right)^{1/q} \\ \leq& 2^q \limsup_{k\to\infty} \left(\fint_{B_k} |f(y) - f_{B_k}|^q d\nu(y) \right)^{1/q} \\ &+ 2^q \limsup_{k\to\infty} \left(\fint_{B_k} |f_{B_k} - f_{B_1}|^q d\nu(y) \right)^{1/q} \\ =& I_1 + I_2,\end{aligned}$$

where I_1 and I_2 are defined by the last equality.

It is easy to show that $I_1 = 0$ for every Lebesgue point x of g. This can be seen by the Poincaré inequality (H1):

$$\begin{aligned}I_1 =& \limsup_{k\to\infty} \left(\fint_{B_k} |f(y) - f_{B_k}|^q \, d\nu(y) \right)^{1/q} \\ \leq& C \limsup_{k\to\infty} \frac{r(B_k)}{\mu(B_k)} \int_{B_k} g(y) \, d\mu(y) = 0 \cdot g(x) = 0.\end{aligned}$$

We now estimate I_2. By observing that $f_{B_{j+1}} - f_{B_j}$ is a constant function, we have

$$\begin{aligned}I_2 =& \limsup_{k\to\infty} |f_{B_k} - f_{B_1}| \\ \leq& \limsup_{k\to\infty} \sum_{j=1}^{k-1} |f_{B_{j+1}} - f_{B_j}| \\ =& \limsup_{k\to\infty} \sum_{j=1}^{k-1} \left(\fint_{S_j} |f_{B_{j+1}} - f_{B_j}|^q \, d\nu \right)^{1/q} \\ \leq& 2^q \sum_{j=1}^{\infty} \left(\fint_{S_j} |f_{B_{j+1}} - f|^q \, d\nu \right)^{1/q} + 2^q \sum_{j=1}^{\infty} \left(\fint_{S_j} |f_{B_j} - f|^q \, d\nu \right)^{1/q} \\ \leq& 2^q \sum_{j=1}^{\infty} \left(\fint_{B_{j+1}} |f_{B_{j+1}} - f|^q \, d\nu \right)^{1/q} + 2^q \sum_{j=1}^{\infty} \left(\fint_{B_j} |f_{B_j} - f|^q \, d\nu \right)^{1/q}\end{aligned}$$

since $S_j \subset B_j \cap B_{j+1}$ and $\nu(S_j) \approx \nu(B_j) \approx \nu(B_{j+1})$ by Theorem 2.5. Combining estimates and applying (H1) to the terms of each of the last two sums, we obtain

$$I_2 \leq C \sum_{j=1}^{\infty} r(B_j) \fint_{B_j} g(y) \, d\mu(y).$$

Now, as arguing in [LW2], if $y \in B_j$, then

$$\frac{r(B_j)}{\mu(B_j)} \approx \frac{\rho(x,y)}{\mu(B(y,\rho(x,y)))} \approx \frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}$$

by part (3) of Theorem 2.5 and the fact that μ is a doubling measure. Thus we obtain

$$\begin{aligned} I_2 \leq & C\sum_{j=1}^{\infty} \int_{B_j} g(y)\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\,d\mu(y) \\ \leq & C\int_{B_0} g(y)\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\,d\mu(y) \end{aligned}$$

by properties (6) and (1) of Theorem 2.5. This completes the proof of Theorem 2.2 by combining estimates for I_1 and I_2.

Proof of Theorem 2.3. By observing that if $x, y \in B_0$, then $\rho(x,y) \leq 2r(B_0)$ and consequently by (H2), we have

$$\frac{r(B_0)}{\mu(B_0)} \leq C\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))} \text{ if } x, y \in B_0.$$

Thus, the second term on the right in the conclusion of Theorem 2.2 is bounded by the first term.

Proof of Theorem 2.4. Let $x \in \Omega$. By the definition of weak Boman chain domain, we may select B^* with $x \in B^*$ and a chain $\{B_j\}_{j=0}^k$ connecting $B^* = B_k$ to the central ball B_0. We have

$$|f(x) - f_{B_0}| \leq |f(x) - f_{B^*}| + |f_{B^*} - f_{B_0}|. \tag{2.8}$$

For the first term on the right side of (2.8), we have by Theorem B that

$$|f(x) - f_{B^*}| \leq C\int_{B^*} g(y)\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\,d\mu(y).$$

This holds for $\nu-$a.e. point of B^*, and we may assume it holds for our fixed x by initially excluding from Ω the set of measure zero formed by taking the union of the exceptional sets of measure zero in each Boman ball. Since $B^* \subset \Omega$, we obtain the desired estimate

$$|f(x) - f_{B^*}| \leq C\int_{\Omega} g(y)\frac{\rho(x,y)}{\mu(B(x,\rho(x,y)))}\,d\mu(y).$$

Thus we only need to estimate $|f_{B^*} - f_{B_0}|$. By using the chain $\{B_j\}$ connecting B_0 and $B_k = B^*$, we have

$$|f_{B^*} - f_{B_0}| \leq \sum_{j=1}^{k} |f_{B_j} - f_{B_{j-1}}|.$$

If D_j is a ball with $D_j \subset B_j \cap B_{j-1} \subset MB_j$ and $r(D_j) \approx r(B_j) \approx r(B_{j-1})$, then the last sum is

$$
\begin{aligned}
&= C \sum_{j=1}^{k} \left(\fint_{D_j} |f_{B_j} - f_{B_{j-1}}|^q d\nu \right)^{1/q} \\
&\le C \sum_{j=1}^{k} \left(\fint_{D_j} |f_{B_j} - f(y)|^q d\nu(y) \right)^{1/q} + C \sum_{j=1}^{k} \left(\fint_{D_j} |f_{B_{j-1}} - f(y)|^q d\nu(y) \right)^{1/q} \\
&\le C \sum_{j=1}^{k} \left(\fint_{B_j} |f_{B_j} - f(y)|^q d\nu(y) \right)^{1/q} + C \sum_{j=1}^{k} \left(\fint_{B_{j-1}} |f_{B_{j-1}} - f(y)|^q d\nu(y) \right)^{1/q} \\
&\le C \sum_{j=0}^{k} \left(\fint_{B_j} |f_{B_j} - f(y)|^q d\nu(y) \right)^{1/q}
\end{aligned}
$$

By Poincaré's inequality (H1),

$$
\begin{aligned}
&\le C \sum_{j=0}^{k} \frac{r(B_j)}{\mu(B_j)} \int_{B_j} g(y)\, d\mu(y) \\
&= C \int_{\Omega} \left\{ \sum_{j=0}^{k} \frac{r(B_j)}{\mu(B_j)} \chi_{B_j}(y) \right\} g(y)\, d\mu(y).
\end{aligned}
$$

The proof will be complete if we show that the sum above in curly brackets is bounded by a fixed multiple of $\rho(x,y)/\mu(B(x,\rho(x,y)))$ for each $y \in \Omega$. Indeed, this is the case as shown in [LW2].

§3 Definitions of Sobolev spaces of first order in metric spaces

We first recall the first order Sobolev spaces in metric space $(\mathcal{S}, \rho, d\mu)$ defined in [H], and further developed in [FLW2] and [FHK].

Let $\Omega \subset \mathcal{S}$. In [H], the first order Sobolev space $W^{1,p}(\Omega, d\mu)$ $(p > 1)$ in metric spaces is defined as follows.

Definition 3.1. *The first order Sobolev space $W^{1,p}(\Omega, d\mu)$ $(p > 1)$ in metric spaces is defined to be the collection of $f \in L^p(\Omega, d\mu)$ such that there is a nonnegative function $g \in L^p(\Omega, d\mu)$ satisfying for all $x, y \in \Omega$*

$$|f(x) - f(y)| \le \rho(x,y)\,(g(x) + g(y))\,.$$

Subsequently, an alternate definition by using Poincaré inequality is found in [FLW2]. The definition given in [FLW2] is as follows: $f \in L^p(\Omega, d\mu)$ is said to be in $W^{1,p}(\Omega, d\mu)$, if there exists some $g \in L^p(\Omega, d\mu)$ such that for all balls $B \subset \Omega$

$$\frac{1}{\mu(B)} \int_B |f - f_B|\, d\mu \le C r(B) \left(\frac{1}{\mu(B)} \int_B g\, d\mu \right),$$

with C independent of B.

It was also assumed in [FLW2] that μ has to satisfy reverse doubling of order $1+\epsilon$, and it was improved in [FW] that μ being reverse doubling of order 1 is sufficient.

This definition by using Poincaré inequality to define Sobolev spaces in metric spaces has been subsequently improved in [FHK] and [LW2] by dropping any additional assumption on the reverse doubing for the measure μ. By the result in [FHK], and theorem A^* in [LW2], that μ is only doubling is sufficient to define the first order Sobolev spaces by using the Poincaré inequality. We state this as

Definition 3.2. *(see [FHK], [LW2]) Let $p > 1$ be given and μ is doubling. $f \in L^p(\Omega, d\mu)$ is said to be in $W^{1,p}(\Omega, d\mu)$, if there exists some $g \in L^p(\Omega, d\mu)$ such that for all balls $B \subset \Omega$*

$$\frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu \leq Cr(B) \frac{1}{\mu(B)} \int_B g \, d\mu,$$

with C independent of B.

One of the advantage of using the Poincaré inequality to define Sobolev spaces in metric spaces is that it enables us to define Sobolev spaces of any higher order by using Poincaré inequality of high order. This is done in [LLW1].

By using the results obtained in Section 2 of this paper, we can give a seemingly weaker, but actually equivalent, definition to all known ones of first order Sobolev spaces in metric spaces.

Definition 3.3. *Let $p > 1$ be given and μ is doubling. $f \in L^p(\Omega, d\mu)$ is said to be in $W^{1,p}(\Omega, d\mu)$, if there exist some $0 < q < 1$ and some $g \in L^p(\Omega, d\mu)$ such that for all balls $B \subset \Omega$*

$$\left(\frac{1}{\mu(B)} \int_B |f - f_B|^q \, d\mu \right)^{1/q} \leq Cr(B) \left(\frac{1}{\mu(B)} \int_B g \, d\mu \right),$$

with C independent of B.

Definition 3.4. *Let $p > 1$ be given and μ is doubling. $f \in L^p(\Omega, d\mu)$ is said to be in $W^{1,p}(\Omega, d\mu)$, if there exists some $g \in L^p(\Omega, d\mu)$ such that for all balls $B \subset \Omega$*

$$|f - f_B|_{L^{1,\infty}(B,\nu} \leq C\, r(B) \left(\frac{1}{\mu(B)} \int_B g \, d\mu \right),$$

with C independent of B.

Remark. As shown in [FHK] and [LLW1], we can replace the L^1 average of g on the right hand side by any L^r average of g for $1 \leq r < p$. We shall not discuss this here.

Thus, our main results in Section 2 have implied the following

Theorem 3.5. *All the above definitions (3.1)-(3.4) are equivalent.*

§4 Representation formulas and Sobolev spaces of high order in metric spaces

Before we proceed, we first need to modify the notion of polynomial functions introduced in [LW2]. Let m be a positive integer, Ω be a domain in $(\mathcal{S}, \rho)$, and ν, μ be doubling measures. The two main properties that we require of a polynomial function $P(x), x \in \Omega$, are

(P1) Given any $0 < q < \infty$, there is a constant $C_1 > 0$ such that for every metric ball $D \subset \Omega$,

$$esssup_{x\in D}|P(x)| \leq C_1 \left(\frac{1}{\nu(D)} \int_D |P(y)|^q d\nu(y)\right)^{1/q},$$

where the essential supremum is taken with respect to ν and C_1 depends on q and μ; and

(P2) If D is any metric ball in Ω and E is a subball of D with $\nu(E) > \gamma\nu(D)$, $\gamma > 0$, then

$$\| P \|_{L^\infty_\nu(E)} \geq C_2(\gamma) \| P \|_{L^\infty_\nu(D)} .$$

The difference between this notion of polynomials here and the one introduced in [LW2] is that we require (P1) to hold for all $q > 0$ rather than for $q = 1$ only. However, for polynomials in Euclidean spaces and Carnot groups, polynomials do satisfy our (P1) here.

We mention in passing that the condition $\nu(E) > \gamma\nu(D)$ in (P2) is equivalent to that $r(E) > \gamma_1 r(D)$ for some $\gamma_1 > 0$ only depending on γ and the doubling constant of ν (see [R] and [LPR]). In [LPR], we also refined some results in [LW2] on polynomials and higher order Sobolev spaces, and defined higher order gradiants in metric space and established distribution theory in metric spaces by using higher order Sobolev spaces in metric spaces. We shall not discuss this in detail here.

Definition 4.1. *Given m, f and Ω, we say that functions $P_m(f, B)$ are polynomial functions associated with m, balls $B \subset \Omega, f, g, \nu$ and μ if for some $0 < q < 1$ the following Poincaré inequality*

$$\left(\fint_B |f(x) - P_m(B, f)(x)|^q d\nu\right)^{1/q} \leq Cr(B)^m \fint_B g(x)\, d\mu(x)$$

holds and if (P1), (P2) hold with $P = P_m(B, f)$ and also with $P = P_m(B_1, f) - P_m(B_2, f)$ for every $B \subset \Omega$ and $B_1 \subset B_2 \subset \Omega$, for constants $C, C_1, C_2(\gamma)$ which are uniform in B, B_1, B_2.

We denote such polynomials by $P_m(B, f, g, \nu, \mu)$, and usually write them simply as $P_m(B, f)$. In practice, the constants $C_1, C_2(\gamma)$ are also independent of f, but we do not need to make this assumption.

We mention here that Poincaré inequalities of higher order do hold on Carnot groups (see [N], [L3], [L4], [L5], [LW2]) and such Poincaré inequalities are useful in proving Sobolev extension theorems and interpolation inequalities on Carnot groups.

Similar to the first order case, we impose the following assumptions, not all needed in every theorem, in the metric space $(\mathcal{S}, \rho)$ as in [LW2]. Let μ and ν be doubling measures with respect to metric balls, and let Ω be a domain in $\mathcal{S}$. In addition to (H3) and (H4) given in Section 2, we impose the following assumptions:

(A1) f is a function satisfying definition (4.1), i.e., f satisfies the $L^1 \to L^q$ Poincaré estimate there for all $B \subset \Omega$ with polynomial functions $P_m(B, f) = P_m(B, f, g, \nu, \mu)$ which satisfy (P1) and (P2), and the differences $P_m(B_1, f) - P_m(B_2, f), B_1 \subset B_2 \subset \Omega$, also satisfy (P1) and (P2), with constants independent of B, B_1 and B_2.

(A2) The measure μ in (H1) satisfies a reverse doubling condition of order m, i.e., there is a constant $C > 0$ such that if B and $\tilde{B}$ are balls with centers in Ω and with $B \subset \tilde{B}$, then

$$\mu(\tilde{B}) \geq C \left(\frac{r(\tilde{B})}{r(B)} \right)^m \mu(B).$$

We now state our main results.

Theorem 4.2. *Let ν, μ be doubling measures on a metric space $(\mathcal{S}, \rho)$. Let B_0 be a ball and suppose that (A1) and (H3) hold with $\Omega = B_0$. Then for ν-a.e. $x \in B_0$,*

$$\begin{aligned} &|f(x) - P_m(B_0, f)(x)| \\ &\qquad \leq C \int_{B_0} g(y) \frac{\rho(x,y)^m}{\mu(B(x, \rho(x,y)))} \, d\mu(y) + C \frac{r(B_0)^m}{\mu(B_0)} \int_{B_0} g(y) \, d\mu(y), \end{aligned}$$

where C depends only on ν, μ and the constants in (A1).

Theorem 4.3. *Let ν, μ be doubling measures on a metric space $(\mathcal{S}, \rho)$. Let B_0 be a ball and suppose that (A1), (A2) and (H3) hold with $\Omega = B_0$. Then for ν-a.e. $x \in B_0$,*

$$|f(x) - P_m(B_0, f)(x)| \leq C \int_{B_0} g(y) \frac{\rho(x,y)^m}{\mu(B(x, \rho(x,y)))} \, d\mu(y),$$

where C depends only on ν, μ and the constants in (A1), (A2).

Theorem 4.4. *Suppose that ν and μ are doubling measures on a metric space $(\mathcal{S}, \rho)$ and that hypotheses (A1), (A2), (H3) and (H4) hold for a domain $\Omega \subset \mathcal{S}$. Then for ν-a.e. $x \in \Omega$,*

$$|f(x) - P_m(B_0, f)(x)| \leq C \int_{\Omega} g(y) \frac{\rho(x,y)^m}{\mu(B(x, \rho(x,y)))} \, d\mu(y),$$

where B_0 is the central ball in Ω, $P_m(B_0, f)$ is the polynomial associated with m, B_0, f, g, ν and μ in (A1), and C depends only on ν, μ and the constants in (A1), (A2) and (H4).

Proof of Theorem 4.2. We will use again Theorem 2.5 to prove Theorem 2.2. Let B_0 be a ball in $\mathcal{S}$ and suppose that (A1) and the segment property (H3) hold for B_0. Given $x \in B_0$, let $\{B_k\}_{k\geq 1}$ be a sequence of balls with the properties guaranteed by Theorem 2.5. Then

$$\begin{aligned}&|f(x) - P_m(B_0, f)(x)|\\ \leq&|f(x) - P_m(B_1, f)(x)| + |P_m(B_1, f)(x) - P_m(B_0, f)(x)|.\end{aligned} \tag{4.5}$$

For the second term on the right in (4.5), we get for $\nu-$a.e. $x \in B_0$ that

$$\begin{aligned}&|P_m(B_1, f)(x) - P_m(B_0, f)(x)|\\ \leq&||P_m(B_1, f)(x) - P_m(B_0, f)(x)||_{L^\infty_\nu(B_0)}\\ \leq&||P_m(B_1, f)(x) - P_m(B_0, f)(x)||_{L^\infty_\nu(B_1)}\end{aligned}$$

by (P2) since $B_1 \subset B_0$ and $\nu(B_1) \approx \nu(B_0)$ by property (1) of Theorem 2.5 (ν is doubling). Using (P1) this is bounded by

$$\begin{aligned}&C\left(\fint_{B_1} |P_m(B_1, f) - P_m(B_0, f)|^q d\nu\right)^{1/q}\\ \leq&C\left(\fint_{B_1} |f(y) - P_m(B_1, f)(y)|^q\, d\nu(y)\right)^{1/q} + C\left(\fint_{B_1} |f(y) - P_m(B_0, f)(y)|^q\, d\nu(y)\right)^{1/q}\\ \leq&C\left(\fint_{B_1} |f(y) - P_m(B_1, f)(y)|^q\, d\nu(y)\right)^{1/q} + C\left(\fint_{B_0} |f(y) - P_m(B_0, f)(y)|^q\, d\nu(y)\right)^{1/q}\\ \leq&C\frac{r(B_1)^m}{\mu(B_1)}\int_{B_1} g\, d\mu + C\frac{r(B_0)^m}{\mu(B_0)}\int_{B_0} g\, d\mu \,\text{by the Poincaré inequality (A1)}\\ \leq&C\frac{r(B_0)^m}{\mu(B_0)}\int_{B_0} gd\mu\end{aligned}$$

since $B_1 \subset B_0$, $r(B_1) \approx r(B_0)$ and $\mu(B_1) \approx \mu(B_0)$.

Assuming as we may that x is a Lebesgue point for both $|f - P_m(B_1, f)|^q$ and g with respect to ν and using properties (1)–(3) from Theorem 2.5, we have for the first term on the right in (4.5) that

$$\begin{aligned}|f(x) - P_m(B_1, f)(x)| =& \lim_{k\to\infty}\left(\fint_{B_k} |f(y) - P_m(B_1, f)(y)|^q d\nu(y)\right)^{1/q}\\ \leq&2^q \limsup_{k\to\infty}\left(\fint_{B_k} |f(y) - P_m(B_k, f)(y)|^q d\nu(y)\right)^{1/q}\\ &+ 2^q \limsup_{k\to\infty}\left(\fint_{B_k} |P_m(B_k, f)(y) - P_m(B_1, f)(y)|^q d\nu(y)\right)^{1/q}\\ =&I_1 + I_2.\end{aligned}$$

It is clear to show that $I_1 = 0$ for every Lebesgue point x of g. Indeed, this can be seen by the Poincaré inequality (A1):

$$\begin{aligned} I_1 =& 2^q \limsup_{k\to\infty} \left(\fint_{B_k} |f(y) - P_m(B_k, f)(y)|^q \, d\nu(y) \right)^{1/q} \\ \leq& C \limsup_{k\to\infty} \frac{r(B_k)^m}{\mu(B_k)} \int_{B_k} g(y) d\mu(y) = 0 \cdot g(x) = 0. \end{aligned}$$

Since L^∞ norm obeys the triangle inequality for arbitrary terms, we can estimate I_2 as follows.

$$\begin{aligned} I_2 \leq& 2^q \limsup_{k\to\infty} ||P_m(B_k, f) - P_m(B_1, f)||_{L^\infty_\nu(B_k)} \\ \leq& 2^q \limsup_{k\to\infty} \sum_{j=1}^{k-1} ||P_m(B_{j+1}, f) - P_m(B_j, f)||_{L^\infty_\nu(B_k)} \\ \leq& 2^q \limsup_{k\to\infty} \sum_{j=1}^{k-1} ||P_m(B_{j+1}, f) - P_m(B_j, f)||_{L^\infty_\nu(cB_j)} \end{aligned}$$

by property (5) of Theorem 2.5. The last expression equals

$$\begin{aligned} & \sum_{j=1}^{\infty} ||P_m(B_{j+1}, f) - P_m(B_j, f) \; ||_{L^\infty_\nu(cB_j)} \\ \leq& C \sum_{j=1}^{\infty} ||P_m(B_{j+1}, f) - P_m(B_j, f) \; ||_{L^\infty_\nu(S_j)} \text{ by property (4) and (P2)} \\ \leq& C \sum_{j=1}^{\infty} \left(\fint_{S_j} |P_m(B_{j+1}, f) - P_m(B_j, f) \; |^q d\nu \right)^{1/q} \quad \text{by (P1).} \end{aligned}$$

This is bounded by

$$\begin{aligned} & C2^q \sum_{j=1}^{\infty} \left(\fint_{S_j} |P_m(B_{j+1}, f) - f|^q d\nu \right)^{1/q} + C2^q \sum_{j=1}^{\infty} \left(\fint_{S_j} |P_m(B_j, f) - f|^q d\nu \right)^{1/q} \\ \leq& C2^q \sum_{j=1}^{\infty} \left(\fint_{B_{j+1}} |P_m(B_{j+1}, f) - f|^q d\nu \right)^{1/q} + C2^q \sum_{j=1}^{\infty} \left(\fint_{B_j} |P_m(B_j, f) - f|^q d\nu \right)^{1/q} \end{aligned}$$

since $S_j \subset B_j \cap B_{j+1}$ and $\nu(S_j) \approx \nu(B_j) \approx \nu(B_{j+1})$ by Theorem 2.5. Combining estimates and applying Poincaré inequality (A1) to the terms of each of the last two sums, we obtain

$$I_2 \leq C \sum_{j=1}^{\infty} r(B_j)^m \fint_{B_j} g(y) \, d\mu(y).$$

Arguing as in [LW2], if $y \in B_j$, then

$$\frac{r(B_j)^m}{\mu(B_j)} \approx \frac{\rho(x,y)^m}{\mu(B(y, \rho(x,y)))} \approx \frac{\rho(x,y)^m}{\mu(B(x, \rho(x,y)))}$$

by part (3) of Theorem 2.5 and the fact that μ is a doubling measure. Thus, we get

$$\begin{aligned} I_2 \leq & C\sum_{j=1}^{\infty}\int_{B_j} g(y)\frac{\rho(x,y)^m}{\mu(B(x,\rho(x,y)))}\,d\mu(y) \\ \leq & C\int_{B_0} g(y)\frac{\rho(x,y)^m}{\mu(B(x,\rho(x,y)))}\,d\mu(y) \end{aligned}$$

by properties (6) and (1) of Theorem 2.5. The proof of Theorem 2.2 now is complete.

Remark. We omit the proof of Theorem 4.3 since is similar to that of Theorem 2.3 by using (A2) instead of (H2).

Proof of Theorem 4.4. Let $x \in \Omega$. By the definition of weak Boman chain domain, we may select B^* with $x \in B^*$ and a chain $\{B_j\}_{j=0}^k$ connecting $B^* = B_k$ to the central ball B_0. We have

$$\begin{aligned} |f(x) - P_m(B_0, f)(x)| \leq & |f(x) - P_m(B^*, f)(x)| \\ & + |P_m(B^*, f)(x) - P_m(B_0, f)(x)|. \end{aligned} \tag{4.6}$$

For the first term on the right side of (4.6), we have by Theorem B that

$$|f(x) - P_m(B^*, f)(x)| \leq C\int_{B^*} g(y)\frac{\rho(x,y)^m}{\mu(B(x,\rho(x,y)))}\,d\mu(y).$$

This holds for $\nu-$a.e. point of B^*, and we may assume it holds for our fixed x by initially excluding from Ω the set of measure zero formed by taking the union of the exceptional sets of measure zero in each Boman ball. Since $B^* \subset \Omega$, we obtain

$$|f(x) - P_m(B^*, f)(x)| \leq C\int_{\Omega} g(y)\frac{\rho(x,y)^m}{\mu(B(x,\rho(x,y)))}\,d\mu(y).$$

We now estimate $|P_m(B^*, f)(x) - P_m(B_0, f)(x)|$. By using the chain $\{B_j\}$ connecting B_0 and $B_k = B^*$ and noticing that $B^* \subset MB_j$ and $x \in B^*$, we have

$$\begin{aligned} & |P_m(B^*, f)(x) - P_m(B_0, f)(x)| \\ \leq & \sum_{j=1}^{k} |P_m(B_j, f)(x) - P_m(B_{j-1}, f)(x)| \\ \leq & \sum_{j=1}^{k} ||P_m(B_j, f) - P_m(B_{j-1}, f)||_{L^\infty_\nu(B^*)} \\ \leq & \sum_{j=1}^{k} ||P_m(B_j, f) - P_m(B_{j-1}, f)||_{L^\infty_\nu(MB_j)}. \end{aligned}$$

If D_j is a ball with $D_j \subset B_j \cap B_{j-1} \subset MB_j$ and $r(D_j) \approx r(B_j) \approx r(B_{j-1})$, then by (P1) and (P2), the last sum is majorized by

$$\begin{aligned} &C \sum_{j=1}^{k} ||P_m(B_j, f) - P_m(B_{j-1}, f)||_{L^\infty_\nu(D_j)} \\ \leq &C \sum_{j=1}^{k} \left(\fint_{D_j} |P_m(B_j, f) - P_m(B_{j-1}, f)|^q d\nu \right)^{1/q}, \end{aligned}$$

which by the inequality (2.6) and doubling is bounded by

$$\begin{aligned} &C2^q \sum_{j=1}^{k} \left(\fint_{D_j} |P_m(B_j, f)(y) - f(y)|^q \, d\nu(y) \right)^{1/q} \\ &+ C2^q \sum_{j=1}^{k} \left(\fint_{D_j} |P_m(B_{j-1}, f)(y) - f(y)|^q \, d\nu(y) \right)^{1/q} \\ \leq &C2^q \sum_{j=1}^{k} \left(\fint_{B_j} |P_m(B_j, f)(y) - f(y)|^q \, d\nu(y) \right)^{1/q} \\ &+ C2^q \sum_{j=1}^{k} \left(\fint_{B_{j-1}} |P_m(B_{j-1}, f)(y) - f(y)|^q \, d\nu(y) \right)^{1/q} \\ \leq &C \sum_{j=0}^{k} \left(\frac{1}{\nu(B_j)} \int_{B_j} |P_m(B_j, f)(y) - f(y)|^q \, d\nu(y) \right)^{1/q} \end{aligned}$$

By Poincaré's inequality, the last expression above is at most

$$\begin{aligned} &C \sum_{j=0}^{k} \frac{r(B_j)^m}{\mu(B_j)} \int_{B_j} g(y) \, d\mu(y) \\ = &C \int_\Omega \left\{ \sum_{j=0}^{k} \frac{r(B_j)^m}{\mu(B_j)} \chi_{B_j}(y) \right\} g(y) \, d\mu(y). \end{aligned}$$

As shown in [LW2], the sum above in curly brackets is bounded by a fixed multiple of $\rho(x, y)^m / \mu(B(x, \rho(x, y)))$ for each $y \in \Omega$. Thus, we have completed the proof.

By using Theorem (4.2), we will be able to weaken the hypotheses in defining high order Sobolev spaces in metric spaces given in [LLW1] (see also [LLW2]).

Definition 4.7. *Given a positive integer m and $1 < p < \infty$, we define the Sobolev class $A^{m,p}(\Omega)$ to be the set of functions $f \in L^p(\Omega)$ so that for each $k = 1, \cdots, m$, there exist r_k with $1 \leq r_k < p$ and q_k with $0 < q_k < 1$, functions $g_k(x)$ with $0 \leq g_k \in L^p(\Omega)$, and polynomials $P_k(B, f)$ with*

$$\left(\fint_B |f(x) - P_k(B, f)(x)|^{q_k} d\mu(x) \right)^{\frac{1}{q_k}} \leq r(B)^k \left(\fint_B g_k^{r_k}(x) d\mu(x) \right)^{\frac{1}{r_k}} \tag{4.8}$$

for every ball $B \subset \Omega$. The polynomials $P_k(B,f)$ are assumed to belong to a linear class which satisfies (P1) and (P2) with constants depending only on k,γ,μ. If $f \in A^{m,p}(\Omega)$, we define

$$||f||_{A^{m,p}(\Omega)} = ||f||_{L^p(\Omega)} + \inf_{\{g_k\}} \sum_{k=1}^{m} ||g_k||_{L^p(\Omega)},$$

where the infimum is taken over all sequences such that (4.8) holds for f for $k = 1,\ldots,m$.

It is easy to see that $A^{m,p}(\Omega)$ is a linear space.

The reason we can impose the L^{r_k} norm rather than the L^1 norm is because we can show that definition (4.7) is equivalent to the following definitions (4.9) and (4.11) given in [LLW1]. The proof of equivalence follows from our Theorem (4.2) in this section by combining the proofs given in [LLW1]. We shall omit the details here.

Definition 4.9 [LLW1]. *Given a positive integer m and $1 < p < \infty$, we define the Sobolev class $B^{m,p}(\Omega)$ to be the set of functions $f \in L^p(\Omega)$ so that for each $k = 1,\cdots,m$, there exist functions $0 \le g_k \in L^p(\Omega)$ and polynomials $P_k(B,f)$ such that*

$$|f(x) - P_k(B,f)(x)| \le \int_B \frac{\rho(x,y)^k g_k(y)}{\mu(B(x,\rho(x,y)))} d\mu(y) + r(B)^k \fint_B g_k(y) d\mu(y) \tag{4.10}$$

for $\mu-$a.e. $x \in B$ for every ball $B \subset \Omega$. The polynomials $P_k(B,f)$ are assumed to belong to a linear class which satisfies (P1) and (P2) with constants depending only on k,γ,μ. If $f \in B^{m,p}(\Omega)$, we define

$$||f||_{B^{m,p}(\Omega)} = ||f||_{L^p(\Omega)} + \inf_{\{g_k\}} \sum_{k=1}^{m} ||g_k||_{L^p(\Omega)},$$

where the infimum is taken over all sequences such that (4.10) holds for f for $k = 1,\cdots,m$.

The class $B^{m,p}(\Omega)$ is clearly a Banach space with norm $||\cdot||_{B^{m,p}(\Omega)}$.

Definition 4.11 [LLW1]. *Given a positive integer m and $1 < p < \infty$, we define the Sobolev class $C^{m,p}(\Omega)$ to be the set of functions $f \in L^p(\Omega)$ so that for each $k = 1,\cdots,m$ there exist functions $0 \le g_k \in L^p(\Omega)$ and polynomials $P_k(B,f)$ such that*

$$|f(x) - P_k(B,f)(x)| \le r(B)^k g_k(x) \tag{4.12}$$

for $\mu-$a.e. $x \in B$ for every metric ball $B \subset \Omega$. The polynomials $P_k(B,f)$ are assumed to belong to a linear class which satisfies (P1) and (P2) with constants depending only on k,γ,μ. If $f \in C^{m,p}(\Omega)$, let

$$||f||_{C^{m,p}(\Omega)} = ||f||_{L^p(\Omega)} + \inf_{\{g_k\}} \sum_{k=1}^{m} ||g_k||_{L^p(\Omega)}.$$

The class $C^{m,p}(\Omega)$ is a Banach space with norm $||\cdot||_{C^{m,p}}$.

To show that definitions (4.7), (4.9) and (4.11) are all equivalent, we will need the following theorem.

Theorem 4.13. *Let $1 \le r < \infty$, m be a positive integer, $B_0 \subset \Omega$ be a fixed ball, and suppose that the segment property (H3) holds for B_0. Let f be a locally integrable function in Ω for which there exist a function $0 \le g \in L^r(\Omega)$ and polynomials $P_m(B,f)$, and $0 < q < 1$ such that the Poincaré inequality*

$$\left(\fint_B |f(x) - P_m(B,f)(x)|^q \, d\mu(x) \right)^{1/q} \le cr(B)^m \left(\fint_B |g(x)|^r d\mu(x) \right)^{1/r}$$

holds for every ball $B \subset \Omega$. The polynomials $P_m(B,f)$ are assumed to belong to a linear class which satisfies (P1) and (P2) with constants depending only on m, γ, μ. Then for $\mu-$a.e. $x \in B_0$,

$$|f(x) - P_m(B_0,f)(x)| \le Cr(B_0)^m M(g^r)(x)^{1/r}$$

with C independent of x.

Proof of Theorem 4.13. Let $x \in B_0$. We will use the chain of subballs $\{B_j\}$ of B_0 constructed from Theorem 2.5. The chain depends on x. We may assume without loss of generality that x is a Lebesgue point for both $|f - P_m(B_0,f)|^q$ and $|g|^r$ with respect to μ. Then by properties (1), (2) and (3) of the chain,

$$\begin{aligned}
|f(x) - P_m(B_0,f)(x)| = & \lim_{j\to\infty} \left(\fint_{B_j} |f(y) - P_m(B_0,f)(y)|^q \, d\mu(y) \right)^{1/q} \\
\le & \limsup_{j\to\infty} 2^q \left(\fint_{B_j} |f(y) - P_m(B_j,f)(y)|^q \, d\mu(y) \right)^{1/q} \\
& + \limsup_{j\to\infty} 2^q \left(\fint_{B_j} |P_m(B_j,f)(y) - P_m(B_0,f)(y)|^q \, d\mu(y) \right)^{1/q} \\
= & I_1 + I_2.
\end{aligned}$$

By the Poincaré inequality, for every Lebesgue point x of $|g|^r$

$$\begin{aligned}
I_1 &\le c \limsup_{j\to\infty} r(B_j)^m \left(\fint_{B_j} |g(y)|^r d\mu(y) \right)^{1/r} \\
&= 0 \cdot |g(x)| = 0
\end{aligned}$$

by properties (1), (2) and (3) of the chain.

We have for I_2

$$
\begin{aligned}
I_2 \leq & \limsup_{j\to\infty} ||P_m(B_j,f)(y) - P_m(B_0,f)(y)||_{L^\infty_\mu(B_j)} \\
\leq & \limsup_{j\to\infty} \sum_{\ell=0}^{j-1} ||P_m(B_{\ell+1},f) - P_m(B_\ell,f)||_{L^\infty_\mu(B_j)} \\
\leq & \limsup_{j\to\infty} \sum_{\ell=0}^{j-1} ||P_m(B_{\ell+1},f) - P_m(B_\ell,f)||_{L^\infty_\mu(cB_\ell)} \text{ by (5)} \\
\leq & C \sum_{\ell=0}^{\infty} ||P_m(B_{\ell+1},f) - P_m(B_\ell,f)||_{L^\infty_\mu(S_\ell)} \text{ by (4) and (P2)} \\
\leq & C \sum_{\ell=0}^{\infty} \left(\fint_{S_l} |P_m(B_{\ell+1},f) - P_m(B_\ell,f)|^q d\mu \right)^{1/q} \text{ by (P1)} \\
\leq & C \sum_{\ell=0}^{\infty} 2^q \left(\fint_{B_\ell} |P_m(B_\ell,f)(y) - f(y)|^q \, d\mu(y) \right)^{1/q} \\
& + C \sum_{\ell=0}^{\infty} 2^q \left(\fint_{B_{\ell+1}} |P_m(B_{\ell+1},f)(y) - f(y)|^q \, d\mu(y) \right)^{1/q} \text{ by (4)} \\
\leq & C \sum_{\ell=0}^{\infty} r(B_\ell)^m \left(\fint_{B_\ell} |g(y)|^q d\mu(y) \right)^{1/q} \\
\leq & C \sum_{\ell=0}^{\infty} r(B_\ell)^m M(|g|^r)(x)^{1/r} \text{ by (3)} \\
= & C \sum_{\ell=0}^{\infty} 2^{-\ell m} r(B_0)^m M(|g|^r)(x)^{1/r} \text{ by (2)} \\
\leq & C r(B_0)^m M(|g|^r)(x)^{1/r}.
\end{aligned}
$$

This completes the proof of Theorem 4.13.

By using Theorem 4.13, and arguing similarly as in [LLW1], we will be able to show the equivalence of definitions (4.7), (4.9) and (4.11). We shall omit the details here and refer the reader to [LLW1].

We end this section by mentioning that above definitions of Sobolev spaces of higher order coincide with those of classical Sobolev spaces in Euclidean space and non-isotropic Sobolev spaces on stratified nilpotent Lie groups (see [LLW1] for detailed proof).

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