Logarithmic Cohomology of the Complement of a Plane Curve

Francisco Calderón Moreno[*], David Mond,
Luis Narváez Macarro and Francisco Castro Jiménez[†]

December 10, 1999

Abstract

Let D, x be a plane curve germ. We prove that the complex $\Omega^\bullet(\log D)_x$ computes the cohomology of the complement of D, x *only if* D is quasihomogeneous. This is a partial converse to a theorem of [5], which asserts that this complex does compute the cohomology of the complement, whenever D is a locally weighted homogeneous free divisor (and so in particular when D is a quasihomogeneous plane curve germ). We also give an example of a free divisor in $D \subset \mathbb{C}^3$ which is not locally weighted homogeneous, but for which this (second) assertion continues to hold.

1 Introduction

In [5] the last three authors showed that if D is a locally quasi-homogeneous free divisor in the complex manifold X then locally the complex $\Omega^\bullet(\log D)$ of holomorphic differential forms with logarithmic poles along D calculates the cohomology of the complement of D in X. More precisely, the following two equivalent statements hold:

Theorem 1.1.– With D as above,

1. If $V \subset X$ is a Stein open set then the de Rham map (integration of forms over cycles) gives rise to an isomorphism

$$h^k(\Gamma(V, \Omega^\bullet(\log D))) \xrightarrow{\sim} H^k(V \setminus D; \mathbb{C}).$$

[*]Supported by MEC of Spain and EPSRC of United Kingdom.
[†]The first and the last two authors are partially supported by PB97-0723.

2. Denoting by U the complement of D in X and by $j : U \hookrightarrow X$ the inclusion,, the de Rham morphism gives rise to an isomorphism

$$\Omega^\bullet(\log D) \xrightarrow{\sim} \mathbf{R}j_*(\mathbb{C}_U).$$

□

By analogy with Grothendieck's Comparison Theorem [7], in which the complex $\Omega^\bullet(\log D)$ is replaced in these two statements by $\Omega^\bullet(*D)$, but which holds for an arbitrary divisor, we summarise this with a slogan: if $D \hookrightarrow X$ is a locally quasi-homogeneous free divisor then the *logarithmic comparison theorem* holds.

The definition of local quasi-homogeneity, (called *strong* quasi-homogeneity in [5]), is as follows:

Definition 1.2.–

1. The polynomial $h(z_1, \cdots, z_n) = \sum a_{i_1,\cdots,i_n} z_1^{i_1} \cdots z_n^{i_n} \in \mathcal{O}_{\mathbb{C}^n}$ is *weighted homogeneous* if there exist positive integer weights $w_1, \cdots, w_n$ such that $h(z_1^{w_1}, \cdots, z_n^{w_n})$ is homogeneous.

2. The divisor $D \subset X$ is *locally quasi-homogeneous* if for all $x \in D$ there are local coordinates on X, centered at x, with respect to which D has a weighted homogeneous defining equation.

Every plane curve is a free divisor, since the module of logarithmic vector fields $\mathcal{D}\mathrm{er}(\log D)$ is reflexive and thus has depth at least 2. In [4, Cor. 4.2.2] the first author showed that if D is a plane curve then the logarithmic de Rham complex $\Omega^\bullet(\log D)$ is perverse, a necessary condition for the logarithmic comparison theorem.

In [6] the logarithmic comparison theorem has been tested for the following non locally quasi-homogeneous plane curve (cf. [8]): $D = \{f = x_1^4 + x_2^5 + x_2^4 x_1 = 0\} \subset X = \mathbb{C}^2$. A basis for $\mathcal{D}\mathrm{er}(\log D)$ is given by:

$$\delta_1 = (16x_1^2 + 20x_1x_2)\frac{\partial}{\partial x_1} + (12x_1x_2 + 16x_2^2)\frac{\partial}{\partial x_2}$$
$$\delta_2 = (16x_1x_2^2 + 4x_2^3 - 12x_1x_2)\frac{\partial}{\partial x_1} + (12x_2^3 - 4x_1^2 + 5x_1x_2 - 100x_2^2)\frac{\partial}{\partial x_2}.$$

Let $\mathcal{D}_X$ be the sheaf of linear differential operators with holomorphic coefficients on X and I the left $\mathcal{D}_X$-ideal generated by δ_1, δ_2. By [4, Th. 4.2.1], we have a (canonical) isomorphism (in the derived category)

$$\Omega^\bullet(\log D) \simeq \mathbf{R}\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/I, \mathcal{O}_X),$$

and so we can compute the characteristic cycle $CC(\Omega^\bullet(\log D))$ as the cycle Z in T^*X determined by the ideal $J = \sigma(I)$ generated by the principal symbols of elements in I. The symbols $\sigma_1 = \sigma(\delta_1), \sigma_2 = \sigma(\delta_2)$ form a regular sequence in $\mathcal{O}_{T^*X}$ and so, by [4, Prop. 4.1.2], the ideal J is generated by σ_1, σ_2. An easy computation shows that the multiplicity of the conormal at 0 in Z is 4. On the other hand, the multiplicity of the conormal at 0 in $CC(\mathbf{R}j_*(\mathbb{C}_U))$ is equal to $\mathrm{mult}_0(D) - 1 = 3$ (cf. [3]), and so the logarithmic comparison theorem does not hold for D.

For the family of non locally quasi-homogeneous plane curves (cf. [8])

$$x_1^q + x_2^p + x_2^{p-1}x_1 = 0, \quad p \geq q + 1 \geq 5,$$

the multiplicities of the conormal at 0 in $CC(\Omega^\bullet(\log D))$ and in $CC(\mathbf{R}j_*(\mathbb{C}_U))$ are $2(q-2)$ and $q-1$ respectively, and so these curves also do not satisfy the logarithmic comparison theorem.

A natural question is therefore whether or not the logarithmic comparison theorem holds for a given free divisor.

The purpose of this paper is to prove a partial converse to theorem 1.1. We prove:

Theorem 1.3.– Let D be a plane curve. If the logarithmic comparison theorem holds for D, then D is locally quasi-homogeneous.

Our proof shows that if h is a local equation of D, and the logarithmic comparison theorem holds, then there is a vector field germ χ such that $\chi \cdot h = h$. As a reduced curve has isolated singularities, we can then apply the theorem of K. Saito [9]: if $h \in \mathcal{O}_{\mathbf{C}^n,0}$ has isolated singularity and h belongs to its Jacobian ideal J_h then in suitable coordinates h is weighted homogeneous.

We conjecture that in higher dimensions the following version of our theorem 1.3 holds:

Conjecture 1.4.– If $D \hookrightarrow X$ is a free divisor and if the logarithmic comparison theorem holds, then for all $x \in D$ there is a local equation h for D around x, and a germ of vector field χ vanishing at x such that $\chi \cdot h = h$.

A singular free divisor of dimension greater than 1 has non-isolated singularities, so even if this conjecture is true, Saito's theorem cannot be used to deduce local quasi-homogeneity. Indeed, it is *not* true in higher dimensions that if the logarithmic comparison theorem holds for a free divisor D then D is necessarily locally quasi-homogeneous. This is shown by an example in Section 4 below: the logarithmic comparison theorem holds for the free divisor

$$D = \{(x, y, z) : xy(x + y)(zx + y) = 0\},$$

(the total space of a family of four lines in the plane with varying cross-ratio, cf. [4]), in the neighbourhood of $(0, 0, \lambda)$, with $\lambda \in \mathbb{C} \setminus \{0, 1\}$; however it is well known that this divisor is not locally quasi-homogeneous. On the other hand, it does satisfy Conjecture 1.4.

2 Preliminary results

In this section we recall the spectral sequence argument used in [5] to compare the cohomology of the logarithmic complex $\Omega^\bullet(\log D)$ with the cohomology of $X \setminus D$. Except for referring to "local" rather than "strong" quasi-homogeneity, we will use the same notation as [5].

Without loss of generality we assume $X = \mathbb{C}^n$ and $x_0 = 0$. Let V be a Stein neighbourhood (sufficiently small) of 0, let $\mathcal{U}$ be the open cover of $V \setminus \{0\}$ consisting of the sets $U_i = V \cap \{z_i \neq 0\}$, and let $\mathcal{U}'$ be the open cover of $V \setminus D$ consisting of the open sets $U_i' = (V \setminus D) \cap \{z_i \neq 0\} = U_i \setminus D$.

We consider the two double complexes

$$K^{p,q} = \check{C}^q(\mathcal{U}, \Omega^p(\log D))$$

and

$$\tilde{K}^{p,q} = \check{C}^q(\mathcal{U}', \Omega^p),$$

equipped with the exterior derivative d (the horizontal differential) and the Čech differential δ (the vertical differential). There is an obvious restriction morphism $\rho_{p,q} : K^{p,q} \to \tilde{K}^{p,q}$ which commutes with both differentials, and thus gives rise to morphisms of the two spectral sequences arising from each double complex. These spectral sequences have E_1 terms

$$''E_1^{p,q} = \check{H}^q(\mathcal{U}, \Omega^p(\log D))$$

$$''\tilde{E}_1^{p,q} = \check{H}^q(\mathcal{U}', \Omega^p)$$

$$'E_1^{p,q} = \oplus_{1 \leq i_1 < \cdots < i_{q+1} \leq n} h^p(\Gamma(\bigcap_j U_{i_j}, \Omega^\bullet(\log D)))$$

$$'\tilde{E}_1^{p,q} = \oplus_{1 \leq i_1 < \cdots < i_{q+1} \leq n} h^p(\Gamma(\bigcap_j U'_{i_j}, \Omega^\bullet)).$$

As both $\mathcal{U}$ and $\mathcal{U}'$ are Stein covers,

$$\check{H}^q(\mathcal{U}, \Omega^p(\log D)) = \check{H}^q(V \setminus \{0\}, \Omega^p(\log D))$$

and

$$\check{H}^q(\mathcal{U}',\Omega^p)) = \check{H}^q(V\setminus D,\Omega^p))$$

As $V\setminus D$ is Stein, $\check{H}^q(V\setminus D,\Omega^p)=0$ if $q>0$. It follows that

$$''\tilde{E}_2^{p,q} = \begin{cases} H^p(V\setminus D;\mathbb{C}) & \text{if } q=0\\ 0 & \text{if } q\neq 0\end{cases},$$

and in particular the spectral sequence $''\tilde{E}$ converges to the cohomology of $V\setminus D$. Now assume that outside 0, D is locally quasi-homogeneous, so that by 1.1 $\mathbf{R}j_*(\mathbb{C}_U)\simeq\Omega^\bullet(\log D)$, again outside 0. As $\mathcal{U}$ and $\mathcal{U}'$ are Stein covers, by 1.1 the quotient of the restriction $\rho_{p,q}$ defines an isomorphism $'\rho_{p,q}:\ 'E_1^{p,q}\to'\ \tilde{E}_1^{p,q}$ for all p,q. This isomorphism persists to give an isomorphism of the cohomology of the total complexes K^{tot} and $\tilde{K}^{\text{tot}}$ as calculated by the spectral sequences. It follows that the spectral sequence $''E$, like $''\tilde{E}$, also converges to the cohomology of $V\setminus D$:

$$H^k(V\setminus D;\mathbb{C})\simeq\oplus_{p+q=k}\ ''E_\infty^{p,q}.$$

As D is a free divisor, $\check{H}^q(V\setminus\{0\},\Omega^p(\log D))=0$ for $q\neq 0,n-1$, so $''E_1$ has only two non-null rows; writing for the moment $\Omega^p(D)$ and V^* in place of $\Omega^p(\log D)$ and $V\setminus\{0\}$, $''E_1$ thus looks like

$$\begin{array}{ccccccccc}
\check{H}^{n-1}(V^*,\Omega^0(D)) & \stackrel{d_1}{\to} & \cdots & \stackrel{d_1}{\to} & \check{H}^{n-1}(V^*,\Omega^p(D)) & \stackrel{d_1}{\to} & \cdots & \stackrel{d_1}{\to} & \check{H}^{n-1}(V^*,\Omega^n(D))\\
0 & & \cdots & & 0 & & \cdots & & 0\\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots\\
0 & & \cdots & & 0 & & \cdots & & 0\\
\Gamma(V,\Omega^0(D)) & \stackrel{d_1}{\to} & \cdots & \stackrel{d_1}{\to} & \Gamma(V,\Omega^p(\log D)) & \stackrel{d_1}{\to} & \cdots & \stackrel{d_1}{\to} & \Gamma(V,\Omega^n(\log D)).
\end{array}$$

(Note that as $n\geq 2$ and as the $\Omega^p(\log D)$ are free modules, we have $\Gamma(V^*,\Omega^p(D))=\Gamma(V,\Omega^p(D))$.)

As this spectral sequence converges to the cohomology of $V\setminus D$, we have

$$H^{n-1}(V\setminus D;\mathbb{C})\simeq E_\infty^{0,n-1}\oplus\cdots\oplus E_\infty^{n-1,0} = E_{n+2}^{0,n-1}\oplus h^{n-1}(\Gamma(V,\Omega^\bullet(\log D)))$$

$$H^n(V\setminus D;\mathbb{C}) = E_\infty^{0,n}\oplus\cdots\oplus E_\infty^{0,n} = E_{n+2}^{1,n-1}\oplus\frac{h^n(\Gamma(V,\Omega^\bullet(\log D)))}{d_{n+1}(E_{n+2}^{0,n-1})},$$

where

$$E_{n+2}^{0,n-1} = \text{Ker } d_1 : \check{H}^{n-1}(V^*,\Omega^0(D))\to\check{H}^{n-1}(V^*,\Omega^1(D)).$$

In [5], the main theorem was proved by showing that if D is locally quasi-homogeneous then the complex

$$(\check{H}^{n-1}(V\setminus\{0\},\Omega^\bullet(\log D)),d_1)$$

is exact.

3 Proof of the theorem

We continue with the discussion of the last paragraph. If the natural morphism $\Omega^\bullet(\log D) \to \mathbf{R}j_*(\mathbb{C}_U)$ is a quasi-isomorphism (i.e. if the logarithmic comparison theorem holds for D) then by the formulae of the last section, $d_1 : \check{H}^{n-1}(V \setminus \{0\}, \Omega^0(\log D)) \to \check{H}^{n-1}(V \setminus \{0\}, \Omega^1(\log D))$ is injective.

Let $\{\omega_1, \cdots, \omega_n\}$ be a free basis of $\Omega^1(\log D)$ as $\mathcal{O}_V$-module, and let $\delta_1, \cdots, \delta_n$ be the dual basis of $\mathcal{D}\mathrm{er}(\log D)$. Then $\check{H}^{n-1}(V \setminus \{0\}, \Omega^0(\log D)) = \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})$ and $\check{H}^{n-1}(V \setminus \{0\}, \Omega^1(\log D)) \simeq \oplus_1^n \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})$. The morphism $d_1 : \check{H}^{n-1}(V \setminus \{0\}, \Omega^0(\log D)) \to \check{H}^{n-1}(V \setminus \{0\}, \Omega^1(\log D))$ now becomes

$$\begin{array}{ccc} \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n}) & \xrightarrow{d_1} & \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})^n \\ [g] & \mapsto & ([\delta_1 \cdot g], \cdots, [\delta_n \cdot g]). \end{array}$$

where $g \in \Gamma(V \setminus \cup_i \{z_i = 0\}, \mathcal{O}_{\mathbb{C}^n}) = \Gamma(\mathbb{C}^n \setminus \cup_i \{z_i = 0\}, \mathcal{O}_{\mathbb{C}^n})$ represents the class $[g]$ in $\check{H}^{n-1}(\mathbb{C}^n \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})$.

For $\delta \in \mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n})$, we denote by d_δ the homomorphism

$$d_\delta : \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n}) \to \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n}), \quad d_\delta([g]) = [\delta \cdot g].$$

Proposition 3.5.– Let $\delta \in \mathbf{m}_{\mathbb{C}^n,0} \mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n})$,

$$\delta = (x_1, \cdots, x_n) \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n \end{pmatrix} + \delta_{\geq 1}$$

with the $a_{i,j} \in \mathbb{C}$ and $\delta_{\geq 1} \in \mathbf{m}^2_{\mathbb{C}^n,0} \mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n})$. If d_δ is injective, then the eigenvalues of A do not satisfy any relation with positive integer coefficients (in this case, we will say that δ satisfies condition (I)).

Proof: By a coordinate change we can make A lower triangular. Its eigenvalues $a_1, \cdots, a_n$ are then the elements of the diagonal. The group $\check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})$ is isomorphic to the space of Laurent series, convergent for all $\underline{x} = (x_1, \cdots, x_n)$ with $\underline{x} \neq 0$, whose non-zero coefficients are those with strictly negative indices in all variables, i.e.

$$\sum_{i_1, \cdots, i_n < 0} a_{i_1, \cdots, i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

For $p \geq n$, we set

$$G^p = \{ \sum_{\substack{i_1, \cdots, i_n < 0 \\ i_1 + \cdots + i_n = -p}} c_{\mathbf{i}} x_1^{i_1} \cdots x_n^{i_n} \}, \quad F^p = \{ \sum_{\substack{i_1, \cdots, i_n < 0 \\ i_1 + \cdots + i_n \geq -p}} c_{\mathbf{i}} x_1^{i_1} \cdots x_n^{i_n} \}.$$

Then $F^p = G^p \oplus G^{p-1} \oplus \cdots \oplus G^n$. Each G^p is a finite-dimensional $\mathbb{C}$-vector space, whose dimension we denote by r_p, and d_δ restricts to morphisms of vector spaces

$$d_{\delta\ |F^p} : F^p \to F^p$$

and

$$d_{\delta\ |G^p} : G^p \to F^p.$$

Let us denote by $d^p_{\delta,p}$ the component of this second restriction lying in G^p. Then $d^p_{\delta,p}$ depends only on the weight 0 part δ_0 of δ. We claim that with respect to a suitable ordered basis of G^p, its matrix $[d^p_{\delta,p}]$ is lower triangular.

As basis for G^p we take the monomials

$$\frac{1}{x_1^{i_1} \cdots x_n^{i_n}}$$

with $i_1 + \cdots + i_n = p$.

We have

$$d_\delta(x_1^{-i_1} \cdots x_n^{-i_n}) = -\sum_{j,k} i_k\ a_{j,k}\ x_1^{-i_1} \cdots x_k^{-(i_k-1)} \cdots x_j^{-(i_j+1)} \cdots x_n^{-i_n}. \quad (1)$$

Thus, if we give our basis of G^p the lexicographic order corresponding to the order of the coordinates $x_1, \cdots, x_n$, then since $a_{j,k} = 0$ if $j < k$ (recall that we have chosen our coordinates so that A is lower triangular), the matrix $[d^p_{\delta,p}]$ is lower triangular.

Let $q \leq p$. Then $d_\delta(G^q) \subset G^q + G^{q-1} + \cdots G^n$. Thus, it follows from the above that if we give F^p the ordered basis consisting of the ordered bases for each $G^q, n \leq q \leq p$ that we have chosen, and order these by descending value of q, then the matrix of $d_{\delta\ |F^p}$ is also lower triangular.

What are its diagonal elements? In the right hand side of equation (1), the coefficient of $x_1^{-i_1} \cdots x_n^{-i_n}$ is equal to

$$i_1 a_{1,1} + \cdots + i_n a_{n,n};$$

this is the diagonal element in the matrix of $d_{\delta\ |F^p}$ in the row and column corresponding to the basis element $x_1^{-i_1} \cdots x_n^{-i_n}$. Note that the diagonal elements of A are its eigenvalues; thus, the diagonal elements in the matrix of $d_{\delta\ |F^p}$ with respect to the chosen basis are all linear combinations $i_1\lambda_1 + \cdots + i_n\lambda_n$ of the eigenvalues $\lambda_1, \cdots, \lambda_n$ of A, with the i_j positive integers and $i_1 + \cdots + i_n \leq p$. As this matrix is lower triangular, $d_{\delta\ |F^p}$ is injective only if the product of these diagonal elements is non-zero. □

Remark 3.6.– We have used in the proof of this lemma the fact that if d_δ is injective then so is its restriction to each F^p. We do not know if the opposite implication holds. It seems likely that an argument involving faithful flatness would prove it. However, we do not need it in what follows.

Let D be a plane curve. We suppose as above that 0 is the singular point of D. In this case the upper non-zero row in the E_2 page of the spectral sequence ${}'\tilde{E}$ begins

$$d_1 : \check{H}^1(\mathbb{C}^2 \setminus \{0\}, \mathcal{O}_{\mathbb{C}^2}) \to \oplus_1^2 \check{H}^1(\mathbb{C}^2 \setminus \{0\}, \mathcal{O}_{\mathbb{C}^2})$$

Theorem 3.7.– Let D be a plane curve, singular at 0. If d_1 is injective, then there is a local equation h for D around 0, and a germ of vector field χ at 0 such that $\chi \cdot h = h$.

Proof: Any reduced plane curve whose equation has non-zero quadratic part is quasihomogeneous, by the classification of singularities of functions of two variables: such a curve is equivalent to A_k, $x^2 + y^{k+1} = 0$, for some k. For a quasihomogeneous curve, the conclusion of the theorem of course holds. Thus, we may assume that the equation h of D lies in $\mathbf{m}^3_{\mathbb{C}^2,0}$. As the determinant of the coefficients of a free basis of $\mathcal{D}\mathrm{er}(\log D)$ is a local defining equation for D ([10]), we may therefore choose a free basis δ, γ for $\mathcal{D}\mathrm{er}(\log D)$ such that γ has zero linear part. In fact the supposition that d_1 is injective implies that at least one member of the basis has non-zero linear part, as otherwise $d_1([1/xy]) = [\delta \cdot 1/xy], [\gamma \cdot 1/xy]) = 0$.

We may thus take

$$\delta = \delta_0 + \delta_1 + \delta_2 + \cdots = \sum_{k \geq 0} \sum_{i+j=k+1} (\alpha_{ij} x^i y^j \frac{\partial}{\partial x} + \beta_{ij} x^i y^j \frac{\partial}{\partial y})$$

where $\delta_0 = \underline{x} A \underline{\partial_x}^t$, with $A \neq 0$ and in Jordan normal form, i.e.

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Let h be the reduced equation of D:

$$h = h_n + h_{n+1} + h_{n+2} + \cdots = \sum_{k \geq n} h_k = \sum_{k \geq n} \sum_{i+j=k} a_{ij} x^i y^j,$$

where the polynomials h_i are homogeneous of degree i.

Let us now suppose that δ is not an Euler vector field for h, we will see that (up to multiplication by a non-zero constant) the only possibility for h and δ is

$$h_1 = \cdots = h_{n-1} = 0, h_n = x^a y^b \quad \text{and} \quad \delta_0 = qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}.$$

First case: $h_n = \sum_{i+j=n} a_{ij}x^i y^j$ and $\delta_0 = \lambda_1 x\frac{\partial}{\partial x} + \lambda_2 y\frac{\partial}{\partial y}$. Then

$$0 = \delta_0(h_n) = \sum_{i+j=n} (i\lambda_1 + j\lambda_2)a_{ij}x^i y^j.$$

So, $a_{ij} = 0$ if $i\lambda_1 + j\lambda_2 \neq 0$; thus, since by assumption $h_n \neq 0$, we have $q\lambda_1 = -p\lambda_2$ and $p + q = n$ $(p, q \in \mathbb{N})$. In this case,

$$h_n = x^p y^q, \quad \delta_0 = qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}.$$

Second case: $h_n = \sum_{i+j=n} a_{ij}x^i y^j$ and $\delta_0 = (\lambda_1 x + y)\frac{\partial}{\partial x} + \lambda_1 y\frac{\partial}{\partial y}$. Then

$$0 = \delta_0(h_n) = n\lambda_1 a_{n0}x^n + \sum_{i+j=n, j\geq 1} (n\lambda_1 a_{ij} + i a_{i+1,j-1})x^i y^j.$$

So, if $\lambda_1 \neq 0$, then we must have $a_{n0} = 0$, then $a_{n-1,1} = 0, \cdots, a_{1,n-1} = 0, a_{0n} = 0$, so that $h_n = 0$. This is absurd, by hypothesis.

If $\lambda_1 = 0$, then d_1 is not inyective, because

$$d_1([1/xy]) = (d_\delta([1/xy]), d_\gamma([1/xy])) = (0, 0).$$

Then, we have

$$h = x^p y^q + h_{n+1} + h_{n+2} + \cdots, \quad \delta_0 = qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}.$$

We will prove that, in this case, after a coordinate change h can be reduced to $h = x^p y^q$ with $p + q = n \geq 3$. This contradicts our supposition that h is reduced. Then our initial supposition about δ is false, and δ is an Euler vector field for h.

Inductively, for all $k \geq 0$, we construct coordinates $(x_{(k)}, y_{(k)})$ and functions $h^{(k)}$ such that

$$h(x, y) = h^{(k)}(x_{(k)}, y_{(k)}) = x_{(k)}^p y_k^q + \sum_{s\geq n+k} h_s^{(k)}(x_{(k)}, y_{(k)}) \equiv x_{(k)}^p y_{(k)}^q (\mathbf{m}_{\mathbb{C}^2,0}^{n+k}),$$

where $h_i^{(k)}$ is homogeneous of degree i. Then, by Artin approximation [1, theorem 1.2], there exist coordinates z_1, z_2 solving the equation

$$h(x,y) - z_1^p z_2^q = 0.$$

Let us construct the $x_{(k)}, y_{(k)}$, $h^{(k)}$. We suppose that we have $x_{(k)}, y_{(k)}$ and $h^{(k)} \in \mathbb{C}\{x_{(k)}, y_{(k)}\}$, such that

$$h(x,y) = h^{(k)}(x_{(k)}, y_{(k)}) = x_{(k)}^p y_{(k)}^q + \sum_{s \geq n+k} h_s^{(k)},$$

$$\delta_0^{(k)} = q x_{(k)} \frac{\partial}{\partial x_{(k)}} - p y_{(k)} \frac{\partial}{\partial y_{(k)}}.$$

We define $x_{(k+1)}, y_{(k+1)}$ and $h^{(k+1)} \in \mathbb{C}\{x_{(k+1)}, y_{(k+1)}\}$, such that

$$h(x,y) = h^{(k+1)}(x_{(k+1)}, y_{(k+1)}) = x_{(k+1)}^p y_{(k+1)}^q + \sum_{s \geq n+k+1} h_s^{(k+1)},$$

$$\delta_0^{(k+1)} = q x_{(k+1)} \frac{\partial}{\partial x_{(k+1)}} - p y_{(k+1)} \frac{\partial}{\partial y_{(k+1)}}.$$

Let $h_{n+k}^{(k)} = \sum_{i+j=n+k} a_{i,j}^{(k)} x_{(k)}^i y_{(k)}^j$, then

$$\delta_0^{(k)}(h_{n+k}) = \sum_{i+j=n+k} (iq - jp) a_{i,j}^{(k)} x_{(k)}^i y_{(k)}^j.$$

As the part of $h^{(k)}$ of degree less than $n+k$ is $x_{(k)}^p y_{(k)}^q$, it follows that the part of degree $n+k$ of $\delta^{(k)}(h^{(k)}) \in \mathbf{m}_{\mathbb{C}^2,0} h^{(k)}$ belongs to $(x_{(k)}^p y_{(k)}^q)$:

$$[\delta^{(k)}(h^{(k)})]_{n+k} = \delta_0^{(k)}(h_{n+k}^{(k)}) + \delta_k^{(k)}(x_{(k)}^p y_{(k)}^q) \in (x_{(k)}^p y_{(k)}^q),$$

but

$$\delta_k^{(k)}(x_{(k)}^p y_{(k)}^q) \in (x_{(k)}^{p-1} y_{(k)}^q, x_{(k)}^p y_{(k)}^{q-1}),$$

then

$$\delta_0^{(k)}(h_{n+k}^{(k)}) \in (x_{(k)}^{p-1} y_{(k)}^q, x_{(k)}^p y_{(k)}^{q-1}),$$

so

$$(iq - jp) a_{i,j}^{(k)} = 0 \ (i + j = n + k) \text{ if } \ i < p-1 \text{ or } j < q-1,$$

but if $iq - jp = 0$, then $(i,j) = \frac{n+k}{n}(p,q)$, and $i > p$, $j > q$. So $h_{n+k}^{(k)} \in (x_{(k)}^{p-1} y_{(k)}^q, x_{(k)}^p y_{(k)}^{q-1})$:

$$h_{n+k}^{(k)} = x_{(k)}^{p-1} y_{(k)}^q f_{k+1}(x_{(k)}, y_{(k)}) + x_{(k)}^p y_{(k)}^{q-1} g_{k+1}(x_{(k)}, y_{(k)}).$$

Let

$$x_{(k+1)} = x_{(k)} + \frac{1}{p} f_{k+1}(x_{(k)}, y_{(k)}) \qquad y_{(k+1)} = y_{(k)} + \frac{1}{q} g_{k+1}(x_{(k)}, y_{(k)}),$$

We have

$$h(x,y) = x^p_{(k+1)} y^q_{(k+1)} + \sum_{r \geq k+1} \sum_{i+j=n+r} a^{(k+1)}_{i,j} x^i_{(k+1)} y^j_{(k+1)}.$$

We define $h^{(k+1)}$ by the equation $h(x,y) = h^{(k+1)}(x_{(k+1)}, y_{(k+1)})$, where

$$h^{(k+1)} = x^p_{(k+1)} y^q_{(k+1)} + \sum_{s \geq n+k+1} h^{(k+1)}_s,$$

with $h^{(k+1)}_s = \sum_{i+j=s} a^{(k+1)}_{i,j} x^i_{(k+1)} y^j_{(k+1)}$ homegeneous polynomials of degree $s \geq n+k+1$. Moreover, as

$$x_{(k+1)} = x_{(k)}; \quad y_{(k+1)} = y_{(k)} \pmod{\mathbf{m}^2_{\mathbb{C}^2,0}},$$

we have $\delta = \sum_{q \geq 0} \delta^{(k+1)}_q$, where each $\delta^{(k+1)}_q$ is homogeneous of degree q, and

$$\delta^{(k+1)}_0 = q x_{(k+1)} \frac{\partial}{\partial x_{(k+1)}} - p y_{(k+1)} \frac{\partial}{\partial y_{(k+1)}}.$$

□

Proposition 3.8.– Let D a plane curve, singular at 0. If there exists $\delta \in \mathcal{D}\mathrm{er}(\log D)$ satisfying condition I, then there exists a unit α such that $\alpha\delta \cdot h = h$, and so D is Euler homogeneous.

Proof: The proof is similar to the proof of Theorem 3.7. There, we consider the case where $h_n = x^p y^q$ and $\delta_0 = qx\partial/\partial x - py\partial/\partial y$, with $p, q \in \mathbb{N}$. Condition I forces one of p and q to be 0. The proof now proceeds as before, with this additional hypothesis. □

Theorem 3.9.– Let $(D,0) \in (\mathbb{C}^2,0)$ be a plane curve. The following conditions are equivalent:

a) There exits $\delta \in \mathcal{D}\mathrm{er}(\log D)_0$ such that d_δ is injective.

b) There exists $\delta \in \mathcal{D}\mathrm{er}(\log D)_0$ satisfying condition (I).

c) d_1 is injective.

d) $(D,0)$ is Euler homogeneous.

e) $(D,0)$ is quasi-homogeneous.

f) The logarithmic comparison theorem holds for $(D,0)$ on a neighbourhood of 0.

Proof: By theorem 3.7, if d_1 is injective, then $(D,0)$ is Euler homogeneous. By Saito's theorem [9] (for a function h with isolated singularity, $h \in J_h$ is equivalent to the quasihomogeneity of h) to be Euler homogeneous or quasi-homogeneous is the same. Theorem 1.1 proves that if $(D,0)$ is quasi-homogeneous, the logarithmic comparison theorem holds for $(D,0)$ on a neighborough of 0. From the results of section 2 we can easily deduce that logarithmic comparison theorem implies the injectivity oj d_1. Then, the last four conditions are equivalent. If $\chi = w_1\frac{\partial}{\partial x} + w_2\frac{\partial}{\partial y}$ is the Euler vector field then d_χ is injective. Proposition 3.5 shows that if d_δ is injective, then δ satisfies (I) and, finally, by proposition 3.8, $\delta \in \mathcal{D}\mathrm{er}(\log D)$ implies that D is Euler homogeneous. □

4 Example

In this section we give an example of a free divisor $D \subset \mathbb{C}^3$ which is Euler homogeneous but not locally quasi-homogeneous, and for which the logarithmic comparison theorem does hold. This example is studied in [4], where the perversity of $\Omega^\bullet(\log D)$ is proved.

D is defined by the equation

$$h(x,y,z) = xy(x+y)((z-\lambda)x+y) = h_1h_2h_3h_4, \quad \lambda \in \mathbb{C} \setminus \{0,1\}$$

$\mathcal{D}\mathrm{er}(\log D)$ has free basis $\{\delta_1, \delta_2, \delta_3\}$

$$\begin{array}{rclcccl}
\delta_1 & = & x\frac{\partial}{\partial x} & + & y\frac{\partial}{\partial y} & & \\
\delta_2 & = & & & & + & ((z-\lambda)x+y)\frac{\partial}{\partial z} \\
\delta_3 & = & x^2\frac{\partial}{\partial x} & - & y^2\frac{\partial}{\partial y} & - & (z-\lambda)(x+y)\frac{\partial}{\partial z}
\end{array}$$

Note that $\delta_1 \cdot h = 4h$, so that h is Euler homogeneous. Note also that it is easy to check that each of these vector fields is logarithmic, and that the determinant of their coefficients is a reduced equation for D. From this it follows by a theorem of K. Saito ([10]) that they really do form a basis for

$\mathcal{D}$er$(\log D)$; as no linear combination of them has non-singular linear part, it follows that D cannot be quasihomogeneous.

This example of free divisor is interesting also as it provides a counterexample to the "logarithmic Sard's theorem": every point of $\mathbb{C} = z$–axis is a logarithmic critical value with respect to the projection $(x, y, z) \mapsto z$.

The basis of $\Omega^1(\log D)$ dual to $\{\delta_1, \delta_2, \delta_3\}$ is

$$\begin{array}{rcl} \omega_1 & = & \frac{y^2\ dx + x^2\ dy}{xy(x+y)} \\ \omega_2 & = & \frac{y(z-\lambda)\ dx - x(z-\lambda)\ dy + xy\ dz}{xy(x(z-\lambda)+y)} \\ \omega_3 & = & \frac{y\ dx - x\ dy}{xy(x+y)} \end{array}$$

We have to calculate homology groups of the stalk at 0 of the logarithmic de Rham complex

$$0 \to \Omega^0(\log D) \stackrel{d_0}{\to} \Omega^1(\log D) \stackrel{d_1}{\to} \Omega^2(\log D) \stackrel{d_2}{\to} \Omega^3(\log D) \stackrel{d_3}{\to} 0.$$

Although D is not weighted homogeneous in the strict sense, it is homogeneous if we assign weights $1, 1, 0$ to the variables x, y, z. The Lie derivative with respect to the vector field δ_1,

$$L_{\delta_1}(\omega) = \iota_{\delta_1}(d\omega) + d(\iota_{\delta_1}(\omega)),$$

then defines a contracting homotopy from $\Omega^\bullet(\log D)$ to its weight-zero part $\Omega^\bullet_0(\log D)$. For if $\omega \in \Omega^k(\log D)$ is a sum of homogenenous parts ω_i, and if $d\omega = 0$, then $d\omega_i = 0$ for all i. Since $L_{\delta_1}(\omega_i) = i\omega_i$, each ω_i, for $i \neq 0$, is then exact, and ω is cohomologous to $\omega - \iota_{\delta_1}(\sum_{i\neq 0}(1/i)\omega_i)$.

Thus we consider only the weight 0 subcomplex

$$0 \to \Omega^0_0(\log D) \stackrel{d^0_0}{\to} \Omega^1_0(\log D) \stackrel{d^0_1}{\to} \Omega^2_0(\log D) \stackrel{d^0_2}{\to} \Omega^3_0(\log D) \stackrel{d^0_3}{\to} 0.$$

- We have $\Omega^0_0(\log D) = \mathbb{C}\{z\}$, and $d_0(z^k) = kz^{k-1}[((z-\lambda)x + y)\omega_2 - (z-\lambda)(x+y)\omega_3]$ $(k \geq 0)$, so

$$\mathrm{Im}(d^0_0) = \mathbb{C}\{z\}dz = \mathbb{C}\{z\}\langle((z-\lambda)x + y)\omega_2 - (z-\lambda)(x+y)\omega_3\rangle.$$

- $\Omega^1_0(\log D) = \mathbb{C}\{z\}\langle\omega_1, x\omega_2, y\omega_2, x\omega_3, y\omega_3\rangle$, and we find

$$d_1(\omega_1) = d_1(x\omega_2) = d_1(x\omega_3) = d_1(y\omega_3) = 0$$

$$\begin{array}{rcl} d_1(z^k\omega_1) & = & kz^{k-1}((x(\lambda - z) - y)\omega_1 \wedge \omega_2 + (z-\lambda)(x+y)\omega_1 \wedge \omega_3) \\ d_1(y\omega_2) & = & (xy + y^2)\omega_2 \wedge \omega_3 \\ d_1(z^k x\omega_2) & = & kz^{k-1}((z-\lambda)(x+y)x\omega_2 \wedge \omega_3) \\ d_1(z^k y\omega_2) & = & ((k+1)z^k - k\lambda z^{k-1})(x+y)y\omega_2 \wedge \omega_3 \\ d_1(z^k x\omega_3) & = & kz^{k-1}x(x(z-\lambda) + y)\omega_2 \wedge \omega_3 \\ d_1(z^k y\omega_3) & = & kz^{k-1}y(x(z-\lambda) + y)\omega_2 \wedge \omega_3 \end{array}$$

It follows that $\mathrm{Ker}(d_1^0) = \mathbb{C}\langle \omega_1, x\omega_2, x\omega_3, y\omega_3\rangle \oplus \mathrm{Im}(d_0^0)$, so

$$H^1(\Omega^\bullet(\log D)_0) = \mathbb{C}\langle \omega_1, x\omega_2, x\omega_3, y\omega_3\rangle$$

is 4-dimensional. Also we have

$$\mathrm{Im}(d_1^0) = \mathbb{C}\{z\}\langle((\lambda - z)x - y)\omega_1 \wedge \omega_2 + (z-\lambda)(x+y)\omega_1 \wedge \omega_3)\rangle \oplus$$

$$\mathbb{C}\{z\}\langle x^2, xy, y^2\rangle \omega_2 \wedge \omega_3.$$

- $\Omega_0^2(\log D)$ is generated over $\mathbb{C}\{z\}$ by

  $x\omega_1 \wedge \omega_2, y\omega_1 \wedge \omega_2, x\omega_3 \wedge \omega_1, y\omega_3 \wedge \omega_1, x^2\omega_2 \wedge \omega_3, xy\omega_2 \wedge \omega_3, y^2\omega_2 \wedge \omega_3.$

  We find

$$\begin{array}{lllllll} d_2(x\omega_1 \wedge \omega_2) & = & d_2(x\omega_1 \wedge \omega_3) & = & d_2(y\omega_1 \wedge \omega_3) & = & 0 \\ d_2(z^k x^2 \omega_2 \wedge \omega_3) & = & d_2(z^k xy\omega_2 \wedge \omega_3) & = & d_2(z^k y^2 \omega_2 \wedge \omega_3) & = & 0. \end{array}$$

$$\begin{array}{rcl} d_2(z^k x\omega_1 \wedge \omega_2) & = & kz^{k-1}(\lambda - z)(x+y)x\omega_1 \wedge \omega_2 \wedge \omega_3 \\ d_2(y\omega_1 \wedge \omega_2) & = & (xy + y^2)\omega_1 \wedge \omega_2 \wedge \omega_3 \\ d_2(z^k y\omega_1 \wedge \omega_2) & = & z^{k-1}(x+y)(ky(\lambda - z) - zy)\omega_1 \wedge \omega_2 \wedge \omega_3) \\ d_2(z^k x\omega_1 \wedge \omega_3) & = & -kz^{k-1}x((z-\lambda)x + y)\omega_1 \wedge \omega_2 \wedge \omega_3 \\ d_2(z^k y\omega_1 \wedge \omega_3) & = & -kz^{k-1}y((z-\lambda)x + y)\omega_1 \wedge \omega_2 \wedge \omega_3 \end{array}$$

  We deduce that $\mathrm{Ker}(d_2^0) = \mathbb{C}\langle x\omega_1 \wedge \omega_2, x\omega_1 \wedge \omega_3, y\omega_1 \wedge \omega_3\rangle \oplus \mathrm{Im}(d_1^0)$, and thus that

$$H^2(\Omega^\bullet(\log D)_0) = \mathbb{C}\langle x\omega_1 \wedge \omega_2, x\omega_1 \wedge \omega_3, y\omega_1 \wedge \omega_3\rangle$$

  is 3-dimensional.

- Finally,

$$\mathrm{Im}(d_2^0) = \mathbb{C}\{z\}\langle x^2, xy, y^2\rangle \omega_1 \wedge \omega_2 \wedge \omega_3 = \Omega_0^3(\log D),$$

  and, consequently,

$$H^3(\Omega^\bullet(\log D)_0) = 0.$$

Now consider the intersection $D_0 = D \cap \{z = 0\}$, which has equation

$$h^0 = h_1^0 h_2^0 h_3^0 h_4^0 = xy(x+y)(-\lambda x + y).$$

It is a line arrangement, and the cohomology of its complement is therefore given by the Brieskorn complex, the exterior algebra generated over $\mathbb{C}$ by the forms dh_i^0/h_i^0, with trivial differential ([2]). This is of course a subcomplex of $\Omega^\bullet(\log D_0)$. Let $V \subset \mathbb{C}^3$ be a neighbourhood of 0. Restriction from $\mathbb{C}^3$ to $\mathbb{C}^2 = \{z=0\}$ gives rise to a commutative diagram

$$\begin{array}{ccccc}
\wedge^p \sum_{1\leq i\leq 4}\mathbb{C}\langle\frac{dh_i}{h_i}\rangle & \stackrel{a}{\longrightarrow} & H^p(\Omega^\bullet(\log D)(V)) & \stackrel{b}{\longrightarrow} & H^p(V\setminus D;\mathbb{C}) \\
\downarrow & & \downarrow & & \downarrow\cong \\
\wedge^p \sum_{1\leq i\leq 4}\mathbb{C}\langle\frac{dh_i^0}{h_i^0}\rangle & \stackrel{\cong}{\longrightarrow} & H^p(\Omega^\bullet(\log D_0)(V_0)) & \stackrel{\cong}{\longrightarrow} & H^p(V_0\setminus D_0;\mathbb{C}).
\end{array}$$

in which the left-hand horizontal morphisms are induced by the inclusion of the Brieskorn complex in the logarithmic complex, and the right-hand horizontal morphisms are de Rham maps. The lower horizontal morphisms are isomorphisms by the theorem of Brieskorn and by 1.1. The right hand vertical morphism is an isomorphism because D is a topologically trivial deformation of D_0, so inclusion induces an isomorphism of the homology groups of the complements. The left hand vertical morphism is evidently surjective, and thus the de Rham map $H^p(\Omega^\bullet(\log D)(V)) \to H^p(V\setminus D;\mathbb{C})$ is surjective. As $H^p(\Omega^\bullet(\log D)_0) = \lim_{U\ni 0} H^p(\Omega^\bullet(\log D)(V))$ and $\lim_{U\ni 0} H^p(V\setminus D;\mathbb{C}) = H^p(\mathbb{C}^3\setminus D;\mathbb{C})$, then the de Rham map $H^p(\Omega^\bullet(\log D)) \to H^p(\mathbb{C}^3\setminus D;\mathbb{C})$ is surjective. To see that it is an isomorphism we compare dimensions. A calculation (for example, using the Brieskorn complex) gives

$$\begin{array}{l}
\dim_{\mathbb{C}} H^1(\mathbb{C}^2\setminus D_0;\mathbb{C}) = 4 \\
\dim_{\mathbb{C}} H^2(\mathbb{C}^2\setminus D_0;\mathbb{C}) = 3 \\
\dim_{\mathbb{C}} H^3(\mathbb{C}^2\setminus D_0;\mathbb{C}) = 0
\end{array}$$

As these are the same as the dimension of $H^p(\Omega^\bullet(\log D)_0)$, this completes the proof that the logarithmic comparison theorem holds for D. □

Remark 4.10.– The calculations whose results we summarise here are not so simple as might be supposed. We have presented each image $d_i^0(\Omega_0^i(\log D))$ as a module over $\mathbb{C}\{z\}$ with algebraic generators, obscuring the fact that because D is not quasihomogeneous, the anti-derivatives of an algebraic exact logarithmic form are in general transcendental. For example,

$$\begin{array}{rcl}
z^k(x^2+xy)\omega_1\wedge\omega_2\wedge\omega_3 & = & d(\sum_{s=1}^\infty (z^{k+s}/\lambda^s(k+s))x\omega_1\wedge\omega_2) \\
& = & d(-(\log(1-\frac{z}{\lambda})+\sum_{s=1}^k (z^s/\lambda^s s))\lambda^k x\omega_1\omega_2)
\end{array}$$

and

$$\begin{array}{rcl}
z^k xy\omega_1\wedge\omega_2\wedge\omega_3 & = & d(\sum_{s=1}^\infty (z^{k+s}/(\lambda+1)^s(k+s))x(\omega_1\wedge\omega_2+\omega_1\wedge\omega_3)) \\
& = & d(-((\lambda+1)^k\log(1-(z/(\lambda+1)))+ \\
& & \sum_{s=1}^k (z^s(\lambda+1)^{k-s}s))x(\omega_1\wedge\omega_2+\omega_1\wedge\omega_3)).
\end{array}$$

References

[1] M. Artin, On the solutions of analytic equations, *Invent. Math.* 5 (1968), 277-291.

[2] E. Brieskorn, Sur le groupe de tresses (d'apres V.I. Arnol'd), *Sem. Bourbaki 1971/72*, Lecture Notes in Math. 317, Springer Verlag, Berlin, 1973, 21-44.

[3] J.L. Brylinski, A.S. Dubson and M. Kashiwara. Formule de l'indice pour modules holonomes et obstruction d'Euler locale. *C. R. Acad. Sci. Paris Sér. I Math.*, 293 (1981), 573–576.

[4] F.J. Calderón-Moreno. Logarithmic Differential Operators and Logarithmic De Rham Complexes Relative to a Free Divisor. *Ann. Sci. E.N.S.*, to appear.

[5] F.J. Castro-Jiménez, D. Mond and L. Narváez-Macarro. Cohomology of the complement of a free divisor. *Transactions of the A.M.S.*, 348 (1996), 3037–3049.

[6] F.J. Castro-Jiménez, D. Mond and L. Narváez-Macarro. Unpublished, 1997.

[7] A. Grothendieck, On the de Rham cohomology of algebraic varieties, Publ. Math. de l'I.H.E.S. 29 (1966), 95-103.

[8] H.J. Reiffen, Das Lemma von Poincaré fur holomorphe Differential-formen auf komplexen Raumen, *Math. Z.* 101 (1967), 269–284.

[9] K. Saito, Quasihomogene isolierte Singularitäten von Hyperflächen, *Invent. Math.* 14 (1971), 123-141.

[10] K. Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo*, 27 (1980), 265–291.

Departamento de Álgebra, Facultad de Matemáticas, Universidad de Sevilla, Ap 1160, 41080 Sevilla, Spain.
E-mail: frcalder@cica.es, castro@cica.es, narvaez@algebra.us.es

Mathematics Institute, University of Warwick, Coventry CV4 7AL, England.
E-mail: mond@maths.warwick.ac.uk