The fixed point property and unbounded sets in CAT(0) spaces

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Abstract In this work we study the fixed point property for nonexpansive self-mappings defined on convex and closed subsets of a CAT(0) space. We will show that a positive answer to this problem is very much linked with the Euclidean geometry of the space while the answer is more likely to be negative if the space is more hyperbolic. As a consequence we extend a very well known result of W.O. Ray on Hilbert spaces.

Keywords: Nonexpansive mappings, Fixed points, Unbounded convex sets, CAT(0) spaces, Banach-Steinhaus theorem

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1 Introduction

Let X be a Banach space and $Y \subseteq X$ nonempty closed and convex. Then Y is said to have the fixed point property (FPP) for nonexpansive mappings if, for any nonempty closed and convex $Z \subseteq Y$, any nonexpansive self-mapping (i.e., $T: Z \to Z$ such that $\|T(x) - T(y)\| \leq \|x - y\|$) has a fixed point (i.e., an $x \in Z$ such that $T(x) = x$). The problem of the fixed point property for bounded subsets has been extensively studied for the last fifty years and the literature about it is huge (see, for instance, [9, 18] and references therein). The same problem for unbounded subsets Y is trivial as long as Y contains a half-line (that is, if Y is *linearly unbounded*), as you can always apply a translation (shift-operator) in the half-ray, but far to be well understood if Y is supposed to lack this property. Without any doubt, one of the most relevant results in this direction was obtained by W.O. Ray in [26].

Theorem 1.1. (Ray's theorem) *Let K be a nonempty closed and convex subset of a (real) Hilbert space. Then K has the fixed point property for nonexpansive mappings if and only if K is bounded.*

In fact, in [12] two antagonistic questions were raised. The first one asked if Ray's theorem characterized Hilbert spaces among the class of Banach spaces. The second one asked if Ray's theorem characterized any Banach space at all, that is, if it holds for any Banach space. Recently the first question has been answered in the negative by T. Domínguez-Benavides [5] where it is shown that the space of real sequences converging to 0, that is, the space c_0, satisfies Ray's theorem. Regarding the second question it is still to be found a Banach space where Ray's theorem does not hold. More successful attempts to approach this problem in a more general way have tried to establish the –trivial in the bounded case [9, pg. 28]– approximate fixed point property (AFPP), that is, to establish whether

$$\inf\{d(x, T(x)) : x \in Y\} = 0.$$

Some remarkable results pointing in this direction were obtained by I. Shafrir, E. Kopecká, T. Kuczumov and S. Reich [21, 24, 27, 30], among others.

We study here the problem of the fixed point property for unbounded sets in CAT(0) spaces, also known as geodesic metric spaces of global nonpositive curvature in the sense of M. Gromov (see preliminaries for more details or [1, 2]). CAT(0) spaces have called the attention of many authors working in metric fixed point theory in the last years (see [6, 7, 13, 14] and references therein). It is very well-known that (real) Hilbert spaces are the only Banach spaces which are CAT(0). Also, the fact that CAT(0) spaces satisfy the so-called C-N inequality (see [1, pg. 163]) makes them share very relevant properties with Hilbert spaces. Therefore, it is natural to wonder whether Ray's theorem still holds true for CAT(0) spaces. The answer to this question is known to be negative since two examples of CAT(0) spaces failing Ray's theorem are known: the (complex) Hilbert ball with the hyperbolic metric (see Theorem 32.2 in [10]) and $\mathbb{R}$-trees [7]. In fact, it is known that, in both cases, a nonempty closed and convex subset of these spaces has the fixed point property if and only if it is geodesically bounded and so not necessarily bounded (see [7] for the $\mathbb{R}$-tree case and [10, Theorem 32.2] for the Hilbert ball case). A few years after Ray's result appeared, R. Sine [31] found a new and much shorter proof of the same result. This proof basically relies on two facts: Banach-Steinhaus theorem and the extraordinarily good properties of the metric projection onto closed and convex subsets of Hilbert spaces. As it will be explained in Section 2, metric projections on CAT(0) spaces behave in a very similar way as they do in Hilbert spaces. The goal of this paper is therefore to study farther to which extend

Ray's theorem remains true in CAT(0) spaces. For related works on fixed point on unbounded sets the reader may check [17, 23] or, for recent developments, [5, 29].

This work is organized as follows. In Section 2 we introduce some preliminary notions and notations jointly with auxiliary results which will be used by our main results. In Section 3 we enlarge the known collection of CAT(0) spaces failing Ray's theorem. In Section 4 we define and study a geometrical condition inspired in the Banach-Steinhaus theorem, which we call property U, about the structure of unbounded sets in geodesic spaces. Next we study this property on different spaces. In Section 5 we obtain a counterpart of Ray's theorem in CAT(0) spaces by showing that any nonempty closed and convex subset of a complete CAT(0) space with property U has the fixed point property if and only if it is bounded. Although based on the same two facts as in Sine's proof, our result will require a much more involved proof where the Busemann convexity of CAT(0) spaces plays a very distinguished role. We close this work with an appendix about the modulus of convexity at infinity of CAT(κ) spaces with $\kappa < 0$.

2 Preliminaries

Let (X, d) be a metric space and $x, y \in X$. A *geodesic path* from x to y is a mapping $c : [0, l] \subseteq \mathbb{R} \to X$ with $c(0) = x, c(l) = y$ and $d\,(c(t), c(t')) = |t - t'|$ for every $t, t' \in [0, l]$. The image $c\,([0, l])$ of c forms a *geodesic segment* which joins x and y and is not necessarily unique unless the space is uniquely geodesic. If no confusion arises, we will use $[x, y]$ to denote a geodesic segment joining x and y. (X, d) is a *(uniquely) geodesic space* if every two points $x, y \in X$ can be joined by a (unique) geodesic path. A point $z \in X$ belongs to the geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(z, x) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$, and, abusing of notation, we will write $z = (1 - t)x + ty$ if no confusion arises. A subset K of X is *convex* if it contains any geodesic segment that joins every two points of it. A *geodesic* will be an isometric set to the real line and a *geodesic ray* an isometric set to a half-line. A geodesic metric space will be *geodesically bounded* if it does not contain any geodesic ray. More details about geodesic metric spaces can be found in [1, 2].

Given a point $x \in X$ and a positive number r, the notation $B(x, r)$ will stand for the closed ball of center x and radius r. If $Y \subseteq X$ then

$$\operatorname{dist}(x, Y) = \inf\{d(x, y) \colon y \in Y\}.$$

The same definition and notation will be used for distances between sets.

Let us assume that (X, d) is a geodesic space. A *geodesic triangle* $\Delta(A, B, C)$ consists of three points A, B and C in X (the *vertices* of the triangle) and three geodesic segments joining each pair of vertices (the *edges* of the triangle). For the geodesic triangle $\Delta{=}\Delta(A, B, C)$, a *comparison triangle* is a triangle $\bar{\Delta} = \Delta(\bar{A}, \bar{B}, \bar{C})$ in the Euclidean plane $\mathbb{E}^2$ such that $d(A, B) = d_{\mathbb{E}^2}(\bar{A}, \bar{B})$, $d(A, C) = d_{\mathbb{E}^2}(\bar{A}, \bar{C})$ and $d(B, C) = d_{\mathbb{E}^2}(\bar{B}, \bar{C})$. As an elementary fact of the Euclidean geometry such a triangle always exists and is unique up to isometries. Corresponding angles at corresponding vertices are called comparison angles (for more precise definitions, and in particular definition of angle in (X, d), see [1, Pg. 8]).

A geodesic triangle Δ satisfies the *CAT*(0) *inequality* if for a comparison triangle $\bar{\Delta}$ of Δ and for every $x, y \in \Delta$ we have

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}),$$

where $\bar{x}, \bar{y} \in \bar{\Delta}$ are the comparison points of x and y, i.e., for x, if $x = (1 - t)A + tB$ then $\bar{x} = (1 - t)\bar{A} + t\bar{B}$. From now on we will omit the notation $d_{\mathbb{E}^2}$ to refer to the Euclidean metric since the context will make it clear.

Definition 2.1. *A metric space (X, d) is said to be a CAT(0) (or global nonpositive curvature) space if it is a geodesic space and all its triangles satisfy the CAT(0) inequality.*

A very thorough exposition of CAT(0) spaces can be found in Chapter II of [1] (the reader can find different equivalent definitions of CAT(0) spaces in this monograph). We summarize next some of the properties of CAT(0) spaces that will be needed in our work and which can be found in [1, Chapter II]. CAT(κ) spaces for $\kappa \in \mathbb{R}$ are defined by comparing in an analogous way with model spaces of constant curvature κ (see [1, Chapter II] for more details). We omit details for these spaces, however model spaces with constant negative curvature will be described in the next section.

Proposition 2.2. *Let (X, d) be a CAT(0) space, then the following properties hold:*

1. *(X, d) is uniquely geodesic.*
2. *If Δ=$\Delta(A, B, C)$ is a triangle in (X, d) and $\bar{\Delta} = \Delta(\bar{A}, \bar{B}, \bar{C})$ is its Euclidean comparison triangle, then for any vertex of Δ, let us say A,*
$$\gamma = \angle_A(B, C) \leq \angle_{\bar{A}}(\bar{B}, \bar{C}).$$
3. *(Law of cosines.) If γ is as above and $a = d(B, C)$, $b = d(A, C)$ and $c = d(A, B)$ then*
$$a^2 \geq b^2 + c^2 - 2bc \cos \gamma.$$
In particular, if $\gamma \geq \pi/2$ then the largest side of Δ is the opposite to γ.

A geodesic space (X, d) is *Busemann convex* (firstly introduced in [3] but also known by other authors as hyperbolic metric spaces [28]) if given any pair of geodesic paths $c_1 : [0, l_1] \to X$ and $c_2 : [0, l_2] \to X$ with $c_1(0) = c_2(0)$ one has

$$d(c_1(tl_1), c_2(tl_2)) \leq td(c_1(l_1), c_2(l_2)) \text{ for all } t \in [0, 1].$$

Applying a simple reasoning in the definition of Busemann convexity it is possible to renounce to the condition $c_1(0) = c_2(0)$ as it is shown next

$$d(c_1(tl_1), c_2(tl_2)) \leq (1-t)d(c_1(0), c_2(0)) + td(c_1(l_1), c_2(l_2)) \text{ for all } t \in [0, 1].$$

It is a very well-known fact that CAT(0) spaces are Busemann convex. The next proposition, intrinsic to the definition of Busemann convexity, will be repeatedly applied in Section 5.

Proposition 2.3. *Let X be a geodesic Busemann convex space. Consider x and y distinct points in X. If $z \in X$ then, for any $t \in (0, 1)$,*

$$d(z, tx + (1-t)y) < \max\{d(x, z), d(y, z)\}.$$

Another important feature of CAT(0) spaces is the behavior of the metric projection.

Definition 2.4. *Given a metric space X and a nonempty subset K of X, the metric projection (or nearest point map) from X onto K is denoted as P_K and defined by*

$$P_K(x) = \{y \in K : d(x, y) = \text{dist}(x, K)\}.$$

The next proposition [1, Proposition 2.4, Chapter II] summarizes the properties of the metric projection onto closed and convex subsets of CAT(0) spaces.

Proposition 2.5. *Let X be a complete CAT(0) space and $K \subseteq X$ nonempty, closed and convex. Then the metric projection onto K is well-defined (single-valued) and nonexpansive. Moreover, if $x \notin K$ and $y \in K$ with $y \neq P_K(x)$ then*

$$\angle_{P_K(x)}(x, y) \geq \frac{\pi}{2}.$$

An $\mathbb{R}$-*tree* is a uniquely geodesic metric space X such that if $[y, x] \cap [x, z] = \{x\}$ then $[y, x] \cup [x, z] = [y, z]$ for each $x, y, z \in X$. A very well-known example of an $\mathbb{R}$-tree is given by $\mathbb{R}^2$ with the river metric as follows:

$$d(v_1, v_2) = \begin{cases} \mid y_1 - y_2 \mid, & \text{if} \quad x_1 = x_2, \\ \mid y_1 \mid + \mid y_2 \mid + \mid x_1 - x_2 \mid, & \text{if} \quad x_1 \neq x_2, \end{cases}$$

where $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in \mathbb{R}^2$.

From the definition of $\mathbb{R}$-tree, it immediately follows that if $x, y, z \in X$, then $[x, y] \cap [x, z] = [x, w]$ for some $w \in X$. Likewise, if K is a closed and convex subset of an $\mathbb{R}$-tree X, then for every $x \in X$, $P_K(x)$ is a singleton and for any $y \in K$, $d(x, y) = d(x, P_K(x)) + d(P_K(x), y)$. This property is also known as the *gate property.* In general, given M and N two subsets of a geodesic metric space, we say that K is a *gate* between M and N if for any $x \in M$ and $y \in N$ there exists $z \in K$ such that $d(x, y) = d(x, z) + d(z, y)$. If the space is uniquely geodesic then such a z is unique.

It is known that $\mathbb{R}$-trees are CAT(0) spaces. In [7] it was shown that any geodesically bounded complete $\mathbb{R}$-tree has the fixed point property for nonexpansive mapping and so Ray's theorem does not hold on $\mathbb{R}$-trees in general. For extensions of this result the interested reader may also check [16, 25].

In [30], I. Shafrir considered the fixed point problem for unbounded sets in Banach spaces and Busemann convex geodesic spaces. More precisely, the approximate fixed point property for unbounded sets was studied in [30]. A set K of a metric space is said to have the AFPP (*approximate fixed point property*) for nonexpansive mappings if

$$\inf\{d(x, Tx) \colon x \in K\} = 0$$

for every nonexpansive mapping $T \colon K \to K$. The following definition was also introduced in [30].

Definition 2.6. *Let X be a geodesic space. A curve $\gamma : [0, \infty) \to X$ is said to be directional if there exists $b \geq 0$ such that*

$$t - s - b \leq d(\gamma(s), \gamma(t)) \leq t - s$$

for all $t \geq s \geq 0$. A convex subset K of X is called directionally bounded if it contains no directional curves.

Then the following result, which will be used in this work, was shown.

Theorem 2.7. *A convex subset of a Busemann convex space has the AFPP if and only if it is directionally bounded.*

3 Spaces failing Ray's theorem

We start this section by describing an abstract CAT(0) space for which Ray's theorem fails. Then we will show that this kind of spaces can be found as subsets of concrete spaces.

Proposition 3.1. *Let C be a complete $CAT(0)$ space which can be written as*

$$C = C_0 \cup \left(\bigcup_{n=1}^{\infty} C_n \right),$$

where:

i) C_0 *is bounded closed and convex,*

ii) C_n *is closed and bounded with* $C_0 \cup C_n$ *convex for any* n*,*

iii) $\{C_n\}$ *is a family of pairwise disjoint sets such that* $\mathrm{diam}(C_n)$ *tends to infinity as* $n \to \infty$*,*

iv) $W_n = C_0 \cap C_n$ *is nonempty,* $\mathrm{diam}(W_n) \leq \alpha$ *for each* n *and* $\mathrm{dist}(W_n, W_m) \geq \alpha$ *for a certain* $\alpha \geq 0$ *and any* $n \neq m$*.*

Then, C is geodesically bounded and unbounded, and has the fixed point property.

Proof. Notice first that the sets W_n are gates to get from C_n to anywhere out of C_n. Therefore, since C_0 is bounded, it follows that C is geodesically bounded. Moreover, C is unbounded because $\mathrm{diam}(C_n)$ tends to infinity as $n \to \infty$.

Now, let us take a nonexpansive mapping $T\colon C \to C$. Since each unbounded curve must cut the set C_0 infinitely many times, on account of Theorem 2.7.,

$$\inf\{d(x, T(x))\colon x \in C\} = 0.$$

Therefore, any approximate fixed point sequence (x_n) must be unbounded since, otherwise, a standard reasoning (see, for instance, [6]) would lead that the asymptotic center of (x_n) is a fixed point of T. In particular,

$$\inf\{d(x, T(x))\colon x \in C_0\} = p > 0,$$

Let (x_n) be an approximate fixed point sequence, without loss of generality we assume that $x_n \in C_n$ for each n.

Let us choose n so large that $d(x_n, T(x_n)) =: \varepsilon < p$ and denote $l = \mathrm{dist}(x_n, W_n) + \mathrm{diam}(W_n)$. Then make

$$D = C_n \cup B(x_n, l).$$

From *ii), iii)* and *iv)* we trivially have that $D \subseteq C_n \cup C_0$ and is closed. By construction D is also bounded. Let us see that, in addition, it is convex too. Indeed, let us fix $x, y \in C_n$ such that the metric segment $[x, y] \not\subset C_n$. Since W_n is the gate of C_n, there are two points $u, v \in [x, y]$ such that $u, v \in W_n$. So $u, v \in B(x_n, l)$ and so $[u, v] \subset D$. A similar proof works for $x \in C_n$ and $y \in B(x_n, l) \setminus C_n$.

Consider $P_D \circ T\colon D \to D$ which is a nonexpansive mapping. Therefore, it has a fixed point $\bar{x} = P_D(T(\bar{x}))$. If $T(\bar{x}) \in D$ then there is nothing to prove. Otherwise $\bar{x}$ must be in the topological boundary of D and so, by construction, in C_0. In particular, $d(\bar{x}, T(\bar{x})) \geq$

p. Moreover, since the topological boundary of D is contained in the boundary of $B(x_n, l)$, $d(\bar{x}, x_n) = l$ and so

$$d(T(\bar{x}), x_n) \leq d(T(\bar{x}), T(x_n)) + d(T(x_n), x_n) \leq l + \varepsilon.$$

Therefore, $\operatorname{dist}(T(\bar{x}), W_n) \leq \varepsilon$ and so

$$d(\bar{x}, T(\bar{x})) = d(P_D(T(\bar{x})), T(\bar{x})) \leq \varepsilon,$$

which is a contraction since $\varepsilon < p$. Therefore, $\bar{x}$ is a fixed point of T.

To complete the proof we need to show that the same happens for any nonempty closed and convex subset C' of C. However, this follows directly from the facts that C' is a nonexpansive retract of C and that any nonexpansive mapping from C into C has a fixed point.

One may wonder if this kind of CAT(0) spaces actually happens. The easiest example of this kind may be the following $\mathbb{R}$-tree.

Example 3.2. *Consider $\mathbb{R}^2$ with the radial metric, that is, $d(x, y) = \|x\| + \|y\|$ if $x, y \in \mathbb{R}^2$ are not collinear with the origin and $d(x, y) = \|x - y\|$ otherwise, where $\|\cdot\|$ stands for the Euclidean norm. Let $C \subset \mathbb{R}^2$ be made of segments of finite length starting at the origin in such a way that C is unbounded. Let $C_0 = B(0, 1)$, then*

$$C = C_0 \cup \left(\bigcup_{y \notin C_0} C_y \right),$$

where C_y is the largest geodesic segment starting in $P_{C_0}(y)$ and passing through y. Notice that for this particular example $\alpha = 0$.

More sophisticated and interesting examples can be built considering the Reshetnyak gluing technique (see Theorem 11.1 in Chapter II of [1]).

Example 3.3. *Let C be the closed unit ball in ℓ^2 and $\{e_n\}$ the elements of its standard basis. For each $n \in \mathbb{N}$ let us consider*

$$C_n = \overline{\operatorname{conv}} \left(B\left((1 - \frac{1}{n}) e_n, \frac{1}{n} \right) \cup \{n e_n\} \right),$$

where the $\overline{\operatorname{conv}}$ stands for the closed and convex hull. Now take $N \in \mathbb{N}$ so that

$$\operatorname{dist}(C_n, C_m) = \inf\{d(x, y) : x \in C_n, y \in C_m\} > 1$$

for any $n, m \geq N$. Consider now the gluings of C with C_n for $n \geq N$. By the basic gluing theorem, these gluings are all CAT(0) spaces and, gluing again all them,

$$X = C \cup \left(\bigcup_{n \geq N} C_n \right)$$

is a complete CAT(0) space as those given by Proposition 3.1..

Proposition 3.1. can also be used to deduce the main result from [7].

Corollary 3.4. *Let X be an unbounded but geodesically bounded complete $\mathbb{R}$-tree, then it has the fixed point property for nonexpansive mappings.*

Proof. First, it is easy to see that if an $\mathbb{R}$-tree is geodesically bounded then is directionally bounded too. Therefore, if $T\colon X \to X$ is a nonexpansive mapping then we may assume that there exists an unbounded approximate fixed point sequence since otherwise T has a fixed point. Let (x_n) be one of those sequences. Fix $x_0 \in X$ and denote as B_R the closed ball of center x_0 and radius R. Then there must exist $R > 0$ such that the cardinality of $\{P_{B_R}(x_n)\colon n \in \mathbb{N}\}$ is not finite. If this is not the case then there would exist a sequence of points (p_n) such that $d(p_n, x_0) = n$ and a subsequence of (x_n), which we denote equally as (x_n), such that

$$P_{B_R}(x_n) = p_N$$

for each $n \geq N$. Now consider the segment $I_1 = [x_0, x_1]$ and then

$$x^2 = P_{I_1}(x_2) \in [p_1, x_1].$$

Now consider the segment $[x_0, x^2] \subseteq [x_0, x_2]$. Repeating the same process with I_2, x_3 and p_2 we obtain x^3 such that $d(x_0, x^3) \geq 3$ and $[x_0, x^2] \subseteq [x_0, x^3]$. Continuing in this way we can geodesically extend $[x_0, x^2]$ as a geodesic ray which is a contradiction.

So we can take x_0, for simplicity $R = 1$ and (x_n) an approximate fixed point sequence such that $P_{B_1}(x_n) \neq P_{B_1}(x_m)$ for $n \neq m$. Consider

$$Y = B_1 \cup \left(\bigcup_{n \in \mathbb{N}} C_{x_n} \right)$$

where $C_{x_n} = [P_{B_1}(x_n), x_n]$. Let $\tilde{T} = P_Y \circ T : Y \to Y$. From Proposition 3.1., there exists $x \in Y$ such that $\tilde{T}x = x$. If $Tx \in Y$ then $x = Tx$. Otherwise x is the gate of Tx to Y and so, from the nonexpansivity of T, $d(z, Tz) \geq d(x, Tx)$ for $z \in Y$. In particular, this implies that (x_n) cannot be an approximate fixed point sequence.

The two very well-known cases for which Ray's theorem does not hold are $\mathbb{R}$-trees [7] and the complex Hilbert ball with the hyperbolic metric [10, Theorem 32.2]. In both cases more than the negation of Ray's theorem is known, more precisely, a nonempty closed and convex subset has the fixed point property if and only if is geodesically bounded. We show next that any complete locally compact CAT(0) space also falls into this class although they also satisfy Ray's theorem.

Proposition 3.5. *Let X be a complete and unbounded locally compact CAT(0) space. Then X contains a geodesic ray.*

Proof. Let $x \in X$ and consider $\{x_n\}$ an unbounded sequence of points in X. For $k \in \mathbb{N}$ consider n_k large enough so that $d(x, x_n) \geq k$ for $n \geq n_k$. For $n \geq n_k$, take

$$y_n^k \in [x, x_n]$$

such that $d(x, y_n^k) = k$. According to the Hopf-Rinow theorem, closed and bounded subsets of complete and locally compact length spaces are compact. Hence, by a diagonalization process if needed, we can assume that all sequences $\{y_n^k\}_{n \geq n_k}$ are convergent to respective points y_k. Now, the CAT(0) condition (applied to angles) implies that all the points y_k lie in a same geodesic ray emanating from x. Finally, since $d(x, y_k) = k$ for each k, the segments joining the points of the sequence $\{y_k\}$ define a geodesic ray.

As a consequence of this proposition we obtain the following theorem.

Theorem 3.6. *Let X be a complete locally compact $CAT(0)$ space. Then the following statements are equivalent:*

i) A nonempty closed and convex subset Y of X has the fixed point property for nonexpansive mappings.

ii) Y is geodesically bounded.

iii) Y is bounded.

Proof. It was already pointed in the introduction that if Y has the fixed point property then it must be geodesically bounded. On the other hand, if it is geodesically bounded, from Proposition 3.5., it must be bounded and so, it is a very well-known fact [12, 14], it has the fixed point property for nonexpansive mappings.

We will make use next of the Hilbert ball model to show that a nonempty closed and convex subset of a space of negative constant curvature has the fixed point property if and only it it is geodesically bounded. Spaces of negative constant curvature can be obtained one from another by introducing a factor in the distance (see [1, pg. 23]). We describe next the model space of the infinite dimensional space with -1 constant curvature. Notice that the finite dimensional ones would fall into the scope of the Theorem 3.6..

First, let us recall the finite dimensional space $\mathbb{H}^n$. Let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $u = (u_1, \cdots, u_{n+1})$ and $v = (v_1, \cdots, v_{n+1})$ the real number $\langle u|v\rangle$ defined by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_i v_i.$$

Then the *real hyperbolic n-space* $\mathbb{H}^n$ is

$$\{u \in \mathbb{E}^{n,1} : \langle u|u\rangle = -1, u_{n+1} \geq 1\}.$$

Proposition 3.7. *Let $d : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ be the function that assigns to each pair $(A, B) \in \mathbb{H}^n \times \mathbb{H}^n$ the unique non-negative number $d(A, B)$ such that*

$$\cosh d(A, B) = -\langle A, B\rangle.$$

Then $(\mathbb{H}^n, d)$ is a uniquely geodesic metric space of constant curvature -1.

The analogous infinite dimensional space, which we will denote as $\mathbb{H}^\infty$, is defined in the same way but for elements of ℓ^2 with the condition

$$\langle u|u\rangle = -u_1^2 + \sum_{i=2}^{\infty} u_i^2 = -1,$$

meaning in this occasion

$$\langle u|v\rangle = -u_1 v_1 + \sum_{i=2}^{\infty} u_i v_i$$

which is well defined for elements in ℓ^2 thanks to the Cauchy-Schwarz inequality.

There are several models for the above hyperbolic space. We will need the Klein model in our next result. Let $\mathbf{B}$ be the open unit ball of ℓ^2. The following proposition can be found in [1, Pg. 83] for the finite dimensional case, we reproduce it here for the infinite dimensional case for completeness.

Proposition 3.8. *Let h_K be the homeomorphism from $\mathbf{B}$ to $\mathbb{H}^\infty$ such that*

$$h_K(x) = h_K((x_1, x_2, \cdots)) = \frac{1}{\sqrt{1 - \|x\|}}(1, x_1, x_2, \cdots) \in \mathbb{H}^\infty.$$

The ball $\mathbf{B}$, equipped with the pull-back by h_K of the hyperbolic metric of $\mathbb{H}^\infty$ is the Klein model. The distance between two points x, y in this model is given by the formula:

$$\cosh d(x, y) = \frac{1 - (x|y)}{\sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2}},$$

where $(x|y)$ is the scalar product and $\|\cdot\|$ is the Hilbert norm.

Given two distinct points $x, y \in \mathbf{B}$, the unique hyperbolic geodesic line containing x and y is the intersection of $\mathbf{B}$ with the affine line in ℓ^2 through x and y. Let x_∞ and y_∞ be the two intersection points of this line with the boundary of $\mathbf{B}$, arranged so that x_∞, x, y, y_∞ occur in order on the line through x and y. The hyperbolic distance $d(x, y)$ is then given by

$$d(x, y) = \frac{1}{2} \log \left(\frac{\|x - y_\infty\| \cdot \|y - x_\infty\|}{\|x - x_\infty\| \cdot \|y - y_\infty\|} \right).$$

Our next goal is to show that $\mathbb{H}^\infty$ can be isometrically embedded into the Hilbert ball with the hyperbolic metric (notice also the discussion in [10, Pg. 148]). For that we recall some basic details about this space taken from [10, pg. 97-102]. Let $\mathbf{C}$ be the open unit ball of a complex Hilbert space H of dimension larger than 1. Given $a \in \mathbf{C}$ with $a \neq 0$, let P_a be the orthogonal projection of H onto the one-dimensional subspace spanned by a. P_a is then given by

$$P_a(z) = \frac{(z, a)a}{|a|^2},$$

where (z, a) denotes the inner product of z and a, and let $Q_a = I - P_a$, where I stands for the identity map. Motivated by the Möbius transformation in one dimensional spaces, let

$$m_a(z) = \frac{z + a}{1 + (z, a)}.$$

The proof of the next theorem can be found in [10, pg. 97-99] and [11].

Theorem 3.9. *Consider the mapping $M_a : \mathbf{C} \to \mathbf{C}$ given by*

$$M_a(z) = (\sqrt{1 - |a|^2} Q_a + P_a) m_a(z).$$

Then M_a is a holomorphic mapping and

$$\rho(x, y) = \operatorname{arctanh} |M_{-x}(y)|$$

defines a metric on $\mathbf{C}$. The metric space $(\mathbf{C}, \rho)$ is the Hilbert ball with the hyperbolic metric. Furthermore, $(\mathbf{C}, \rho)$ is a $CAT(-1)$ space.

Remark 3.10. *The fact that $(\mathbf{C}, \rho)$ is a $CAT(-1)$ space is not stated in any of the previous references. This can be shown, however, following a similar argument to the one on [10, pg. 106] leading to the computation of the modulus of convexity of $(\mathbf{C}, \rho)$. Now the idea would be to sharpen this argument to show the counterpart to the CN inequality for $CAT(-1)$ spaces in $(\mathbf{C}, \rho)$. We omit details since the fact that $(\mathbf{C}, \rho)$ is a $CAT(-1)$ space is a well-known one. More on the modulus of convexity of a $CAT(\kappa)$ space will be shown in the appendix at the end of this work.*

Next we show the immersion result.

Theorem 3.11. *For $n \in \{2, 3, \ldots, \infty\}$, there is an isometric embedding $\varphi \colon \mathbb{H}^n \to B^n$, where B^n denotes the unit ball of $\mathbb{C}^n$ with the hyperbolic metric.*

Proof. Let $B \subset \mathbb{C}^n$ such that

$$x \in B \;\Leftrightarrow\; \|x\| < 1 \wedge \text{ each coordinate of } x \text{ is a real number.}$$

Consider any n-dimensional unit ball of ℓ^2 (denoted again by B), then the Klein model is obtained when we provide this ball with the metric

$$d(x, y) = \frac{1}{2} \ln \frac{\|x - y_0\| \|y - x_0\|}{\|x - x_0\| \|y - y_0\|},$$

as given by Proposition 3.8..

Let us consider the identity map of B and take two point $x, \bar{x} \in B$. It is enough if we show that

$$d(x, \bar{x}) = \rho(x, \bar{x}),$$

where ρ denotes the hyperbolic metric on B.

Take now the geodesic line in B passing through x and $\bar{x}$ and denote by u the point of this geodesic with minimal distance to the origin. Points x, $\bar{x}$ and the origin span a 2-dimensional subspace of B. Clearly, the geodesic must belong to this space.

From Euclidean geometry, we may find two orthonormal vectors k and l such that

$$u = a \cdot k + 0 \cdot l.$$

Then it must be the case that

$$x = a \cdot k + b \cdot l$$

and

$$\bar{x} = a \cdot k + c \cdot l.$$

Moreover, M_{-u} maps the geodesic segment onto the diameter parallel to l.

First let us suppose that $c = 0$ (what means that $u = \bar{x}$) and $a \neq 0$. Then

$$(x, u) = (a \cdot k + b \cdot l, a \cdot k + 0 \cdot l) = a^2 + 0b > 0$$

and moreover

$$\|u\| = a.$$

Hence

$$M_{-u}(x) = y = \frac{a - a}{1 - a^2} k + \sqrt{1 - a^2} \frac{b - 0}{1 - a^2} l = \frac{b}{\sqrt{1 - a^2}} l$$

and

$$\|M_{-u}(x)\| = \frac{|b|}{\sqrt{1 - a^2}},$$

what leads to

$$\rho(x, \bar{x}) = \rho(x, u) = \operatorname{arctanh} \|y\| = \operatorname{arctanh} \frac{|b|}{\sqrt{1 - a^2}}.$$

At the same time

$$d(x,\bar{x}) = d(x,u) = \frac{1}{2}\ln\frac{(\|x-u\|+\sqrt{1-a^2})(\sqrt{1-a^2})}{\sqrt{1-a^2}(\sqrt{1-a^2}-\|x-u\|)}$$

$$= \frac{1}{2}\ln\frac{1+\dfrac{|b|}{\sqrt{1-a^2}}}{1-\dfrac{|b|}{\sqrt{1-a^2}}} = \operatorname{arctanh}\frac{|b|}{\sqrt{1-a^2}}.$$

So

$$d(x,\bar{x}) = \rho(x,\bar{x}).$$

In each other case $u \in [x,\bar{x}]$ or $\bar{x} \in [x,u]$ or $x \in [u,\bar{x}]$ and we may repeat our considerations to obtain that

$$d(x,\bar{x}) = d(x,u) + d(u,\bar{x}) \quad \text{and} \quad \rho(x,\bar{x}) = \rho(x,u) + \rho(u,\bar{x})$$

or

$$d(x,\bar{x}) = |d(x,u) - d(u,\bar{x})| \quad \text{and} \quad \rho(x,\bar{x}) = |\rho(x,u) - \rho(u,\bar{x})|.$$

As an immediate consequence of this immersion result we obtain the following corollary.

Corollary 3.12. *A nonempty closed and convex subset of $\mathbb{H}^\infty$ has the fixed point property if and only if it is geodesically bounded.*

Remark 3.13. *Proposition 3.5. shows that there is no unbounded and geodesically bounded closed and convex subset of $\mathbb{H}^n$ for any positive entire n. To find such sets in $\mathbb{H}^\infty$ the reader may check, after the previous theorem, Example 32.1 in [10].*

4 Property U

Next we introduce a geometrical condition. This condition refers to the structure of unbounded sets in geodesic spaces.

Definition 4.1. *Let X an unbounded geodesic space. Then we say that X has the property of the far unbounded convex set (property U, for short) if for any convex closed and unbounded subset Y of X either Y is geodesically unbounded or for each closed convex and unbounded $K \subseteq Y$ and $x \in K$ there exists a closed convex and unbounded subset K_1 of K such that* $\operatorname{dist}(x, K_1) \geq 1$.

Property U holds in reflexive Banach spaces from the Banach-Steinhaus theorem as the following proposition shows.

Proposition 4.2. *Let X be a reflexive Banach space. Then X has property U.*

Proof. Let $Y \subset X$ unbounded closed and convex. Let $x \in Y$ be a given point. By the Banach-Steinhaus theorem weakly bounded sets are also norm bounded. Hence there exists a continuous functional f such that

$$Y_n = \{y \in Y : f(y) \geq n\}$$

is a nonempty subset of Y for any $n \in \mathbb{N}$. Of course, Y_n is closed convex and unbounded for any $n \in \mathbb{N}$. Consider the sets

$$K_n = B(x, 1) \cap Y_n$$

for $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $K_n = \emptyset$ then the proposition follows. Otherwise we have that $\{K_n\}$ is a decreasing sequence of weakly compact subsets of X. Therefore their intersection is nonempty which would contradict the finiteness of the functional f.

A natural question in our context, since Hilbert spaces (as well as any of its unbounded closed and convex subsets) enjoy property U, is whether CAT(0) spaces do it too. $\mathbb{R}$-trees provide an easy answer in the negative.

Proposition 4.3. *If an unbounded $\mathbb{R}$-tree is geodesically bounded then it does not have property U.*

Proof. Let X be a geodesically bounded but unbounded $\mathbb{R}$-tree and take $x_1 \in X$. Suppose X has property U. Then, by property U, there exists K_1 an unbounded convex subset of X such that $\operatorname{dist}(x_1, K_1) \geq 1$. Take x_2 as the metric projection of x_1 onto K_1. Repeat this argument with x_2 and K_1, then we find x_3 such that $d(x_2, x_3) \geq 1$ and, since $\mathbb{R}$-trees are gated spaces as explained in Section 2, $x_2 \in (x_1, x_3)$. Proceeding in this way we obtain $\{x_n\}$ an unbounded sequence of points of X within the same ray, which contradicts our hypothesis.

Corollary 4.4. *Since it is easy to construct unbounded but geodesically bounded $\mathbb{R}$-trees, CAT(0) spaces do not have property U in general.*

On the other hand, Proposition 3.5. can be understood in the following way.

Proposition 4.5. *Any unbounded locally compact complete CAT(0) space has property U.*

Next we show an example of a non Hilbertian neither locally compact CAT(0) space with property U.

Example 4.6. *Let us consider the space of real sequences ℓ^2 and $\{e_1, e_2, \cdots\}$ the elements of its canonical basis. Then consider the following two convex sets:*

$$X_1 = \overline{\operatorname{conv}}\{0, 2e_1, 2e_2, 3e_3, 4e_4, 5e_5, \cdots\}$$

and

$$X_2 = \overline{\operatorname{conv}}\{0, e_1, 3e_2, 3e_3, 4e_4, 5e_5, \cdots\}.$$

Clearly, each of them is unbounded and geodesically bounded. Since $X_1 \cap X_2 \neq \emptyset$, we can glue X_1 and X_2 through their intersection, as shown by the basic gluing theorem in [1, Pg. 347], to obtain a new non-Hilbertian CAT(0) space

$$X = X_1 \sqcup_{X_1 \cap X_2} X_2.$$

To see that X enjoys property U it suffices to notice that for each closed convex and unbounded subset K of X at least one of the sets $K \cap X_1$ or $K \cap X_2$ must be unbounded. Let it be $K \cap X_1$. Hence, taking $x \in K$ we may consider the subset $K \cap X_1$ in the Hilbert space. From Proposition 4.2., it follows the existence of an unbounded subset K' of $K \cap X_1$ such that

$$\operatorname{dist}(x, K') \geq 1.$$

There are some other ways to use the basic gluing theorem to obtain new examples of CAT(0) spaces with property U. In fact the above reasoning works for any finite family of convex and unbounded sets $\{X_n\}$ keeping similar relations among them as in the previous example. More complicated is the case in which an infinite family of such sets is considered, as the next example shows. For this example we first recall the definition of convex hull in a geodesic space.

Definition 4.7. *Let C be a subset of a geodesic space X. Then the convex hull of C is the smallest convex set containing C, that is, the intersection of all the convex sets containing the set C.*

The convex hull of a set can also be built in the following inductive way (see [1, pg. 112] or [20, 22]): Let $G_1(C)$ denote the union of all geodesics segments with endpoints in A. Notice that C is convex if and only if $G_1(C) = C$. Recursively, for $n \geq 2$ we set $G_n(C) = G_1(G_{n-1}(C))$. Then the convex hull of C will be

$$\operatorname{conv}(C) = \bigcup_{n\in\mathbb{N}} G_n(C).$$

By $\overline{\operatorname{conv}}(C)$ we shall denote the closure of the convex hull. It is easy to see that in a CAT(0) space, the closure of the convex hull will be convex and hence it is the smallest closed convex set containing C.

Example 4.8. *As in Example 4.6., let us consider the space of real sequences ℓ^2 and $\{e_1, e_2, \cdots\}$ the elements of its canonical basis. Then consider the following sequence of unbounded but geodesically bounded closed and convex sets:*

$$\begin{aligned}
X_1 &= \overline{\operatorname{conv}}\{0, 2e_1, 2e_2, 3e_3, 4e_4, 5e_5, \cdots\}\\
X_2 &= \overline{\operatorname{conv}}\{0, e_1, 3e_2, 3e_3, 4e_4, 5e_5, \cdots\}\\
&\vdots \qquad\qquad \vdots\\
X_n &= \overline{\operatorname{conv}}\{0, e_1, 2e_2, 3e_3, \cdots, (n-2)e_{n-2}, (n-1)e_{n-1}, (n+1)e_n, (n+1)e_{n+1}, \cdots\}\\
&\vdots \qquad\qquad \vdots
\end{aligned}$$

It is immediate that

$$X_0 := \overline{\operatorname{conv}}\{0, e_1, 2e_2, 3e_3, 4e_4, 5e_5, \cdots\}$$

is contained in X_n for any n. Consider X the gluing of all the elements of $\{X_n\}$ through X_0 ([1, Pg. 347]). Then X is a CAT(0) space. Notice also that

$$\operatorname{dist}(u, X_0) \leq 1$$

for any $u \in X$. We want to show that X has property U. Let $K \subseteq X$ be closed convex and unbounded although geodesically bounded and take $x \in K$. It is enough to show that there exists $K' \subseteq K$ unbounded closed and convex such that $\operatorname{dist}_X(x, K') \geq 1$. If there exists n such that $K \cap X_n$ is unbounded then there is nothing to prove. Suppose this is not the case. Then, by Banach-Steinhaus theorem, we can fix a functional f with $\|f\| = 1$ and a value M such that $f(x) \leq M$ and the set

$$F_{M+3} = \{y \in \ell^2 \colon f(y) \geq M + 3\} \cap K$$

is unbounded. It will be enough if we show that

$$\operatorname{conv}^X(F_{M+3}) \subseteq \{y \in \ell^2 \colon f(y) \geq M + 1\},$$

where the convex hull is considered with respect to X.

Since $\mathrm{conv}^X(F_{M+3}) = \bigcup_{n\in\mathbb{N}} G_n(F_{M+3})$, *where* $G_1(F_{M+3})$ *denotes the union of all metric segments with endpoints in* F_{M+3} *and inductively* $G_{n+1}(F_{M+3}) = G_1(G_n(F_{M+3}))$, *we can assume without loss generality that* $y \in G_2(F_{M+3})$. *Then* $y \in [p, q]$, *where* $p \in [v_1, v_2]$, $q \in [v_3, v_4]$ *and* $v_i \in F_{M+3}$, $i \in \{1, 2, 3, 4\}$. *Then there is a point* p' *of metric segment* $[P_{X_0}(v_1), P_{X_0}(v_2)]$ *such that*

$$\frac{d(p, v_2)}{d(v_1, v_2)} = \frac{d(p', P_{X_0}(v_2))}{d(P_{X_0}(v_1), P_{X_0}(v_2))}$$

with $p' = P_{X_0}(v_1)$ *if* $P_{X_0}(v_1) = P_{X_0}(v_2)$. *In a similar way we can define* $q' \in [P_{X_0}(v_3), P_{X_0}(v_4)]$ *and* $y' \in [p', q']$. *Since* $\mathrm{dist}(v_i, X_0) \leq 1$, $i \in \{1, 2, 3, 4\}$, *and* $y' \in \mathrm{conv}^{\mathrm{X}}\{\mathrm{P}_{\mathrm{X}_0}(\mathrm{v_i}) : \mathrm{i} = 1, 2, 3, 4\} = \mathrm{conv}^{\ell^2}\{\mathrm{P}_{\mathrm{X}_0}(\mathrm{v_i}) : \mathrm{i} = 1, 2, 3, 4\}$, *we get that* $f(y') \geq M + 2$. *But from the Busemann convexity it follows that* $d(y, y') \leq 1$ *and hence* $f(y) \geq M + 1$, *what completes the proof.*

Another interesting class of CAT(0) spaces is provided by the so-called 0-cones of CAT(1) spaces. We show in Example 4.10. below that these spaces enjoy property U. Before introducing 0-cones let us take a look at the cones on the plane with the Euclidean metric. In particular, consider e a point in the unit sphere of the plane and S_e the intersection of $B(e, 1)$ with the unit sphere. Define next C_e as the cone generated by S_e and the origin. Consider F the orthogonal line to the line passing through the origin and e and which distance to the origin is $1/2$. Then it only requires basic trigonometry to state that

$$\mathrm{conv}(C_e \cap \{x \colon \|x\| \geq 1\}) \subseteq F^+,$$

where F^+ is the half-space determined by F not containing the origin. As a consequence, also by elementary geometry, we can deduce that

$$\mathrm{dist}(0, \mathrm{conv}(C_e \cap \{x \colon \|x\| \geq n\})) \to \infty \tag{4.1}$$

as $n \to \infty$ by just repeating the same argument under the condition $\|x\| \geq 1$.

Next we give the definition of 0-cone.

Definition 4.9. *Given a metric space Y, consider X the quotient of $[0, \infty) \times Y$ given by the equivalence relation: $(t, y) \sim (t', y')$ if $t = t' = 0$ or $t = t' > 0$ and $y = y'$ otherwise. For $x = ty$ and $x' = t'y'$ in X, define*

$$d^2(x, x') = t^2 + t'^2 - 2tt' \cos(d_\pi(y, y'))$$

where $d_\pi(y, y') := \min\{\pi, d(y, y')\}$.

Then (X, d) is the 0-cone of Y. The 0-cone of Y will be denoted as C_0Y.

A very good feature of 0-cones is that they have many 2-dimensional flat subspaces. In particular, given any geodesic segment $[y_1, y_2] \subseteq Y$ joining y_1 and y_2, the subcone $C_0[y_1, y_2]$ is isometric to the 2-dimensional Euclidean cone defined by the origin and a segment in the unit sphere of length $d(y_1, y_2)$.

Example 4.10. *Consider Y any CAT(1) space of diameter 1. Then the 0-cone of Y is a CAT(0) space with property U. Indeed, take $K \subseteq C_0Y$ geodesically bounded. Then, given any two points ty and $t'y'$ in K, the segment $[ty, t'y']$ would be in an isometric set to $C_0[y, y']$ and so, by (4.1), separated from the origin in function of* $\min\{t, t'\}$.

Remark 4.11. *The condition that diameter of Y is 1 is mainly for simplicity. Larger diameters would be allowed in the previous reasoning although not larger diameters than π.*

5 Fixed point results

In this section we show that a complete CAT(0) space furnished with property U satisfies Ray's theorem.

Theorem 5.1. *Let X be a complete CAT(0) space. Suppose also that X satisfies property U. Then a nonempty closed convex subset $Y \subset X$ has the fixed point property for nonexpansive mappings if and only if Y is bounded.*

Proof. If Y is bounded then it is a very well-known fact (see [13, 14], for instance) that Y has the fixed point property for nonexpansive mappings. Therefore, let us assume that Y is not bounded and let us see that then we can find a nonempty convex and closed subset Y' of Y with a fixed point free nonexpansive mapping $T: Y' \to Y'$.

First, if Y is geodesically unbounded then there exists a ray $\gamma \subset Y$. Make $Y' = \gamma$. Let γ be parametrized by arc-length in the usual way as $\gamma(t)$ for $t \geq 0$. Define $T: \gamma \to \gamma$ as $T(\gamma(t)) = \gamma(t+1)$, then T is a fixed point free injective isometry from Y' into Y'.

Suppose now that Y is geodesically bounded. We will construct a fixed point free nonexpansive mapping $T: Y \to Y$.

We first show that given $A \in Y$ we can find a decreasing sequence of unbounded closed and convex subsets $\{K_n\}$ of Y such that

$$\operatorname{dist}^2(A, K_n) = \inf\{d^2(A,x): x \in K_n\} \geq n \tag{5.2}$$

for any $n \in \mathbb{N}$. Indeed, fixed $A \in Y$. Then, since Y is geodesically bounded and X has property U, we can find an unbounded closed and convex subset K_1 of Y such that $\mathrm{dist(A, K_1)} \geq 1$. Take $P_1 = P_{K_1}(A)$. Notice that from the uniqueness of the metric projection onto K_1 and its definition, we have that $d(x, A) > d(A, P_1)$ for $x \in K_1$ with $x \neq P_1$. Now, apply again property U to P_1 to find another set $K_2 \subseteq K_1$, closed convex and unbounded, such that $\operatorname{dist}(P_1, K_2) \geq 1$.

Now let us estimate the distance between any point $x \in K_2$ and A. Since, by Proposition 2.5., the Alexandrov angle $\angle_{P_1}(A, x)$ is not smaller then $\pi/2$, then, for the angle $\angle_{\bar{P}_1}(\bar{A}, \bar{x})$ in the comparison triangle $\Delta(\bar{A}, \bar{P}_1, \bar{x})$ in $\mathbb{E}^2$, Proposition 2.2. implies that

$$\frac{\pi}{2} \leq \angle_{P_1}(A, x) \leq \angle_{\bar{P}_1}(\bar{A}, \bar{x})$$

holds. Therefore, again by Proposition 2.2.,

$$d^2(A, x) \geq d^2(A, P_1) + d^2(P_1, x) \geq d^2(A, P_1) + 1 \geq 2.$$

Proceeding in this way, by picking at each step the sets K_n and $P_n = P_{K_n}(A)$ as above, we can define a decreasing sequence of sets $\{K_n\}$ each of them closed convex and unbounded such that (5.2) holds, as we wished.

We claim next that

$$d(P_k, P_n) < d(A, P_n) \tag{5.3}$$

for each $k \in \{1, \ldots, n-1\}$. Let $n > 2$ and $k \in \{1, \ldots, n-1\}$. Notice that $P_n \neq P_k$ for $n \neq k$. To prove our claim it is enough to recall that $P_n \in K_k$ and $P_k = P_{K_k}(A)$. So $\angle_{P_k}(A, P_n) \geq \pi/2$ and so, attending to the comparison triangle of $\triangle(A, P_k, P_n)$ in the Euclidean plane and the CAT(0) condition, the claim follows (see Proposition 2.2.).

Now we start the construction of our fixed point free mapping. Choose a sequence of positive numbers $\{c_n\}$ such that

$$\sum_{n=1}^{\infty} c_n d(A, P_n) < \infty$$

and

$$\sum_{n=1}^{\infty} c_n = 1.$$

First let us suppose that T_1 is the identity map on $K_0 = Y$. A sequence of mappings $\{T_n\}$ can be defined in an inductive way as shown next:

$$T_{n+1}(x) = (1 - c_n)T_n(x) + c_n P_{K_n}(x).$$

Notice that the fact that the mappings $\{T_n\}$ defined in this way are nonexpansive follows from the facts that they are convex combination of two nonexpansive mappings (by induction) and the Busemann convexity of CAT(0) spaces applied at each inductive step. Notice also that if $x \in K_n$ then $T_i(x) = x$ for $i \le n + 1$. We want to show that $\{T_n\}$ is pointwise convergent on Y. Indeed, let $x \in Y$ then it is sufficient to show that

$$\sum_{k=n}^{m-1} d(T_k(x), T_{k+1}(x)) \to 0$$

as $m, n \to \infty$. Let us show first that this is the case for $x = A$. By construction,

$$T_n(A) \in \overline{\text{conv}}\{A, P_1, \dots, P_{n-1}\}.$$

We claim now that

$$d(T_n(A), P_n) < d(A, P_n) \tag{5.4}$$

for $n \ge 2$.

For $n = 2$ we need to show that $d(T_2(A), P_2) < d(A, P_2)$. But

$$T_2(A) = (1 - c_1)A + c_1 P_1$$

with $c_1 > 0$ and, of course, $A \ne P_1$. By (5.3), $d(P_1, P_2) < d(A, P_2)$ and so a direct application of Proposition 2.3. implies that $d(T_2(A), P_2) < d(A, P_2)$.

Let us consider a general n now. Then, again by (5.3), $d(P_1, P_n) < d(A, P_n)$. Furthermore, $T_2(A) \in [P_1, A]$, therefore, again by Proposition 2.3.,

$$d(T_2(A), P_n) < d(A, P_n).$$

Recalling (5.3), we can continue in this way to deduce that $d(T_3(A), P_n) < d(A, P_n)$. Continuing in this way till we reach $T_n(A)$ completes the proof of the claim.

The next follows from our claim,

$$\sum_{k=n}^{m-1} d(T_k(A), T_{k+1}(A)) = \sum_{k=n}^{m-1} c_k d(T_k(A), P_k) \tag{5.5}$$

$$< \sum_{k=n}^{m-1} c_k d(A, P_k) \to 0 \text{ if } n, m \to \infty, \tag{5.6}$$

which, in particular, shows that $\{T_n(A)\}$ is a Cauchy sequence and so convergent.

Take $x \in Y$, from the nonexpansiveness of the metric projection, we have that

$$d(P_k, P_{K_k}(x)) \leq d(A, x)$$

and, as a direct application of Busemann convexity, by induction we can assure that

$$d(T_k(A), T_k(x)) \leq d(A, x)$$

holds for any k. Combining this with (5.4) and (5.6), we obtain

$$\sum_{k=n}^{m-1} d(T_k(x), T_{k+1}(x)) = \sum_{k=n}^{m-1} c_k d(T_k(x), P_{K_k}(x))$$

$$\leq \sum_{k=n}^{m-1} c_k d(T_k(x), T_k(A)) + \sum_{k=n}^{m-1} c_k d(T_k(A), P_k) + \sum_{k=n}^{m-1} c_k d(P_k, P_{K_k}(x))$$

$$\leq \sum_{k=n}^{m-1} c_k d(A, P_k) + 2\sum_{k=n}^{m-1} c_k d(A, x) \to 0 \text{ if } n, m \to \infty.$$

Since Y is a closed subset of a complete metric space, $T_n(x) \to T(x)$ for each $x \in Y$. Furthermore, since each T_n is a nonexpansive mapping, so is T.

Now we will show that T is fixed point free. Let us take any $x \in Y$. Since, by (5.2), $d(A, P_n) \to \infty$ as $n \to \infty$, x must belong to a finite number of subsets K_n. Let $N \in \mathbb{N}$ be the smallest natural number such that $x \notin K_N$ and let us denote $P_{K_N}(x) = P_N(x)$. Then $T_{N+2}(x)$ is in $[T_{N+1}(x), P_{N+1}(x)]$. Remember that $T_{N+1}(x) \in [x, P_N(x)] \setminus \{x, P_N(x)\}$ and let us consider $\triangle(\bar{T}_{N+1}(x), \bar{P}_{N+1}(x), \bar{P}_N(x))$ the comparison triangle of $\triangle(T_{N+1}(x), P_{N+1}(x), P_N(x))$. Then it is clear that

$$\angle_{\bar{P}_N(x)}(\bar{T}_{N+1}(x), \bar{P}_{N+1}(x)) \geq \pi/2.$$

So, for $y \in [T_{N+1}(x), P_{N+1}(x)]$, we have that

$$\begin{aligned} \operatorname{dist}(y, K_N) &\leq \operatorname{dist}(y, [P_N(x), P_{N+1}(x)]) \leq \operatorname{dist}(\bar{y}, [\bar{P}_N(x), \bar{P}_{N+1}(x)]) \\ &< \operatorname{dist}(\bar{x}, [\bar{P}_N(x), \bar{P}_{N+1}(x)]) = d(\bar{x}, \bar{P}_N(x)) \\ &= d(x, P_N(x)) = \operatorname{dist}(x, K_N), \end{aligned}$$

therefore

$$\operatorname{dist}(T_{N+2}(x), K_N) < \operatorname{dist}(x, K_N).$$

Now, if we repeat the same argument in an inductive way for $n \geq N+2$, we obtain that

$$\begin{aligned} \operatorname{dist}(T(x), K_N) &\leq \operatorname{dist}(T_{n+1}(x), K_N) \\ &\quad < \operatorname{dist}(T_n(x), K_N) < \operatorname{dist}(x, K_N) \end{aligned}$$

at least at a certain moment $T_n(x) \in K_N$. Either case, by taking limit, we obtain that

$$\operatorname{dist}(T(x), K_N) < \operatorname{dist}(x, K_N)$$

and so $T(x) \neq x$.

Since each space with all unbounded subsets being geodesically unbounded obviously satisfies property U, we will focus on spaces with geodesically bounded subsets. More precisely, in the sequel we suppose that for each space X there is at least one $Y \subset X$ which is unbounded but geodesically bounded.

As a corollary of the proof of Theorem 5.1. we obtain the following.

Corollary 5.2. *Let X be an unbounded and complete CAT(0) space with property U. Let $x \in X$ and $C \subseteq X$ unbounded closed and convex. Then, if X is geodesically bounded, for $R > 0$ there exist $D \subseteq C$ unbounded closed and convex such that* $\operatorname{dist}(x, D) \geq R$.

The next corollary also follows as an immediate consequence of Theorem 5.1. and results in Section 3.

Corollary 5.3. *The Hilbert ball with the hyperbolic metric and infinite dimensional negative constant curvature spaces fail property U.*

Our next goal is to show that this latter corollary actually holds in CAT(κ) spaces with $\kappa < 0$. We first need a technical result which will follow as a corollary of the next fact on $\mathbb{H}^2$. The Poincaré model of the 2-dimensional hyperbolic model space of constant curvature -1 [1, pg. 18], which we describe next, will be required. Let B be the open unit ball of $\mathbb{R}^2$, then the Poincaré ball model is B endowed with the hyperbolic distance given by

$$d(x,y) = \log\left(\frac{\|x - y_\infty\| \cdot \|y - x_\infty\|}{\|x - x_\infty\| \cdot \|y - y_\infty\|}\right),$$

where $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^2$ and the points x_∞ and y_∞ are chosen so that both have norm 1 and the circle joining all four points is orthogonal to the unit circle of $\mathbb{R}^2$. Points are also considered to be arranged in the order x_∞, x, y and y_∞ in this circle. One of the main features of this model is that angles between geodesics (circles and lines orthogonal to the unit circle) are the same as angles in $\mathbb{H}^2$ (for more details on this model see [1, p. 86-88]).

Lemma 5.4. *In the Poincaré ball model consider 0 as the origin. Then, for $\gamma > 0$ given, there exists $R > 0$ such that*

$$B(0,R) \cap [u_1, u_2] \neq \emptyset$$

for any u_1 and u_2 such that $\angle_0(u_1, u_2) \geq \gamma$.

Proof. Let v_1 and v_2 be norm one points such that $u_i \in [0, v_i]$ for each i. Since $\angle_0(u_1, u_2) \geq \gamma$, the uniform convexity of the Hilbert spaces implies that there exists $\varepsilon > 0$ such that if $w = (v_1 + v_2)/2$ then

$$\|w\| \leq 1 - \varepsilon := M.$$

From where it follows that

$$\begin{aligned}\operatorname{dist}(0, [u_1, u_2]) &\leq \operatorname{dist}(0, [v_1, v_2]) \leq d(0, w)\\ &\leq \log\frac{1+M}{1-M}.\end{aligned}$$

Corollary 5.5. *Let X be a CAT(κ) space with $\kappa < 0$ and let x_0, x and $y \in X$ such that there exists $r, \varepsilon > 0$ with $d(u,v) \geq \varepsilon$, where u and v are, respectively, the metric projection of x and y onto $B(x_0, r)$, then there exists $R > 0$, depending only on r and ε, such that*

$$B(x_0, R) \cap [x, y] \neq \emptyset.$$

Proof. We can assume that X is a CAT(-1) space. Let x_0, y, x, u and v be as in the statement. Consider the triangle $\Delta = \Delta(x_0, x, y)$ and its comparison one $\bar{\Delta} = \Delta(\bar{x}_0, \bar{x}, \bar{y})$ in $\mathbb{H}^2$. Hence, from CAT(-1) inequality, $d(\bar{u}, \bar{v}) \geq d(u,v) \geq \varepsilon$ where, as customary, $\bar{u}$ and $\bar{v}$ are the corresponding point of u and v, respectively, in $\bar{\Delta}$. Therefore,

$$\gamma = \angle_{\bar{x}_0}(\bar{x}, \bar{y}) = \angle_{\bar{x}_0}(\bar{u}, \bar{v}) > 0.$$

Now, on account of Lemma 5.4., we have that there exists $R > 0$, depending only on r and ε, such that

$$[\bar{x}, \bar{y}] \cap B(\bar{x}_0, R) \neq \emptyset.$$

So there exists $\bar{p} \in [\bar{x}, \bar{y}] \cap B(\bar{x}_0, R)$ and, by taking $p \in [x, y]$ as the corresponding point to $\bar{p}$ in Δ, we finally have that $B(x_0, R) \cap [x, y] \neq \emptyset$ with R depending only on r and ε.

Next we reach to the announced result.

Theorem 5.6. *Let X be an unbounded and complete CAT(κ) space, $\kappa < 0$, containing unbounded but geodesically bounded subsets, then X fails property U.*

Proof. From the definition of property U we reduce ourselves to the case in which X is geodesically bounded. Fix $x_0 \in X$ and assume that X has property U, then, from Corollary 5.2., we can consider (A_n) as a decreasing sequence of unbounded closed and convex subsets of X such that $\lim_{n\to\infty} \operatorname{dist}(x_0, A_n) = \infty$. Without loss of generality, we can assume that $\operatorname{dist}(x_0, A_n) > n$.

For each $N \in \mathbb{N}$ consider the sequence of sets, for $n \geq N$,

$$E_n^N = \overline{P_{B(x_0,N)}(A_n)},$$

that is, the closed closure of $P_{B(x_0,N)}(A_n)$ where $P_{B(x_0,N)}$ stands for the metric projection onto $B(x_0, N)$, and the limits

$$d_N = \lim_{n\to\infty} \operatorname{diam}(E_n^N).$$

Notice that, since the sequence (A_n) is decreasing, limits d_N are well-defined.

Assume first that $d_N = 0$ for each N. Then, from completeness, there exists a point x_N such that

$$\{x_N\} = \bigcap_{n\geq N} E_n^N$$

for each $N \in \mathbb{N}$. Let $N > 1$ arbitrary, then, for $n \geq N$, there exists $y_n \in A_n$ such that $u_n = P_{B(x_0,N)}(y_n)$ is such that the sequence (u_n) converges to x_N as n goes to infinity. Now, if we consider the triangles $\Delta(x_0, x_N, u_n)$ and their comparison ones in $\mathbb{H}^2$ we obtain that if $u_n^m \in [x_0, u_n]$, for $1 \leq m \leq N$, is such that $d(x_0, u_n^m) = m$ then the sequence (u_n^m) converges to x_m. Therefore, all the points $x_0, x_1, \cdots, x_N$ stand on a same geodesic segment. Since N was arbitrary we obtain the X contains a geodesic ray which is a contradiction with our assumptions.

Therefore there must be $N \geq 1$ such that $d_N > 0$. Now, it is enough to recall Corollary 5.5. to reach to a contradiction to the fact that the sets A_n are as far from x_0 as one may wish, which completes the proof of the theorem.

After this theorem we raise the following conjecture.

Conjecture. Let $\kappa < 0$, then it is always possible to find an unbounded but geodesically bounded closed and convex subset in any non locally compact and complete CAT(κ) spaces with the fixed point property of nonexpansive mappings. That is, Ray's theorem does not hold in unbounded non locally compact and complete CAT(κ) spaces with $\kappa < 0$.

The following questions are also natural after the results presented in this work.

Remark 5.7. *i) Example 3.3. shows that it is not necessary for a CAT(0) space to be a CAT(κ) space for some $\kappa < 0$ in order to fail Ray's theorem. Still the space provided by Example 3.3. is δ-hyperconvex (see [4, Chapter 1] for definition and properties). Therefore* *one step farther in the above problem is to consider whether Ray's theorem fails on any non locally compact and complete CAT(0) space which is δ-hyperbolic for some $\delta \geq 0$. Or, even more, on any Busemann convex and δ-hyperbolic non locally compact and complete geodesic space.*

ii) Any nonexpansive self-mapping defined on an unbounded closed and convex subset of an $\mathbb{R}$-tree, the Hilbert ball or a space of constant negative curvature has a fixed point if and only if it is geodesically bounded. May this same result be obtained for CAT(κ) spaces with $\kappa < 0$?

iii) Is property U a necessary condition for a CAT(0) space to satisfy Ray's theorem?

Appendix: Modulus of convexity in CAT(κ) spaces with $\kappa < 0$.

Lemma 5.4. and Corollary 5.5. basically tell that very large triangles look like triangles of $\mathbb{R}$-trees in a certain uniform way in CAT(κ) spaces with $\kappa < 0$ (for simplicity, from now on, we will refer to these spaces as *negative CAT(κ) spaces*). Motivated by this fact we propose the following study the modulus of convexity at infinity of negative CAT(κ) spaces. We want to show that, in fact, the modulus of convexity of any unbounded negative CAT(κ) space tends at infinity to the modulus of convexity of $\mathbb{R}$-trees. For that we first need to properly introduce the notion of modulus of convexity in geodesic spaces. This kind of modulus (see [9, Chapter 6]) has been a very useful tool in geometry of Banach spaces and it basically gives information about how square or rotund balls are. In particular, it measures the uniform convexity of a Banach space. Next we give its definition in geodesic spaces (see also [10, 27]).

Definition 5.8. *A geodesic metric space (X, d) is said to be uniformly convex if for any $r > 0$ and any $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $a, x, y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \geq \varepsilon r$ it is the case that*

$$d(m, a) \leq (1 - \delta)r$$

where m stands for any midpoint of any geodesic segment $[x, y]$. A mapping $\delta : (0, +\infty) \times (0, 2] \to (0, 1]$ providing such a $\delta = \delta(r, \varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0, 2]$ is called a modulus of uniform convexity.

While in Banach spaces there exists a natural modulus of convexity for each space which only depends on ε, the situation is much more complicated in general geodesic spaces where in general we need to assume that the modulus depends on three variables: the center of the ball, the radius of the ball and the separation condition given by ε. The *best* modulus of convexity of a geodesic space is the one that gives the largest δ for each a, r and ε. From the CN-inequality for CAT(0) spaces (see [1, pg. 163]) it is immediate that the modulus of convexity of a Hilbert space is a modulus of convexity for any CAT(0) space and so, in particular, any CAT(0) space is uniformly convex (see [6, 7, 8, 11, 15, 19] for more on this fact). That is, if $\delta(a, r, \varepsilon)$ is the best modulus of convexity of a CAT(0) space then it must be the case that

$$\delta(a, r, \varepsilon) \geq \delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \tag{5.7}$$

where δ_H is the modulus of convexity of a (real) Hilbert space (see also [27, pg. 541]).

The best modulus of convexity can be easily calculated for $\mathbb{R}$-trees and spaces of constant curvature. We only consider the case of spaces of nonpositive constant curvature because it is the one we are interested in, notice however that things are dramatically different if the space were of positive constant curvature (see for instance [6]).

Theorem 5.9. *The following holds:*

i) The best modulus of convexity of an $\mathbb{R}$-tree coincides with the modulus of the real line, that is, it can be written as

$$\delta_{\mathbb{R}}(\varepsilon) = \frac{\varepsilon}{2}.$$

ii) If X is a space of constant curvature $\kappa < 0$ then its best modulus of convexity is given by

$$\delta_\kappa(r,\varepsilon) = 1 - \frac{1}{r}\operatorname{arccosh}\frac{\cosh\sqrt{-\kappa}r}{\cosh\frac{\varepsilon\sqrt{-\kappa}r}{2}}.$$

Therefore the best universal modulus of convexity for a CAT(κ) space is the above.

Proof. The case of $\mathbb{R}$-trees is immediate. The case of spaces of constant curvature $\kappa < 0$ follows in an straightforward way from the cosine law [1, pg. 24] for these spaces. We omit details.

The best modulus of convexity was also estimated for the Hilbert ball with the hyperbolic metric in [10, pg. 107] (see also [10, pg. 72] for the Poincaré disk model). It takes only some rudimentary trigonometric manipulations to show that the modulus of the Hilbert ball found in [10] actually coincides with δ_{-1} in the above theorem. Some misleading remarks were given in [10, pg. 72,107] about the shape of balls in the Hilbert ball model which were already noticed in [28]. It is also claimed in [10] that

$$\lim_{r\to\infty} \delta_{-1}(r,\varepsilon) = 0$$

for each fixed ε. From (5.7) it is obvious that this estimation is not correct. The final result of this work shows that this limit can actually be calculated for any unbounded negative CAT(κ) space as we show next.

Theorem 5.10. *Let X be an unbounded negative CAT(κ) space. Let $\delta(a,r,\varepsilon)$ be the best modulus of convexity for X. Then, for any a and ε fixed,*

$$\lim_{r\to\infty} \delta(a,r,\varepsilon) = \frac{\varepsilon}{2}.$$

Proof. It is clear that $\delta(a,r,\varepsilon) \le \frac{\varepsilon}{2}$ and so it is enough to show that

$$\liminf_{r\to\infty} \delta(a,r,\varepsilon) \ge \frac{\varepsilon}{2}.$$

For simplicity we may assume that X is a CAT(-1) space, for other cases we just obtain a factor which cancels after following the same reasoning. Since X is CAT(-1) a simple comparison procedure shows that δ_{-1} is a modulus of convexity for X, therefore

$$\delta(a,r,\varepsilon) \ge \delta_{-1}(r,\varepsilon)$$

and so it suffices to show that $\lim_{r\to\infty} \delta_{-1}(r,\varepsilon) = \frac{\varepsilon}{2}$. Let

$$h(r) = \operatorname{arccosh} \frac{\cosh r}{\cosh \frac{\varepsilon r}{2}}.$$

Then,

$$\cosh h(r) = \frac{\cosh r}{\cosh \frac{\varepsilon r}{2}}.$$

Since $\cosh x \sim \frac{e^x}{2}$ when x goes to infinity, we have

$$\frac{e^{h(r)}}{2} \sim \frac{e^r}{e^{\frac{\varepsilon r}{2}}}$$

and so

$$h(r) \sim r(1 - \frac{\varepsilon}{2}) + \ln 2$$

what finally leads to the conclusion.

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