ARITHMETIC PROPERTIES OF PARTITIONS AND HECKE-ROGERS TYPE IDENTITIES

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DECLARATION

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

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Summary

This thesis contains four parts on various types of partitions, Hecke-Rogers type identities and false theta functions.

In the first part, we give explicit formulas for the number of partition pairs and triples with 3 cores. The main tools that we use are Ramanujan's ${}_1\psi_1$ summation formula and Bailey's ${}_6\psi_6$ formula. By using these formulas, we establish many arithmetic identities satisfied by these two partition functions, which simplified the proofs of some identities in the literature. The results described in this part appears in [82].

In the second part, following the strategies of A.O.L. Atkin and B. Gordon, we prove three infinite families of congruences modulo arbitrary powers of 11 for some partition functions, including 11-regular partitions and 11-core partitions. We also confirm a conjecture of H.H. Chan and P.C. Toh on the ordinary partition function $p(n)$. The results described in this part appears in [85].

In the third part, we introduce a unified modular approach to find q-product representations for the generating functions of k-colored generalized Frobenius partitions. Let $c\phi_k(n)$ denote the number of k-colored generalized Frobenius partitions of n and $\mathrm{C}\Phi_k(q)$ be its generating function. We give various representations of $\mathrm{C}\Phi_k(q)$ for $k \leq 17$. Moreover, we discover new surprising properties of $\mathrm{C}\Phi_k(q)$.

This part is based on the joint work with Chan and Y.F. Yang [36].

In the fourth part, we turn to study some identities on basic hypergeometric series. Using some elegant formulas of Liu, we prove an intriguing identity, which involves a double series of Hecke-Rogers type associated with definite quadratic forms. In addition, two similar identities will be given. We also provide new proofs of five identities of Ramanujan associated with false theta functions. Our proofs do not use the Rogers-Fine identity or Bailey transforms. This part is based on [86] and a joint work with A.J. Yee [87].

Chapter 1

Introduction

A partition of an integer n is a sequence of non-increasing positive integers which add up to n. We denote the number of partitions of n by $p(n)$. For example, there are 5 partitions of 4, namely

$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1.$$

Hence $p(4) = 5$. L. Euler found that the generating function of $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}, \tag{1.1}$$

where we agree that $p(0) = 1$. Since the generating function of $p(n)$ is an infinite product, it will be convenient to introduce some q-series notation, which will be used throughout this thesis. We define

$$(a;q)_0 := 1, \tag{1.2}$$

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad n \geq 1, \tag{1.3}$$

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \quad |q| < 1, \tag{1.4}$$

$$(a_1, a_2, \ldots, a_m; q)_n := (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n, \tag{1.5}$$

$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1;q)_\infty(a_2;q)_\infty \cdots (a_m;q)_\infty, \quad |q| < 1. \tag{1.6}$$

We also define $(a;q)_n$ for negative integers n as

$$(a;q)_n := \frac{1}{(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{-n})} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n q^{n(n-1)/2}}{(q/a;q)_n}. \tag{1.7}$$

Moreover, we define an ${}_r\phi_s$ basic hypergeometric series by

$$\begin{aligned} &{}_r\phi_s\left(\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ & b_1, & \ldots, & b_s \end{matrix}; q, z\right) \\ &:= \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left((-1)^n q^{n(n-1)/2}\right)^{1+s-r} z^n \end{aligned} \tag{1.8}$$

where $q \neq 0$ when $r > s+1$. The general bilateral basic hypergeometric series in base q with r numerator and s denominator parameters is defined by

$$\begin{aligned} &{}_r\psi_s\left(\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ & b_1, & \ldots, & b_s \end{matrix}; q, z\right) \\ &:= \sum_{n=-\infty}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}(-1)^{(s-r)n} q^{(s-r)n(n-1)/2} z^n. \end{aligned} \tag{1.9}$$

In (1.9) we assume that q, z and the parameters are such that each term of the series is well-defined.

Throughout this thesis, unless otherwise stated, we will treat numerous functions as Laurent series in one variable q and will assume that $|q| < 1$. In particular, since the infinite product in (1.4) diverges when $a \neq 0$ and $|q| \geq 1$, the condition $|q| < 1$ is necessary. For more discussions about the convergence of infinite series, and in particular the convergence of the ${}_r\phi_s$ and ${}_r\psi_s$ series, see [42].

Using the notation in (1.4), (1.1) can be rewritten as

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{1.10}$$

One of S. Ramanujan's famous works is the study of the arithmetic properties of $p(n)$. In 1919, Ramanujan [72, 74] found three simple congruences satisfied by $p(n)$, namely,

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.11}$$

$$p(7n+5) \equiv 0 \pmod{7}, \quad \text{and} \tag{1.12}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.13}$$

He gave a few proofs of (1.11). In one of his proofs, he deduced (1.11) from his identity

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{1.14}$$

Identity (1.14) was later regarded by G.H. Hardy as "Ramanujan's most beautiful identity". Ramanujan also stated without proof the following similar identity for $p(7n+5)$,

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}, \tag{1.15}$$

and deduced (1.12) from (1.15). Although Ramanujan also announced that he has a proof for (1.13), the first elementary proof was published by L. Winquist [89].

Based on the congruences (1.11)–(1.13), Ramanujan [72, 74] made a general conjecture for congruences modulo arbitrary powers of 5, 7 and 11. His original conjecture is not quite correct. However, much to Ramanujan's credit, a slighted modified correct version is

$$p(5^j n + \delta_{5,j}) \equiv 0 \pmod{5^j}, \tag{1.16}$$

$$p(7^j n + \delta_{7,j}) \equiv 0 \pmod{7^{\lfloor j/2 \rfloor + 1}}, \tag{1.17}$$

$$p(11^j n + \delta_{11,j}) \equiv 0 \pmod{11^j}, \tag{1.18}$$

where $j \geq 1$ and $\delta_{\ell,j}$ is the reciprocal modulo ℓ^j of 24. The proofs of congruences (1.16) and (1.17) were attributed to G.N. Watson [88]. M.D. Hirschhorn and D.C. Hunt [49] also gave a simple proof of (1.16), and using similar strategy, later F. Garvan [39] gave an elementary proof to (1.17). The congruence (1.18) was proved by A.O.L. Atkin [16]. For more results and complete history about the partition function $p(n)$, see B.C. Berndt's book [26].

Motivated by Ramanujan's work on partitions, arithmetic properties of numerous kinds of partitions have been investigated. In recent years, there are several types of restricted partitions which have drawn much attention. These include t-core partitions, ℓ-regular partitions, and k-colored generalized Frobenius partitions.

To understand the definition of t-core partitions, we need to explain the concept of hook numbers. Given a partition $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ of a positive integer n, where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1, \quad \lambda_1 + \lambda_2 + \cdots + \lambda_k = n,$$

one can represent it by the Ferrers diagram, which is a set of left justified rows of equi-spaced dots wherein the i-th row has λ_i dots for each $1 \leq i \leq k$. Figure 1.1 shows the Ferrers diagram of the partition $\lambda = \{3, 2, 1\}$. The dots are labeled by row

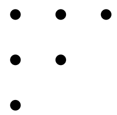

Figure 1.1

and column coordinates in the same way as we label the entries of a matrix. Let λ'_j denote the number of dots in column j. For each dot with label (i, j), we associate it with a hook number $H(i, j)$ which is defined as the number of dots directly below and to the right of the dot (i, j) including the dot itself. That is,

$$H(i, j) = \lambda_i + \lambda'_j - j - i + 1. \tag{1.19}$$

For the partition represented by Figure 1.1, the dots $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$ and $(3, 1)$ have hook numbers 5, 3, 1, 3, 1 and 1, respectively.

A partition λ of n is said to be a t-core partition or partition with t cores if it has no hook numbers which are multiples of t. For example, the partition $\lambda = \{3, 2, 1\}$ is a 2-core partition since all its hook numbers are odd. It is neither a 3-core partition nor a 5-core partition since there are some hook numbers divisible by 3 and 5, respectively. We denote by $a_t(n)$ the number of partitions of n with t-cores. From [41, Eq. (2.1)], the generating function of $a_t(n)$ is given by

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}. \tag{1.20}$$

A partition k-tuple $(\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(k)})$ of n is a k-tuple of partitions $\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(k)}$ such that the sum of all the parts equals n. For example, let $\lambda^{(1)} = \{3,1\}, \lambda^{(2)} = \{4,1\}, \lambda^{(3)} = \{1\}$. Then $(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ is a partition triple of 10 since $(3+1)+(4+1)+1 = 10$. A partition k-tuple of n with t-cores is a partition k-tuple $(\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(k)})$ of n where each $\lambda^{(i)}$ is t-core partition for $i = 1, 2, \cdots, k$.

Let $A_t(n)$ (resp. $B_t(n)$) denote the number of partition pairs (resp. triples) of n with t-cores. From (1.20), it is not difficult to see that the generating functions for $A_t(n)$ and $B_t(n)$ are given by

$$\sum_{n=0}^{\infty} A_t(n)q^n = \frac{(q^t;q^t)_\infty^{2t}}{(q;q)_\infty^2} \tag{1.21}$$

and

$$\sum_{n=0}^{\infty} B_t(n)q^n = \frac{(q^t;q^t)_\infty^{3t}}{(q;q)_\infty^3} \tag{1.22}$$

respectively.

In 1996, using the theory of modular forms, A. Granville and K. Ono [46] found an explicit formula for $a_3(n)$:

$$a_3(n) = d_{1,3}(3n+1) - d_{2,3}(3n+1), \tag{1.23}$$

where $d_{r,3}(n)$ denotes the number of divisors of n congruent to r modulo 3. Subsequently, other mathematicians such as N.D. Baruah, B.C. Berndt, Hirschhorn, B.L.S. Lin, K. Nath, J. Sellers, L. Wang and E.X.W. Xia found many new arithmetic identities satisfied by $a_3(n)$, $A_3(n)$ and $B_3(n)$. For instance, Baruah and Nath [22] established three infinite families of arithmetic identities involving $A_3(n)$. They discovered that for any integers $k \geq 0$ and $n \geq 0$,

$$A_3\left(2^{2k+2}n + \frac{2(2^{2k}-1)}{3}\right) = \frac{2^{2k+2}-1}{3} A_3(4n), \tag{1.24}$$

$$A_3\left(2^{2k+2}n+\frac{2(2^{2k+2}-1)}{3}\right)=\frac{2^{2k+2}-1}{3}A_3(4n+2)-\frac{2^{2k+2}-4}{3}A_3(n), \tag{1.25}$$
$$A_3\left(2^{2k+1}n+\frac{5\cdot 2^{2k}-2}{3}\right)=\big(2^{2k+1}-1\big)A_3(2n+1). \tag{1.26}$$

In view of (1.23), it is natural to ask if we can find explicit formulas for $A_3(n)$ and $B_3(n)$. The main goal of Chapter 2 is to give an affirmative answer to this question. By using Ramanujan's ${}_1\psi_1$ summation formula and Bailey's ${}_6\psi_6$ formula, we give a new elementary proof of (1.23) and find explicit formulas of $A_3(n)$ and $B_3(n)$. From these formulas, we derive numerous arithmetic relations satisfied by $A_3(n)$ and $B_3(n)$. We also generalize many known identities associated with these partition functions.

In Chapter 3 we apply the theory of modular forms to establish some congruences modulo arbitrary powers of 11 for three partition functions $a_{11}(n)$, $b_{11}(n)$ and $p_{[1^1 11^1]}(n)$. Here $a_{11}(n)$ denotes the number of 11-core partitions of n, $b_{11}(n)$ denotes the number of 11-regular partitions of n and $p_{[1^1 11^1]}(n)$ is defined as

$$\sum_{n=0}^{\infty} p_{[1^1 11^1]}(n)q^n=\frac{1}{(q;q)_\infty (q^{11};q^{11})_\infty}.$$

Here for any positive integer ℓ, a partition of n is called ℓ-regular if none of its parts are divisible by ℓ. We use $b_\ell(n)$ to denote the number of ℓ-regular partitions of n and its generating function is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n=\frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}. \tag{1.27}$$

Following the strategy of Atkin [16] and B. Gordon [44], we establish congruences modulo arbitrary powers of 11 for these functions. For instance, for any integers $n\geq 0$ and $k\geq 1$, we prove that

$$b_{11}\left(11^{2k-1}n+\frac{7\cdot 11^{2k-1}-5}{12}\right)\equiv 0 \pmod{11^k}.$$

We have so far encountered different types of partitions. These partitions, which are interesting combinatorial objects, also connect to other areas of mathematics.

For instance, they play an important role in representation theory of the symmetric groups (see [52, 75], for example). Here we give a few examples. Let $\mathfrak{S}_n$ be the symmetric group on the set $X = \{1, 2, \cdots, n\}$. It is known that [52, p. 6] the number of conjugacy classes of $\mathfrak{S}_n$ equals $p(n)$. This implies that $p(n)$ enumerates the number of ordinary irreducible representations of $\mathfrak{S}_n$. Moreover, a conjugacy class of a group is called a ℓ-regular class if the order of an element in that class is not divisible by ℓ. It was shown in [52, p. 36, Lemma 10.2] that the number of ℓ-regular partitions of n equals the number of partitions of n where no parts of the partition appear ℓ or more times, which is also the same as the number of ℓ-regular classes of $\mathfrak{S}_n$. Some interesting discussions about hook numbers, t-core partitions and their applications in representation theory can be found in [46, 52, 75].

Chapter 4 is devoted to the study of the k-colored generalized Frobenius partitions. The k-colored generalized Frobenius partitions were first introduced by G.E. Andrews [4]. Let $c\phi_k(n)$ denote the number of k-colored generalized Frobenius partitions of n and

$$\mathrm{C}\Phi_k(q) := \sum_{n=0}^{\infty} c\phi_k(n) q^n. \tag{1.28}$$

Andrews showed that

$$\mathrm{C}\Phi_k(q) = \frac{1}{(q;q)_\infty^k} \sum_{m_1, \cdots, m_{k-1} \in \mathbf{Z}} q^{Q(m_1, \cdots, m_{k-1})}$$

where

$$Q(m_1, m_2, \cdots, m_{k-1}) = \sum_{i=1}^{k-1} m_i^2 + \sum_{1 \le i < j \le k-1} m_i m_j.$$

He also found beautiful alternative representations for $\mathrm{C}\Phi_k(q)$ for $k = 1, 2, 3$ and 5 [4]. Andrews commented that similar identity exists for $k = 7$, and the identity was published later by L.W. Kolitsch [57]. Kolitsch [57, 60], N.D. Baruah and B.K. Sarmah [23, 24] subsequently found alternative representations of $\mathrm{C}\Phi_k(q)$ for $k = 4, 5, 6$ and 11. However, it is not easy to apply their methods to find alternative representations of $\mathrm{C}\Phi_k(q)$ for other k. The main goal of Chapter 4 is to extend the list of formulas of $C\Phi_k(q)$. We discuss the modular properties of $C\Phi_k(q)$ and then use the theory of modular forms to find alternative representations. We give new proofs of the existing formulas of $C\Phi_k(q)$ and discover new representations of $C\Phi_k(q)$ for $k < 18$. For instance, we prove that

$$C\Phi_9(q) = 324q\frac{(q^3;q^3)_\infty^8}{(q;q)_\infty^9} + 19683q^4\frac{(q^9;q^9)_\infty^{12}}{(q;q)_\infty^9(q^3;q^3)_\infty^4} - 240q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^4} - 1458q^2\frac{(q^9;q^9)_\infty^6}{(q;q)_\infty^3(q^3;q^3)_\infty^4} + \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty^4}. \tag{1.29}$$

We also prove some interesting congruences associated with $c\phi_k(n)$. For example, let p be a prime and N be a positive integer which is not divisible by p. For any integers $\alpha \geq 1$ and $n \geq 0$, we prove that

$$c\phi_{p^\alpha N}(n) \equiv c\phi_{p^{\alpha-1}N}(n/p) \pmod{p^{2\alpha}}, \tag{1.30}$$

with the convention that $c\phi_k(x) = 0$ whenever x is not an integer.

In Chapter 5 we turn to study some interesting basic hypergeometric series identities. It was E. Hecke who first systematically investigated theta series related to indefinite quadratic forms [48]. For example, Hecke [48, p. 425] found that

$$\prod_{n=1}^{\infty}(1-q^n)^2 = \sum_{n=-\infty}^{\infty}\sum_{|m|\leq n/2}(-1)^{n+m}q^{(n^2-3m^2)/2+(n+m)/2}, \tag{1.31}$$

which is originally due to L.J. Rogers [76, p. 323].

In the study of q-series, Hecke-type sums are interesting objects themselves. In early works of Andrews [5, 6], they appear as bridges between mock theta functions and Jacobi forms. Such an essential role still holds even after S.P. Zwegers' seminal work on the modularity theory of mock theta functions [96]. However, unlike Hecke-type sums associated with indefinite quadratic forms, there are few results on definite quadratic forms (c.f. [34, 80]). One of the main results of this chapter is the following identity associated with definite quadratic forms:

$$\sum_{n=1}^{\infty}\frac{q^n(q;q^2)_n}{(-q;q^2)_n(1+q^{2n})} = \sum_{n=1}^{\infty}\sum_{|m|\leq n}(-1)^m q^{n^2+m^2} - \sum_{n=1}^{\infty}(-1)^n q^{2n^2}. \tag{1.32}$$

Two other interesting identities will also be given. Moreover, as an application of (1.32), we give new proofs of two congruences which appeared in [11].

In the later part of Chapter 5, we will prove some identities of Ramanujan on False theta functions. False theta functions are series that are instances of classical theta series except for an alteration of the signs of some of the terms in the series. They were first introduced by Rogers [77]. By the method used in proving (1.32), we provide new proofs of the following identities of Ramanujan [73]: for $|q| < 1$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n\, q^{n(n+1)}}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}, \tag{1.33}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n^2\, q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)}, \tag{1.34}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n\, q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)/2}, \tag{1.35}$$

$$\sum_{n=0}^{\infty} \frac{(q;-q)_{2n}\, q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)}, \tag{1.36}$$

$$\sum_{n=0}^{\infty} \frac{(q;-q)_n (-q^2;q^2)_n\, q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)}. \tag{1.37}$$

These identities were first proved by Andrews [3, Section 6] by employing the Rogers-Fine identity as well as some other identities. Recently, Andrews and S.O. Warnaar [13] provided new proofs of (1.33)–(1.36) by using symmetric bilateral Bailey transforms. However, their methods apparently do not yield proof of (1.37). Using other bilateral Bailey transformation formulaes, W.C. Chu and W. Zhang [37] proved many identities on false theta functions including (1.33)–(1.37). Our proofs do not use the Rogers-Fine identity or Bailey transforms and it is evident that our method can be applied to discover and prove many identities similar to (1.33)–(1.37).

Chapter 2

Partitions with 3 Cores

2.1 Introduction

Let $a_t(n)$ be the number of partitions of n that are t-cores. From [41, Eq. (2.1)], the generating function of $a_t(n)$ is given by

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}.$$

In particular, when $t = 3$, we have

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}. \tag{2.1.1}$$

Let $A_t(n)$ (resp. $B_t(n)$) denote the number of bipartitions (resp. partition triples) of n with t-cores. The generating functions for $A_t(n)$ and $B_t(n)$ are given by (1.21) and (1.22), respectively.

In 1996, A. Granville and K. Ono [46] found the explicit formula (1.23) for $a_3(n)$ based on the theory of modular forms. N.D. Baruah and B.C. Berndt [19] showed that for any nonnegative integer n,

$$a_3(4n+1) = a_3(n).$$

In 2009, M.D. Hirschhorn and J.A. Sellers [50] provided an elementary proof of (1.23) and as corollaries, they proved some arithmetic identities. For example, let

$p \equiv 2 \pmod 3$ be prime and let k be a positive even integer. Then, for all $n \geq 0$,

$$a_3\Big(p^k n + \frac{p^k - 1}{3}\Big) = a_3(n).$$

Let $u(n)$ denote the number of representations of a nonnegative integer n in the form x^2+3y^2 with $x, y \in \mathbf{Z}$. By using Ramanujan's theta function identities, Baruah and K. Nath [21] proved that

$$u(12n+4) = 6a_3(n).$$

In 2014, B.L.S. Lin [62] discovered some arithmetic identities associated with $A_3(n)$. For example, he proved that $A_3(8n+6) = 7A_3(2n+1)$. Let $v(n)$ denote the number of representations of a nonnegative integer n in the form $x_1^2 + x_2^2 + 3y_1^2 + 3y_2^2$ with $x_1, x_2, y_1, y_2 \in \mathbf{Z}$. Lin showed that

$$v(6n+5) = 12A_3(2n+1). \tag{2.1.2}$$

Again, Baruah and Nath [22] generalized (2.1.2) and established three infinite families of arithmetic identities involving $A_3(n)$, which are listed in (1.24)–(1.26). E.X.W. Xia [90] found several infinite families of congruences modulo 4, 8 for $A_3(n)$. For example, he showed that for all integers $n \geq 0$,

$$A_3(8n+4) \equiv 0 \pmod 4, \quad A_3(16n+4) \equiv 0 \pmod 8.$$

He also proposed the following conjecture.

Conjecture 2.1.1. For any positive integer j and prime $p \geq 3$, there exists a positive integer k_0 such that for all $n \geq 0$ and $\alpha \geq 0$,

$$A_3\Big(4^{k_0(\alpha+1)} n + \frac{2^{2k_0(\alpha+1)-1} - 2}{3}\Big) \equiv 0 \pmod{p^j}.$$

Recently, we [81] studied the arithmetic properties of partition triples with 3-cores. By using some identities of q series, we proved some analogous results. For example, we proved that

$$B_3(4n+1) = 3B_3(2n), \quad B_3(3n+2) = 9B_3(n), \quad \text{and}$$

$$B_3(4n+3) = 3B_3(2n+1) + 4B_3(n).$$

From these relations we deduce three infinite families of arithmetic identities as well as some Ramanujan-type congruences involving $B_3(n)$. For instance, for any integer $k \geq 1$, we proved that

$$B_3(3^k n + 3^k - 1) = 3^{2k} B_3(n), \tag{2.1.3}$$

$$B_3\big(2^{k+1}n + 2^k - 1\big) = \frac{2^{2k+2} + (-1)^k}{5} B_3(2n), \tag{2.1.4}$$

and

$$B_3\big(2^{k+1}n + 2^{k+1} - 1\big) = \frac{2^{2k+2} + (-1)^k}{5} B_3(2n+1) + \frac{2^{2k+2} - 4(-1)^k}{5} B_3(n). \tag{2.1.5}$$

Furthermore, let $\omega(n)$ denote the number of representations of a nonnegative integer n in the form

$$n = x_1^2 + x_2^2 + x_3^2 + 3y_1^2 + 3y_2^2 + 3y_3^2, \quad x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbf{Z}.$$

We found some interesting arithmetic relations between $\omega(n)$ and $B_3(n)$:

$$\omega(6n+5) = 4B_3(6n+4),$$

$$\omega(12n+2) = 12B_3(6n),$$

$$\omega(12n+10) = 6B_3(6n+4).$$

For more results and details about $a_3(n)$ and $A_3(n)$, see [19, 20, 21, 22, 50, 62, 93].

In view of the similar arithmetic identities satisfied by $a_3(n)$, $A_3(n)$ and $B_3(n)$, it is natural to guess that explicit formulas similar to (1.23) for $A_3(n)$ and $B_3(n)$ may also exist. In Sections 2.2 and 2.3, we confirm our guess. By using Ramanujan's ${}_1\psi_1$ summation formula and Bailey's ${}_6\psi_6$ formula, we give a new simple proof of (1.23) and find explicit formulas for $A_3(n)$ and $B_3(n)$. With these formulas in mind, most of the results mentioned above become direct consequences. In particular, we will confirm Conjecture 2.1.1 and give some generalizations of the identities which appeared in [62, 81].

2.2 Explicit formula for $A_3(n)$

Before we present the explicit formula for $A_3(n)$, we provide a new elementary proof of (1.23). The main tool in this section is Ramanujan's ${}_1\psi_1$ summation formula [26, Theorem 1.3.12].

Lemma 2.2.1 (Ramanujan's ${}_1\psi_1$ Summation)**.** *For* $|b/a| < |z| < 1$ *and* $|q| < 1$,

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(az, q/(az), q, b/a; q)_\infty}{(z, b/(az), b, q/a; q)_\infty}. \tag{2.2.1}$$

Proof of (1.23). Recall from (2.1.1) that

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.$$

Note that since

$$(q;q)_\infty = (q;q^3)_\infty (q^2;q^3)_\infty (q^3;q^3)_\infty, \tag{2.2.2}$$

we deduce that

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{(q^3;q^3)_\infty^2}{(q;q^3)_\infty (q^2;q^3)_\infty}. \tag{2.2.3}$$

Taking $(a, b, z, q) \to (q, q^4, q, q^3)$ in (2.2.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{(q;q^3)_n}{(q^4;q^3)_n} \cdot q^n = \frac{(q^2, q, q^3, q^3; q^3)_\infty}{(q, q^2, q^4, q^2; q^3)_\infty}.$$

Dividing both sides by $1 - q$, after simplification, we deduce that

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{3n+1}} = \frac{(q^3;q^3)_\infty^2}{(q;q^3)_\infty (q^2;q^3)_\infty}. \tag{2.2.4}$$

Combining (2.2.3) with (2.2.4), we obtain

$$\sum_{n=0}^{\infty} a_3(n)q^n = \sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{3n+1}}.$$

Replacing q by q^3 and multiplying both sides by q, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} a_3(n)q^{3n+1} &= \sum_{m=0}^{\infty} \frac{q^{3m+1}}{1-q^{3(3m+1)}} + \sum_{m=-\infty}^{-1} \frac{q^{3m+1}}{1-q^{3(3m+1)}} \\ &= \sum_{m=0}^{\infty} \frac{q^{3m+1}}{1-q^{3(3m+1)}} + \sum_{m=0}^{\infty} \frac{q^{-3m-2}}{1-q^{3(-3m-2)}} \\ &= \sum_{m=0}^{\infty} \frac{q^{3m+1}}{1-q^{3(3m+1)}} - \sum_{m=0}^{\infty} \frac{q^{2(3m+2)}}{1-q^{3(3m+2)}} \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} q^{(3m+1)(3k+1)} - \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} q^{(3m+2)(3k+2)},\end{aligned} \tag{2.2.5}$$

where the second equality follows by replacing m by $-m-1$ in the second summation.

Now (1.23) follows by comparing the coefficients of q^{3n+1} on both sides of (2.2.5).

□

Let $\sigma(n)$ denote the sum of positive divisors of n. Applying the method used in proving (1.23), we can find the explicit formula for $A_3(n)$.

Theorem 2.2.2. *For any integer $n \geq 0$, we have $A_3(n) = \frac{1}{3}\sigma(3n+2)$. If we write $3n+2 = \prod_{i=1}^{s} p_i^{\alpha_i}$ as the unique prime factorization, then*

$$A_3(n) = \frac{1}{3}\prod_{i=1}^{s} \frac{p_i^{\alpha_i+1}-1}{p_i-1}.$$

Proof. Setting $t = 3$ in (1.21), and applying (2.2.2) we obtain that

$$\sum_{n=0}^{\infty} A_3(n)q^n = \frac{(q^3;q^3)_\infty^4}{(q;q^3)_\infty^2(q^2;q^3)_\infty^2}. \tag{2.2.6}$$

Taking $(a,b,q) \to (q,q^4,q^3)$ in (2.2.1), and dividing both sides by $1-\frac{q^2}{z}$, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{(q;q^3)_n}{(q^4;q^3)_n} \cdot \frac{z^n}{1-q^2/z} = \frac{(qz,q^5/z,q^3,q^3;q^3)_\infty}{(z,q^3/z,q^4,q^2;q^3)_\infty}. \tag{2.2.7}$$

Let $z \to q^2$. By L'Hospital's rule, we deduce that

$$\sum_{n=-\infty}^{\infty} \frac{(1-q)\cdot nq^{2n}}{1-q^{3n+1}} = \frac{(q^3;q^3)_\infty^4}{(q^2;q^3)_\infty^2(q;q^3)_\infty(q^4;q^3)_\infty}.$$

Dividing both sides by $1-q$ and combining with (2.2.6), we obtain

$$\sum_{n=0}^{\infty} A_3(n)q^n = \frac{(q^3;q^3)_\infty^4}{(q;q^3)_\infty^2(q^2;q^3)_\infty^2} = \sum_{n=-\infty}^{\infty} \frac{nq^{2n}}{1-q^{3n+1}}. \tag{2.2.8}$$

Replacing q by q^3 and multiplying both sides by q^2, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} A_3(n)q^{3n+2} &= \sum_{m=0}^{\infty} \frac{mq^{2(3m+1)}}{1-q^{3(3m+1)}} + \sum_{m=-\infty}^{-1} \frac{mq^{2(3m+1)}}{1-q^{3(3m+1)}} \\ &= \sum_{m=0}^{\infty} \frac{mq^{2(3m+1)}}{1-q^{3(3m+1)}} + \sum_{m=0}^{\infty} \frac{(-m-1)q^{2(-3m-2)}}{1-q^{3(-3m-2)}} \\ &= \sum_{m=0}^{\infty} \frac{mq^{2(3m+1)}}{1-q^{3(3m+1)}} + \sum_{m=0}^{\infty} \frac{(m+1)q^{3m+2}}{1-q^{3(3m+2)}} \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} mq^{(3m+1)(3k+2)} + \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} (m+1)q^{(3m+2)(3k+1)} \\ &= \frac{1}{3}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\Big((3m+1)q^{(3m+1)(3k+2)} + (3m+2)q^{(3m+2)(3k+1)}\Big) \\ &\quad + \frac{1}{3}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\Big(q^{(3m+2)(3k+1)} - q^{(3m+1)(3k+2)}\Big).\end{aligned} \tag{2.2.9}$$

Interchanging the roles of k and m, we find that

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} q^{(3m+2)(3k+1)} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q^{(3k+2)(3m+1)} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} q^{(3m+1)(3k+2)}.$$

Thus the second sum in the right hand side of (2.2.9) vanishes. Now we observe that for any factorization $3n+2=ab$ where both a and b are positive integers, one of the residues of a and b modulo 3 must be 1 and the other is 2. Comparing the coefficients of q^{3n+2} on both sides of (2.2.9), we prove the first assertion of the theorem. The second assertion then follows immediately. $\square$

Once we know the explicit formula for $A_3(n)$, we can verify the identities (1.24)–(1.26) by simple arguments. For example, since $\sigma(n)$ is multiplicative, by Theorem 2.2.2, we deduce that

$$A_3(4n) = \frac{1}{3}\sigma(2(6n+1)) = \frac{1}{3}\sigma(2)\sigma(6n+1) = \sigma(6n+1),$$

$$A_3\Big(2^{2k+2}n + \frac{2(2^{2k}-1)}{3}\Big) = \frac{1}{3}\sigma\big(2^{2k+1}(6n+1)\big) = \frac{1}{3}\sigma(2^{2k+1})\sigma(6n+1).$$

Note that $\sigma(2^{2k+1}) = 2^{2k+2} - 1$. This proves (1.24). Identities (1.25) and (1.26) can be proved in a similar way.

Moreover, we can extend the identities (1.24)–(1.26) to some larger families of arithmetic identities.

Theorem 2.2.3. *Let p be a prime, and let k, n be nonnegative integers.*
(1) If $p \equiv 1 \pmod 3$, we have

$$A_3\Big(p^k n + \frac{2p^k-2}{3}\Big) = \frac{p^k-1}{p-1}A_3\Big(pn + \frac{2p-2}{3}\Big) - \frac{p^k-p}{p-1}A_3(n).$$

(2) If $p \equiv 2 \pmod 3$, we have

$$A_3\Big(p^{2k} n + \frac{2p^{2k}-2}{3}\Big) = \frac{p^{2k}-1}{p^2-1}A_3\Big(p^2 n + \frac{2p^2-2}{3}\Big) - \frac{p^{2k}-p^2}{p^2-1}A_3(n).$$

Proof. We write $3n+2 = p^m N$, where N is an integer not divisible by p.

(1) By Theorem 2.2.2, we deduce that

$$A_3(n) = \frac{1}{3}\sigma(p^m N) = \frac{1}{3}\sigma(p^m)\sigma(N) = \frac{1}{3}\cdot\frac{p^{m+1}-1}{p-1}\sigma(N). \tag{2.2.10}$$

Similarly, we have

$$A_3\big(pn + \frac{2p-2}{3}\big) = \frac{1}{3}\sigma(p^{m+1}N) = \frac{1}{3}\cdot\frac{p^{m+2}-1}{p-1}\sigma(N), \tag{2.2.11}$$

$$A_3\Big(p^k n + \frac{2p^k-2}{3}\Big) = \frac{1}{3}\sigma(p^{k+m}N) = \frac{1}{3}\cdot\frac{p^{k+m+1}-1}{p-1}\sigma(N). \tag{2.2.12}$$

Now the assertion follows from (2.2.10)–(2.2.12) by direct verification.

(2) In the same way, we have

$$A_3\big(p^2 n + \frac{2p^2-2}{3}\big) = \frac{1}{3}\sigma(p^{m+2}N) = \frac{1}{3}\cdot\frac{p^{m+3}-1}{p-1}\sigma(N), \tag{2.2.13}$$

$$A_3\Big(p^{2k} n + \frac{2p^{2k}-2}{3}\Big) = \frac{1}{3}\sigma(p^{2k+m}N) = \frac{1}{3}\cdot\frac{p^{2k+m+1}-1}{p-1}\sigma(N). \tag{2.2.14}$$

Combining (2.2.10), (2.2.13) and (2.2.14), we prove the assertion by direct verification. □

As some special cases, by setting $p = 2, 5, 7$ in Theorem 2.2.3, we obtain the following arithmetic identities for $k, n \geq 0$,

$$\begin{aligned} A_3\Big(2^{2k}n + \frac{2^{2k+1}-2}{3}\Big) &= \frac{2^{2k}-1}{3}A_3(4n+2) - \frac{2^{2k}-4}{3}A_3(n), \\ A_3\Big(5^{2k}n + \frac{2\cdot 5^{2k}-2}{3}\Big) &= \frac{5^{2k}-1}{24}A_3(25n+16) - \frac{5^{2k}-25}{24}A_3(n), \\ A_3\Big(7^{k}n + \frac{2\cdot 7^{k}-2}{3}\Big) &= \frac{7^{k}-1}{6}A_3(7n+4) - \frac{7^{k}-7}{6}A_3(n). \end{aligned}$$

Theorem 2.2.4. *Let p be a prime, and let k, n be nonnegative integers such that $p \nmid 3n+2$.*

(1) If $p \equiv 1 \pmod 3$, then

$$A_3\Big(p^k n + \frac{2p^k-2}{3}\Big) = \frac{p^{k+1}-1}{p-1}A_3(n).$$

(2) If $p \equiv 2 \pmod 3$, then

$$A_3\Big(p^{2k} n + \frac{2p^{2k}-2}{3}\Big) = \frac{p^{2k+1}-1}{p-1}A_3(n).$$

Proof. From Theorem 2.2.2, we deduce that

$$A_3\Big(p^k n + \frac{2p^k-2}{3}\Big) = \frac{1}{3}\sigma\big(p^k(3n+2)\big) = \frac{1}{3}\sigma(p^k)\sigma(3n+2) = \frac{p^{k+1}-1}{p-1}A_3(n).$$

This implies (1). Assertion (2) can be proved in a similar way. □

For example, if we set $p = 2$ and replace n by $2n+1$ in (2), we obtain (1.26). If we set $p = 5$ (resp. $p = 7$) and replace n by $5n+r$ (resp. $7n+r$), we deduce that for $k, n \geq 0$,

$$A_3\Big(5^{2k}(5n+r) + \frac{2\cdot 5^{2k}-2}{3}\Big) = \frac{5^{2k+1}-1}{4}A_3(5n+r), \quad r \in \{0, 2, 3, 4\}$$

and

$$A_3\Big(7^{k}(7n+r) + \frac{2\cdot 7^{k}-2}{3}\Big) = \frac{7^{k+1}-1}{6}A_3(7n+r), \quad r \in \{0, 1, 2, 3, 5, 6\}.$$

We conclude this section by proving Conjecture 2.1.1.

Proof of Conjecture 2.1.1. By Theorem 2.2.2, we get

$$\begin{aligned} A_3\Big(4^{k_0(\alpha+1)}n + \frac{2^{2k_0(\alpha+1)-1}-2}{3}\Big) &= \frac{1}{3}\sigma\big(2^{2k_0(\alpha+1)-1}(6n+1)\big) \\ &= \frac{2^{2k_0(\alpha+1)}-1}{3}\sigma(6n+1). \end{aligned} \tag{2.2.15}$$

Let $k_0 = \frac{1}{2}p^j(p-1)$. Since $2k_0(\alpha+1) \equiv 0 \pmod{p^j(p-1)}$, by Euler's theorem, we have $2^{2k_0(\alpha+1)} \equiv 1 \pmod{p^{j+1}}$. From (2.2.15) the conjecture follows immediately. □

Indeed, most of the congruences found by Xia [91] can be proved by using Theorem 2.2.2. We omit the details here.

2.3 Explicit formula for $B_3(n)$

In order to find the explicit formula for $B_3(n)$, we need the following formula.

Lemma 2.3.1 (Bailey's ${}_6\psi_6$ formula)**.** *For $|qa^2/(bcde)| < 1$,*

$$\begin{aligned} &{}_6\psi_6\left(\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e \end{matrix}; q, \frac{qa^2}{bcde}\right) \\ &= \frac{(aq, aq/(bc), aq/(bd), aq/(be), aq/(cd), aq/(ce), aq/(de), q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/(bcde); q)_\infty}. \end{aligned} \tag{2.3.1}$$

For the proof of this lemma, see [42, Sec. 5.3].

Theorem 2.3.2. *For any integer $n \geq 0$, we have*

$$B_3(n) = \sum_{\substack{d|n+1 \\ d\equiv 1 \pmod 3}} \left(\frac{n+1}{d}\right)^2 - \sum_{\substack{d|n+1 \\ d\equiv 2 \pmod 3}} \left(\frac{n+1}{d}\right)^2.$$

Furthermore, if we write

$$n+1 = 3^\alpha \prod_{p_i\equiv 1 \,(\mathrm{mod}\, 3)} p_i^{\alpha_i} \prod_{q_j\equiv 2 \,(\mathrm{mod}\, 3)} q_j^{\beta_j}$$

as the unique prime factorization of $n+1$ with $\alpha, \alpha_i, \beta_j \geq 0$, then

$$B_3(n) = 3^{2\alpha} \prod_{p_i\equiv 1 \,(\mathrm{mod}\, 3)} \frac{p_i^{2(\alpha_i+1)}-1}{p_i^2-1} \prod_{q_j\equiv 2 \,(\mathrm{mod}\, 3)} \frac{q_j^{2(\beta_j+1)}+(-1)^{\beta_j}}{q_j^2+1}.$$

Proof. Setting $t = 3$ in (1.22) and applying (2.2.2), we see that

$$\sum_{n=0}^{\infty} B_3(n)q^n = \frac{(q^3;q^3)_\infty^6}{(q;q^3)_\infty^3(q^2;q^3)_\infty^3}. \tag{2.3.2}$$

Taking $(a,b,c,d,e,q) \to (q^2,q,q,q,q,q^3)$ in (2.3.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{(q^4,-q^4,q,q,q,q;q^3)_n q^{3n}}{(q,-q,q^4,q^4,q^4,q^4;q^3)_n} = \frac{(q^5,q;q^3)_\infty(q^3;q^3)_\infty^7}{(q^3;q^3)_\infty(q^4,q^2;q^3)_\infty^4}. \tag{2.3.3}$$

Simplifying this identity and multiplying both sides by $\frac{q(1-q^2)}{(1-q)^4}$, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{(1+q^{3n+1})q^{3n+1}}{(1-q^{3n+1})^3} = q \cdot \frac{(q^3;q^3)_\infty^6}{(q;q^3)_\infty^3(q^2;q^3)_\infty^3}.$$

Combining this with (2.3.2), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} B_3(n)q^{n+1} &= \sum_{m=0}^{\infty} \frac{q^{3m+1}(1+q^{3m+1})}{(1-q^{3m+1})^3} + \sum_{m=-\infty}^{-1} \frac{q^{3m+1}(1+q^{3m+1})}{(1-q^{3m+1})^3} \\ &= \sum_{m=0}^{\infty} \frac{q^{3m+1}(1+q^{3m+1})}{(1-q^{3m+1})^3} - \sum_{m=0}^{\infty} \frac{q^{3m+2}(1+q^{3m+2})}{(1-q^{3m+2})^3},\end{aligned} \tag{2.3.4}$$

where the second equality follows by replacing m by $-m-1$ in the second sum.

It is well known that

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad |x| < 1.$$

Applying the operator $x\frac{d}{dx}$ twice to both sides, we get

$$\frac{x(1+x)}{(1-x)^3} = \sum_{k=1}^{\infty} k^2 x^k, \quad |x| < 1.$$

Applying this identity to (2.3.4), we obtain

$$\sum_{n=0}^{\infty} B_3(n)q^{n+1} = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} k^2\left(q^{(3m+1)k} - q^{(3m+2)k}\right).$$

The first assertion of this theorem now follows immediately by comparing the coefficients of q^{n+1} on both sides.

For $n \geq 1$, let

$$f(n) = \sum_{\substack{d|n \\ d\equiv 1 \pmod 3}} \left(\frac{n}{d}\right)^2 - \sum_{\substack{d|n \\ d\equiv 2 \pmod 3}} \left(\frac{n}{d}\right)^2,$$

so that $f(n+1) = B_3(n)$. Suppose m and n are integers which are coprime to each other. It is not hard to see that

$$\begin{aligned}
f(mn) &= \sum_{\substack{d|mn \\ d\equiv 1 \pmod 3}} \left(\frac{mn}{d}\right)^2 - \sum_{\substack{d|mn \\ d\equiv 2 \pmod 3}} \left(\frac{mn}{d}\right)^2 \\
&= \sum_{\substack{d_1|m \\ d_1\equiv 1 \pmod 3}} \sum_{\substack{d_2|n \\ d_2\equiv 1 \pmod 3}} + \sum_{\substack{d_1|m \\ d_1\equiv 2 \pmod 3}} \sum_{\substack{d_2|n \\ d_2\equiv 2 \pmod 3}} \left(\frac{mn}{d_1 d_2}\right)^2 \\
&\quad - \sum_{\substack{d_1|m \\ d_1\equiv 1 \pmod 3}} \sum_{\substack{d_2|n \\ d_2\equiv 2 \pmod 3}} - \sum_{\substack{d_1|m \\ d_1\equiv 2 \pmod 3}} \sum_{\substack{d_2|n \\ d_2\equiv 1 \pmod 3}} \left(\frac{mn}{d_1 d_2}\right)^2 \\
&= \Bigg(\sum_{\substack{d_1|m \\ d_1\equiv 1 \pmod 3}} \left(\frac{m}{d_1}\right)^2 - \sum_{\substack{d_1|m \\ d_1\equiv 2 \pmod 3}} \left(\frac{m}{d_1}\right)^2\Bigg) \\
&\quad \cdot \Bigg(\sum_{\substack{d_2|n \\ d_2\equiv 1 \pmod 3}} \left(\frac{n}{d_2}\right)^2 - \sum_{\substack{d_2|n \\ d_2\equiv 2 \pmod 3}} \left(\frac{n}{d_2}\right)^2\Bigg) \\
&= f(m)f(n).
\end{aligned}$$

This implies that $f(n)$ is multiplicative. For any prime p, from the definition of $f(n)$ and by direct calculations, we obtain that

$$f(p^k) = \begin{cases} 3^{2k} & \text{if } p = 3, \\ \frac{p^{2(k+1)}-1}{p^2-1} & \text{if } p \equiv 1 \pmod 3, \\ \frac{p^{2(k+1)}+(-1)^k}{p^2+1} & \text{if } p \equiv 2 \pmod 3. \end{cases} \tag{2.3.5}$$

The second assertion of this theorem then follows since $f(n)$ is multiplicative and $B_3(n) = f(n+1)$. □

Theorem 2.3.3. *Let p be a prime, and let k, n be nonnegative integers.*

(1) If $p \equiv 1 \pmod 3$, then

$$B_3\left(p^k n + p^k - 1\right) = \frac{p^{2k}-1}{p^2-1} B_3(pn+p-1) - \frac{p^{2k}-p^2}{p^2-1} B_3(n).$$

(2) If $p \equiv 2 \pmod 3$*, then*

$$B_3(p^k n + p^k - 1) = \frac{p^{2k} - (-1)^k}{p^2+1} B_3(pn + p - 1) + \frac{p^{2k} + (-1)^k p^2}{p^2+1} B_3(n).$$

Proof. Let $n + 1 = p^m N$, where N is not divisible by p.

(1) Since $f(n)$ is multiplicative, by (2.3.5) we have

$$B_3(n) = f(n+1) = f(p^m)f(N) = \frac{p^{2(m+1)} - 1}{p^2 - 1} f(N), \tag{2.3.6}$$

$$B_3(pn + p - 1) = f(p(n+1)) = f(p^{m+1})f(N) = \frac{p^{2(m+2)} - 1}{p^2 - 1} f(N), \tag{2.3.7}$$

$$B_3(p^k n + p^k - 1) = f(p^k(n+1)) = f(p^{k+m})f(N) = \frac{p^{2(m+k+1)} - 1}{p^2 - 1} f(N). \tag{2.3.8}$$

From those identities (2.3.6)–(2.3.8), we prove (1) by direct verification.

(2) Similarly, by (2.3.5) we have

$$B_3(n) = f(n+1) = f(p^m)f(N) = \frac{p^{2(m+1)} + (-1)^m}{p^2 + 1} f(N), \tag{2.3.9}$$

$$B_3(pn + p - 1) = f(p(n+1)) = f(p^{m+1})f(N) = \frac{p^{2(m+2)} + (-1)^{m+1}}{p^2 + 1} f(N), \tag{2.3.10}$$

$$B_3(p^k n + p^k - 1) = f(p^k(n+1)) = f(p^{k+m})f(N) = \frac{p^{2(m+k+1)} + (-1)^{m+k}}{p^2 + 1} f(N). \tag{2.3.11}$$

From those identities (2.3.9)–(2.3.11), we prove (2) by direct verification. □

By setting $p = 2$ in this theorem we obtain (2.1.5) immediately. For more examples, by setting $p = 5, 7$ in this theorem, we obtain for $k, n \geq 0$,

$$B_3(5^k n + 5^k - 1) = \frac{5^{2k} - (-1)^k}{26} B_3(5n + 4) + \frac{5^{2k} + 25(-1)^k}{26} B_3(n)$$

and

$$B_3(7^k n + 7^k - 1) = \frac{7^{2k} - 1}{48} B_3(7n + 6) - \frac{7^{2k} - 49}{48} B_3(n).$$

In some special cases, we can obtain some relations between $B_3(p^k n + p^k - 1)$ and $B_3(n)$.

Theorem 2.3.4. *Let p be a prime, and let k, n be nonnegative integers.*
(1) If $p = 3$, then $B_3(3^k n + 3^k - 1) = 3^{2k} B_3(n)$.
(2) If $p \equiv 1 \pmod 3$ and $p \nmid n + 1$, then

$$B_3(p^k n + p^k - 1) = \frac{p^{2(k+1)} - 1}{p^2 - 1} B_3(n).$$

(3) If $p \equiv 2 \pmod 3$ and $p \nmid n + 1$, then

$$B_3(p^k n + p^k - 1) = \frac{p^{2(k+1)} + (-1)^k}{p^2 + 1} B_3(n).$$

Proof. Throughout the proofs of (1)–(3), let $n + 1 = p^m N$, where N is not divisible by p. By Theorem 2.3.2 and the fact that $f(n)$ is multiplicative, we find that

$$B_3(n) = f(p^m N) = f(p^m) f(N) \tag{2.3.12}$$

and

$$B_3(p^k n + p^k - 1) = f(p^{k+m} N) = f(p^{k+m}) f(N). \tag{2.3.13}$$

(1) By (2.3.5), (2.3.12) and (2.3.13), we deduce that

$$B_3(3^k n + 3^k - 1) = 3^{2k+2m} f(N) = 3^{2k} B_3(n).$$

(2) Since $p \nmid n + 1$, we have $m = 0$. By (2.3.5), (2.3.12) and (2.3.13), we deduce that

$$B_3(p^k n + p^k - 1) = f(p^k) f(N) = \frac{p^{2(k+1)} - 1}{p^2 - 1} B_3(n).$$

(3) Since $p \nmid n + 1$, we have $m = 0$. By (2.3.5), (2.3.12) and (2.3.13), we deduce that

$$B_3(p^k n + p^k - 1) = f(p^k) f(N) = \frac{p^{2(k+1)} + (-1)^k}{p^2 + 1} B_3(n).$$

□

Note that in this theorem, (1) is (2.1.3) exactly. By setting $p = 2$ and replacing n by $2n$ in (3), we obtain (2.1.4) at once. For more examples, by setting $p = 5$ (resp.

$p = 7$) and replacing n by $5n + r$ (resp. $7n + r$) in (3) (resp. (2)) we obtain for $k, n \geq 0$,

$$B_3\Big(5^{k+1}n + 5^k(r+1) - 1\Big) = \frac{5^{2k+2} + (-1)^k}{26} B_3(5n + r), \quad r \in \{0, 1, 2, 3\}$$

and

$$B_3\Big(7^{k+1}n + 7^k(r+1) - 1\Big) = \frac{7^{2k+2} - 1}{48} B_3(7n + r), \quad r \in \{0, 1, 2, 3, 4, 5\}$$

respectively.

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Chapter 3

Congruences Modulo Powers of 11 for Some Partitions

3.1 Introduction

In this chapter, we will present congruences modulo arbitrary powers of 11 for several other partitions. We begin with a remark on notation. Let m be an integer. Suppose we have two Laurent series $f(q) = \sum_{n=-\infty}^{\infty} a_n q^n$ and $g(q) = \sum_{n=-\infty}^{\infty} b_n q^n$ where all the coefficients a_n and b_n are rational numbers. If $a_n \equiv b_n \pmod{m}$ for every integer n, then we say that the following congruence holds:

$$f(q) \equiv g(q) \pmod{m}.$$

We go back to the S. Ramanujan's congruences (1.16)–(1.18). The ideas for G.N. Watson's proof of (1.16)–(1.17) and A.O.L. Atkin's proof of (1.18) are similar.

Let ℓ be a prime and n a nonnegative integer. We define

$$L_{n,\ell} := \begin{cases} (q^\ell; q^\ell)_\infty \sum\limits_{m=0}^{\infty} p(\ell^n m + \delta_{\ell,n}) q^{m+1}, & \text{if } n \text{ is odd}, \\ (q;q)_\infty \sum\limits_{m=0}^{\infty} p(\ell^n m + \delta_{\ell,n}) q^{m+1}, & \text{if } n \text{ is even}. \end{cases} \tag{3.1.1}$$

One can show that $L_{n,\ell}$ are modular functions on $\Gamma_0(\ell)$ for $\ell \in \{5, 7, 11\}$. Here for any positive integer N, the congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbf{Z})$ is defined as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big| a, b, c, d \in \mathbf{Z}, ad - bc = 1, c \equiv 0 \pmod{N} \right\}.$$

Therefore, we can express $L_{n,\ell}$ using linear basis for the space of modular functions on $\Gamma_0(\ell)$. Examining the ℓ-adic orders of the coefficients will lead to (1.16)–(1.18).

Let

$$\Delta = q(q;q)_\infty^{24}, \quad E_8 = 1 + 480 \sum_{n=1}^{\infty} \frac{n^7 q^n}{1-q^n}.$$

H.H. Chan and P.C. Toh [33] observed empirically that there exist integers a_n, b_n and c_n with $(5, a_n) = (7, b_n) = (11, c_n) = 1$ such that

$$L_{n,5} \equiv 5^n a_n \Delta \pmod{5^{n+1}}, \tag{3.1.2}$$

$$L_{n,7} \equiv 7^{[n/2]+1} b_n \Delta \pmod{7^{[n/2]+2}}, \tag{3.1.3}$$

and

$$L_{n,11} \equiv 11^n c_n \Delta E_8 \pmod{11^{n+1}}. \tag{3.1.4}$$

Both (3.1.2) and (3.1.3) follow immediately from Watson's work (see [55]). Chan and Toh [33] suggested that (3.1.4) could be proved using Atkin's method given in [16]. The first goal of this chapter is to show that (3.1.4) indeed follows from Atkin's work [16]. So we can rewrite it as

Theorem 3.1.1. *For any integer $n \geq 1$, there exists an integer c_n with $(11, c_n) = 1$ such that*

$$L_{n,11} \equiv 11^n c_n \Delta E_8 \pmod{11^{n+1}}.$$

Following the notation of Chan and Toh [33], we define

$$\sum_{n=0}^{\infty} p_{[1^c t^d]}(n) q^n = \frac{1}{(q;q)_\infty^c (q^t;q^t)_\infty^d}, \quad c, d, t \in \mathbf{Z}, t \geq 1.$$

It is then clear that in this notation, the t-core partition function $a_t(n)$ discussed in Chapter 2 satisfies $a_t(n) = p_{[1^1 t^{-t}]}(n)$. Moreover, we denote the number of ℓ-regular partitions of n by $b_\ell(n)$. Recall that in (1.27) we gave the generating function of $b_\ell(n)$ as

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}. \tag{3.1.5}$$

This means that $b_\ell(n) = p_{[1^1\ell^{-1}]}(n)$.

For some particular integer triples (c,d,t), arithmetic properties of $p_{[1^c t^d]}(n)$ have been extensively investigated. See [33, 41, 43, 49, 65, 66, 70, 83, 84], for example. For more comprehensive reference lists about t-core partitions and t-regular partitions, we refer the reader to [82] and [83].

It should be noted that so far almost all works concentrated on discovering congruences modulo small powers of primes for those partition functions. There are only a few works where congruences modulo arbitrary prime powers appear, see [17, 30, 31, 33, 45, 58, 66, 70, 83, 84] for example. By using Ramanujan's cubic continued fraction, H.C. Chan [30] proved that

$$p_{[1^1 2^1]}(3^j n + c_j) \equiv 0 \pmod{3^{2[j/2]+1}},$$

where $c_j \equiv 1/8 \pmod{3^j}$. Similarly, let $d_j \equiv 1/8 \pmod{5^j}$, Chan and Toh [33] showed that for any integer $n \geq 0$,

$$p_{[1^1 2^1]}(5^j n + d_j) \equiv 0 \pmod{5^{[j/2]}}.$$

Recently, using the modular equation of fifth order, we [83] proved that for any integers $k \geq 1$ and $n \geq 0$,

$$b_5\Big(5^{2k-1}n + \frac{5^{2k}-1}{6}\Big) \equiv 0 \pmod{5^k}.$$

We [84] also proved that

$$p_{[1^1 5^1]}\Big(5^k n + \frac{3\cdot 5^k + 1}{4}\Big) \equiv 0 \pmod{5^k}.$$

While congruences modulo arbitrary powers of 2, 3, 5 or 7 appeared in a few literature, we observed that after the work of Atkin [16], congruences modulo powers of 11 for partition functions other than $p(n)$ are seldom discussed. One of the few examples known to us is the work of B. Gordon [44], where Gordon established many congruences modulo arbitrary powers of 11 for the function $p_k(n)$ defined by

$$\sum_{n=0}^{\infty} p_k(n)q^n = (q;q)_\infty^k. \tag{3.1.6}$$

In view of this phenomenon, the second goal of this chapter is to provide more partition congruences modulo arbitrary powers of 11. In Sections 3.2 and 3.3, we will follow the strategy of Atkin [16] and Gordon [44] to establish those congruences for three different types of partition functions.

Theorem 3.1.2. *For any integers $n \geq 0$ and $k \geq 1$, we have*

$$a_{11}\Big(11^k n + 11^k - 5\Big) \equiv 0 \pmod{11^k}.$$

Theorem 3.1.3. *For any integers $n \geq 0$ and $k \geq 1$, we have*

$$b_{11}\Big(11^{2k-1} n + \frac{7 \cdot 11^{2k-1} - 5}{12}\Big) \equiv 0 \pmod{11^k}.$$

Theorem 3.1.4. *For any integers $n \geq 0$ and $k \geq 1$, we have*

$$p_{[1^1 11^1]}\Big(11^k n + \frac{11^k + 1}{2}\Big) \equiv 0 \pmod{11^k}.$$

We remark here that an equivalent form of Theorem 3.1.2 was discovered by Garvan [43, Eq. (1.9)].

The method used here can be applied to obtain similar results for $p_{[1^c 11^d]}(n)$ for other values of $c, d \in \mathbf{Z}$. Since the partition functions in Theorems 3.1.2–3.1.4 are more popular, we will illustrate the method by studying these examples.

3.2 Useful facts for establishing congruences modulo powers of 11

In the remaining part of this chapter, we are going to prove Theorem 3.1.1 and Theorems 3.1.2–3.1.4. To present the proofs, we collect some facts which are essential in proving our results. We will follow the notation of Gordon [44].

Let $\mathbf{H}$ be the upper half complex plane. Recall that the Dedekind eta function is

$$\eta(\tau) = q^{1/24}(q;q)_\infty, \quad q = e^{2\pi i\tau}, \quad \tau \in \mathbf{H}.$$

Let $R_0(N)$ be the Riemann surface of $\Gamma_0(N)$. Let $K_0(N)$ be the field of meromorphic functions on $R_0(N)$. It is known that $R_0(N)$ has a cusp at $\tau = i\infty$, and $q = e^{2\pi i\tau}$ is a uniformizing parameter there. If $f(\tau) \in K_0(N)$, then the Laurent expansion about $\tau = i\infty$ has the form

$$f(\tau) = \sum_{n\geq n_0} a_n q^n.$$

Let

$$\phi(\tau) = \frac{\eta(121\tau)}{\eta(\tau)} = q^5\frac{(q^{121};q^{121})_\infty}{(q;q)_\infty}.$$

It is known that $\phi(\tau) \in K_0(121)$. This function will play a key role in our proofs.

We define the U-operator as

$$Uf(\tau) = \sum_{11n\geq n_0} a_{11n}q^n.$$

It is known that (see [14, pp. 80–82], for example) if $f(\tau) \in K_0(121)$, then $Uf(\tau) \in K_0(11)$.

If $f(\tau) \in K_0(11)$ and p is a point of $R_0(11)$, we use $\mathrm{ord}_p f(\tau)$ to denote the order of $f(\tau)$ at p.

Let V be the vector space of functions $g(\tau) \in K_0(11)$ which are holomorphic except possibly at 0 and ∞. Atkin [16] constructed a linear basis for V. In [16, Appendix A], Atkin first defined $g_n(\tau)$ and $G_n(\tau)$ for $2 \leq n \leq 6$ by

$$10g_2(\tau)(q;q)_\infty^5 = -\sum_{n=0}^{\infty}\Big(1 + (\frac{n-3}{11})\Big)p_5(n)q^n + 11^2q^{25}(q^{121};q^{121})_\infty^5, \tag{3.2.1}$$

$$14\Big(g_3(\tau) + g_2(\tau)\Big)(q;q)_\infty^7 = -\sum_{n=0}^{\infty}\Big(1 + (\frac{2-n}{11})\Big)p_7(n)q^n + 11^3q^{35}(q^{121};q^{121})_\infty^7, \tag{3.2.2}$$

$$\Big(11^2 + 10G_2(\tau)\Big)(q^{11};q^{11})_\infty^5 = \sum_{n=-2}^{\infty} p_5(11n+25)q^n, \tag{3.2.3}$$

$$\Big(11^3 + 14G_3(\tau) + 154G_2(\tau)\Big)(q^{11};q^{11})_\infty^7 = \sum_{n=-3}^{\infty} p_7(11n+35)q^n, \tag{3.2.4}$$

$$G_4(\tau) = G_2^2(\tau) - 11G_3(\tau), \quad g_4(\tau) = g_2^2(\tau) - g_3(\tau), \tag{3.2.5}$$

$$G_6(\tau) = G_2(\tau)G_4(\tau), \quad g_6(\tau) = g_2(\tau)g_4(\tau), \tag{3.2.6}$$

$$G_5(\tau) = \frac{\eta^{12}(\tau)}{\eta^{12}(11\tau)}, \quad g_5(\tau) = \frac{\eta^{12}(11\tau)}{\eta^{12}(\tau)}, \tag{3.2.7}$$

where $p_k(n)$ was defined in (3.1.6). Next, he defined inductively for $n \geq 7$,

$$G_n(\tau) = G_{n-5}(\tau)G_5(\tau), \quad g_n(\tau) = g_{n-5}(\tau)g_5(\tau). \tag{3.2.8}$$

Following the notation of Gordon [44], for $k \neq 0, -1$, let $J_k(\tau)$ be the element of Atkin's basis whose order at ∞ is k. We define $J_0(\tau) = 1$ and $J_{-1}(\tau) = J_{-6}(\tau)J_5(\tau)$. In terms of the notation of Atkin, we have for $k \geq 1$,

$$J_k(\tau) = \begin{cases} g_k(\tau) & \text{if } k \equiv 0 \pmod 5, \\ g_{k+2}(\tau) & \text{if } k \equiv 4 \pmod 5, \\ g_{k+1}(\tau) & \text{otherwise} \end{cases} \tag{3.2.9}$$

and $J_k(\tau) = G_{-k}(\tau)$ for $k \leq -2$.

Lemma 3.2.1. *(Cf. [44, Lemma 3].) For all $k \in \mathbf{Z}$, the following holds:*

(i) $J_{k+5}(\tau) = J_k(\tau)J_5(\tau)$.

(ii) $\{J_k(\tau) | k \in \mathbf{Z}\}$ *is a basis of* V.

(iii) $\mathrm{ord}_\infty J_k(\tau) = k$.

(iv) $\mathrm{ord}_0 J_k(\tau) = \begin{cases} -k & \text{if } k \equiv 0 \pmod 5, \\ -k-1 & \text{if } k \equiv 1, 2 \text{ or } 3 \pmod 5, \\ -k-2 & \text{if } k \equiv 4 \pmod 5. \end{cases}$

(v) *The Fourier series of $J_k(\tau)$ has integer coefficients, and is of the form $J_k(\tau) = q^k + O(q^{k+1})$ where the symbol $O(q^{k+1})$ stands for a series wherein the powers of q are higher than k.*

From [44], we know that V is mapped into itself by the linear transformation

$$T_\lambda : g(\tau) \to U(\phi(\tau)^\lambda g(\tau))$$

for any integer λ. Following Atkin, we write the elements of V as row vectors and let matrices act on the right. Let $C^{(\lambda)} = (c^{(\lambda)}_{\mu,\nu})$ be the matrix of T_λ with respect to the basis $\{J_k\}$ of V. We have

$$U(\phi(\tau)^\lambda J_\mu(\tau)) = \sum_{\nu\in\mathbf{Z}} c^{(\lambda)}_{\mu,\nu} J_\nu(\tau). \tag{3.2.10}$$

For any nonnegative integer n, let $\pi(n)$ be the 11-adic order of n with the convention that $\pi(0) = \infty$. As shown in [44], we have

$$\pi(c^{(\lambda)}_{\mu,\nu}) \geq [(11\nu - \mu - 5\lambda + \delta)/10], \tag{3.2.11}$$

where $\delta = \delta(\mu,\nu)$ depends on the residues of μ and ν (mod 5) according to Table 3.1.

Table 3.1

μ \ ν	0	1	2	3	4
0	-1	8	7	6	15
1	0	9	8	2	11
2	1	10	4	3	12
3	2	6	5	4	13
4	3	7	6	5	9

From Table 3.1, we see that $\delta(\lambda,\mu) \geq -1$ for any λ, μ. Therefore, (3.2.11) implies

$$\pi(c^{(\lambda)}_{\mu,\nu}) \geq [(11\nu - \mu - 5\lambda - 1)/10]. \tag{3.2.12}$$

By Lemma 3.2.1(v) and (3.2.10) we know that the Fourier series of $U(\phi^\lambda J_\mu)$ has all coefficients divisible by 11 if and only if

$$c^{(\lambda)}_{\mu,\nu} \equiv 0 \pmod{11} \quad \text{for all } \nu. \tag{3.2.13}$$

We define a function $\theta(\lambda,\mu)$ as follows. If (3.2.13) holds we put $\theta(\lambda,\mu) = 1$ and $\theta(\lambda,\mu) = 0$ otherwise. From [44], we know that

$$\theta(\lambda,\mu) = \theta(\lambda - 11,\mu), \quad \theta(\lambda + 12, \mu - 5) = \theta(\lambda,\mu). \tag{3.2.14}$$

This implies that $\theta(\lambda,\mu)$ is completely determined by its values in the range $0 \le \lambda \le 10$, $0 \le \mu \le 4$, which are listed in Table 3.2.

Table 3.2

μ \ λ	0	1	2	3	4	5	6	7	8	9	10
0	0	1	0	1	0	1	0	1	1	0	0
1	1	1	0	1	0	0	0	1	1	0	0
2	1	1	1	0	0	0	0	1	1	0	0
3	1	0	1	0	1	0	0	1	1	0	0
4	1	0	1	0	1	0	1	1	0	0	0

Let $M_k(\Gamma_0(N),\chi)$ denote the space of modular forms of weight k on $\Gamma_0(N)$ with Dirichlet character χ (see [69]). In particular, if χ is the trivial Dirichlet character, we also write $M_k(\Gamma_0(N),\chi)$ as $M_k(\Gamma_0(N))$. The following result, known as Sturm's criterion [79], will be used in proving Theorem 3.1.1.

Lemma 3.2.2. *Let p be a prime and $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_0(N))$ where $a_n \in \mathbf{Q}$ for all $n \ge 0$. If $a_n \equiv 0 \pmod p$ for*

$$n \le \frac{kN}{12}\prod_{\ell \mid N}\Big(1+\frac{1}{\ell}\Big),$$

where the product is over the distinct prime divisors of N, then $f(\tau) \equiv 0 \pmod p$, i.e., $a_n \equiv 0 \pmod p$ for any $n \ge 0$.

3.3 Proofs of congruences modulo powers of 11

Proof of Theorem 3.1.1. As in [16, p. 20], we define $a(1) = 0, a(2) = 1$ and for $n \ge 3$,

$$a(n) = \begin{cases} n-1 & \text{if } n \equiv 4 \pmod 5, \\ n-2 & \text{otherwise.} \end{cases}$$

Similarly, let $b(1) = 0, b(2) = 1$ and $b(n) = a(n) + 1$ $(n \geq 3)$.

Let X^0 be the class of functions $f(\tau)$ with

$$f(\tau) = \sum_{n=1}^{N} \lambda_n 11^{a(n)} J_n(\tau), \quad \pi(\lambda_1) = 0, \quad N \geq 1$$

and let Y^0 be the class of functions $f(\tau)$ with

$$f(\tau) = \sum_{n=1}^{M} \mu_n 11^{b(n)} J_n(\tau), \quad \pi(\mu_1) = 0, \quad M \geq 1$$

where λ_n and μ_n are integers for any integer n. Note that we have changed Atkin's original definitions in terms of $g_n(\tau)$ to expressions involving $J_n(\tau)$ according to (3.2.9). We also change the sequences $\xi(n)$ and $\eta(n)$ in [16] to $a(n)$ and $b(n)$ accordingly.

In the proof of (1.18), Atkin [16, p. 26] showed that

$$11^{1-2n} L_{2n-1,11}(\tau) \in X^0, \quad 11^{-2n} L_{2n,11}(\tau) \in Y^0. \tag{3.3.1}$$

For $n \geq 2$, we have $a(n) \geq 1$ and $b(n) \geq 1$. By Lemma 3.2.1, the Fourier expansion of $J_n(\tau)$ has integer coefficients. We deduce from (3.3.1) that

$$11^{1-2n} L_{2n-1,11}(\tau) \equiv \lambda_1 J_1(\tau) \pmod{11}, \quad 11^{-2n} L_{2n,11}(\tau) \equiv \mu_1 J_1(\tau) \pmod{11} \tag{3.3.2}$$

for some integers λ_1 and μ_1 which depend on n and are relatively prime with 11. Thus we have shown that there exists integer c_n such that $(11, c_n) = 1$ and

$$11^{-n} L_{n,11}(\tau) \equiv c_n J_1(\tau) \pmod{11}.$$

To prove (3.1.4), it suffices to show that

$$J_1(\tau) \equiv \Delta E_8 \pmod{11}. \tag{3.3.3}$$

By Lemma 3.2.1 we know $\mathrm{ord}_\infty J_1(\tau) = 1$ and $\mathrm{ord}_0 J_1(\tau) = -2$. Let

$$f(\tau) = \frac{\eta^{11}(\tau)}{\eta(11\tau)}.$$

From [69, Theorems 1.64 and 1.65], we know that $f(\tau) \in M_5(\Gamma_0(11), \left(\frac{-11}{\cdot}\right))$. Moreover, $\mathrm{ord}_0 f(\tau) = 5$. Hence $f^4(\tau)J_1(\tau)$ has no poles at the cusps and hence is holomorphic. This implies $f^4(\tau)J_1(\tau) \in M_{20}(\Gamma_0(11))$.

Note that $\Delta E_8 \in M_{20}(\Gamma_0(11))$, hence $f^4(\tau)J_1(\tau) - \Delta E_8 \in M_{20}(\Gamma_0(11))$. Write

$$f^4(\tau)J_1(\tau) - \Delta E_8 = \sum_{n=0}^{\infty} c(n)q^n.$$

By the definition of $f(\tau)$, we know that $f(\tau)$ has integral coefficients as a power series in $q = e^{2\pi\tau}$. By Lemma 3.2.1 we know $J_1(\tau)$ also has integral coefficients. Now from the definitions of Δ and E_8, we see that $c(n) \in \mathbf{Z}$ for any $n \geq 0$. Using the definition of $J_1(\tau)$, it is easy to verify that $c(n) \equiv 0 \pmod{11}$ for $n \leq 20$. Hence by Lemma 3.2.2, we deduce that

$$f^4(\tau)J_1(\tau) \equiv \Delta E_8 \pmod{11}. \tag{3.3.4}$$

By the binomial theorem, we have $f(\tau) \equiv 1 \pmod{11}$. Therefore, (3.3.4) implies (3.3.3) and we have completed the proof of Theorem 3.1.1. □

Before we proceed to proofs of Theorem 3.1.2-3.1.4, we observe that (1.13) gives

$$p(11n+6) \equiv 0 \pmod{11}. \tag{3.3.5}$$

It is then clear that Theorems 3.1.2-3.1.4 are true for the case $k = 1$. Therefore, we only need to give proofs for $k \geq 2$.

Proof of Theorem 3.1.2. Recall that

$$\sum_{n=0}^{\infty} a_{11}(n)q^n = \frac{(q^{11};q^{11})_\infty^{11}}{(q;q)_\infty}.$$

Let

$$L_0(\tau) := \frac{\eta^{11}(11\tau)\eta(121\tau)}{\eta(\tau)\eta^{11}(1331\tau)} = \frac{(q^{121};q^{121})_\infty}{(q^{1331};q^{1331})_\infty^{11}} \sum_{n\geq 0} a_{11}(n)q^{n-600}.$$

We have

$$UL_0(\tau) = \frac{(q^{11};q^{11})_\infty}{(q^{121};q^{121})_\infty^{11}} \sum_{n\geq 0} a_{11}(11n+6)q^{n-54}.$$

Let

$$L_1(\tau) := U^2 L_0(\tau) = \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty^{11}} \sum_{n\geq 0} a_{11}(11^2 n + 116) q^{n-4}.$$

Note that $L_0(\tau) \in K_0(1331)$, hence $UL_0(\tau) \in K_0(121)$ and $L_1(\tau) \in K_0(11)$. For $r \geq 2$, we define

$$L_r(\tau) := U(\phi(\tau)^{\lambda_{r-1}} L_{r-1}(\tau)) \tag{3.3.6}$$

where

$$\lambda_r = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ -11 & \text{if } r \text{ is even.} \end{cases}$$

By induction on r it is not difficult to see that for $r \geq 1$, $L_r(\tau) \in V$ and

$$L_r(\tau) = \begin{cases} (q;q)_\infty (q^{11};q^{11})_\infty^{-11} \sum_{n\geq 0} a_{11}(11^{r+1} n + 11^{r+1} - 5) q^{n-4} & \text{if } r \text{ is odd,} \\ (q^{11};q^{11})_\infty (q;q)_\infty^{-11} \sum_{n\geq 0} a_{11}(11^{r+1} n + 11^{r+1} - 5) q^{n+1} & \text{if } r \text{ is even.} \end{cases} \tag{3.3.7}$$

Let

$$\mu_r = \begin{cases} -4 & \text{if } r \text{ is odd,} \\ 1 & \text{if } r \text{ is even.} \end{cases}$$

Since $L_r(\tau) \in V$, from (3.3.7) we may write

$$L_r(\tau) = \sum_{\nu \geq \mu_r} a_{r,\nu} J_\nu(\tau), \quad a_{r,\nu} \in \mathbf{Z}. \tag{3.3.8}$$

We will prove that for any $r \geq 1$,

$$\pi(a_{r,\nu}) \geq r + 1 + [\frac{\nu - \mu_r}{2}], \quad \forall \nu \geq \mu_r. \tag{3.3.9}$$

If $r = 1$, we find that

$$L_1(\tau) = 167948 J_{-4}(\tau) + 3529812 J_{-3}(\tau) + 19501812 J_{-2}(\tau) + 214358881 J_0(\tau).$$

Therefore, we have

$$\pi(a_{1,-4}) = 2, \quad \pi(a_{1,-3}) = 3, \quad \pi(a_{1,-2}) = 4, \quad \pi(a_{1,-1}) = \infty, \quad \pi(a_{1,0}) = 8$$

and $\pi(a_{1,\nu}) = \infty$ for any $\nu \geq 1$. Hence (3.3.9) is true for $r = 1$.

Now suppose (3.3.9) holds for $r-1$ ($r \geq 2$). From (3.2.10) we see that

$$a_{r,\nu} = \sum_{\mu=\mu_{r-1}}^{\infty} a_{r-1,\mu} c_{\mu,\nu}^{(\lambda_{r-1})}.$$

Thus

$$\pi(a_{r,\nu}) \geq \min_{\mu \geq \mu_{r-1}} \left(\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})})\right). \tag{3.3.10}$$

To complete the induction, it suffices to prove that

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1 + \left[\frac{\nu - \mu_r}{2}\right], \quad \text{for all } \mu \geq \mu_{r-1}, \nu \geq \mu_r. \tag{3.3.11}$$

By induction hypothesis and (3.2.12), we deduce that

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + \left[\frac{\mu - \mu_{r-1}}{2}\right] + \left[\frac{11\nu - \mu - 5\lambda_{r-1} - 1}{10}\right]. \tag{3.3.12}$$

If we increase μ by 2, then $\left[\frac{\mu-\mu_{r-1}}{2}\right]$ increases by at least 1 and $\left[\frac{11\nu-\mu-5\lambda_{r-1}-1}{10}\right]$ decreases by at most 1. Hence the value of the right hand side of (3.3.12) does not decrease. Therefore, its minimum value occurs when $\mu = \mu_{r-1} + 1$. Thus

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + \left[\frac{11\nu - \mu_{r-1} - 5\lambda_{r-1} - 2}{10}\right]. \tag{3.3.13}$$

If r is odd, then $\mu_{r-1} = 1$ and $\lambda_{r-1} = -11$. For $\nu \geq -3$, we have

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1 + \left[\frac{11\nu + 42}{10}\right] \geq r + 1 + \left[\frac{\nu + 4}{2}\right], \tag{3.3.14}$$

For $\nu = -4$, (3.3.11) reduces to

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1, \quad \mu \geq \mu_{r-1}. \tag{3.3.15}$$

This inequality holds for $\mu = \mu_{r-1}$ since $\pi(a_{r-1,\mu_{r-1}}) \geq r$ and

$$\pi(c_{\mu_{r-1},\nu}^{(\lambda_{r-1})}) \geq \theta(\lambda_{r-1}, \mu_{r-1}) = \theta(-11, 1) = 1.$$

Similarly it holds for $\mu = \mu_{r-1} + 1$. If $\mu \geq \mu_{r-1} + 2$, then by induction hypothesis we have

$$\pi(a_{r-1,\mu}) \geq r + \left[\frac{\mu - \mu_{r-1}}{2}\right] \geq r + 1.$$

Thus (3.3.15) holds.

Combining (3.3.14) with (3.3.15), we see that (3.3.11) holds for any odd r.

If r is even, then $\mu_{r-1} = -4$ and $\lambda_{r-1} = 1$. For $\nu \geq 2$, from (3.3.13) we have

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1 + \left[\frac{11\nu - 13}{10}\right] \geq r + 1 + \left[\frac{\nu - 1}{2}\right]. \tag{3.3.16}$$

For $\nu = 1$, (3.3.11) reduces to

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1, \quad \mu \geq \mu_{r-1}. \tag{3.3.17}$$

This inequality holds for $\mu = \mu_{r-1}$ since $\pi(a_{r-1,\mu_{r-1}}) \geq r$ and

$$\pi(c_{\mu_{r-1},\nu}^{(\lambda_{r-1})}) \geq \theta(\lambda_{r-1}, \mu_{r-1}) = \theta(1, -4) = 1.$$

Similarly it holds for $\mu = \mu_{r-1} + 1$. If $\mu \geq \mu_{r-1} + 2$, then by induction hypothesis, we have

$$\pi(a_{r-1,\mu}) \geq r + \left[\frac{\mu - \mu_{r-1}}{2}\right] \geq r + 1.$$

Thus (3.3.17) holds.

Combining (3.3.16) with (3.3.17), we see that (3.3.11) holds for any even r.

By induction on r, we complete the proof of (3.3.9). Therefore, for any $\nu \geq \mu_r$ we have $\pi(a_{r,\nu}) \geq r + 1$. By (3.3.8) we see that for any $r \geq 1$,

$$L_r(\tau) \equiv 0 \pmod{11^{r+1}}.$$

From this congruence and (3.3.7) we complete the proof of the theorem. □

Proof of Theorem 3.1.3. Recall that

$$\sum_{n=0}^{\infty} b_{11}(n) q^n = \frac{(q^{11}; q^{11})_\infty}{(q; q)_\infty}.$$

Let

$$L_0(\tau) := \frac{\eta(11\tau)\eta(121\tau)}{\eta(\tau)\eta(1331\tau)} = \frac{(q^{121}; q^{121})_\infty}{(q^{1331}; q^{1331})_\infty} \sum_{n=0}^{\infty} b_{11}(n) q^{n-50}.$$

We have

$$UL_0(\tau) = \frac{(q^{11}; q^{11})_\infty}{(q^{121}; q^{121})_\infty} \sum_{n=0}^{\infty} b_{11}(11n + 6) q^{n-4}.$$

Let

$$L_1(\tau) := U^2 L_0(\tau) = \frac{(q;q)_\infty}{(q^{11};q^{11})_\infty} \sum_{n=0}^{\infty} b_{11}(11^2 n + 50) q^n.$$

Note that $L_0(\tau) \in K_0(1331)$, hence $UL_0(\tau) \in K_0(121)$ and $L_1(\tau) \in K_0(11)$. For $r \geq 2$, we define

$$L_r(\tau) := U(\phi(\tau)^{\lambda_{r-1}} L_{r-1}(\tau)) \tag{3.3.18}$$

where

$$\lambda_r = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ -1 & \text{if } r \text{ is even.} \end{cases}$$

By induction on r, it is not difficult to see that for $r \geq 1$, $L_r(\tau) \in V$ and

$$L_r(\tau) = \begin{cases} (q;q)_\infty (q^{11};q^{11})_\infty^{-1} \sum\limits_{n=0}^{\infty} b_{11}(11^{r+1} n + \frac{5 \cdot 11^{r+1} - 5}{12}) q^n & \text{if } r \text{ is odd,} \\ (q^{11};q^{11})_\infty (q;q)_\infty^{-1} \sum\limits_{n=0}^{\infty} b_{11}(11^{r+1} n + \frac{7 \cdot 11^{r+1} - 5}{12}) q^{n+1} & \text{if } r \text{ is even.} \end{cases} \tag{3.3.19}$$

Let

$$\mu_r = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ 1 & \text{if } r \text{ is even.} \end{cases}$$

Since $L_r(\tau) \in V$, from (3.3.19) we may write

$$L_r(\tau) = \sum_{\nu \geq \mu_r} a_{r,\nu} J_\nu, \quad a_{r,\nu} \in \mathbf{Z}. \tag{3.3.20}$$

We will prove that for any $r \geq 1$,

$$\pi(a_{r,\nu}) \geq 1 + [\frac{r}{2}] + [\frac{\nu - \mu_r}{2}], \quad \forall \nu \geq \mu_r. \tag{3.3.21}$$

If $r = 1$, we find that

$$L_1(\tau) = \sum_{\nu=0}^{50} a_{1,\nu} J_\nu(\tau).$$

We have $\pi(a_{1,0}) = 1$ and the 11-adic orders of $a_{1,\nu}$ $(1 \leq \nu \leq 50)$ are given in Table 3.3, from which it is easy to verify that (3.3.21) holds for $r = 1$.

Now suppose (3.3.21) holds for $r-1$ $(r \geq 2)$. For the same reason as in the proof of Theorem 3.1.2, to complete the induction, it suffices to prove that

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq 1 + [\frac{r}{2}] + [\frac{\nu - \mu_r}{2}], \quad \text{for all } \mu \geq \mu_{r-1}, \nu \geq \mu_r. \tag{3.3.22}$$

Table 3.3

ν	1	2	3	4	5	6	7	8	9	10
$\pi(a_{1,\nu})$	3	4	4	7	6	8	9	10	12	12
ν	11	12	13	14	15	16	17	18	19	20
$\pi(a_{1,\nu})$	14	14	15	17	17	19	21	22	24	24
ν	21	22	23	24	25	26	27	28	29	30
$\pi(a_{1,\nu})$	26	26	27	29	29	31	32	34	36	36
ν	31	32	33	34	35	36	37	38	39	40
$\pi(a_{1,\nu})$	37	38	39	41	41	43	44	45	49	48
ν	41	42	43	44	45	46	47	48	49	50
$\pi(a_{1,\nu})$	50	51	52	55	54	56	57	58	∞	58

By induction hypothesis and (3.2.12), we deduce that

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq 1 + \left[\frac{r-1}{2}\right] + \left\lceil\frac{\mu-\mu_{r-1}}{2}\right\rceil + \left\lceil\frac{11\nu-\mu-5\lambda_{r-1}-1}{10}\right\rceil.$$

Note that if we increase μ by 2, the value of the right hand side does not decrease. Therefore, its minimum value occurs when $\mu = \mu_{r-1}+1$. Thus

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq 1 + \left[\frac{r-1}{2}\right] + \left\lceil\frac{11\nu-\mu_{r-1}-5\lambda_{r-1}-2}{10}\right\rceil. \tag{3.3.23}$$

If r is odd, then $\mu_{r-1}=1$ and $\lambda_{r-1}=-1$. We have

$$\pi(a_{r-1,\mu})+\pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq 1+\left[\frac{r-1}{2}\right]+\left\lceil\frac{11\nu+2}{10}\right\rceil \geq 1+\left[\frac{r}{2}\right]+\left\lceil\frac{\nu}{2}\right\rceil, \quad \forall \nu \geq 0. \tag{3.3.24}$$

Thus (3.3.22) holds for any odd r.

If r is even, then $\mu_{r-1}=0$ and $\lambda_{r-1}=1$. For $\nu \geq 2$, by (3.3.23) we have

$$\pi(a_{r-1,\mu})+\pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq 1+\left[\frac{r-1}{2}\right]+1+\left\lceil\frac{11\nu-17}{10}\right\rceil \geq 1+\left[\frac{r}{2}\right]+\left\lceil\frac{\nu-1}{2}\right\rceil. \tag{3.3.25}$$

For $\nu=1$, (3.3.22) reduces to

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq 1 + \left[\frac{r}{2}\right], \quad \mu \geq \mu_{r-1}. \tag{3.3.26}$$

This inequality holds for $\mu = \mu_{r-1}$ since $\pi(a_{r-1,\mu_{r-1}}) \geq 1 + \left[\frac{r-1}{2}\right]$ and

$$\pi(c^{(\lambda_{r-1})}_{\mu_{r-1},\nu}) \geq \theta(\lambda_{r-1}, \mu_{r-1}) = \theta(1,0) = 1.$$

Similarly, it holds for $\mu = \mu_{r-1} + 1$. If $\mu \geq \mu_{r-1} + 2$, then by induction hypothesis, we have

$$\pi(a_{r-1,\mu}) \geq 1 + \left[\frac{r-1}{2}\right] + \left[\frac{\mu - \mu_{r-1}}{2}\right] \geq 1 + \left[\frac{r}{2}\right].$$

Thus (3.3.26) holds for any even r.

Combining (3.3.25) with (3.3.26) we see that (3.3.22) holds for r.

By induction on r, we deduce (3.3.21). Therefore, for any $r \geq 1$ and $\nu \geq \mu_r$ we have

$$\pi(a_{r,\nu}) \geq 1 + \left[\frac{r}{2}\right].$$

From (3.3.20) we deduce that

$$L_r(\tau) \equiv 0 \pmod{11^{1+[r/2]}}.$$

From this congruence and (3.3.19) we complete the proof of the theorem. □

Proof of Theorem 3.1.4. Let

$$L_0(\tau) := \frac{\eta(121\tau)\eta(1331\tau)}{\eta(\tau)\eta(11\tau)} = q^{60}\frac{(q^{121};q^{121})_\infty(q^{1331};q^{1331})_\infty}{(q;q)_\infty(q^{11};q^{11})_\infty}.$$

We have

$$L_0(\tau) = (q^{121};q^{121})_\infty(q^{1331};q^{1331})_\infty \sum_{n=0}^{\infty} p_{[1^1 11^1]}(n) q^{n+60}.$$

Applying the U-operator twice, we get

$$L_1(\tau) := U^2 L_0(\tau) = (q;q)_\infty(q^{11};q^{11})_\infty \sum_{n=0}^{\infty} p_{[1^1 11^1]}(11^2 n + 61) q^{n+1}.$$

Since $L_0(\tau) \in K_0(1331)$, we have $UL_0(\tau) \in K_0(121)$ and $L_1(\tau) \in K_0(11)$. For $r \geq 2$, we define

$$L_r(\tau) := U(\phi^{\lambda_{r-1}}(\tau) L_{r-1}(\tau)) \tag{3.3.27}$$

where $\lambda_r = 1$ for any $r \geq 1$. By induction on r it is not difficult to see that for $r \geq 1$, $L_r(\tau) \in V$ and

$$L_r(\tau) = (q;q)_\infty (q^{11};q^{11})_\infty \sum_{n=0}^{\infty} p_{[1^1 11^1]}\Big(11^{r+1}n + \frac{11^{r+1}+1}{2}\Big) q^{n+1}. \qquad (3.3.28)$$

Let $\mu_r = 1$ for all $r \geq 1$. For any integer $r \geq 1$, since $L_r(\tau) \in V$ we can write

$$L_r(\tau) = \sum_{\nu \geq \mu_r} a_{r,\nu} J_\nu(\tau), \quad a_{r,\nu} \in \mathbf{Z}. \qquad (3.3.29)$$

We will prove that for any $r \geq 1$,

$$\pi(a_{r,\nu}) \geq r + 1 + [\frac{\nu - \mu_r}{2}], \quad \forall \nu \geq \mu_r. \qquad (3.3.30)$$

If $r = 1$, we find that

$$L_1(\tau) = \sum_{\nu=1}^{60} a_{1,\nu} J_\nu(\tau).$$

The 11-adic orders of $a_{1,\nu}$ ($1 \leq \nu \leq 60$) are given in Table 3.4, from which it is not difficult to verify that (3.3.30) holds for $r = 1$.

Table 3.4

ν	1	2	3	4	5	6	7	8	9	10
$\pi(a_{1,\nu})$	2	3	3	5	5	8	8	10	11	11
ν	11	12	13	14	15	16	17	18	19	20
$\pi(a_{1,\nu})$	13	14	14	17	16	18	19	20	22	22
ν	21	22	23	24	25	26	27	28	29	30
$\pi(a_{1,\nu})$	24	25	25	27	27	29	31	32	34	34
ν	31	32	33	34	35	36	37	38	39	40
$\pi(a_{1,\nu})$	36	36	37	39	39	41	42	44	46	46
ν	41	42	43	44	45	46	47	48	49	50
$\pi(a_{1,\nu})$	47	48	49	51	51	53	54	55	59	57
ν	51	52	53	54	55	56	57	58	59	60
$\pi(a_{1,\nu})$	59	60	61	64	63	65	66	67	∞	68

Now suppose (3.3.30) holds for $r-1$ ($r \geq 2$). For the same reason as in the proof of Theorem 3.1.2, to complete the induction, it suffices to prove that

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1 + \left[\frac{\nu - \mu_r}{2}\right], \quad \text{for all } \mu \geq \mu_{r-1}, \nu \geq \mu_r. \tag{3.3.31}$$

By induction hypothesis and (3.2.12), we deduce that

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + \left[\frac{\mu - \mu_{r-1}}{2}\right] + \left[\frac{11\nu - \mu - 5\lambda_{r-1} - 1}{10}\right]. \tag{3.3.32}$$

Note that if we increase μ by 2, the value of the right hand side does not decrease. Therefore, its minimum value occurs when $\mu = \mu_{r-1} + 1$. Thus

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + \left[\frac{11\nu - \mu_{r-1} - 5\lambda_{r-1} - 2}{10}\right].$$

Since $\mu_{r-1} = \lambda_{r-1} = 1$, for $\nu \geq 2$ we have

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1 + \left[\frac{11\nu - 18}{10}\right] \geq r + 1 + \left[\frac{\nu - 1}{2}\right]. \tag{3.3.33}$$

For $\nu = 1$, (3.3.31) reduces to

$$\pi(a_{r-1,\mu}) + \pi(c_{\mu,\nu}^{(\lambda_{r-1})}) \geq r + 1. \tag{3.3.34}$$

By induction hypothesis, we have $\pi(a_{r-1,\mu_{r-1}}) \geq r$. Since

$$\pi(c_{\mu_{r-1},\nu}^{(\lambda_{r-1})}) \geq \theta(\lambda_{r-1}, \mu_{r-1}) = \theta(1,1) = 1,$$

we see that (3.3.34) holds for $\mu = \mu_{r-1}$. Similarly it holds for $\mu = \mu_{r-1} + 1$. If $\mu \geq \mu_{r-1} + 2$, then by induction hypothesis we have

$$\pi(a_{r-1,\mu}) \geq r + \left[\frac{\mu - \mu_{r-1}}{2}\right] \geq r + 1.$$

Thus (3.3.34) holds for all r.

Combining (3.3.33) with (3.3.34), we see that (3.3.31) holds for all r.

By induction on r, we deduce (3.3.30). Therefore, for any $r \geq 1$ and $\nu \geq \mu_r$, we have

$$\pi(a_{r,\nu}) \geq r + 1.$$

By (3.3.29) we see that

$$L_r(\tau) \equiv 0 \pmod{11^{r+1}}.$$

From this congruence and (3.3.28), we complete the proof of the theorem. □

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Chapter 4

Generalized Frobenius Partitions with k Colors

4.1 Introduction

It is known that a partition π of n can be visualized using a Ferrers diagram by representing the positive integer m of the s-th part by m dots on s-th row. For example, the pictorial representation of the partition $4+4+4+2$ of the integer 14 is given in Figure 4.1.

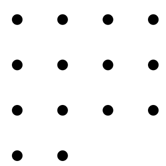

Figure 4.1

From the Ferrers diagram of a partition, we can construct a 2 by d matrix by carrying out the following steps:

1. Remove all the dots lying on the diagonal of the diagram.

2. Fill the first row of the matrix with entries $r_{1,j}$, where $r_{1,j}$ is the number of

dots on the j-th row that are above the diagonal.

3. Fill the second row of the matrix with entries $r_{2,j}$, where $r_{2,j}$ is the number of dots on the j-th column that are below the diagonal.

For example, after Step 1, we obtained Figure 4.2 from Figure 4.1.

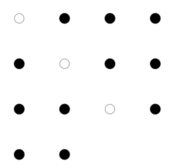

Figure 4.2

Carrying out Steps 2 and 3, we arrive at the matrix

$$\begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix}.$$

It is clear that we can always construct a 2 by d matrix from any partition π with d dots along the diagonal of its Ferrers diagram. The matrix obtained from a partition π using the above procedures is called a Frobenius symbol for the partition π. A Frobenius symbol, by construction, has strictly decreasing entries on each row. Frobenius symbols have been introduced primarily in representation theory of the symmetric groups (see [75], for example).

One way to find new functions that are similar to the partition function $p(n)$ is to start with a modified version of the Frobenius symbol. In his 1984 AMS Memoir, Andrews [4, Section 4] introduced a generalized Frobenius symbol with at most k repetitions for each integer by relaxing the "strictly decreasing" property and allowing at most k-repetitions of each positive integer in each row. Andrews then used the generalized Frobenius symbol to define the generalized Frobenius partition of n. For a generalized Frobenius symbol with entries $r_{i,j}, i = 1, 2, 1 \leq j \leq d$, the generalized Frobenius partition of n is given by

$$n = d + \sum_{j=1}^{d} (r_{1,j} + r_{2,j}).$$

Andrews used the symbol $\phi_k(n)$ to denote the number of such partitions of n. As an example, we observe that $\phi_2(3) = 5$ and these are given by the following generalized Frobenius symbols with at most 2 repetitions on each row:

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Note that with this definition,

$$\phi_1(n) = p(n).$$

In order to restore the "strictly decreasing" property of a Frobenius symbol from a generalized Frobenius symbol with at most k repetitions, Andrews colored the repeated parts using "colors" denoted by $1, 2, \cdots, k$ and imposed an ordering on these parts as follows:

$$0_1 \prec 0_2 \prec \cdots \prec 0_k \prec 1_1 \prec 1_2 \prec \cdots \prec 1_k \prec 2_1 \prec 2_2 \prec \cdots \prec 2_k \prec \cdots. \tag{4.1.1}$$

Here, we use "$\prec$" to differentiate the inequality from the usual inequality "$<$". Andrews referred to a matrix $(r_{2,j})_{2\times d}$ as a k-colored generalized Frobenius symbol if the entries

$$r_{i,j} \in \{\ell_c | \ell \text{ and } c \text{ are non-negative integers with } 1 \leq c \leq k\}$$

and

$$r_{i,j+1} \prec r_{i,j}, i = 1, 2 \quad \text{and} \quad 1 \leq j \leq d - 1.$$

Andrews associated a k-colored generalized Frobenius partition of n to a k-colored generalized Frobeinus symbol $(r_{i,j})_{2\times d}$ by setting

$$n = d + \sum_{j=1}^{d} (r_{1,j} + r_{2,j}),$$

where only the numerical value ℓ is added in the sum if $r_{i,j} = \ell_c$. He used the symbol $c\phi_k(n)$ to denote the number of such partitions of n. Observe that when $k = 1$, the 1-colored generalized Frobenius symbols coincide with the Frobenius symbols and $c\phi_1(n) = p(n)$. To help the reader to understand k-colored generalized Frobenius symbols, we list the following 2-colored generalized Frobenius symbols which give rise to 2-colored generalized Frobenius partitions of 2:

$$\begin{pmatrix}1_1\\0_1\end{pmatrix}, \quad \begin{pmatrix}1_1\\0_2\end{pmatrix}, \quad \begin{pmatrix}1_2\\0_1\end{pmatrix}, \quad \begin{pmatrix}1_2\\0_2\end{pmatrix}, \tag{4.1.2}$$
$$\begin{pmatrix}0_1\\1_1\end{pmatrix}, \quad \begin{pmatrix}0_2\\1_1\end{pmatrix}, \quad \begin{pmatrix}0_1\\1_2\end{pmatrix}, \quad \begin{pmatrix}0_2\\1_2\end{pmatrix}, \quad \begin{pmatrix}0_2 & 0_1\\0_2 & 0_1\end{pmatrix}.$$

Note that there are altogether nine 2-colored generalized Frobenius partitions of 2 and hence, $c\phi_2(2) = 9$.

Let

$$\mathrm{C}\Phi_k(q) := \sum_{n=0}^{\infty} c\phi_k(n) q^n.$$

In [4, Theorem 5.2], Andrews showed that

$$\mathrm{C}\Phi_k(q) = \frac{1}{(q;q)_\infty^k} \sum_{m_1,\cdots,m_{k-1}\in\mathbf{Z}} q^{Q(m_1,\cdots,m_{k-1})} \tag{4.1.3}$$

where

$$Q(m_1, m_2, \cdots, m_{k-1}) = \sum_{i=1}^{k-1} m_i^2 + \sum_{1\le i<j\le k-1} m_i m_j. \tag{4.1.4}$$

Using (4.1.3), Andrews [4, Corollary 5.2] discovered alternative expressions for $\mathrm{C}\Phi_k(q)$ when $k = 2, 3$ and 5. To describe Andrews' identities, let $q = e^{2\pi i\tau}$ throughout this chapter. We define

$$\Theta_3(q) = \vartheta_3(0|2\tau) = \sum_{j=-\infty}^{\infty} q^{j^2}, \text{ and } \Theta_2(q) = \vartheta_2(0|2\tau) = \sum_{j=-\infty}^{\infty} q^{(j+1/2)^2},$$

where

$$\vartheta_2(u|\tau) = \sum_{j=-\infty}^{\infty} e^{\pi i\tau(j+1/2)^2} e^{(2j+1)iu}$$

and

$$\vartheta_3(u|\tau) = \sum_{j=-\infty}^{\infty} e^{\pi i\tau j^2} e^{2jiu}.$$

Andrews showed that

$$\mathrm{C}\Phi_2(q) = \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty^4 (q^4;q^4)_\infty}, \tag{4.1.5}$$

$$\mathrm{C}\Phi_3(q) = \frac{1}{(q;q)_\infty^3}\left(\Theta_3(q)\Theta_3(q^3) + \Theta_2(q)\Theta_2(q^3)\right) \tag{4.1.6}$$

$$= \frac{1}{(q;q)_\infty^3}\left(1 + 6\sum_{j=0}^{\infty}\left(\frac{j}{3}\right)\frac{q^j}{1-q^j}\right) \tag{4.1.7}$$

and

$$\mathrm{C}\Phi_5(q) = \frac{1}{(q;q)_\infty^5}\left(1 + 25\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{q^j}{(1-q^j)^2} - 5\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^j}\right) \tag{4.1.8}$$

where $\left(\frac{j}{\cdot}\right)$ is the Kronecker symbol. For (4.1.8), we have recorded the equivalent version of Andrews' identity found in the work of Kolitsch [57, Lemma 1]. Andrews [4, pp. 13–15] used Jacobi triple product identity (see for example [4, (3.1)]) and properties of theta series to prove (4.1.5) and (4.1.6). The proofs of (4.1.7) and (4.1.8) [4, pp. 26–27] are dependent on the work of H.D. Kloosterman [54, p. 362, p. 358]. In a paragraph before the proofs of (4.1.7) and (4.1.8), Andrews [4, p. 26] mentioned that similar identity exists for $k = 7$, but this identity was not given in [4]. This missing identity, namely,

$$\mathrm{C}\Phi_7(q) = \frac{1}{(q;q)_\infty^7}\left(1 + \frac{343}{8}\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{q^j + q^{2j}}{(1-q^j)^3} - \frac{7}{8}\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{j^2q^j}{1-q^j}\right), \tag{4.1.9}$$

was later published by Kolitsch [57, Lemma 2].

Recently, Baruah and Sarmah [23, 24] used the method illustrated in Z. Cao's work [29] and found representations of $\mathrm{C}\Phi_k(q)$ for $k = 4, 5$ and 6. They showed that

$$\mathrm{C}\Phi_4(q) = \frac{1}{(q;q)_\infty^4}\left(\Theta_3^3(q^2) + 3\Theta_3(q^2)\Theta_2^2(q^2)\right), \tag{4.1.10}$$

$$\begin{aligned}\mathrm{C}\Phi_5(q) = \frac{1}{(q;q)_\infty^5}\Big(&\Theta_3(q^{10})\Theta_3^3(q^2) + 3\Theta_3(q^{10})\Theta_3(q^2)\Theta_2^2(q^2) + \frac{1}{2}\Theta_2(q^{5/2})\Theta_2^3(q^{1/2})\\ &+ 3\Theta_2(q^{10})\Theta_2(q^2)\Theta_3^2(q) + \Theta_2(q^{10})\Theta_2^3(q^2)\Big)\end{aligned} \tag{4.1.11}$$

and

$$\mathrm{C}\Phi_6(q) = \frac{1}{(q;q)_\infty^6}\Big(\Theta_3^3(q)\Theta_3(q^2)\Theta_3(q^6) + \frac{3}{4}\Theta_2^3(q^{1/2})\Theta_2(q)\Theta_2(q^{3/2})$$

$$+ \Theta_3^2(q)\Theta_2(q^2)\Theta_2(q^6)\Big). \tag{4.1.12}$$

Identities (4.1.10) and (4.1.11) can be found in [23, (2.2)] and [23, (2.13)] respectively while (4.1.12) can be found in [24, (2.1)].

For $k > 7$, it is not clear if new identities associated with $\mathrm{C}\Phi_k(q)$ could be derived using the methods of Andrews and Baruah-Sarmah. In fact, Andrews [4, p. 15] commented that as k increases, "the expressions quickly become long and messy". The main goal of this chapter is to discuss ways of finding new representations of $\mathrm{C}\Phi_k(q)$. Using the theory of modular forms, we will derive all the identities mentioned above. In addition to providing new proofs to known identities, we will also construct new representations for $\mathrm{C}\Phi_k(q)$ for the first time for $8 \leq k \leq 17$,. In Section 4.2, we discuss the behavior of $\mathrm{C}\Phi_k(q)$ as modular form for each integer $k > 2$. In Section 4.3, we derive alternative representations of $\mathrm{C}\Phi_k(q)$ for primes $k = 3, 5, 7, 11, 13$ and 17 and prove Kolitsch's identities [57, p. 223]

$$c\phi_5(n) = p(n/5) + 5p(5n - 1) \tag{4.1.13}$$

and

$$c\phi_7(n) = p(n/7) + 7p(7n - 2). \tag{4.1.14}$$

We also discover and prove the identities

$$c\phi_{11}(n) = p(n/11) + 11p(11n - 5), \tag{4.1.15}$$

and

$$c\phi_{13}(n) = p(n/13) + 13p(13n - 7) + 26a(n) \tag{4.1.16}$$

where $p(x) = 0$ when x is not an integer and

$$q\prod_{n=1}^{\infty} \frac{(1 - q^{13n})}{(1 - q^n)^2} = \sum_{n=0}^{\infty} a(n)q^n.$$

It turns out that (4.1.15) is equivalent to Kolitsch's identity for 11-colored generalized Frobenius partition with order 11 [60, Theorem 3] which was first established using the results of F.G. Garvan, D. Kim and D. Stanton [41]. Identity (4.1.16), on the other hand, is new. The proof of (4.1.16) motivates the discovery of a uniform method in treating identities such as (4.1.16). We discuss this method in Section 4.4 and derive analogues of (4.1.16) for $\ell = 17, 19$ and 23. This method also leads to the discovery of interesting modular functions that satisfy mysterious congruences. For example, if

$$h_\ell(\tau) = (q^\ell;q^\ell)_\infty \mathrm{C}\Phi_\ell(q) - 1 - \ell(q^\ell;q^\ell)_\infty \sum_{j=1}^{\infty} p\left(\ell j - \frac{\ell^2-1}{24}\right) q^j - 2\ell^{(\ell-11)/2} \frac{\eta^{\ell-11}(\ell\tau)}{\eta^{\ell-11}(\tau)},$$

where $\eta(\tau)$ is the Dedekind eta function given by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n),$$

then for $\ell = 17, 19$ and 23,

$$h_\ell(\tau) \equiv 0 \pmod{\nu_\ell}$$

where

$$\nu_\ell = \ell^2 - \ell p\left(\ell n - \frac{\ell^2-1}{24}\right).$$

In Section 4.5, we discuss the cases for $k = 9$ and 15, the two composite odd integers less than 17. We derive the following congruence satisfied by $c\phi_k(n)$:

$$c\phi_{p^\alpha N}(n) \equiv c\phi_{p^{\alpha-1}N}(n/p) \pmod{p^{2\alpha}}, \tag{4.1.17}$$

where $c\phi_k(m) = 0$ if m is not an integer, p a prime, N and α are positive integers with $(N,p) = 1$. The discovery of congruence (4.1.17) is motivated by congruences found in the study of $\mathrm{C}\Phi_{10}(q)$ and $\mathrm{C}\Phi_{14}(q)$ in Section 4.6 where identities associated with $k = 4, 6, 8, 10, 12$ and 16 are given. More precisely, we discovered that

$$\mathrm{C}\Phi_{2p}(q) \equiv \frac{\Theta_3(q^p)}{(q^p;q^p)_\infty} = \mathrm{C}\Phi_2(q^p) \pmod{p^2}, \tag{4.1.18}$$

which holds for any odd prime p. The second equality follows from Andrews' identity (4.1.5) for $\mathrm{C}\Phi_2(q)$ (see also (4.3.1)). Congruence (4.1.18) can be viewed as an extension of Andrews' congruence [4, Corollary 10.2]

$$\mathrm{C}\Phi_p(q) \equiv \frac{1}{(q^p;q^p)_\infty} \pmod{p^2} \tag{4.1.19}$$

if we rewrite (4.1.19) as

$$\mathrm{C}\Phi_p(q) \equiv \mathrm{C}\Phi_1(q^p) \pmod{p^2} \tag{4.1.20}$$

using the fact that

$$c\phi_1(n) = p(n).$$

The discovery of (4.1.18) leads to the congruence

$$\mathrm{C}\Phi_{\ell p}(q) \equiv \mathrm{C}\Phi_\ell(q^p) \pmod{p^2}, \tag{4.1.21}$$

which holds for any distinct primes ℓ and p. Congruence (4.1.21) eventually leads to the discovery of (4.1.17).

There may be more surprising properties to be discovered for $c\phi_k(n)$ and we hope that this discussion will be helpful to future researchers who are interested in knowing more about these functions.

4.2 Modular properties of $\mathrm{C}\Phi_k(q)$

In this section, we determine the modular properties of the function

$$\mathfrak{A}_k(q) := (q;q)_\infty^k \mathrm{C}\Phi_k(q) = \sum_{m_1,\cdots,m_{k-1}\in\mathbf{Z}} q^{Q(m_1,\cdots,m_{k-1})}, \quad k > 1.$$

Let χ be a Dirichlet character and $M_k(\Gamma_0(N),\chi)$ be the space of modular forms on $\Gamma_0(N)$ with weight k and multiplier χ. When χ is the trivial Dirichlet character, we write $M_k(\Gamma_0(N))$ for $M_k(\Gamma_0(N),\chi)$.

Let

$$A_n = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}_{n\times n}. \tag{4.2.1}$$

Then $\det(A_n) = n+1$ and

$$A_n^{-1} = \begin{pmatrix} \frac{n}{n+1} & -\frac{1}{n+1} & -\frac{1}{n+1} & \cdots & -\frac{1}{n+1} \\ -\frac{1}{n+1} & \frac{n}{n+1} & -\frac{1}{n+1} & \cdots & -\frac{1}{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{n+1} & -\frac{1}{n+1} & -\frac{1}{n+1} & \cdots & \frac{n}{n+1} \end{pmatrix}. \tag{4.2.2}$$

Let n be a positive even integer and

$$\chi_{A_n}(\cdot) = \left(\frac{(-1)^{n/2}\det(A_n)}{\cdot}\right) = \left(\frac{(-1)^{n/2}(n+1)}{\cdot}\right).$$

Since all the diagonal components of A_n and $(n+1)A_n^{-1}$ are even, we deduce from [68, Corollary 4.9.5 (3)] that if

$$\theta(\tau; A_n) = \sum_{m\in\mathbf{Z}^n} e^{\pi i\tau\cdot m^t A_n m} = \sum_{m\in\mathbf{Z}^n} q^{\frac{1}{2}m^t A_n m} = \sum_{m\in\mathbf{Z}^n} q^{Q(m_1,\cdots m_n)},$$

then

$$\theta(\tau; A_n) = \sum_{m\in\mathbf{Z}^n} q^{Q(m_1,\cdots m_n)} = \mathfrak{A}_{n+1} \in M_{n/2}\big(\Gamma_0(n+1), \chi_{A_n}\big). \tag{4.2.3}$$

Next, let $n > 1$ be an odd positive integer and

$$B_n = \begin{pmatrix} A_n & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $\det B_n = 2(n+1)$. We have

$$\begin{aligned} \theta(\tau; B_n) &= \sum_{m\in\mathbf{Z}^{n+1}} e^{\pi i\tau\cdot m^t B_n m} = \sum_{m\in\mathbf{Z}^n} q^{\frac{1}{2}m^t A_n m} \sum_{m_{n+1}\in\mathbf{Z}} q^{m_{n+1}^2} \\ &= \sum_{m\in\mathbf{Z}^n} q^{Q(m_1,\cdots m_n)} \sum_{m_{n+1}\in\mathbf{Z}} q^{m_{n+1}^2} \\ &= \mathfrak{A}_{n+1}(q)\Theta_3(q). \end{aligned}$$

Note that

$$B_n^{-1} = \begin{pmatrix} A_n^{-1} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Let

$$\chi_{B_n}(\cdot) = \left(\frac{(-1)^{(n+1)/2}\det(B_n)}{\cdot}\right) = \left(\frac{2(-1)^{(n+1)/2}(n+1)}{\cdot}\right).$$

Since all the diagonal components of B_n and $2(n+1)B_n^{-1}$ are even, we deduce from [68, Corollary 4.9.5 (3)] that

$$\theta(\tau; B_n) = \mathfrak{A}_{n+1}(q)\Theta_3(q) \in M_{(n+1)/2}\big(\Gamma_0(2(n+1)), \chi_{B_n}\big). \tag{4.2.4}$$

Similarly, let

$$C_n = \begin{pmatrix} A_n & 0 \\ 0 & 4 \end{pmatrix}.$$

Then $\det C_n = 4(n+1)$. Note that

$$C_n^{-1} = \begin{pmatrix} A_n^{-1} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

Let

$$\chi_{C_n}(\cdot) = \left(\frac{(-1)^{(n+1)/2}\det(C_n)}{\cdot}\right) = \left(\frac{(-1)^{(n+1)/2}(n+1)}{\cdot}\right).$$

Since all the diagonal components of C_n and $4(n+1)C_n^{-1}$ are even, we find from [68, Corollay 4.9.5 (3)] that

$$\theta(\tau; C_n) = \mathfrak{A}_{n+1}(q)\Theta_3(q^2) \in M_{(n+1)/2}\big(\Gamma_0(4(n+1)), \chi_{C_n}\big). \tag{4.2.5}$$

From (4.2.3), (4.2.4) and (4.2.5), we deduce the following theorem:

Theorem 4.2.1. *If $k = 2r+1$ is odd, then*

$$\mathfrak{A}_k(q) \in M_{(k-1)/2}\Big(\Gamma_0(k), \big(\frac{(-1)^r \cdot k}{\cdot}\big)\Big).$$

If $k = 2r$ is even, then

$$\Theta_3(q)\mathfrak{A}_k(q) \in M_{k/2}\Big(\Gamma_0(2k), \big(\frac{2(-1)^r \cdot k}{\cdot}\big)\Big),$$

and

$$\Theta_3(q^2)\mathfrak{A}_k(q) \in M_{k/2}\Big(\Gamma_0(4k), \big(\frac{(-1)^r \cdot k}{\cdot}\big)\Big).$$

4.3 Generating function of $c\phi_k(n)$ when k is a prime

In this section, we will derive expressions for $\mathrm{C}\Phi_k(q)$ when k is a prime number less than 18.

4.3.1 Case $k = 2$

Our proof for $k = 2$ is exactly the same as that of Andrews' proof of (4.1.5) and we include it for the sake of completeness. From (4.1.3), we find that

$$\mathrm{C}\Phi_2(q) = \frac{\Theta_3(q)}{(q;q)_\infty^2}. \tag{4.3.1}$$

Using Jacobi triple product identity (see [4, (3.1)]), we deduce that

$$\Theta_3(q) = (-q;q^2)_\infty^2 (q^2;q^2)_\infty. \tag{4.3.2}$$

Substituting (4.3.2) into (4.3.1) and simplifying, we complete the proof of (4.1.5).

4.3.2 Case $k = 3$

From Theorem 4.2.1, we deduce that $\mathfrak{A}_3(q)$ is a modular form of weight 1 on $\Gamma_0(3)$ with multiplier $\left(\frac{-3}{\cdot}\right)$. From [38, Theorem 4.8.1] we know that

$$E_{3,1} := 1 + 6\sum_{j=1}^{\infty} \left(\frac{3}{j}\right) \frac{q^j}{1-q^j}. \tag{4.3.3}$$

is an Eisenstein series in $M_1(\Gamma_0(3), \left(\frac{-3}{\cdot}\right))$. Thus $E_{3,1}(\tau)\mathfrak{A}_3(q) \in M_2(\Gamma_0(3))$ and $E_{3,1}^2 \in M_2(\Gamma_0(3))$. From [69, Theorem 1.34] we compute that $\dim M_2(\Gamma_0(3)) = 1$. By comparing the Fourier coefficients of $E_{3,1}(\tau)\mathfrak{A}_3(q)$ and $E_{3,1}^2$, we deduce that

$$E_{3,1}(\tau)\mathfrak{A}_3(q) = E_{3,1}^2.$$

Hence

$$\mathfrak{A}_3(q) = E_{3,1}$$

and we have completed the proof of (4.1.7).

Another proof of (4.1.7) can also be found, for example, in the article by J.M. Borwein, P.B. Borwein and Garvan [28, p. 43].

We next show that (4.1.6) follows from a general identity. Let $\omega = (1+\sqrt{-d})/2$, with $d \equiv 3 \pmod 4$. Observe that the set

$$S = \{m + n\omega | m, n \in \mathbf{Z}\}$$

is a disjoint union of

$$S_0 = \{m + n\omega | m, n \in \mathbf{Z}, n \equiv 0 \pmod 2\} \text{ and } S_1 = \{m + n\omega | m, n \in \mathbf{Z}, n \equiv 1 \pmod 2\}.$$

Let

$$N(m + n\omega) = m^2 + mn + \left(\frac{d+1}{4}\right) n^2.$$

Then

$$\sum_{v \in S} q^{N(v)} = \sum_{v \in S_0} q^{N(v)} + \sum_{v \in S_1} q^{N(v)}.$$

Simplifying the above, we deduce that

$$\sum_{m,n \in \mathbf{Z}} q^{m^2+mn+\left(\frac{d+1}{4}\right)n^2} = \Theta_3(q)\Theta_3(q^d) + \Theta_2(q)\Theta_2(q^d). \tag{4.3.4}$$

Identity (4.1.6) follows from (4.3.4) with $d = 3$.

4.3.3 Case $k = 5$

We first establish three representations of $\mathrm{C}\Phi_5(q)$:

Theorem 4.3.1. *The following identities hold:*

$$\mathrm{C}\Phi_5(q) = \frac{1}{(q;q)_\infty^5}\left(1 + 25\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{q^j}{(1-q^j)^2} - 5\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^j}\right), \tag{4.3.5}$$

$$\mathrm{C}\Phi_5(q) = \frac{1}{(q^5;q^5)_\infty} + 25q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{4.3.6}$$

$$\mathrm{C}\Phi_5(q) = \frac{1}{(q^5;q^5)_\infty} + 5\sum_{j=1}^{\infty} p(5j-1)q^j. \tag{4.3.7}$$

Proof. From Theorem 4.2.1, we deduce that

$$\mathfrak{A}_5(q) \in M_2\left(\Gamma_0(5), \left(\frac{5}{\cdot}\right)\right).$$

Since [69, Theorem 1.34]

$$\dim M_2\left(\Gamma_0(5), \left(\frac{5}{\cdot}\right)\right) = 2,$$

we deduce that the two modular forms

$$E_{5,1} := \sum_{m=1}^{\infty}\sum_{d=1}^{\infty}\left(\frac{d}{5}\right) m q^{md} = \sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{q^j}{(1-q^j)^2} \tag{4.3.8}$$

and

$$E_{5,2} := 1 - 5\sum_{m=1}^{\infty}\sum_{d=1}^{\infty}\left(\frac{d}{5}\right) d^2 q^{md} = 1 - 5\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^j},$$

which are in $M_2\left(\Gamma_0(5), \left(\frac{5}{\cdot}\right)\right)$ (see [38, Sec. 4.6]), form a basis for this space of modular forms. By comparing Fourier coefficients of $\mathfrak{A}_5(q), E_{5,1}$ and $E_{5,2}$, we deduce that

$$\mathfrak{A}_5(q) = 25E_{5,1} + E_{5,2}$$

and the proof of (4.3.5) is complete.

Before we begin with our proof of (4.3.6), we observe that if $p > 3$ is a prime, then by Theorem 4.2.1,

$$\mathrm{C}\Phi_p(q)(q^p;q^p)_\infty$$

is a modular function on $\Gamma_0(p)$. This implies that the function can be expressed in terms of combinations of infinite products. For more details, see for example the paper by Chan, H. Hahn, R.P. Lewis and S.L. Tan [32]. In [4, Corollary 10.2], Andrews showed that if p is a prime, then

$$\mathrm{C}\Phi_p(q) = \frac{1}{(q^p;q^p)_\infty} + p^2 G_p(q)$$

for some $G_p(q)$ analytic inside $|q| < 1$ with integral power series coefficients. He then asked [4, Problem 6] for explicit closed forms for $G_p(q)$. Since

$$G_p(q)(q^p;q^p)_\infty = \frac{1}{p^2}\left(\mathrm{C}\Phi_p(q)(q^p;q^p)_\infty - 1\right),$$

we conclude that $G_p(q)$ is a modular function on $\Gamma_0(p)$ for $p > 3$. This provides an answer to Andrews' question. The above discussion also gives us a way to derive alternative expressions for $C\Phi_p(q)$ whenever the functions invariant under $\Gamma_0(p)$ can be expressed as a rational function of a single modular function. This happens for $p = 5, 7$, and 13. We now use this fact to derive an expression for $C\Phi_5(q)$. It is known from T. Kondo's work [61] that every modular function on $\Gamma_0(5)$ is a rational function of $\eta^6(5\tau)/\eta^6(\tau)$, where

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{j=1}^{\infty}\left(1 - e^{2\pi ij\tau}\right).$$

Since $C\Phi_5(q)(q^5;q^5)_\infty$ is a modular function on $\Gamma_0(5)$, by comparing the Fourier coefficients and direct calculations, we deduce that

$$C\Phi_5(q)(q^5;q^5)_\infty = 1 + 25\frac{\eta^6(5\tau)}{\eta^6(\tau)}.$$

This completes the proof of (4.3.6).

Using the fact that

$$\frac{1}{(q;q)_\infty} = \sum_{j=0}^{\infty} p(j)q^j$$

and Ramanujan's identity [26, Theorem 2.3.4],

$$\sum_{j=1}^{\infty} p(5j-1)q^j = 5q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{4.3.9}$$

we deduce (4.3.7) from (4.3.6). □

Remark 4.3.1. Identity (4.3.5) is Andrews' (4.1.8), which was first proved using results found in Kloosterman's work [54]. Identity (4.3.7) immediately implies (4.1.13). We emphasize here that our proof of (4.1.13) is different from Kolitsch's proof as we have used (4.3.6) instead of (4.1.8).

As shown in (4.1.11), there is a fourth representation of $C\Phi_5(q)$ due to Baruah and Sarmah. This identity can be proved by realizing that

$$\mathfrak{A}_5(q) \in M_2\left(\Gamma_0(40), \left(\frac{5}{\cdot}\right)\right),$$

together with the fact that the space $M_2\Big(\Gamma_0(40), (\frac{5}{\cdot})\Big)$ has a basis consisting of the modular forms

$$\begin{aligned}
&\Theta_3(q)\Theta_3^3(q^5), \quad \Theta_3^3(q)\Theta_3(q^5), \quad \Theta_3(q^2)\Theta_3^3(q^{10}), \quad \Theta_3^3(q^2)\Theta_3(q^{10}),\\
&\Theta_3(q)\Theta_3(q^5)\Theta_2^2(q^2), \quad \Theta_3(q^2)\Theta_3(q^{10})\Theta_2^2(q^2), \quad \Theta_2^3(q^{1/2})\Theta_2(q^{5/2}), \qquad (4.3.10)\\
&\Theta_3^2(q)\Theta_2(q^2)\Theta_2(q^{10}), \quad \Theta_2^3(q^2)\Theta_2(q^{10}), \quad \text{and} \quad \Theta_3^2(q^5)\Theta_2(q^2)\Theta_2(q^{10}).
\end{aligned}$$

Remark 4.3.2. From [26, Corollary 1.3.4] we know that

$$\Theta_2(q) = 2q^{1/4}\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}, \quad \Theta_3(q) = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}. \qquad (4.3.11)$$

Hence all the functions in (4.3.10) can be expressed as eta-quotients. By [69, Theorems 1.64 and 1.65], it is straightforward to check that these functions belong to the space $M_2\Big(\Gamma_0(40), (\frac{5}{\cdot})\Big)$. Moreover, by comparing their Fourier coefficients, we can verify that they are linearly independent. Now by [69, Theorem 1.41] we compute that

$$\dim M_2\Big(\Gamma_0(40), (\frac{5}{\cdot})\Big) = 10.$$

Therefore, the functions in (4.3.10) form a basis of $M_2\Big(\Gamma_0(40), (\frac{5}{\cdot})\Big)$. In the sequel, we will omit those details of verifying that the functions we constructed are basis of certain spaces of modular forms.

4.3.4 Case $k = 7$

Theorem 4.3.3. *The following identities are true:*

$$\mathrm{C}\Phi_7(q) = \frac{1}{(q;q)_\infty^7}\left(1 - \frac{7}{8}\sum_{k=1}^{\infty}\left(\frac{k}{7}\right)\frac{k^2q^k}{1-q^k} + \frac{343}{8}\sum_{k=1}^{\infty}\left(\frac{k}{7}\right)\frac{q^k+q^{2k}}{(1-q^k)^3}\right), \qquad (4.3.12)$$

$$\mathrm{C}\Phi_7(q) = \frac{1}{(q^7;q^7)_\infty} + 49q\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 343q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}, \qquad (4.3.13)$$

$$\mathrm{C}\Phi_7(q) = \frac{1}{(q^7;q^7)_\infty} + 7\sum_{j=1}^{\infty}p(7j-2)q^j. \qquad (4.3.14)$$

Proof. Before giving the proof of (4.3.12), we observe that (4.3.12) is the same as (4.1.9). We will prove (4.3.12) using the theory of modular forms. Note that by Theorem 4.2.1, we have $\mathfrak{A}_7(q) \in M_3\left(\Gamma_0(7), \left(\frac{-7}{\cdot}\right)\right)$. From [69, Theorem 1.34] we deduce that

$$\dim M_3\left(\Gamma_0(7), \left(\frac{-7}{\cdot}\right)\right) = 3.$$

From [38, Theorem 4.5.2] we find that the two Eisenstein series

$$E_{7,1} := \sum_{m=1}^{\infty}\sum_{d=1}^{\infty}\left(\frac{d}{7}\right) m^2 q^{md} = \sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{q^j + q^{2j}}{(1-q^j)^3},$$

and

$$E_{7,2} := 1 - \frac{7}{8}\sum_{m=1}^{\infty}\sum_{d=1}^{\infty}\left(\frac{d}{7}\right) d^2 q^{md} = 1 - \frac{7}{8}\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{j^2 q^j}{1-q^j}$$

belong to $M_3\left(\Gamma_0(7), \left(\frac{-7}{\cdot}\right)\right)$. Moreover, by using [69, Theorems 1.64 and 1.65], it is straightforward to check that

$$S_7 := \eta^3(\tau)\eta^3(7\tau) \in M_3\left(\Gamma_0(7), \left(\frac{-7}{\cdot}\right)\right).$$

It is easy to verify that $E_{7,1}$, $E_{7,2}$ and S_7 are linearly independent. Hence they form a basis of $M_3\left(\Gamma_0(7), \left(\frac{-7}{\cdot}\right)\right)$. By comparing Fourier coefficients of these modular forms and $\mathfrak{A}_7(q)$, we deduce that

$$\mathfrak{A}_7(q) = \frac{343}{8}E_{7,1} + E_{7,2}.$$

This complete the proof of (4.3.12).

The proof of (4.3.13) is similar to the proof of (4.3.6). We recall from [61] that modular functions invariant under $\Gamma_0(7)$ is a rational function of

$$\frac{\eta^4(7\tau)}{\eta^4(\tau)}.$$

Since $(q^7;q^7)_\infty \mathrm{C}\Phi_7(q)$ is such a function, by comparing the Fourier coefficients and direct calculations, we deduce that

$$(q^7;q^7)_\infty \mathrm{C}\Phi_7(q) = 1 + 49\frac{\eta^4(7\tau)}{\eta^4(\tau)} + 343\frac{\eta^8(7\tau)}{\eta^8(\tau)}$$

and the proof of (4.3.13) is complete.

Ramanujan discovered that [26, Theorem 2.4.2]

$$\sum_{j=1}^{\infty} p(7j-2)q^j = 7q\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}. \tag{4.3.15}$$

Using (4.3.15) and (4.3.13), we deduce (4.3.14). □

Identity (4.3.14) immediately implies Kolitsch's identity (4.1.14). We emphasize here that our proof of (4.1.14) uses (4.3.13) instead of (4.3.12).

As in the case for $k=5$, we are able to find a representation of $C\Phi_7(q)$ in terms of theta functions. This new identity is an analogue of (4.1.11). We first observe that $\mathfrak{A}_7(q) \in M_3\left(\Gamma_0(28), \left(\frac{-28}{\cdot}\right)\right)$. Furthermore the modular forms

$$\begin{aligned}
&\Theta_3^5(q)\Theta_3(q^7), && \Theta_3^3(q)\Theta_3^3(q^7), && \Theta_3(q)\Theta_3^5(q^7), \quad \Theta_3^4(q)\Theta_2(q^{1/2})\Theta_2(q^{7/2}),\\
&\Theta_3^4(q^7)\Theta_2(q^{1/2})\Theta_2(q^{7/2}), && \Theta_3^4(q)\Theta_2(q)\Theta_2(q^7), && \Theta_3^4(q^7)\Theta_2(q)\Theta_2(q^7),\\
&\Theta_2^3(q^{1/2})\Theta_2^3(q^{7/2}), && \Theta_2^3(q)\Theta_2^3(q^7), && \Theta_2^5(q)\Theta_2(q^7), \quad \text{and} \quad \Theta_2(q)\Theta_2^5(q^7)
\end{aligned}$$

form a basis for $M_3\left(\Gamma_0(28), \left(\frac{-28}{\cdot}\right)\right)$. Hence, by comparing the Fourier coefficients, we deduce that

$$\begin{aligned}
C\Phi_7(q) = \frac{1}{(q;q)_\infty^7}\Bigg(&-\frac{15}{32}\Theta_3^5(q)\Theta_3(q^7) + \frac{55}{16}\Theta_3^3(q)\Theta_3^3(q^7) - \frac{63}{32}\Theta_3(q)\Theta_3^5(q^7)\\
&+\frac{15}{16}\Theta_3^4(q)\Theta_2(q^{1/2})\Theta_2(q^{7/2}) + \frac{105}{16}\Theta_3^4(q^7)\Theta_2(q^{1/2})\Theta_2(q^{7/2})\\
&-\frac{15}{16}\Theta_3^4(q)\Theta_2(q)\Theta_2(q^7) + \frac{525}{16}\Theta_3^4(q^7)\Theta_2(q)\Theta_2(q^7)\\
&+\frac{105}{32}\Theta_2^3(q^{1/2})\Theta_2^3(q^{7/2}) + \frac{95}{8}\Theta_2^3(q)\Theta_2^3(q^7)\\
&+\frac{15}{16}\Theta_2^5(q)\Theta_2(q^7) - \frac{189}{16}\Theta_2(q)\Theta_2^5(q^7)\Bigg).
\end{aligned} \tag{4.3.16}$$

We next prove some congruences satisfied by $c\phi_7(n)$ using (4.3.16) and (4.3.13).

Theorem 4.3.4. *For any integer $n \geq 0$,*

$$c\phi_7(5n+3) \equiv 0 \pmod 5.$$

Proof. From (4.3.16), we deduce that

$$\begin{aligned}\sum_{j=0}^{\infty} c\phi_7(j)q^j &\equiv \frac{1}{(q;q)_\infty^7}\Big(\Theta_3(q)\Theta_3^5(q^7)+\Theta_2(q)\Theta_2^5(q^7)\Big) \pmod 5 \\ &\equiv \frac{1}{(q^5;q^5)_\infty^2}\big((q;q)_\infty^3\Theta_3(q)\Theta_3(q^{35})+(q;q)_\infty^3\Theta_2(q)\Theta_2(q^{35})\big) \pmod 5.\end{aligned} \tag{4.3.17}$$

Using Jacobi's identity for $(q;q)_\infty^3$ [26, Theorem 1.3.9], we find that

$$(q;q)_\infty^3\Theta_3(q)=\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^i(2i+1)q^{i(i+1)/2+j^2}. \tag{4.3.18}$$

Now, observe that

$$m=\frac{i(i+1)}{2}+j^2$$

is equivalent to

$$8m+1=(2i+1)^2+8j^2.$$

Note that $8m\equiv -1 \pmod 5$ if and only if $m\equiv 3 \pmod 5$. Since

$$\left(\frac{-8}{5}\right)=-1,$$

we deduce that

$$(2i+1)^2+8j^2\equiv 0 \pmod 5$$

holds if and only if

$$2i+1\equiv j\equiv 0 \pmod 5.$$

Similarly, we have

$$q^{35/4}(q;q)_\infty^3\Theta_2(q)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^i(2i+1)q^{9+i(i+1)/2+j(j+1)}. \tag{4.3.19}$$

Observe that

$$m=9+\frac{i(i+1)}{2}+j(j+1)$$

is equivalent to

$$8m-69=(2i+1)^2+2(2j+1)^2.$$

Note that $8m - 69 \equiv 0 \pmod 5$ if and only if $m \equiv 3 \pmod 5$. Since

$$\left(\frac{-2}{5}\right) = -1,$$

we deduce that

$$(2i+1)^2 + 2(2j+1)^2 \equiv 0 \pmod 5$$

holds if and only if

$$2i+1 \equiv 2j+1 \equiv 0 \pmod 5.$$

From (4.3.17), (4.3.18) and (4.3.19), we conclude that if $m \equiv 3 \pmod 5$ then

$$c\phi_7(m) \equiv 0 \pmod 5,$$

or equivalently,

$$c\phi_7(5n+3) \equiv 0 \pmod 5$$

for any integer $n \geq 0$. □

Remark 4.3.2. It is possible to deduce Theorem 4.3.4 without using (4.3.16). We first recall a recent result of Garvan and J.A. Sellers [40] which states that if p is a prime number and $0 < r < p$, then the congruence

$$c\phi_k(pn+r) \equiv 0 \pmod p, \quad \text{for all } n \in \mathbf{N}$$

implies that

$$c\phi_{pN+k}(pn+r) \equiv 0 \pmod p, \quad \text{for all } n \in \mathbf{N}.$$

In [4, (10.3)], Andrews showed that for all integers $n \geq 0$,

$$c\phi_2(5n+3) \equiv 0 \pmod 5. \tag{4.3.20}$$

Applying the result of Garvan and Sellers with $p = 5, r = 3, N = 1$ and $k = 2$, we complete the proof of Theorem 4.3.4.

Our next set of congruences are consequences of (4.3.13).

Theorem 4.3.5. *For any integer $n \geq 0$, we have*

$$c\phi_7(7n+3) \equiv c\phi_7(7n+5) \equiv c\phi_7(7n+6) \equiv 0 \pmod{7^3}. \tag{4.3.21}$$

Proof. From (4.3.13), we find that

$$\sum_{k=0}^{\infty} c\phi_7(k)q^k \equiv \frac{1}{(q^7;q^7)_\infty} + 49q\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} \pmod{7^3}. \tag{4.3.22}$$

Let

$$q\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} = \sum_{j=0}^{\infty} a(j)q^j.$$

Then

$$\sum_{n=0}^{\infty} c\phi_7(7n+r)q^n \equiv 49\sum_{n=0}^{\infty} a(7n+r)q^n \pmod{7^3}, \quad 1 \leq r \leq 6. \tag{4.3.23}$$

By the binomial theorem, we find that

$$\sum_{j=0}^{\infty} a(j)q^j \equiv q(q^7;q^7)_\infty^2(q;q)_\infty^3 \equiv (q^7;q^7)_\infty^2\left(\sum_{i=0}^{\infty}(-1)^i(2i+1)q^{i(i+1)/2+1}\right) \pmod{7}. \tag{4.3.24}$$

Since

$$1 + \frac{i(i+1)}{2} \equiv 0, 1, 2 \text{ or } 4 \pmod{7},$$

we deduce that

$$a(7n+3) \equiv a(7n+5) \equiv a(7n+6) \equiv 0 \pmod{7}. \tag{4.3.25}$$

Combining (4.3.23) with (4.3.25) we complete the proof of (4.3.21). □

4.3.5 Case $k = 11$

Theorem 4.3.6. *We have*

$$C\Phi_{11}(q) = \frac{1}{(q^{11};q^{11})_\infty} + 11\sum_{j=1}^{\infty} p(11j-5)q^j. \tag{4.3.26}$$

Proof. By Theorem 4.2.1, we know that $\mathfrak{A}_{11}(q) \in M_5\left(\Gamma_0(11), \left(\frac{-11}{\cdot}\right)\right)$. The dimension of $M_5\left(\Gamma_0(11), \left(\frac{-11}{\cdot}\right)\right)$ is 5 [69, Theorem 1.34] and this space is spanned by the modular forms

$$\frac{\eta^{11}(11\tau)}{\eta(\tau)}, \quad \frac{\eta^{11}(\tau)}{\eta(11\tau)}, \quad (q;q)_\infty^{11}\sum_{j=1}^{\infty} p(11j-5)q^j,$$
$$\sum_{m=1}^{\infty}\sum_{d=1}^{\infty}\left(\frac{d}{11}\right) m^4 q^{md} \quad \text{and} \quad \frac{1275}{11} + \sum_{m=1}^{\infty}\sum_{d=1}^{\infty}\left(\frac{d}{11}\right) d^4 q^{md}.$$

By comparing the coefficients of $\mathfrak{A}_{11}(q)$ with those of the five modular forms above, we deduce that

$$\mathfrak{A}_{11}(q) = \frac{\eta^{11}(\tau)}{\eta(11\tau)} + 11(q;q)_\infty^{11}\sum_{j=1}^{\infty} p(11j-5)q^j.$$

This proves (4.3.26). □

It is immediate that (4.3.26) implies (4.1.15). There is no simple analogue of (4.3.5) and (4.3.12) for $k = 11$. But an analogue for (4.1.11) and (4.3.16) exists. This expression for $\mathrm{C}\Phi_{11}(q)$ is complicated and we omit it. In Section 6, when k is composite, we will give some identities similar to (4.1.11) and (4.3.16) if we do not have other representations for $(q^k;q^k)_\infty \mathrm{C}\Phi_k(q)$.

4.3.6 Case $k = 13$

Theorem 4.3.7. *We have*

$$\begin{aligned}\mathrm{C}\Phi_{13}(q) = \frac{1}{(q^{13};q^{13})_\infty} + 169\Big(q\frac{(q^{13};q^{13})_\infty}{(q;q)_\infty^2} + 36q^2\frac{(q^{13};q^{13})_\infty^3}{(q;q)_\infty^4} + 494q^3\frac{(q^{13};q^{13})_\infty^5}{(q;q)_\infty^6} \\ + 3380q^4\frac{(q^{13};q^{13})_\infty^7}{(q;q)_\infty^8} + 13182q^5\frac{(q^{13};q^{13})_\infty^9}{(q;q)_\infty^{10}} \\ + 28561q^6\frac{(q^{13};q^{13})_\infty^{11}}{(q;q)_\infty^{12}} + 28561q^7\frac{(q^{13};q^{13})_\infty^{13}}{(q;q)_\infty^{14}}\Big)\end{aligned} \tag{4.3.27}$$

$$\mathrm{C}\Phi_{13}(q) = \frac{1}{(q^{13};q^{13})_\infty} + 13\sum_{j=1}^{\infty} p(13j-7)q^j + 26q\frac{(q^{13};q^{13})_\infty}{(q;q)_\infty^2}. \tag{4.3.28}$$

Proof. From the discussion at the end of Section 4.3.3, we know that

$$(q^{13};q^{13})_\infty \mathrm{C}\Phi_{13}(q)$$

is a modular function invariant under $\Gamma_0(13)$ and since modular functions invariant under $\Gamma_0(13)$ are rational functions of $H = \eta^2(13\tau)/\eta^2(\tau)$ [61], by comparing the Fourier coefficients and direct calculations, we deduce that

$$\begin{aligned}&(q^{13};q^{13})_\infty \mathrm{C}\Phi_{13}(q)\\ &= 1+169\Big(H+36H^2+494H^3+3380H^4+13182H^5+28561H^6+28561H^7\Big)\end{aligned}$$

and (4.3.27) follows.

Around 1939, motivated by Ramanujan's identities (4.3.9) and (4.3.15), H. Zuckerman [95, Eq. (1.15)] discovered that

$$\begin{aligned}\sum_{j=1}^{\infty} p(13j-7)q^j &= 11q\frac{(q^{13};q^{13})_\infty}{(q;q)_\infty^2} + 468q^2\frac{(q^{13};q^{13})_\infty^3}{(q;q)_\infty^4} + 6422q^3\frac{(q^{13};q^{13})_\infty^5}{(q;q)_\infty^6}\\ &\quad + 43940q^4\frac{(q^{13};q^{13})_\infty^7}{(q;q)_\infty^8} + 171366q^5\frac{(q^{13};q^{13})_\infty^9}{(q;q)_\infty^{10}}\\ &\quad + 371293q^6\frac{(q^{13};q^{13})_\infty^{11}}{(q;q)_\infty^{12}} + 371293q^7\frac{(q^{13};q^{13})_\infty^{13}}{(q;q)_\infty^{14}}. \qquad (4.3.29)\end{aligned}$$

Using (4.3.29) to simplify (4.3.27), we deduce that

$$\mathrm{C}\Phi_{13}(q) = \frac{1}{(q^{13};q^{13})_\infty} + 13\sum_{j=1}^{\infty} p(13j-7)q^j + 26q\frac{(q^{13};q^{13})_\infty}{(q;q)_\infty^2}$$

and this yields (4.3.28). □

Identity (4.3.28) immediately implies (4.1.16).

We observe that the appearance of

$$(q^{13};q^{13})_\infty \sum_{j=1}^{\infty} p(13j-7)q^j$$

simplifies (4.3.27), leading to (4.3.28) with only three terms on the right hand side. Identity (4.3.28) is clearly an analogue of Kolitsch's identities (4.3.7) and (4.3.14).

In Section 4, we will prove identities involving both $\mathrm{C}\Phi_k(q)$ and

$$\sum_{j=1}^{\infty} p\left(kj - \frac{k^2-1}{24}\right) q^j$$

when $k > 3$ is a prime. This method appears to yield the simplest (in terms of the number of modular forms involved) representation of $\mathrm{C}\Phi_k(q)$ for any prime $k > 3$ and it does not involve the construction of basis for

$$M_{(k-1)/2}\Big(\Gamma_0(k), \big(\frac{(-1)^{(k-1)/2}k}{\cdot}\big)\Big).$$

Constructing such basis could get complicated for large k, as we shall see in the next subsection.

4.3.7 Case $k = 17$

Let

$$\mathcal{E}_a(\tau) = q^{17B_2(a/17)/2} \prod_{m=1}^{\infty} (1 - q^{17(m-1)+a})(1 - q^{17m-a}), \tag{4.3.30}$$

be generalized Dedekind eta functions, where $B_2(x) = x^2 - x + 1/6$. Let

$$\begin{aligned}
&f_{17,1} = \eta^7(\tau)\eta(17\tau), \quad f_{17,2}(\tau) = \eta(\tau)\eta^7(17\tau),\\
&g_{17,1}(\tau) = \frac{1}{8}(17E_2(17\tau) - E_2(\tau)),\\
&g_{17,2}(\tau) = \eta^4(17\tau) \sum_{k=0}^{7} \mathcal{E}_{2\cdot 3^k}(\tau)\mathcal{E}_{14\cdot 3^k}(\tau)\mathcal{E}_{4\cdot 3^k}(\tau)^2\mathcal{E}_{12\cdot 3^k}(\tau)\mathcal{E}_{6\cdot 3^k}(\tau)\mathcal{E}_{10\cdot 3^k}(\tau)^2\mathcal{E}_{8\cdot 3^k}(\tau),\\
&h_{17,1}(\tau) = g_{17,1}^2(\tau), \quad h_{17,2}(\tau) = g_{17,1}(\tau)g_{17,2}(\tau), \quad h_{17,3}(\tau) = g_{17,2}^2(\tau),\\
&h_{17,4}(\tau) = \eta^4(\tau)\eta^4(17\tau), \quad h_{17,5}(\tau) = \frac{1}{24}\big(289E_4(17\tau) - E_4(\tau)\big).
\end{aligned}$$

From Theorem 4.2.1, we know $\mathfrak{A}_{17}(q) \in M_8\big(\Gamma_0(17), (\frac{17}{\cdot})\big)$. By [69, Theorem 1.34], we find that

$$\dim M_8\big(\Gamma_0(17), (\frac{17}{\cdot})\big) = 12.$$

Let

$$B_{17,1} = f_{17,1}h_{17,1},\ B_{17,2} = f_{17,1}h_{17,2},\ B_{17,3} = f_{17,1}h_{17,3},\ B_{17,4} = f_{17,1}h_{17,4},$$

$$B_{17,5}=f_{17,1}h_{17,5},\ B_{17,6}=f_{17,2}h_{17,1},\ B_{17,7}=f_{17,2}h_{17,2},\ B_{17,8}=f_{17,2}h_{17,3},$$
$$B_{17,9}=f_{17,2}h_{17,4},\ B_{17,10}=f_{17,2}h_{17,5},\ B_{17,11}=\frac{\eta^{17}(\tau)}{\eta(17\tau)},\quad \text{and}\quad B_{17,12}=\frac{\eta^{17}(17\tau)}{\eta(\tau)}.$$

One can verify that $\{B_{17,j}|1\le j\le 12\}$ forms a basis of $M_8\big(\Gamma_0(17),\big(\frac{17}{\cdot}\big)\big)$. By comparing the Fourier coefficients of $\mathfrak{A}_{17}(q)$ and $B_{17,j}, 1\le j\le 12$ and with the help of *Mathematica*, we deduce the following identity:

Theorem 4.3.8. *We have*

$$\begin{aligned}\mathrm{C}\Phi_{17}(q)=&\frac{1}{(q;q)_\infty^{17}}\Big(\frac{1491529}{118}B_{17,1}-\frac{20931981}{236}B_{17,2}-\frac{117030839}{236}B_{17,3}\\&+\frac{78308596}{59}B_{17,4}-\frac{988669}{236}B_{17,5}+\frac{424841849}{59}B_{17,6}-\frac{10654955751}{236}B_{17,7}\\&-\frac{17109438979}{236}B_{17,8}+\frac{7515406274}{59}B_{17,9}+\frac{91750275}{236}B_{17,10}\\&+B_{17,11}+6975757441B_{17,12}\Big).\end{aligned}\tag{4.3.31}$$

Note that all the coefficients of $B_{17,j}$, $j\ne 11$, are divisible by 17^2. Therefore,

$$\mathrm{C}\Phi_{17}(q)\equiv\frac{1}{(q^{17};q^{17})_\infty}\pmod{17^2},$$

or equivalently,

$$c\phi_{17}(n)\equiv p(n/17)\pmod{17^2}.$$

This is a special case of Andrews' congruence [4, Theorem 10.2 and Corollary 10.2]

$$c\phi_p(n)\equiv c\phi_1(n/p)\pmod{p^2},\tag{4.3.32}$$

which is true for all primes p.

In the next section, we will provide an analogue for Kolitsch's identities (4.3.7) and (4.3.14) for $\mathrm{C}\Phi_{17}(q)$.

4.4 k-colored generalized Frobenius partitions and ordinary partitions

Kolitsch's identities (4.1.13), (4.1.14), and Andrews' congruence (4.3.32) show a close relation between k-colored generalized Frobenius partitions and ordinary partitions. In this section, we will give a more precise description of the relation and prove (4.3.7), (4.3.14), (4.3.26) and (4.3.28) in a uniform way. We will also give an alternative representation for $C\Phi_{17}(q)$ and illustrate for any prime $\ell > 3$, a general procedure to express $C\Phi_\ell(q)$ in terms of other modular functions, one of which involves generating functions for $p(\ell n - (\ell^2-1)/24)$.

Let

$$F(\tau)\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} := F\left(\frac{a\tau+b}{c\tau+d}\right).$$

Let ℓ be a prime ≥ 5 and let $\mathcal{A}_\ell(\tau)$ denote the function $\mathfrak{A}_\ell(q)$ when the function $\mathfrak{A}_\ell(q)$ is viewed as a function of τ with $q = e^{2\pi i\tau}$. By Theorem 4.2.1,

$$\mathcal{A}_\ell(\tau) = \mathfrak{A}_\ell(q) = (q;q)_\infty^\ell C\Phi_\ell(q) = \sum_{m_1,\ldots,m_{\ell-1}\in\mathbf{Z}} q^{Q(m_1,\ldots,m_{\ell-1})}$$

is a modular form of weight $(\ell-1)/2$ with character $\chi_{(-1)^{(\ell-1)/2}\ell}$ on $\Gamma_0(\ell)$, where $Q(m_1,\ldots,m_{\ell-1})$ is the quadratic form defined by (4.1.4) and χ_d is the character defined by $\chi_d(\cdot) = \left(\frac{d}{\cdot}\right)$. It follows that

$$f_\ell(\tau) := \frac{\eta(\ell\tau)}{\eta(\tau)^\ell}\mathcal{A}_\ell(\tau) \tag{4.4.1}$$

is a modular function on $\Gamma_0(\ell)$. On the other hand, $\eta(\ell^2\tau)/\eta(\tau)$ is a modular function on $\Gamma_0(\ell^2)$ and by a lemma of A.O.L. Atkin and J. Lehner [18, Lemma 7], we find that

$$\frac{\eta(\ell^2\tau)}{\eta(\tau)}\Big|U_\ell := \frac{1}{\ell}\sum_{k=0}^{\ell-1}\frac{\eta(\ell^2\tau)}{\eta(\tau)}\Big|\begin{pmatrix} 1 & k \\ 0 & \ell \end{pmatrix} = (q^\ell;q^\ell)_\infty \sum_{j=1}^{\infty} p\left(\ell j - \frac{\ell^2-1}{24}\right) q^j$$

is also a modular function on $\Gamma_0(\ell)$.

Set

$$g_\ell(\tau) := 1 + \ell \frac{\eta(\ell^2\tau)}{\eta(\tau)}\Big|U_\ell = 1 + \ell(q^\ell;q^\ell)_\infty \sum_{j=1}^{\infty} p\left(\ell j - \frac{\ell^2-1}{24}\right) q^j. \tag{4.4.2}$$

We now compare the analytic behaviors of $f_\ell(\tau)$ and $g_\ell(\tau)$ at cusps associated with $\Gamma_0(\ell)$.

Lemma 4.4.1. *Let $\ell \geq 5$ be a prime and*

$$\delta_\ell = \frac{\ell^2-1}{24}.$$

At the cusp ∞, we have

$$f_\ell(\tau) - g_\ell(\tau) = \begin{cases} \ell(\ell - p(\ell-\delta_\ell))q + O(q^2), & \text{if } \ell \leq 23, \\ \ell^2 q + \left(\frac{1}{4}\ell^2(\ell^2 - 2\ell + 9) - \ell p(2\ell - \delta_\ell)\right) q^2 + O(q^3), & \text{if } 29 \leq \ell \leq 47, \\ \ell^2 q + \frac{1}{4}\ell^2(\ell^2-2\ell+9)q^2 + O(q^3), & \text{if } \ell \geq 53. \end{cases}$$

At the cusp 0, we have

$$(f_\ell(\tau) - g_\ell(\tau))\Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} = \begin{cases} O(q), & \text{if } \ell = 5, 7, 11, \\ 2q^{-1} - 4 + O(q), & \text{if } \ell = 13, \\ 2q^{-(\ell^2-1)/24 + (\ell-1)/2}(1 - q + O(q^2)), & \text{if } \ell \geq 17. \end{cases}$$

Proof. It is clear from the definition of $g_\ell(\tau)$ that

$$\begin{aligned} g_\ell(\tau) &= 1 + \ell(q^\ell;q^\ell)_\infty \sum_{n \geq (\ell^2-1)/24\ell} p(\ell n - \delta_\ell) q^n \\ &= \begin{cases} 1 + \ell p(\ell - \delta_\ell) q + O(q^2), & \text{if } \ell \leq 23, \\ 1 + \ell p(2\ell - \delta_\ell) q^2 + O(q^3), & \text{if } 29 \leq \ell \leq 47, \\ 1 + O(q^3), & \text{if } \ell \geq 53. \end{cases} \end{aligned} \tag{4.4.3}$$

On the other hand, we have

$$\begin{aligned} Q(m_1, \ldots, m_{\ell-1}) &= (m_1^2 + \cdots + m_{\ell-1}^2) + \frac{1}{2} \sum_{i \neq j} m_i m_j \\ &= \frac{1}{2}(m_1^2 + \ldots + m_{\ell-1}^2) + \frac{1}{2}(m_1 + \cdots + m_{\ell-1})^2. \end{aligned}$$

From this, we see that $Q(m_1, \ldots, m_{\ell-1}) = 1$ if and only if exactly one of m_j is ± 1 and the other are all 0, or $m_i = 1$ and $m_j = -1$ for some i, j with $i \neq j$ and all others are 0. Likewise, we can check that $Q(m_1, \ldots, m_{\ell-1}) = 2$ if and only if there are two 1's and two -1's among m_j, or there are two 1's and one -1 among m_j, or there are two -1's and one 1 among m_j. Thus, the number of integer solutions of $Q(m_1, \ldots, m_{\ell-1}) = 2$ is

$$\frac{1}{4}(\ell-1)(\ell-2)(\ell-3)(\ell-4) + 2\cdot\frac{1}{2}(\ell-1)(\ell-2)(\ell-3) = \frac{1}{4}\ell(\ell-1)(\ell-2)(\ell-3).$$

Consequently, we have

$$\mathcal{A}_\ell(\tau) = 1 + \ell(\ell-1)q + \frac{1}{4}\ell(\ell-1)(\ell-2)(\ell-3)q^2 + \cdots$$

and

$$\begin{aligned} f_\ell(\tau) &= \frac{(q^\ell;q^\ell)_\infty}{(1-q-q^2+\cdots)^\ell}\left(1 + \ell(\ell-1)q + \frac{1}{4}\ell(\ell-1)(\ell-2)(\ell-3)q^2 + \cdots\right) \\ &= 1 + \ell^2 q + \frac{1}{4}\ell^2(\ell^2 - 2\ell + 9)q^2 + \cdots . \end{aligned}$$

Together with (4.4.3), this yields the first half of the lemma. We next consider the analytic behavior of $f_\ell(\tau) - g_\ell(\tau)$ at 0.

Recall that if Λ is an even integral lattice of rank n, then the theta series associated with Λ is defined as

$$\theta_\Lambda(\tau) = \sum_{x\in\Lambda} e^{i\pi\tau\|x\|^2}, \quad \tau \in \mathcal{H}. \tag{4.4.4}$$

If we denote the Gram matrix of Λ by G, then this theta series can also be written as

$$\theta_\Lambda(\tau) = \sum_{m\in\mathbf{Z}^n} e^{i\pi\tau m^t G m}. \tag{4.4.5}$$

Moreover, if Λ' is the dual lattice of Λ, then the Gram matrix of Λ' is G^{-1}. Their theta series $\theta_\Lambda(\tau)$ and $\theta_{\Lambda'}(\tau)$ are related by the transformation formula (see [78, Proposition 16, Chapter VII])

$$\theta_{\Lambda'}(-1/\tau) = \left(\frac{\tau}{i}\right)^{n/2}\nu(\Lambda)\theta_\Lambda(\tau), \tag{4.4.6}$$

where $\nu(\Lambda)$ is the volume of the lattice Λ. Here, we let Λ be the lattice of rank $\ell-1$ whose Gram matrix is $\ell A_{\ell-1}^{-1}$, where A_n and A_n^{-1} are given by (4.2.1) and (4.2.2), respectively. The determinant of $\ell A_{\ell-1}^{-1}$ is $\ell^{\ell-1}/\det(A_{\ell-1}) = \ell^{\ell-2}$. Hence

$$\nu(\Lambda) = \ell^{\ell/2-1}.$$

Let $\mathcal{B}_\ell(\tau)$ be the theta series of Λ. Observe that the Gram matrix of Λ' is $A_{\ell-1}/\ell$. Hence from (4.4.5) and (4.2.1) we know that the theta series of Λ' is $\mathcal{A}_\ell(\tau/\ell)$. Thus, by (4.4.6), we have

$$\mathcal{A}_\ell\left(-\frac{1}{\ell\tau}\right) = \ell^{\ell/2-1}\left(\frac{\tau}{i}\right)^{(\ell-1)/2}\mathcal{B}_\ell(\tau). \tag{4.4.7}$$

Now using (4.4.1), (4.4.7) and

$$\eta\left(-\frac{1}{\ell\tau}\right) = \sqrt{\frac{\ell\tau}{i}}\eta(\ell\tau), \quad \text{and} \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau),$$

we deduce that

$$f_\ell(\tau)\Big|\begin{pmatrix}0 & -1\\ \ell & 0\end{pmatrix} = f_\ell\left(-\frac{1}{\ell\tau}\right) = \frac{\eta(-1/\tau)}{\eta(-1/(\ell\tau))^\ell}A_\ell(-\frac{1}{\ell\tau}) = \frac{1}{\ell}\frac{\eta(\tau)}{\eta(\ell\tau)^\ell}\mathcal{B}_\ell(\tau). \tag{4.4.8}$$

We now consider $g_\ell(-\frac{1}{\ell\tau})$.

We have

$$\ell\left(\frac{\eta(\ell^2\tau)}{\eta(\tau)}\Big|U_\ell\right)\Big|\begin{pmatrix}0 & -1\\ \ell & 0\end{pmatrix} = \sum_{k=0}^{\ell-1}\frac{\eta(\ell^2\tau)}{\eta(\tau)}\Big|\begin{pmatrix}k\ell & -1\\ \ell^2 & 0\end{pmatrix}.$$

For $k=0$, the transformation formula for $\eta(\tau)$ yields

$$\frac{\eta(\ell^2\tau)}{\eta(\tau)}\Big|\begin{pmatrix}0 & -1\\ \ell^2 & 0\end{pmatrix} = \frac{1}{\ell}\frac{\eta(\tau)}{\eta(\ell^2\tau)}. \tag{4.4.9}$$

For $1 \le k \le \ell-1$, we find that

$$\eta\left(\ell^2\frac{k\ell\tau-1}{\ell^2\tau}\right) = \eta\left(k\ell - \frac{1}{\tau}\right) = e^{2\pi ik\ell/24}\eta(-1/\tau) = e^{2\pi ik\ell/24}\sqrt{\frac{\tau}{i}}\eta(\tau). \tag{4.4.10}$$

Next, since $1 \le k \le \ell - 1$, we have $(k, \ell) = 1$. Hence there exist integers a and k' such that $kk' - a\ell = 1$. This implies that

$$\eta\left(\frac{k\ell\tau - 1}{\ell^2\tau}\right) = \eta\left(\tau - \frac{k'}{\ell}\right)\Big| \begin{pmatrix} k & a \\ \ell & k' \end{pmatrix} = \left(\frac{k'}{\ell}\right) i^{(1-\ell)/2} e^{2\pi i\ell(k+k')/24}\sqrt{\frac{\ell\tau}{i}}\eta\left(\tau - \frac{k'}{\ell}\right). \tag{4.4.11}$$

It follows from (4.4.10) and (4.4.11) that

$$\begin{aligned}\frac{\eta(\ell^2\tau)}{\eta(\tau)}\Big| \begin{pmatrix} k\ell & -1 \\ \ell^2 & 0 \end{pmatrix} &= \frac{1}{\sqrt{\ell}}\left(\frac{k'}{\ell}\right) i^{(\ell-1)/2} e^{-2\pi i\ell k'/24}\frac{\eta(\tau)}{\eta(\tau - k'/\ell)} \\ &= \frac{1}{\sqrt{\ell}}\left(\frac{k'}{\ell}\right) i^{(\ell-1)/2} e^{-2\pi imk'/\ell} + O(q),\end{aligned}$$

where $m = (\ell^2 - 1)/24$. Hence,

$$\begin{aligned}\sum_{k=1}^{\ell-1}\frac{\eta(\ell^2\tau)}{\eta(\tau)}\Big| \begin{pmatrix} k\ell & -1 \\ \ell^2 & 0 \end{pmatrix} &= \frac{i^{(\ell-1)/2}}{\sqrt{\ell}}\sum_{k=1}^{\ell-1}\left(\frac{k'}{\ell}\right) e^{-2\pi imk'/\ell} + O(q) \\ &= \frac{i^{(\ell-1)/2}}{\sqrt{\ell}}\left(\frac{-m}{\ell}\right)\sum_{n=1}^{\ell-1}\left(\frac{n}{\ell}\right) e^{2\pi in/\ell} + O(q) \\ &= i^{(\ell-1)/2}\left(\frac{-m}{\ell}\right)\begin{cases} 1 + O(q), & \text{if } \ell \equiv 1 \mod 4, \\ i + O(q), & \text{if } \ell \equiv 3 \mod 4, \end{cases} \qquad (4.4.12) \\ &= \left(\frac{8}{\ell}\right)\left(\frac{-m}{\ell}\right) + O(q) \\ &= \left(\frac{12}{\ell}\right) + O(q),\end{aligned}$$

where we have used Gauss' result [15, Section 9.10] in our third equality. Combining (4.4.8), (4.4.9), and (4.4.12), we find that

$$\begin{aligned}(f_\ell(\tau) - g_\ell(\tau))\Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} &= \frac{1}{\ell}q^{-(\ell^2-1)/24}(q;q)_\infty\left(\frac{\mathcal{B}_\ell(\tau)}{(q^\ell;q^\ell)_\infty^\ell} - \frac{1}{(q^{\ell^2};q^{\ell^2})_\infty}\right) \\ &\quad - \left(\frac{12}{\ell}\right) - 1 + O(q).\end{aligned} \tag{4.4.13}$$

We now claim that

$$\mathcal{B}_\ell(\tau) = 1 + 2\ell q^{(\ell-1)/2} + \cdots, \tag{4.4.14}$$

so that

$$\frac{\mathcal{B}_\ell(\tau)}{(q^\ell;q^\ell)_\infty^\ell} - \frac{1}{(q^{\ell^2};q^{\ell^2})_\infty} = \left(1+2\ell q^{(\ell-1)/2}+O(q^\ell)\right)\left(1+O(q^\ell)\right) = 2\ell q^{(\ell-1)/2}+\cdots.$$

Recall that $\mathcal{B}_\ell(\tau)$ is defined to be the theta series associated to the lattice whose Gram matrix is $\ell A_{\ell-1}^{-1}$, where A_n^{-1} is given by (4.2.2). In other words, we have

$$\mathcal{B}_\ell(\tau) = \sum_{m_1,\ldots,m_{\ell-1}\in\mathbf{Z}} q^{Q'(m_1,\ldots,m_{\ell-1})}, \quad q=e^{2\pi i\tau},$$

where

$$\begin{aligned} Q'(m_1,\ldots,m_{\ell-1}) &= \frac{\ell-1}{2}(m_1^2+\cdots+m_{\ell-1}^2) - \frac{1}{2}\sum_{i\neq j} m_i m_j \\ &= \frac{1}{2}\left(\ell(m_1^2+\cdots+m_{\ell-1}^2)-(m_1+\cdots+m_{\ell-1})^2\right). \end{aligned}$$

For each $(m_1,\ldots,m_{\ell-1})\in\mathbf{Z}^{\ell-1}\backslash\{0\}$, let r be the number of nonzero entries in the tuple. By the Cauchy-Schwarz inequality, we have

$$(m_1+\cdots+m_{\ell-1})^2 \le r(m_1^2+\cdots+m_{\ell-1}^2).$$

Then

$$Q'(m_1,\ldots,m_{\ell-1}) \ge \frac{1}{2}(\ell-r)(m_1^2+\cdots+m_{\ell-1}^2) \ge \frac{1}{2}(\ell-r)r \ge \frac{\ell-1}{2}.$$

Therefore, the coefficient of q^j in $\mathcal{B}_\ell(\tau)$ vanishes for $j=1,\ldots,(\ell-1)/2-1$. Also, the contribution to the $q^{(\ell-1)/2}$ term comes from the cases where $r=1$ or $r=\ell-1$ and equality holds for each of the inequality above. In other words, the contribution to $q^{(\ell-1)/2}$ comes from the tuples where exactly one of m_j is ±1 and all the others are 0 or $(m_1,\ldots,m_{\ell-1})=\pm(1,\ldots,1)$. We conclude that the coefficient of $q^{(\ell-1)/2}$ in $\mathcal{B}_\ell(\tau)$ is 2ℓ. This proves the claim (4.4.14).

For the cases $\ell=5$ and $\ell=7$, we have $(\ell^2-1)/24 < (\ell-1)/2$ and $\left(\frac{12}{\ell}\right)=-1$. Therefore,

$$(f_\ell(\tau)-g_\ell(\tau))\Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} = O(q). \tag{4.4.15}$$

When $\ell = 11$, we have $(\ell^2-1)/24 = (\ell-1)/2$ and $\left(\frac{12}{\ell}\right) = 1$. Again, (4.4.13) implies that (4.4.15) holds in this case. For other cases, we note that in general, we have

$$\mathcal{B}_\ell(\tau) = 1 + 2\ell q^{(\ell-1)/2} + \ell(\ell-1)q^{\ell-2} + \cdots,$$

and hence,

$$q^{-(\ell^2-1)/24}(q;q)_\infty \left(\frac{\mathcal{B}_\ell(\tau)}{(q^\ell;q^\ell)_\infty^\ell} - \frac{1}{(q^{\ell^2};q^{\ell^2})_\infty} \right) = 2\ell q^{-(\ell^2-1)/24+(\ell-1)/2}(1-q-q^2+\cdots)$$

for $\ell \geq 11$. When $\ell = 13$, we have $-(\ell^2-1)/24 + (\ell-1)/2 = -1$ and $\left(\frac{12}{13}\right) = 1$. Then from (4.4.13), we deduce that

$$(f_\ell(\tau) - g_\ell(\tau)) \Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} = 2q^{-1} - 4 + O(q).$$

For other primes $\ell \geq 17$, (4.4.13) yields

$$(f_\ell(\tau) - g_\ell(\tau)) \Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} = 2q^{-(\ell^2-1)/24+(\ell-1)/2}(1 - q + O(q^2))$$

instead. This completes the proof of the lemma. □

Theorem 4.4.2. *Let $\ell \geq 5$ be a prime. Let*

$$f_\ell(\tau) = (q^\ell;q^\ell)_\infty \mathrm{C}\Phi_\ell(q),$$

and

$$g_\ell(\tau) = 1 + \ell(q^\ell;q^\ell)_\infty \sum_{n=1}^{\infty} p\left(\ell n - \frac{\ell^2-1}{24}\right) q^n.$$

1. *If $\ell = 5, 7, 11$, then $f_\ell(\tau) = g_\ell(\tau)$.*

2. *If $\ell = 13$, then*
$$f_{13}(\tau) = g_{13}(\tau) + 26\frac{\eta^2(13\tau)}{\eta^2(\tau)}.$$

3. *If $\ell \geq 17$, then*
$$h_\ell(\tau) := f_\ell(\tau) - g_\ell(\tau) - 2\ell^{(\ell-11)/2}\left(\frac{\eta(\ell\tau)}{\eta(\tau)}\right)^{\ell-11} \tag{4.4.16}$$

is a modular function on $\Gamma_0(\ell)$ with a zero at ∞ and a pole of order $(\ell+1)(\ell-13)/24$ at 0 and

$$h_\ell(\tau)(\eta(\tau)\eta(\ell\tau))^{\ell-13}$$

is a holomorphic modular form of weight $\ell-13$ with a zero of order $(\ell-1)(\ell-11)/24$ at ∞.

4. *We have*

$$h_\ell(\tau) \equiv 0 \mod \begin{cases} 170, & \textit{when } \ell = 17, \\ 266, & \textit{when } \ell = 19, \\ 506, & \textit{when } \ell = 23. \end{cases}$$

5. *For any prime $\ell > 11$,*

$$h_\ell(\tau) \equiv \ell F_\ell(\tau) \pmod{\ell^2}$$

where $F_\ell(\tau)$ is a non-zero modular form of weight $\ell - 1$ on $SL(2, \mathbf{Z})$.

Proof. We first remark that the functions $f_\ell(\tau)$ and $g_\ell(\tau)$ are both holomorphic on the upper half-plane. Thus, to prove that $f_\ell(\tau) = g_\ell(\tau)$ for the cases $\ell = 5, 7, 11$, we only need to verify that $f_\ell(\tau) - g_\ell(\tau)$ does not have poles at cusps and $f_\ell(\tau) - g_\ell(\tau)$ vanishes at one particular point in these three cases. Indeed, by Lemma 4.4.1, $f_\ell(\tau) - g_\ell(\tau)$ vanishes at both cusps in the three cases since $p(\ell - \delta_\ell) = \ell$ for $\ell = 5, 7$ and 11. This proves (1). We remark that in fact, it suffices to know that $f_\ell(\tau) - g_\ell(\tau)$ has no pole at the cusp 0 since it would mean that $f_\ell(\tau) - g_\ell(\tau)$ is a constant. Since the expansion at ∞ begins with $\ell\,(\ell - p(\ell - \delta_\ell))\,q$, the only possibility that $f_\ell(\tau) - g_\ell(\tau)$ is a constant is when $p(\ell - \delta_\ell) = \ell$. In other words, without listing out the partitions of $5, 7$ and 11, we know that $p(4) = 5, p(5) = 7$ and $p(6) = 11$.

We next consider the case $\ell = 13$. By Lemma 4.4.1, the Fourier expansion of $f_\ell(\tau) - g_\ell(\tau)$ at 0 is

$$2q^{-1} - 4 + \cdots .$$

Now we observe that $\eta(13\tau)^2/\eta(\tau)^2$ is also a modular function on $\Gamma_0(13)$ and satisfies

$$\frac{\eta^2(13\tau)}{\eta^2(\tau)}\Big| \begin{pmatrix} 0 & -1 \\ 13 & 0 \end{pmatrix} = \frac{1}{13}\frac{\eta^2(\tau)}{\eta^2(13\tau)} = \frac{1}{13}(q^{-1} - 2 + \cdots).$$

Therefore, $f_\ell(\tau) - g_\ell(\tau) - 26\eta(13\tau)^2/\eta(\tau)^2$ is a modular function on $\Gamma_0(13)$ that has no poles and vanishes at the cusps. We conclude that $f_\ell(\tau) - g_\ell(\tau) - 26\eta(13\tau)^2/\eta(\tau)^2$ is identically 0 and the proof of (2) is complete.

Similarly, for primes $\ell \geq 17$, using Lemma 4.4.1 and the transformation formula of $\eta(\tau)$, we find that

$$\begin{aligned} h_\ell(\tau)\Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} &= 2q^{-(\ell^2-1)/24+(\ell-1)/2}((1 - q + O(q^2)) - (1 - (\ell-11)q + O(q^2))) \\ &= 2(\ell-12)q^{-(\ell+1)(\ell-13)/24} + \cdots . \end{aligned}$$

Therefore, $h_\ell(\tau)$ has a pole of order $(\ell+1)(\ell-13)/24$ for $\ell \geq 17$. From Lemma 4.4.1, it is clear that $h_\ell(\tau)$ has a zero at ∞. It follows that $h_\ell(\tau)(\eta(\tau)\eta(\ell\tau))^{\ell-13}$ is a holomorphic modular form of weight $\ell - 13$ on $\Gamma_0(\ell)$ and this completes the proof of (3).

The congruences in (4) can be verified using Sturm's criterion [79] (See also Lemma 3.2.2).

Next, observe from (4.1.19) that

$$f_\ell(\tau) \equiv 1 \pmod{\ell^2}.$$

For $\ell > 13$,

$$h_\ell(\tau) \equiv f_\ell(\tau) - g_\ell(\tau) \equiv -\ell(q^\ell;q^\ell)_\infty \sum_{j=1}^{\infty} p\left(\ell j - \frac{\ell^2-1}{24}\right) q^j \pmod{\ell^2}.$$

It is known that (see [9, p. 157, Corollary 5.15.1] for a proof given by J.P. Serre)

$$(q^\ell;q^\ell)_\infty \sum_{j=1}^{\infty} p\left(\ell j - \frac{\ell^2-1}{24}\right) q^j \equiv F_\ell(\tau) \pmod{\ell}$$

where $F_\ell(\tau)$ is a cusp form on $\mathrm{SL}(2,\mathbf{Z})$ of weight $\ell - 1$. This implies that

$$h_\ell(\tau) \equiv -\ell F_\ell(\tau) \pmod{\ell^2}.$$

The fact that $F_\ell(\tau)$ is non-zero follows from the result of S. Ahlgren and M. Boylan [1, Theorem 1]. □

We now give another representation for $C\Phi_{17}(q)$. Let

$$h_1(\tau) = \eta^8(17\tau)\sum_{k=0}^{7}\mathcal{E}_{3^k}^{-1}(\tau)\mathcal{E}_{2\cdot 3^k}^{-2}(\tau)\mathcal{E}_{5\cdot 3^k}^{-1}(\tau) = q^4+3q^5+8q^6+5q^7+\cdots$$

and

$$h_2(\tau) = \eta^8(17\tau)\sum_{k=0}^{7}\mathcal{E}_{7\cdot 3^k}(\tau)\mathcal{E}_{3^k}^{-2}(\tau)\mathcal{E}_{3^{k+1}}^{-1}(\tau)\mathcal{E}_{5\cdot 3^k}^{-1}(\tau)\mathcal{E}_{8\cdot 3^k}^{-1}(\tau) = q^4+q^5+8q^6+\cdots,$$

where $\mathcal{E}_a(\tau)$ is given by (4.3.30). One can check that both $h_1(\tau)$ and $h_2(\tau)$ belong to $M_4(\Gamma_0(17)$. Now by Theorem 4.4.2 (c) we know that the function $h_{17}(\tau)\eta^4(\tau)\eta^4(17\tau)$ lies in the same space. By [69, Theorem 1.34] we find that $\dim M_4(\Gamma_0(17)) = 4$. Hence by comparing the Fourier coefficients, we deduce that

$$h_{17}(\tau)\eta^4(\tau)\eta^4(17\tau) = 595h_1(\tau) - 425h_2(\tau).$$

Now using (4.4.16) and the definitions of $f_\ell(\tau)$ and $g_\ell(\tau)$ given in Theorem 4.4.2, we obtain

$$C\Phi_{17}(q) = \frac{1}{(q^{17};q^{17})_\infty} + 17\sum_{j=1}^{\infty}p(17j-12)q^j + 2\cdot 17^3 q^4\frac{(q^{17};q^{17})_\infty^5}{(q;q)_\infty^6} + \frac{1}{q^3(q;q)_\infty^4(q^{17};q^{17})_\infty^5}\left(595h_1(\tau)-425h_2(\tau)\right). \tag{4.4.17}$$

Note the simplicity of (4.4.17) as compared to (4.3.31). Identities similar to (4.4.17) exist for $k = 19, 23$ and other primes and they involve generalized Dedekind eta functions similar to $\mathcal{E}_a(\tau)$.

4.5 Generating function of $c\phi_k(n)$ for $k = 9$ and 15

There are two cases to consider in this section, namely, $k = 9$ and 15.

4.5.1 Case $k = 9$

Let

$$E_{9,1} = \frac{1}{240} + \sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k},$$
$$E_{9,2} = \frac{1}{240} + \sum_{k=1}^{\infty} \frac{k^3 q^{3k}}{1-q^{3k}},$$
$$E_{9,3} = \frac{1}{240} + \sum_{k=1}^{\infty} \frac{k^3 q^{9k}}{1-q^{9k}}$$

and

$$E_{9,4} = \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \sum_{d|n} d^3 q^n.$$

These are Eisenstein series of $M_4(\Gamma_0(9))$.

Theorem 4.5.1. *We have*

$$\begin{aligned} \mathrm{C}\Phi_9(q) &= 324q\frac{(q^3;q^3)_\infty^8}{(q;q)_\infty^9} + 19683q^4\frac{(q^9;q^9)_\infty^{12}}{(q;q)_\infty^9(q^3;q^3)_\infty^4} - 240q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^4} \qquad (4.5.1)\\ &\quad - 1458q^2\frac{(q^9;q^9)_\infty^6}{(q;q)_\infty^3(q^3;q^3)_\infty^4} + \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty^4} \\ &= \frac{1}{(q;q)_\infty^9}\left(81E_{9,1} - 84E_{9,2} + 243E_{9,3} - 3E_{9,4} - 6q(q^3;q^3)_\infty^8\right). \qquad (4.5.2)\end{aligned}$$

Proof. By Theorem 4.2.1, we find that $\mathfrak{A}_9(q) \in M_4(\Gamma_0(9))$. Next, from [69, Theorem 1.34], we find that $\dim M_4(\Gamma_0(9)) = 5$ and a basis is given by

$$B_{9,1} = \eta^8(3\tau), \quad B_{9,2} = \frac{\eta^{12}(9\tau)}{\eta^4(3\tau)}, \quad B_{9,3} = \frac{\eta^9(\tau)\eta^3(9\tau)}{\eta^4(3\tau)},$$
$$B_{9,4} = \frac{\eta^6(\tau)\eta^6(9\tau)}{\eta^4(3\tau)}, \quad B_{9,5} = \frac{\eta^{12}(\tau)}{\eta^4(3\tau)}.$$

By comparing Fourier coefficients of $\mathfrak{A}_9(q)$ and $B_{9,j}, 1 \le j \le 5$, we deduce that

$$\mathfrak{A}_9(q) = 324B_{9,1} + 19683B_{9,2} - 240B_{9,3} - 1458B_{9,4} + B_{9,5}. \qquad (4.5.3)$$

This proves (4.5.1).

We can replace the basis $\{B_{9,j}|1 \le j \le 5\}$ by $\{B_{9,1}, E_{9,j}|1 \le j \le 4\}$. Using these modular forms as a basis for $M_4(\Gamma_0(9))$, we deduce (4.5.2). □

Theorem 4.5.2. *For any integer $n \geq 0$, we have*

$$c\phi_9(9n+3) \equiv c\phi_9(9n+6) \equiv 0 \pmod{9}, \tag{4.5.4}$$

$$c\phi_9(3n+1) \equiv 0 \pmod{81} \tag{4.5.5}$$

and

$$c\phi_9(3n+2) \equiv 0 \pmod{729}. \tag{4.5.6}$$

Proof. From [81, Lemma 2.5], we find that

$$(q;q)_\infty^3 = S(q^3) - 3q(q^9;q^9)_\infty^3, \tag{4.5.7}$$

where

$$S(q) = (q;q)_\infty \left(\Theta_3(q)\Theta_3(q^3) + \Theta_2(q)\Theta_2(q^3)\right). \tag{4.5.8}$$

From (4.5.1), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_9(n)q^n &\equiv 2^2 \cdot 3^4 q \frac{(q^3;q^3)_\infty^8}{(q;q)_\infty^9} - 240q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^4} + \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty^4} \pmod{729}\\ &\equiv 2^2 \cdot 3^4 q(q^3;q^3)_\infty^5 - 240q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^4} + \frac{S(q^3)}{(q^3;q^3)_\infty^4} - 3q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^4} \pmod{729}.\end{aligned} \tag{4.5.9}$$

Comparing the coefficients of q^{3n+2} on both sides, we deduce that

$$c\phi_9(3n+2) \equiv 0 \pmod{729}.$$

Extracting the terms of the form q^{3n+1} on both sides of (4.5.9), dividing by q and replacing q^3 by q, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_9(3n+1)q^n &\equiv 2^2 \cdot 3^4 (q;q)_\infty^5 - 240\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty^4} - 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty^4} \pmod{729}\\ &\equiv 2^2 \cdot 3^4 (q;q)_\infty^5 - 243\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty^4} \pmod{729}\\ &\equiv 2^2 \cdot 3^4 (q;q)_\infty^5 - 3^5 (q;q)_\infty^5 \pmod{729}\end{aligned}$$

$$\equiv 3^4(q;q)_\infty^5 \pmod{729}, \tag{4.5.10}$$

which implies (4.5.5).

Extracting the terms of the form q^{3n} on both sides of (4.5.9) and replacing q^3 by q, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_9(3n)q^n &\equiv \frac{S(q)}{(q;q)_\infty^4} \pmod{729}\\ &\equiv \frac{1}{(q;q)_\infty^3}\big(\Theta_3(q)\Theta_3(q^3)+\Theta_2(q)\Theta_2(q^3)\big) \pmod{729}.\end{aligned} \tag{4.5.11}$$

From [81, Lemma 2.6], we deduce that

$$\frac{1}{(q;q)_\infty^3} = \frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^{12}}\left(S^2(q^3)+3qS(q^3)(q^9;q^9)_\infty^3+9q^2(q^9;q^9)_\infty^6\right). \tag{4.5.12}$$

From [25, Corollary (i) and (ii), p. 49], we find that

$$\Theta_3(q) = \Theta_3(q^9) + 2qf(q^3,q^{15}), \tag{4.5.13}$$

and

$$\Theta_2(q) = \Theta_2(q^9) + 2q^{1/4}f(q^6,q^{12}), \tag{4.5.14}$$

where

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Substituting (4.5.12)–(4.5.14) into (4.5.11), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_9(3n)q^n &\equiv \frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^{12}}\Big(S^2(q^3)+3qS(q^3)(q^9;q^9)_\infty^3\Big)\\ \times \bigg(\Theta_3(q^3)\Big(\Theta_3(q^9)+2qf(q^3,q^{15})\Big) &+ \Theta_2(q^3)\Big(\Theta_2(q^9)+2q^{1/4}f(q^6,q^{12})\Big)\bigg) \pmod{9}.\end{aligned} \tag{4.5.15}$$

Extracting the terms of the form q^{3n+1} on both sides of (4.5.15), dividing by q and replacing q^3 by q, applying (4.5.8), we deduce that

$$\sum_{n=0}^{\infty} c\phi_9(9n+3)q^n$$

$$
\begin{aligned}
&\equiv S^2(q)\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty^{12}}\left(3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}+2\left(\Theta_3(q)f(q,q^5)+q^{-1/4}\Theta_2(q)f(q^2,q^4)\right)\right) \pmod 9\\
&\equiv S^2(q)\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty^{12}}\\
&\times\left(3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}+2\frac{(q^2;q^2)_\infty^7(q^3;q^3)_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty^3(q^4;q^4)_\infty^3(q^6;q^6)_\infty}+4\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}\right) \pmod 9,
\end{aligned}
\tag{4.5.16}
$$

where the last congruence follows by converting

$$\Theta_3(q)f(q,q^5)+q^{-1/4}\Theta_2(q)f(q^2,q^4)$$

to infinite products.

From [92, (3.75), (3.38)], we find that

$$\frac{(q^3;q^3)_\infty}{(q;q)_\infty^3}=\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty^9(q^{12};q^{12})_\infty^2}+3q\frac{(q^4;q^4)_\infty^2(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^7} \tag{4.5.17}$$

and

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}=\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}+q\frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty}. \tag{4.5.18}$$

By (4.5.17), we find that

$$2\frac{(q^2;q^2)_\infty^7(q^3;q^3)_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty^3(q^4;q^4)_\infty^3(q^6;q^6)_\infty} \tag{4.5.19}$$

$$
\begin{aligned}
&=2\frac{(q^2;q^2)_\infty^7(q^{12};q^{12})_\infty}{(q^4;q^4)_\infty^3(q^6;q^6)_\infty}\left(\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty^9(q^{12};q^{12})_\infty^2}+3q\frac{(q^4;q^4)_\infty^2(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^7}\right)\\
&=2\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}+6q\frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty}
\end{aligned}
\tag{4.5.20}
$$

and this implies that

$$
\begin{aligned}
&2\frac{(q^2;q^2)_\infty^7(q^3;q^3)_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty^3(q^4;q^4)_\infty^3(q^6;q^6)_\infty}+4\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}\\
&=6\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}+6q\frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty}\\
&=6\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty},
\end{aligned}
\tag{4.5.21}
$$

where we have used (4.5.18) in the last equality. Substituting (4.5.21) into (4.5.16), we deduce that

$$c\phi_9(9n+3) \equiv 0 \pmod{9}.$$

Extracting the terms of the form q^{3n+2} on both sides of (4.5.15), dividing by q^2 and replacing q^3 by q, we deduce that

$$\begin{aligned}&\sum_{n=0}^{\infty} c\phi_9(9n+6)q^n \\ &\equiv S(q)\frac{(q^3;q^3)_\infty^6}{(q;q)_\infty^{12}}\left(6\frac{(q^2;q^2)_\infty^7(q^3;q^3)_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty^3(q^4;q^4)_\infty^3(q^6;q^6)_\infty} + 3\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}\right) \pmod{9} \\ &\equiv 0 \pmod{9},\end{aligned}$$

where we have used (4.5.20) to deduce the last congruence. Hence

$$c\phi_9(9n+6) \equiv 0 \pmod{9}.$$

□

Congruences (4.5.4) and (4.5.5) can also be established using congruences discovered by Kolitsch. In [56], Kolitsch generalized Andrews' congruence (4.3.32) and proved that

$$\sum_{d|(k,n)} \mu(d)c\phi_{\frac{k}{d}}(\frac{n}{d}) \equiv 0 \pmod{k^2}, \tag{4.5.22}$$

where $\mu(n)$ is the Möbius function (see for example [15, Section 2.2]). We now prove a generalization of (4.5.4) and (4.5.5). For any non-negative integer k, we set $c\phi_k(x) = 0$ whenever $x \notin \mathbf{Z}$. We can then rewrite (4.5.22) as

$$\sum_{d|k} \mu(d)c\phi_{\frac{k}{d}}\left(\frac{n}{d}\right) \equiv 0 \pmod{k^2}. \tag{4.5.23}$$

Theorem 4.5.3. *Let p be a prime and N be a positive integer which is not divisible by p. For any integers $\alpha \geq 1$ and $n \geq 0$, we have*

$$c\phi_{p^\alpha N}(n) \equiv c\phi_{p^{\alpha-1}N}(n/p) \pmod{p^{2\alpha}}, \tag{4.5.24}$$

or equivalently,

$$c\phi_{p^\alpha N}(pn+r) \equiv 0 \pmod{p^{2\alpha}}, \quad 1 \le r \le p-1, \tag{4.5.25}$$

and

$$c\phi_{p^\alpha N}(pn) \equiv c\phi_{p^{\alpha-1}N}(n) \pmod{p^{2\alpha}}. \tag{4.5.26}$$

Proof. Let $\Omega(N)$ be the number of prime divisors of N (counting multiplicities). We proceed by induction on $\Omega(N)$. If $\Omega(N)=0$, then $N=1$. Setting $k=p^\alpha$ in (4.5.23), we deduce that

$$c\phi_{p^\alpha}(n) \equiv c\phi_{p^{\alpha-1}}(n/p) \pmod{p^{2\alpha}}. \tag{4.5.27}$$

Thus, (4.5.24) is true if $\Omega(N)=0$. Assume that (4.5.24) is true if $\Omega(N)<h$, where h is a positive integer. When $\Omega(N)=h$, we set $k=p^\alpha N$ in (4.5.23). Since p does not divide N, any positive divisor of $p^\alpha N$ has the form $p^j d$ where $0 \le j \le \alpha$ and $d|N$. In particular, if $j \ge 2$, then $\mu(p^j d')=0$. Hence by (4.5.23), we obtain

$$\sum_{d|N}\left(\mu(d)c\phi_{\frac{p^\alpha N}{d}}\left(\frac{n}{d}\right)+\mu(pd)c\phi_{\frac{p^{\alpha-1}N}{d}}\left(\frac{n}{pd}\right)\right) \equiv 0 \pmod{p^{2\alpha}}. \tag{4.5.28}$$

According to $d=1$ or $d>1$, we separate the summands on the left hand side of (4.5.28) and deduce that

$$\begin{aligned} &c\phi_{p^\alpha N}(n) - c\phi_{p^{\alpha-1}N}\left(\frac{n}{p}\right) \\ &+\sum_{d|N,d>1}\mu(d)\left(c\phi_{\frac{p^\alpha N}{d}}\left(\frac{n}{d}\right)-c\phi_{\frac{p^{\alpha-1}N}{d}}\left(\frac{n}{pd}\right)\right) \equiv 0 \pmod{p^{2\alpha}}. \end{aligned} \tag{4.5.29}$$

Note that in the summand, since $d>1$, we have $\Omega(\frac{N}{d})<h$ and hence by assumption,

$$c\phi_{\frac{p^\alpha N}{d}}\left(\frac{n}{d}\right)-c\phi_{\frac{p^{\alpha-1}N}{d}}\left(\frac{n}{pd}\right) \equiv 0 \pmod{p^{2\alpha}}. \tag{4.5.30}$$

From (4.5.30) and (4.5.29), we deduce that

$$c\phi_{p^\alpha N}(n) - c\phi_{p^{\alpha-1}N}\left(\frac{n}{p}\right) \equiv 0 \pmod{p^{2\alpha}}.$$

Hence (4.5.24) is true when $\Omega(N) = h$. This completes the proof of (4.5.24).

Replacing n in (4.5.24) by $pn + r$, where $0 \le r \le p - 1$, and observing that

$$c\phi_{p^{\alpha-1}N}\left(\frac{pn+r}{p}\right) = 0, \quad 1 \le r \le p-1,$$

we deduce (4.5.25) and (4.5.26). □

Let $(p, \alpha, N) = (3, 2, 1)$ in Theorem 4.5.3. By (4.5.25), we deduce that

$$c\phi_9(3n+1) \equiv c\phi_9(3n+2) \equiv 0 \pmod{81},$$

and this gives another proof of (4.5.5). Similarly, by (4.5.26), we deduce that

$$c\phi_9(3n) \equiv c\phi_3(n) \pmod{81}. \tag{4.5.31}$$

By (4.3.32), we find that

$$c\phi_3(3n+1) \equiv c\phi_3(3n+2) \equiv 0 \pmod{9}.$$

Substituting these congruences into (4.5.31), we complete the proof of (4.5.4).

4.5.2 Case $k = 15$

Let

$$\begin{aligned}
f_{15}(\tau) &= \frac{\eta^2(\tau)\eta^2(15\tau)}{\eta(3\tau)\eta(5\tau)}, \\
h_{15}(\tau) &= \eta^4(\tau)\eta^4(5\tau) - 9\eta^4(3\tau)\eta^4(15\tau), \\
g_{15,1}(\tau) &= -\frac{1}{8}\left(E_2(\tau) + 3E_2(3\tau) - 5E_2(5\tau) - 15E_2(15\tau)\right), \\
g_{15,2}(\tau) &= -\frac{1}{12}\left(E_2(\tau) - 3E_2(3\tau) + 5E_2(5\tau) - 15E_2(15\tau)\right), \\
g_{15,3}(\tau) &= \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau),
\end{aligned}$$

and

$$g_{15,4}(\tau) = \frac{1}{8}\left(E_2(\tau) - 3E_2(3\tau) - 5E_2(5\tau) + 15E_2(15\tau)\right),$$

where

$$E_2(\tau) = 1 - 24\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}.$$

Using dimension formula [69, Theorem 1.34], we find that

$$\dim M_7\Big(\Gamma_0(15), (\frac{-15}{\cdot})\Big) = 14.$$

The modular forms

$$\begin{aligned}
B_{15,1} &= f_{15}g_{15,1}^3, & B_{15,2} &= f_{15}g_{15,1}^2 g_{15,2},\\
B_{15,3} &= f_{15}g_{15,1}g_{15,2}^2, & B_{15,4} &= f_{15}g_{15,2}^3,\\
B_{15,5} &= f_{15}g_{15,1}^2 g_{15,3}, & B_{15,6} &= f_{15}g_{15,1}^2 g_{15,4},\\
B_{15,7} &= f_{15}g_{15,1}g_{15,2}g_{15,3}, & B_{15,8} &= f_{15}g_{15,1}g_{15,2}g_{15,4},\\
B_{15,9} &= f_{15}g_{15,2}^2 g_{15,3}, & B_{15,10} &= f_{15}g_{15,2}^2 g_{15,4}\\
B_{15,11} &= f_{15}g_{15,1}h_{15} & B_{15,12} &= f_{15}g_{15,2}h_{15},\\
B_{15,13} &= \frac{\eta^{14}(3\tau)\eta^{14}(5\tau)}{\eta^7(\tau)\eta^7(15\tau)} \quad \text{and} & B_{15,14} &= \frac{\eta^{17}(\tau)\eta^2(5\tau)}{\eta^4(3\tau)\eta(15\tau)}
\end{aligned}$$

form a basis for $M_7\Big(\Gamma_0(15), \big(\frac{-15}{\cdot}\big)\Big)$.

Using the fact that $\mathfrak{A}_{15}(q) \in M_7\left(\Gamma_0(15), \left(\frac{-15}{\cdot}\right)\right)$, we deduce that

Theorem 4.5.4. *For* $|q| < 1$,

$$\begin{aligned}
\mathrm{C}\Phi_{15}(q) = \frac{1}{(q;q)_\infty^{15}}\Big(&\frac{18125225}{1156}B_{15,1} - \frac{845079}{34}B_{15,2} - \frac{87564447}{1156}B_{15,3}\\
&+ \frac{2491641}{34}B_{15,4} + \frac{147166525}{1156}B_{15,5} + \frac{341957}{68}B_{15,6}\\
&- \frac{483081}{17}B_{15,7} - \frac{28623}{4}B_{15,8} - \frac{9784683}{68}B_{15,9}\\
&- \frac{1168839}{34}B_{15,10} + \frac{7263781}{68}B_{15,11} - \frac{97629}{4}B_{15,12}\\
&+ 3375B_{15,13} - 3374B_{15,14}\Big).
\end{aligned}$$

4.6 Generating function of $c\phi_k(n)$ for even integer $2 < k < 16$

In this section, we derive alternative expressions for $\mathrm{C}\Phi_k(n)$ when $k > 2$ is even.

4.6.1 Case $k = 4$

Theorem 4.6.1. *We have*

$$\mathrm{C}\Phi_4(q) = \frac{1}{(q;q)_\infty^4}\Big(\Theta_3^3(q^2) + 3\Theta_3(q^2)\Theta_2^2(q^2)\Big) \tag{4.6.1}$$

$$= \frac{\Theta_3^4(q)}{(q;q)_\infty^4\Theta_3(q^2)} + \frac{\Theta_3^2(-q)\Theta_2^2(q^2)}{(q;q)_\infty^4\Theta_3(q^2)}. \tag{4.6.2}$$

Proof. Let $k = 4$ in Theorem 4.2.1. We deduce that $\mathfrak{A}_4(q)\Theta_3(q) \in M_2\big(\Gamma_0(8), (\frac{2}{\cdot})\big)$. From [69, Theorem 1.34], we deduce that

$$\dim M_2\big(\Gamma_0(8), (\frac{2}{\cdot})\big) = 3.$$

It can be verified that

$$\Theta_3(q)\Theta_3^3(q^2), \quad \Theta_3(q)^3\Theta_3(q^2), \quad \text{and} \quad \Theta_3(q^2)\Theta_2^2(q^2)\Theta_3(q)$$

form a basis of $M_2\big(\Gamma_0(8), (\frac{2}{\cdot})\big)$. Comparing the Fourier coefficients of $\mathfrak{A}_4(q)\Theta_3(q)$ and the given basis of $M_2\big(\Gamma_0(8), (\frac{2}{\cdot})\big)$, we deduce that

$$\mathfrak{A}_4(q)\Theta_3(q) = \Big(\Theta_3^3(q^2) + 3\Theta_3(q^2)\Theta_2^2(q^2)\Big)\Theta_3(q),$$

which proves (4.6.1).

Theorem 4.2.1 also implies that $\Theta_3(q^2)\mathfrak{A}_4(q) \in M_2\big(\Gamma_0(16)\big)$. From [69, Theorem 1.34], we find that $\dim M_2\big(\Gamma_0(16)\big) = 5$. Identity (4.6.2) then follows from the fact that

$$\Theta_3^4(q), \quad \Theta_3^4(q^2), \quad \Theta_3^4(q^4), \quad \Theta_3^2(-q)\Theta_3^2(-q^2), \quad \text{and} \quad \Theta_3^2(-q)\Theta_2^2(q^2)$$

form a basis of $M_2\big(\Gamma_0(16)\big)$. □

Remark 4.6.1. The representation (4.6.2) was first deduced by W. Zhang and C. Wang [94] from (4.6.1), where they used it to give an elementary proof of the congruence

$$c\phi_4(7n+5) \equiv 0 \pmod 7.$$

Modular proofs of this congruence can be found in [35]. For more congruences satisfied by $c\phi_4(n)$, we refer the reader to [51, 63, 91].

4.6.2 Case $k = 6$

Theorem 4.6.2. *We have*

$$\begin{aligned}
\mathrm{C\Phi}_6(q) =& \frac{4}{9}\frac{(q;q)_\infty^5 (q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5 (q^3;q^3)_\infty^3} - \frac{1}{3}\frac{(q^2;q^2)_\infty^4 (q^4;q^4)_\infty^2}{(q;q)_\infty^4 (q^6;q^6)_\infty^3} + \frac{8}{9}\frac{(q^4;q^4)_\infty^{11}}{(q;q)_\infty^4 (q^2;q^2)_\infty^5 (q^{12};q^{12})_\infty^3} \\
&+ 36q\frac{(q^4;q^4)_\infty^2 (q^3;q^3)_\infty^9}{(q;q)_\infty^7 (q^2;q^2)_\infty^5} + 27q^2\frac{(q^4;q^4)_\infty^2 (q^6;q^6)_\infty^9}{(q;q)_\infty^4 (q^2;q^2)_\infty^8} \\
&+ 72q^4\frac{(q^{12};q^{12})_\infty^9}{(q;q)_\infty^4 (q^2;q^2)_\infty^5 (q^4;q^4)_\infty}. \qquad (4.6.3)
\end{aligned}$$

Proof. Let $k = 6$ in Theorem 4.2.1. We deduce that $\Theta_3(q)\mathfrak{A}_6(q) \in M_3\big(\Gamma_0(12), \big(\frac{-12}{\cdot}\big)\big)$. From [69, Theorem 1.34], we deduce that

$$\dim M_3\Big(\Gamma_0(12), \big(\frac{-12}{\cdot}\big)\Big) = 7.$$

Let

$$B_{6,1} = \frac{\eta^9(\tau)}{\eta^3(3\tau)}, \quad B_{6,2} = \frac{\eta^9(2\tau)}{\eta^3(6\tau)}, \quad B_{6,3} = \frac{\eta^9(4\tau)}{\eta^3(12\tau)}, \quad B_{6,4} = \frac{\eta^9(3\tau)}{\eta^3(\tau)},$$
$$B_{6,5} = \frac{\eta^9(6\tau)}{\eta^3(2\tau)}, \quad B_{6,6} = \frac{\eta^9(12\tau)}{\eta^3(4\tau)} \quad \text{and} \quad B_{6,7} = \eta^3(2\tau)\eta^3(6\tau).$$

The set $\{B_{6,j} | 1 \le j \le 7\}$ forms a basis of $M_3\big(\Gamma_0(12), \big(\frac{-12}{\cdot}\big)\big)$ and by comparing the Fourier coefficients of $\Theta_3(q)\mathfrak{A}_6(q)$ and modular forms in $\{B_{6,j} | 1 \le j \le 7\}$, we deduce that

$$\Theta_3(q)\mathfrak{A}_6(z) = \frac{4}{9}B_{6,1} - \frac{1}{3}B_{6,2} + \frac{8}{9}B_{6,3} + 36B_{6,4} + 27B_{6,5} + 72B_{6,6}. \qquad (4.6.4)$$

This proves (4.6.3). □

Congruences for $c\phi_6(n)$ have drawn much attention in recent years. For example, Baruah and Sarmah [24] established 3-dissections of $\mathrm{C}\Phi_6(q)$ and proved that

$$c\phi_6(3n+1) \equiv 0 \pmod 9 \tag{4.6.5}$$

and

$$c\phi_6(3n+2) \equiv 0 \pmod 9. \tag{4.6.6}$$

We remark here that the congruences above follow directly from (4.5.25) with $(p, \alpha, N) = (3, 1, 2)$. Moreover, setting $(p, \alpha, N) = (2, 1, 3)$ in (4.5.25), we deduce that

$$c\phi_6(2n+1) \equiv 0 \pmod 4. \tag{4.6.7}$$

Congruence (4.6.7) appears to be new.

For more congruences satisfied by $c\phi_6(n)$, see a recent paper of C. Gu, L. Wang and E.X.W. Xia [47] and their list of references.

4.6.3 Case $k = 8$

Theorem 4.6.3. *We have*

$$\begin{aligned}\mathrm{C}\Phi_8(q) = \frac{1}{(q;q)_\infty^8}\Big(&\Theta_3^7(q^4) + 28\Theta_3^6(q^4)\Theta_2(q^4) + 105\Theta_3^5(q^4)\Theta_2^2(q^4) \\ &+ 112\Theta_3^4(q^4)\Theta_2^3(q^4) + 147\Theta_3^3(q^4)\Theta_2^4(q^4) + 84\Theta_3^2(q^4)\Theta_2^5(q^4) + 35\Theta_3(q^4)\Theta_2^6(q^4)\Big).\end{aligned} \tag{4.6.8}$$

Proof. Let $k = 8$ in Theorem 4.2.1. We deduce that $\Theta_3(q)\mathfrak{A}_8(q) \in M_4(\Gamma_0(16))$. From [69, Theorem 1.34], we find that

$$\dim M_4(\Gamma_0(16)) = 9$$

and one can verify that

$B_{8,1} = \Theta_3(q)\Theta_3^7(q^4)$, $\quad B_{8,2} = \Theta_3(q)\Theta_3^6(q^4)\Theta_2(q^4)$, $\quad B_{8,3} = \Theta_3(q)\Theta_3^5(q^4)\Theta_2^2(q^4)$,

$$B_{8,4}=\Theta_3(q)\Theta_3^4(q^4)\Theta_2^3(q^4),\quad B_{8,5}=\Theta_3(q)\Theta_3^3(q^4)\Theta_2^4(q^4),\quad B_{8,6}=\Theta_3(q)\Theta_3^2(q^4)\Theta_2^5(q^4),$$
$$B_{8,7}=\Theta_3(q)\Theta_2^7(q^4),\qquad B_{8,8}=\Theta_2^8(q^4),\qquad \text{and}\qquad B_{8,9}=\Theta_3(q)\Theta_3(q^4)\Theta_2^6(q^4).$$

form a basis for $M_4(\Gamma_0(16))$. By comparing the Fourier coefficients of the basis and those of $\Theta_3(q)\mathfrak{A}_8(q)$, we find that

$$\Theta_3(q)\mathfrak{A}_8(q)=B_{8,1}+28B_{8,2}+105B_{8,3}+112B_{8,4}+147B_{8,5}+84B_{8,6}+35B_{8,9}. \tag{4.6.9}$$

This completes the proof of (4.6.8). □

By (4.5.27), we find that

$$c\phi_8(n)\equiv c\phi_4(n/2)\pmod{64}. \tag{4.6.10}$$

In [23], Baruah and Sarmah proved that

$$c\phi_4(2n+1)\equiv 0\pmod{4^2}, \tag{4.6.11}$$
$$c\phi_4(4n+2)\equiv 0\pmod{4}, \tag{4.6.12}$$

and

$$c\phi_4(4n+3)\equiv 0\pmod{4^4}. \tag{4.6.13}$$

Combining (4.6.11)–(4.6.13) with (4.6.10), we obtain the following congruences for $c\phi_8(n)$:

Theorem 4.6.4. *For any integer $n\geq 0$,*

$$c\phi_8(2n+1)\equiv 0\pmod{64}, \tag{4.6.14}$$
$$c\phi_8(4n+2)\equiv 0\pmod{16}, \tag{4.6.15}$$
$$c\phi_8(8n+4)\equiv 0\pmod{4} \tag{4.6.16}$$

and

$$c\phi_8(8n+6)\equiv 0\pmod{64}. \tag{4.6.17}$$

4.6.4 Case $k = 10$

By Theorem 4.2.1, we have $\Theta_3(q)\mathfrak{A}_{10}(q) \in M_5\big(\Gamma_0(20), (\frac{-20}{\cdot})\big)$. From [69, Theorem 1.34], we deduce that

$$\dim M_5\big(\Gamma_0(20), (\frac{-20}{\cdot})\big) = 14.$$

Let

$$B_{10,1} = \Theta_3^9(q)\Theta_3(q^5), \qquad B_{10,2} = \Theta_3(q)\Theta_2^3(q^{1/2})\Theta_2^3(q)\Theta_2^3(q^{5/2}),$$

$$B_{10,3} = \Theta_3(q)\Theta_3^3(q^5)\Theta_2^2(q^{1/2})\Theta_2^2(q)\Theta_2^2(q^{5/2}), \quad B_{10,4} = \Theta_3(q)\Theta_3(q^5)\Theta_2^2(q^{1/2})\Theta_2^6(q^{5/2}),$$

$$B_{10,5} = \Theta_3(q)\Theta_3(q^5)\Theta_2^8(q), \qquad B_{10,6} = \Theta_3^7(q)\Theta_3^3(q^5),$$

$$B_{10,7} = \Theta_3^6(q^5)\Theta_2^3(q^{1/2})\Theta_2(q^{5/2}), \qquad B_{10,8} = \Theta_3(q)\Theta_3^5(q^5)\Theta_2^4(q),$$

$$B_{10,9} = \Theta_3^3(q^5)\Theta_2(q^{1/2})\Theta_2^5(q)\Theta_2(q^{5/2}), \qquad B_{10,10} = \Theta_3^6(q)\Theta_2^3(q)\Theta_2(q^5),$$

$$B_{10,11} = \Theta_3(q)\Theta_3(q^5)\Theta_2^8(q^{1/2}), \qquad B_{10,12} = \Theta_3(q)\Theta_3(q^5)\Theta_2^6(q)\Theta_2^2(q^5),$$

$$B_{10,13} = \Theta_3^5(q)\Theta_3^5(q^5), \quad \text{and} \quad B_{10,14} = \Theta_3^3(q)\Theta_3^3(q^5)\Theta_2^3(q^{1/2})\Theta_2(q^{5/2}).$$

The set $\{B_{10,j} | 1 \le j \le 14\}$ forms a basis of $M_5\big(\Gamma_0(20), (\frac{-20}{\cdot})\big)$ and we deduce the following:

Theorem 4.6.5. *We have*

$$\mathrm{C}\Phi_{10}(q) = \frac{1}{\Theta_3(q)(q;q)_\infty^{10}}\Big(\frac{13}{8}B_{10,1} + \frac{435}{32}B_{10,2} + \frac{9275}{128}B_{10,3} + \frac{175}{32}B_{10,4} - \frac{31}{8}B_{10,5} - \frac{15}{4}B_{10,6} + \frac{225}{4}B_{10,7} - \frac{775}{32}B_{10,8} + \frac{221}{32}B_{10,10} - \frac{857}{512}B_{10,11} + \frac{155}{8}B_{10,12} + \frac{25}{8}B_{10,13}\Big). \tag{4.6.18}$$

Let

$$f_{10} = \frac{\eta(\tau)\eta(2\tau)\eta(10\tau)\eta(20\tau)}{\eta(4\tau)\eta(5\tau)}$$

$$f_{10,1} = \Theta_3(q)\Theta_3(q^5)\frac{\eta^{10}(\tau)}{\eta^2(5\tau)}$$

and

$$f_{10,2} = \Theta_3^6(q^5)\frac{\eta^5(20\tau)}{\eta(4\tau)}.$$

Let

$$\begin{aligned}
g_{10,1} &= \frac{1}{6}\left(E_2(\tau) - 4E_2(2\tau) + 4E_2(4\tau) + 5E_2(5\tau) - 20E_2(10\tau) + 20E_2(20\tau)\right),\\
g_{10,2} &= \Theta_3^2(q)\Theta_3^2(q^5),\\
g_{10,3} &= \frac{1}{4}\left(-E_2(2\tau) + 5E_2(10\tau)\right),\\
g_{10,4} &= -\frac{1}{24}\left(E_2(\tau) + E_2(2\tau) + 4E_2(4\tau) - 5E_2(5\tau) - 5E_2(10\tau) - 20E_2(20\tau)\right),\\
g_{10,5} &= \eta^2(2\tau)\eta^2(10\tau)
\end{aligned}$$

and

$$g_{10,6} = \frac{5}{4}\Theta_3^4(q^5) - \frac{1}{4}\Theta_3^4(q).$$

Let

$$\begin{aligned}
B_{10,1}^* &= f_{10}g_{10,1}^2, & B_{10,2}^* &= f_{10}g_{10,1}g_{10,2},\\
B_{10,3}^* &= f_{10}g_{10,2}^2, & B_{10,4}^* &= f_{10}g_{10,1}g_{10,3},\\
B_{10,5}^* &= f_{10}g_{10,1}g_{10,4}, & B_{10,6}^* &= f_{10}g_{10,1}g_{10,5},\\
B_{10,7}^* &= f_{10}g_{10,1}g_{10,6}, & B_{10,8}^* &= f_{10}g_{10,2}g_{10,3},\\
B_{10,9}^* &= f_{10}g_{10,2}g_{10,4}, & B_{10,10}^* &= f_{10}g_{10,2}g_{10,5},\\
B_{10,11}^* &= f_{10}g_{10,2}g_{10,6}, & B_{10,12}^* &= f_{10}g_{10,3}^2,\\
B_{10,13}^* &= f_{10,1}, \quad \text{and} & B_{10,14}^* &= f_{10,2}.
\end{aligned}$$

We can replace the basis $\{B_{10,j} | 1 \le j \le 14\}$ by the basis $\{B_{10,j}^* | 1 \le j \le 14\}$ and deduce that

$$\begin{aligned}
\mathrm{C}\Phi_{10}(q) =& \frac{1}{\Theta_3(q)(q;q)_\infty^{10}}\Big(\frac{5075}{2}B_{10,1}^* + \frac{4525}{4}B_{10,2}^* + \frac{29375}{4}B_{10,3}^* + \frac{4525}{2}B_{10,4}^* - 4525B_{10,5}^*\\
&- 6525B_{10,6}^* + \frac{6275}{4}B_{10,7}^* - 4950B_{10,8}^* + 2300B_{10,9}^* - 22375B_{10,10}^*\\
&+ \frac{10325}{4}B_{10,11}^* - 10150B_{10,12}^* + B_{10,13}^* + 200000B_{10,14}^*\Big). \qquad (4.6.19)
\end{aligned}$$

Identity (4.6.19) leads immediately to

$$\mathrm{C}\Phi_{10}(q) \equiv \frac{\Theta_3(q^5)}{(q^5;q^5)_\infty^2} \pmod{5^2}. \qquad (4.6.20)$$

Remark 4.6.2. Congruence (4.6.20) is the motivation behind the discovery of Theorem 4.5.3. Interpreting the congruence (4.5.24) with $(p, \alpha, N) = (p, 1, \ell)$ in terms of generating functions, we obtain the congruence

$$\mathrm{C}\Phi_{p\ell}(q) \equiv \mathrm{C}\Phi_\ell(q^p) \pmod{p^2} \tag{4.6.21}$$

for any distinct primes p and ℓ. Congruence (4.6.20) is a special case of (4.6.21) once we identify the right hand side of (4.6.20) with $\mathrm{C}\Phi_2(q^5)$ (see (4.3.1)).

Theorem 4.6.6. *For any integer $n \geq 0$, we have*

$$c\phi_{10}(2n+1) \equiv 0 \pmod{4}, \tag{4.6.22}$$

$$c\phi_{10}(5n+r) \equiv 0 \pmod{5^2}, \quad 1 \leq r \leq 4 \tag{4.6.23}$$

and

$$c\phi_{10}(25n+15) \equiv 0 \pmod{5}. \tag{4.6.24}$$

Proof. Congruences (4.6.22) and (4.6.23) follow from Theorem 4.5.3 by setting $(p, \alpha, N) = (2, 1, 5)$ and $(5, 1, 2)$, respectively. Congruence (4.6.23) also follows from (4.6.20). Furthermore, from (4.6.20), we deduce that

$$\sum_{n=0}^{\infty} c\phi_{10}(5n) q^n \equiv \frac{\Theta_3(q)}{(q;q)_\infty^2} \equiv \frac{\Theta_3(q)(q;q)_\infty^3}{(q;q)_\infty^5} \pmod{5} \tag{4.6.25}$$

$$\equiv \frac{1}{(q^5;q^5)_\infty} \left(\sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{i^2+j(j+1)/2} \right) \pmod{5}.$$

Note that

$$n = i^2 + \frac{j(j+1)}{2} \text{ if and only if } 8n+1 = 8i^2 + (2j+1)^2.$$

Since $\left(\frac{-8}{5}\right) = -1$, we find that $8n+1 \equiv 0 \pmod{5}$ if and only if $i \equiv 2j+1 \equiv 0 \pmod{5}$. We also observe that $8n+1 \equiv 0 \pmod{5}$ if and only if $n \equiv 3 \pmod{5}$. Hence, by (4.6.25), we deduce that

$$c\phi_{10}(5(5n+3)) \equiv 0 \pmod{5}.$$

□

Remark 4.6.3. One can prove (4.6.24) by first observing that (4.6.20) implies

$$c\phi_{10}(5n) \equiv c\phi_2(n) \pmod{5^2}.$$

Using (4.3.20), we deduce (4.6.24).

4.6.5 Case $k = 12$

By Theorem 4.2.1, we have $\Theta_3(q)\mathfrak{A}_{12}(q) \in M_6\big(\Gamma_0(24), (\frac{24}{\cdot})\big)$. By [69, Theorem 1.34], we deduce that

$$\dim M_6\big(\Gamma_0(24), (\frac{24}{\cdot})\big) = 22.$$

Let

$B_{12,1} = \Theta_3^3(q)\Theta_3^9(q^6)$, $B_{12,2} = \Theta_3^{11}(q)\Theta_3(q^6)$,

$B_{12,3} = \Theta_3(q^2)\Theta_3(q^3)\Theta_2^{10}(q^2)$, $B_{12,4} = \Theta_3^2(q)\Theta_3(q^2)\Theta_3(q^3)\Theta_2^4(q)\Theta_2^4(q^3)$,

$B_{12,5} = \Theta_3^2(q)\Theta_3(q^2)\Theta_3^5(q^3)\Theta_2^2(q)\Theta_2^2(q^3)$, $B_{12,6} = \Theta_3^2(q)\Theta_3(q^2)\Theta_3^9(q^3)$,

$B_{12,7} = \Theta_3(q)\Theta_3^3(q^6)\Theta_2^4(q^2)\Theta_2^4(q^6)$, $B_{12,8} = \Theta_3(q)\Theta_3^5(q^2)\Theta_2^3(q^2)\Theta_2^3(q^6)$,

$B_{12,9} = \Theta_3(q)\Theta_3(q^2)\Theta_2^5(q^2)\Theta_2^5(q^6)$, $B_{12,10} = \Theta_3(q)\Theta_3^5(q^2)\Theta_2^3(q)\Theta_2^3(q^3)$,

$B_{12,11} = \Theta_3(q)\Theta_3(q^2)\Theta_2^5(q)\Theta_2^5(q^3)$, $B_{12,12} = \Theta_3(q)\Theta_3^7(q^6)\Theta_2^2(q)\Theta_2^2(q^3)$,

$B_{12,13} = \Theta_3(q)\Theta_3^3(q^6)\Theta_2^4(q)\Theta_2^4(q^3)$, $B_{12,14} = \Theta_3(q)\Theta_3^5(q^2)\Theta_2(q)\Theta_2^2(q^2)\Theta_2(q^3)\Theta_2^2(q^6)$,

$B_{12,15} = \Theta_3(q)\Theta_3(q^2)\Theta_2(q)\Theta_2^8(q^2)\Theta_2(q^3)$, $B_{12,16} = \Theta_3(q)\Theta_3^3(q^2)\Theta_2(q)\Theta_2^6(q^2)\Theta_2(q^3)$,

$B_{12,17} = \Theta_3(q)\Theta_3^5(q^2)\Theta_2(q)\Theta_2^4(q^2)\Theta_2(q^3)$, $B_{12,18} = \Theta_3(q)\Theta_3^9(q^2)\Theta_2(q)\Theta_2(q^3)$,

$B_{12,19} = \Theta_3(q)\Theta_3^2(q^2)\Theta_3(q^6)\Theta_2^8(q^2)$, $B_{12,20} = \Theta_3(q)\Theta_3^6(q^2)\Theta_3(q^6)\Theta_2^4(q^2)$,

$B_{12,21} = \Theta_3(q)\Theta_3^8(q^2)\Theta_3(q^6)\Theta_2^2(q^2)$, and $B_{12,22} = \Theta_3(q)\Theta_3^{10}(q^2)\Theta_3(q^6)$.

The set $\{B_{12,j} | 1 \le j \le 22\}$ forms a basis of $M_6(\Gamma_0(24), (\frac{24}{\cdot}))$. Using the above basis, we deduce the following identity:

Theorem 4.6.7. *We have*

$$\begin{aligned}\mathrm{C}\Phi_{12}(q) =& \frac{1}{\Theta_3(q)(q;q)_\infty^{12}}\Big(-\frac{36207}{160}B_{12,1} + \frac{923091}{4000}B_{12,4} + \frac{35829}{100}B_{12,5} + \frac{891}{4}B_{12,6} \\ & - \frac{1485}{8}B_{12,7} - \frac{143247}{1000}B_{12,8} - \frac{891}{4}B_{12,9} - \frac{8109}{160}B_{12,10}\end{aligned}$$

$$- \frac{582717}{4000}B_{12,11} + \frac{227691}{200}B_{12,12} + \frac{714249}{8000}B_{12,13} + \frac{8109}{80}B_{12,14}$$
$$+ \frac{33}{8}B_{12,15} + \frac{1179561}{4000}B_{12,16} - \frac{16503}{400}B_{12,17} - \frac{99}{8}B_{12,18} + \frac{10559}{200}B_{12,19}$$
$$- \frac{128807}{100}B_{12,20} + \frac{25647}{160}B_{12,21} + \frac{727}{160}B_{12,22}\Big). \quad (4.6.26)$$

Next, we give some congruences satisfied by $c\phi_{12}(n)$.

Theorem 4.6.8. *We have*

$$c\phi_{12}(2n+1) \equiv 0 \pmod{16}, \quad (4.6.27)$$
$$c\phi_{12}(3n+1) \equiv 0 \pmod{9} \quad (4.6.28)$$

and

$$c\phi_{12}(3n+2) \equiv 0 \pmod{9}. \quad (4.6.29)$$

Proof. This follows directly from Theorem 4.5.3 by setting $(p, \alpha, N) = (2, 2, 3)$ and $(3, 1, 4)$. □

4.6.6 Case $k = 14$

By Theorem 4.2.1, we know $\Theta_3(q)\mathfrak{A}_{14}(q) \in M_7\big(\Gamma_0(28), (\frac{-28}{\cdot})\big)$. By [69, Theorem 1.34], we deduce that
$$\dim M_7\big(\Gamma_0(28), (\frac{-28}{\cdot})\big) = 27.$$

Let

$B_{14,1} = \Theta_3^{13}(q)\Theta_3(q^7)$, $\quad B_{14,2} = \Theta_3^{7}(q)\Theta_3^{3}(q^7)\Theta_2^{2}(q^{1/2})\Theta_2^{2}(q^{7/2})$,

$B_{14,3} = \Theta_3^{5}(q)\Theta_3(q^7)\Theta_2^{4}(q^{1/2})\Theta_2^{4}(q^{7/2})$, $\quad B_{14,4} = \Theta_3(q)\Theta_3^{5}(q^7)\Theta_2^{4}(q^{1/2})\Theta_2^{4}(q^{7/2})$,

$B_{14,5} = \Theta_3^{5}(q)\Theta_3^{3}(q^7)\Theta_2^{2}(q)\Theta_2^{4}(q^{7/2})$, $\quad B_{14,6} = \Theta_3^{3}(q)\Theta_3^{7}(q^7)\Theta_2^{2}(q^{1/2})\Theta_2^{2}(q^{7/2})$,

$B_{14,7} = \Theta_3^{11}(q)\Theta_3^{3}(q^7)$, $\quad B_{14,8} = \Theta_3^{12}(q)\Theta_2(q^{1/2})\Theta_2(q^{7/2})$,

$B_{14,9} = \Theta_3^{8}(q)\Theta_2^{3}(q^{1/2})\Theta_2^{3}(q^{7/2})$, $\quad B_{14,10} = \Theta_3^{4}(q)\Theta_2^{5}(q^{1/2})\Theta_2^{5}(q^{7/2})$,

$B_{14,11} = \Theta_3(q)\Theta_3(q^5)\Theta_2^{8}(q^{1/2})$, $\quad B_{14,12} = \Theta_3^{12}(q)\Theta_2(q)\Theta_2(q^7)$,

$B_{14,13} = \Theta_3^8(q)\Theta_2^3(q)\Theta_2^3(q^7)$, $B_{14,14} = \Theta_3^4(q)\Theta_2^5(q)\Theta_2^5(q^7)$,

$B_{14,15} = \Theta_2^7(q)\Theta_2^7(q^7)$, $B_{14,16} = \Theta_3^8(q)\Theta_3(q^7)\Theta_2^2(q^{1/2})\Theta_2^3(q)$,

$B_{14,17} = \Theta_3^6(q)\Theta_3^3(q^7)\Theta_2^2(q^{1/2})\Theta_2^3(q)$, $B_{14,18} = \Theta_3^8(q)\Theta_2(q)\Theta_2^4(q^2)\Theta_2(q^7)$,

$B_{14,19} = \Theta_3^2(q)\Theta_3^7(q^7)\Theta_2^2(q^{1/2})\Theta_2^3(q)$, $B_{14,20} = \Theta_3^9(q^7)\Theta_2^2(q^{1/2})\Theta_2^3(q)$,

$B_{14,21} = \Theta_3^{10}(q)\Theta_3(q^7)\Theta_2(q)\Theta_2^2(q^{7/2})$, $B_{14,22} = \Theta_3^4(q)\Theta_3(q^7)\Theta_2^3(q)\Theta_2^6(q^{7/2})$,

$B_{14,23} = \Theta_3^2(q)\Theta_3^3(q^7)\Theta_2^3(q)\Theta_2^6(q^{7/2})$, $B_{14,24} = \Theta_3^5(q^7)\Theta_2^3(q)\Theta_2^6(q^{7/2})$,

$B_{14,25} = \Theta_3^{11}(q)\Theta_2(q^{1/2})\Theta_2(q)\Theta_2^3(q^{7/2})$, $B_{14,26} = \Theta_3^2(q)\Theta_3^4(q^7)\Theta_2(q^{1/2})\Theta_2^2(q)\Theta_2^5(q^{7/2})$,

and

$B_{14,27} = \Theta_3^4(q)\Theta_3^2(q^7)\Theta_2(q^{1/2})\Theta_2^2(q)\Theta_2^5(q^{7/2})$.

The set $\{B_{14,j} | 1 \le j \le 27\}$ forms a basis of $M_7\big(\Gamma_0(28), (\frac{-28}{\cdot})\big)$. This basis allows us to derive the following identity:

Theorem 4.6.9. *We have*

$$\begin{aligned}\mathrm{C}\Phi_{14}(q) =& \frac{1}{\Theta_3(q)(q;q)_\infty^{14}}\Big(-\frac{3}{4}B_{14,1} - \frac{332339}{1024}B_{14,2} + \frac{255927}{4096}B_{14,3} - \frac{197519}{4096}B_{14,4} \\ &+ \frac{17325}{64}B_{14,5} + \frac{1407329}{2048}B_{14,6} + \frac{7}{4}B_{14,7} + \frac{3}{4}B_{14,8} - \frac{13765}{256}B_{14,9} \\ &- \frac{52045}{1024}B_{14,10} + \frac{3861}{512}B_{14,12} + \frac{429}{16}B_{14,13} + \frac{6623}{64}B_{14,16} - \frac{79799}{512}B_{14,17} \\ &+ \frac{29407}{512}B_{14,19} - \frac{3989}{64}B_{14,21} + \frac{19803}{128}B_{14,22} - \frac{16807}{256}B_{14,23} \\ &+ \frac{50421}{256}B_{14,26} - \frac{6895}{256}B_{14,27}\Big). \end{aligned} \tag{4.6.30}$$

By setting $(p, \alpha, N) = (7, 1, 2)$ in (4.5.24), we get

$$c\phi_{14}(n) \equiv c\phi_2(n/7) \pmod{49}, \tag{4.6.31}$$

By (4.5.25), we deduce that

$$c\phi_{14}(7n + r) \equiv 0 \pmod{49}, \quad 1 \le r \le 6. \tag{4.6.32}$$

Moreover, setting $(p, \alpha, N) = (2, 1, 7)$ in (4.5.25), we deduce that

$$c\phi_{14}(2n + 1) \equiv 0 \pmod{4}. \tag{4.6.33}$$

4.6.7 Case $k = 16$

By Theorem 4.2.1, we know $\Theta_3(q)\mathfrak{A}_{16}(q) \in M_8\big(\Gamma_0(32), (\frac{-2}{\cdot})\big)$. By [69, Theorem 1.34], we deduce that

$$\dim M_8\big(\Gamma_0(32), (\frac{-2}{\cdot})\big) = 32.$$

Let

$$B_{16,j} = \Theta_3(q)\Theta_3^{j-1}(q^2)\Theta_2^{16-j}(q^8), \quad 1 \le j \le 15,$$
$$B_{16,j} = \Theta_3^3(q)\Theta_3^{j-16}(q^2)\Theta_2^{29-j}(q^8), \quad 16 \le j \le 29,$$
$$B_{16,30} = \Theta_3^5(q)\Theta_3^2(q^2)\Theta_3^9(q^8),$$
$$B_{16,31} = \Theta_3^9(q)\Theta_3(q^4)\Theta_3^6(q^8),$$

and

$$B_{16,32} = \Theta_3^3(q)\Theta_3(q^4)\Theta_3(q^8)\Theta_2^{11}(q^8).$$

The set $\{B_{16,j} | 1 \le j \le 32\}$ forms a basis of $M_8\big(\Gamma_0(32), (\frac{-2}{\cdot})\big)$. Hence, we deduce the following identity:

Theorem 4.6.10. *We have*

$$\begin{aligned}
\mathrm{C}\Phi_{16}(q) =& \frac{1}{\Theta_3(q)(q;q)_\infty^{15}}\Big(- 16384B_{16,1} + 122880B_{16,2} - 431024B_{16,3} \\
&+ 10384B_{16,4} + 3956568B_{16,5} - 12663584B_{16,6} \\
&+ 21477101B_{16,7} - 23125005B_{16,8} + 15986724B_{16,9} \\
&- 6153988B_{16,10} + 108966B_{16,11} + 1259002B_{16,12} - 678464B_{16,13} \\
&+ 162042B_{16,14} - 15218B_{16,15} + 61440B_{16,18} - 337920B_{16,19} \\
&+ 844918B_{16,20} - 870438B_{16,21} - 327528B_{16,22} + 122540544B_{16,23} \\
&- 2366700B_{16,24} + 1511404B_{16,25} - 484664B_{16,26} + 34128B_{16,27} \\
&+ 20722B_{16,28} - 58B_{16,29} + 59B_{16,30}\Big). \qquad (4.6.34)
\end{aligned}$$

By Theorem 4.5.3, we obtain

$$c\phi_{16}(2n+1) \equiv 0 \pmod{256} \qquad (4.6.35)$$

and

$$c\phi_{16}(2n) \equiv c\phi_8(n) \pmod{256}. \tag{4.6.36}$$

4.7 Möbius inversion and Kolitsch's congruence (4.5.22)

In this section, we will use a different notation for k-colored generalized Frobenius symbol λ. The color of a part will be placed on the left hand side of the part. In other words, our symbol λ is now written as

$$\lambda = \begin{pmatrix} c_1(z_1) & c_2(z_2) & \cdots & c_d(z_d) \\ c'_1(z'_1) & c'_2(z'_2) & \cdots & c'_d(z'_d) \end{pmatrix}, \tag{4.7.1}$$

where c_j and c'_j denote colors from the set $\{1, 2, \cdots, k\}$ and z_j, z'_j denote the parts. For example, the 2-colored generalized Frobenius symbol

$$\lambda_0 := \begin{pmatrix} 2_2 & 2_1 \\ 1_2 & 0_1 \end{pmatrix} \tag{4.7.2}$$

is now written as

$$\lambda_0 = \begin{pmatrix} 2(2) & 1(2) \\ 2(1) & 1(0) \end{pmatrix}.$$

Let σ_k be the k-cycle $(1 \quad 2 \quad \cdots \quad k)$. Let the symbol

$$\begin{pmatrix} \cdots\cdots \\ \cdots\cdots \end{pmatrix}^{\text{sort}}$$

denote sorting the resulting rows to be strictly decreasing according to (4.1.1). We say that λ has order ℓ with respect to σ_k if ℓ is the smallest positive integer for which the equality of the following symbols holds:

$$\begin{pmatrix} c_1(z_1) & c_2(z_2) & \cdots & c_d(z_d) \\ c'_1(z'_1) & c'_2(z'_2) & \cdots & c'_d(z'_d) \end{pmatrix} = \begin{pmatrix} \sigma_k^\ell(c_1)(z_1) & \sigma_k^\ell(c_2)(z_2) & \cdots & \sigma_k^\ell(c_d)(z_d) \\ \sigma_k^\ell(c'_1)(z'_1) & \sigma_k^\ell(c'_2)(z'_2) & \cdots & \sigma_k^\ell(c'_d)(z'_d) \end{pmatrix}^{\text{sort}}.$$

For example, applying the 2-cycle $\sigma_2 = (1 \quad 2)$ and the sorting process to λ_0 given in (4.7.2), we get

$$\begin{pmatrix} \sigma_2(2)(2) & \sigma_2(1)(2) \\ \sigma_2(2)(1) & \sigma_2(1)(0) \end{pmatrix}^{\text{sort}} = \begin{pmatrix} 1(2) & 2(2) \\ 1(1) & 2(0) \end{pmatrix}^{\text{sort}} = \begin{pmatrix} 2(2) & 1(2) \\ 1(1) & 2(0) \end{pmatrix}.$$

Applying σ_2 and sorting the resulting rows again, we arrive at

$$\begin{pmatrix} \sigma_2(2)(2) & \sigma_2(1)(2) \\ \sigma_2(1)(1) & \sigma_2(2)(0) \end{pmatrix}^{\text{sort}} = \begin{pmatrix} 1(2) & 2(2) \\ 2(1) & 1(0) \end{pmatrix}^{\text{sort}} = \begin{pmatrix} 2(2) & 1(2) \\ 2(1) & 1(0) \end{pmatrix}.$$

Thus we get back to λ_0 again. This means that λ_0 has order 2 with respect to σ_2.

Let $\Psi_{k,\ell}(n)$ be the number of k-colored generalized Frobenius symbols of n that have order ℓ. When $\ell = k$, we follow Kolitsch and denote $\Psi_{k,k}(n)$ by $\overline{c\phi}_k(n)$. The function $\overline{c\phi}_k(n)$ is implicitly mentioned by Kolitsch in [56] and the following identity was later given by him in [57, p. 220]:

Theorem 4.7.1. *Let k and n be positive integers. Then*

$$\overline{c\phi}_k(n) = \sum_{\ell|(k,n)} \mu(\ell) c\phi_{\frac{k}{\ell}}\left(\frac{n}{\ell}\right). \tag{4.7.3}$$

With (4.7.3), (4.5.22) can be written as

$$\overline{c\phi}_k(n) \equiv 0 \pmod{k^2}. \tag{4.7.4}$$

Congruence (4.7.4) provides an elegant analogue of Andrews' original congruence (4.3.32), which states that

$$c\phi_p(n) \equiv 0 \pmod{p^2}$$

for primes p not dividing n. Using the definition of $\overline{c\phi}_k(n)$, we can rewrite (4.1.13), (4.1.14) and (4.1.15) [60, Theorem 3] as

$$\overline{c\phi}_5(n) = 5p(5n-1), \overline{c\phi}_7(n) = 7p(7n-2) \text{ and } \overline{c\phi}_{11}(n) = 11p(11n-5)$$

where n is any positive integer.

In this section, we prove the following:

Theorem 4.7.2. *Let k and n be positive integers. Then*

$$c\phi_k(n) = \sum_{\ell|k} \overline{c\phi}_\ell\left(\frac{n}{(k/\ell)}\right) = \sum_{\ell|k} \overline{c\phi}_{k/\ell}\left(\frac{n}{\ell}\right), \tag{4.7.5}$$

where we agree that for any integer $m \geq 1$, $\overline{c\phi}_m(x) = 0$ if x is not an integer.

We then establish (4.7.3) using Theorem 4.7.2. We will also take this opportunity to present Kolitsch's proof of (4.7.4) (see Theorem 4.7.4). Our presentation of Kolitsch's proof contains more details than that given in [59]. We feel that it is important for us (and perhaps the reader) to fully understand Koltisch's proof as it is an important congruence and that it is essential in our proof of Theorem 4.5.3.

We now begin our proof of Theorem 4.7.2.

Proof of Theorem 4.7.2. Every k-colored generalized Frobenius symbol has an order ℓ with respect to σ_k. We first show that the order of a k-colored generalized Frobenius symbol λ must divide k. Suppose not. Let $m = ds$ be the order of λ with $d = (m, k)$ and $s > 1$. Observe that σ_k^d splits into a product of d disjoint cycles C_j, $1 \leq j \leq d$, of length k/d. Since $(s, k/d) = 1$, $\left(\sigma_k^d\right)^s$ is again a product of d disjoint cycles C'_j, $1 \leq j \leq d$, and the integers in C_j^s are the same as those in C_j. Hence, if σ_k^m leaves λ invariant, it would have been left invariant under σ_k^d but this contradicts the minimality of m. Therefore, the order of λ must be a divisor of k and we deduce that

$$c\phi_k(n) = \sum_{\ell|k} \Psi_{k,\ell}(n).$$

To prove (4.7.5), it suffices to show that

$$\Psi_{k,\ell}(n) = \overline{c\phi}_\ell\left(\frac{n}{(k/\ell)}\right). \tag{4.7.6}$$

For $\ell|k$, we know that σ_k^ℓ splits into ℓ disjoint cycles C_j, $j = 1, 2, \cdots, \ell$ of length k/ℓ. Now, if λ is a k-colored generalized Frobenius symbol of order ℓ, then it means that if an entry $c_\nu(z)$, with c_ν appears in C_j, appears in λ then $c_\mu(z)$ must appear in λ for every color c_μ that appears in the cycle C_j. We now replace all the colors in this cycle where c_ν belongs by the color represented by the smallest integer, which can be chosen to be less than ℓ. In this way, we will obtain a ℓ-colored generalized Frobenius symbol where each entry $c_j(z)$ appears k/ℓ times. In other words, from

$$n = d + \sum_{i=1}^{d} c_i(z_i) + \sum_{i=1}^{d} c_i'(z_i'),$$

we obtain a ℓ-colored generalized Frobenius symbol giving rise the partition

$$n = d + \frac{k}{\ell}\left(\sum_{i=1}^{d/(k/\ell)} c_{j_i}(z_i) + \sum_{i=1}^{d/(k/\ell)} c_{j_i}'(z_i')\right),$$

which implies that

$$\frac{n}{(k/\ell)} = \frac{d}{(k/\ell)} + \sum_{i=1}^{d/(k/\ell)} c_{j_i}(z_i) + \sum_{i=1}^{d/(k/\ell)} c_{j_i}'(z_i').$$

We have thus constructed from λ the ℓ-colored generalized Frobenius symbol of $n/(k/\ell)$, which we denote as λ^*. We claim that λ^* has order ℓ with respect to

$$\gamma = (1 \quad 2 \quad \cdots \quad \ell).$$

If λ^* is of order m less than ℓ, then this means that

$$\gamma^m = \prod_{j=1}^{m} C_j',$$

where each C_j' is a ℓ/m cycle, leaves λ^* invariant. Since $m < \ell$, at least two of the integers u and v between 1 and ℓ are in some cycle C_j'. When we reverse the above process of obtaining ℓ-colored generalized Frobenius symbol of $n/(k/\ell)$ from a k-colored generalized Frobenius symbol of n of order ℓ, we would obtain a symbol λ which is fixed by a cycle that includes both u and v. But u and v are in disjoint cycles in the decomposition of σ_k^ℓ and this contradicts the fact that λ has order ℓ. Hence, λ^* cannot have order strictly less than ℓ and its order must be ℓ.

Conversely, given a ℓ-colored generalized Frobenius symbol of $n/(k/\ell)$ of order ℓ with respect to γ, we reverse the process to obtain a k-colored generalized Frobenius symbol of n of order ℓ. Hence, we have (4.7.6) and the proof of Theorem 4.7.2 is complete. □

Theorem 4.7.1 now follows from Theorem 4.7.2 by using the following lemma with $F(n,k) = c\phi_k(n)$ and $G(n,k) = \overline{c\phi}_k(n)$:

Lemma 4.7.3. *Let $F(n,k)$ and $G(n,k)$ be two-variable arithmetical functions. Then*

$$F(n,k) = \sum_{\ell|(n,k)} G(n/\ell, k/\ell), \tag{4.7.7}$$

if and only if

$$G(n,k) = \sum_{\ell|(n,k)} \mu(\ell) F(n/\ell, k/\ell). \tag{4.7.8}$$

Proof. To prove (4.7.8), we set $n = dn'$ and $k = dk'$ where $d = (n,k)$. From (4.7.7), we have

$$F(n'd, k'd) = \sum_{\ell|d} G(n'd/\ell, k'd/\ell).$$

Using the Möbius inversion formula, we deduce that

$$G(n'd, k'd) = \sum_{\ell|d} \mu(\ell) F(n'd/\ell, k'd/\ell),$$

or

$$G(n,k) = \sum_{\ell|d} \mu(\ell) F(n/\ell, k/\ell).$$

The converse follows in a similar way from the Möbius inversion formula. □

Remark 4.7.1. We observe that using the above inversion, we can find an expression of Möbius function in terms of Ramanujan's sum $c_q(n)$. We will write Ramanujan's sum as $c(q,n)$. It is known that [15, Section 8.3]

$$c(q,n) = \sum_{\ell|(q,n)} \mu(q/\ell)\ell.$$

Now, we observe that

$$\frac{c(q,n)}{(q,n)} = \sum_{\ell|(q,n)} \mu(q/\ell) \frac{\ell}{(q,n)}.$$

Using the inversion formula with

$$F(q,n) = \frac{c(q,n)}{(q,n)} \quad \text{and} \quad G(q,n) = \frac{\mu(q)}{(q,n)},$$

we deduce that

$$\frac{\mu(q)}{(q,n)} = \sum_{\ell|(q,n)} \frac{c(q/\ell, n/\ell)}{(q,n)} \ell\mu(\ell),$$

or

$$\mu(q) = \sum_{\ell|(q,n)} c(q/\ell, n/\ell)\ell\mu(\ell).$$

We now restate (4.7.4) as the following theorem, which is due to Kolitsch.

Theorem 4.7.4. *Let k and n be positive integers. Then*

$$\overline{c\phi}_k(n) \equiv 0 \pmod{k^2}.$$

Proof of Theorem 4.7.4. Given a k-colored generalized Frobenius symbol λ represented by (4.7.1), we say that the color difference of λ is m when m is the sum of the numerical values of the colors on the first row minus the sum of the numerical values of the colors on the second row of λ. In other words,

$$m = c_1 + c_2 + \cdots + c_d - (c_1' + c_2' + \cdots + c_d').$$

Let $\overline{c\phi}_k(m,n)$ denote the number of k-colored generalized Frobenius symbol λ of n with color difference m and order k. Let $c\phi_k(m,n)$ denote the number of k-colored generalized Frobenius symbol λ of n with color difference m. These functions satisfy the following analogue of (4.7.5):

$$c\phi_k(m,n) = \sum_{\ell|k} \overline{c\phi}_\ell \left(\frac{m}{k/\ell}, \frac{n}{k/\ell} \right). \tag{4.7.9}$$

The proof of (4.7.9) is the same as (4.7.5) by checking that there is a one to one correspondence between a k-colored generalized Frobenius symbol of n with color difference m and order ℓ and a ℓ-colored generalized Frobenius symbol of $n/(k/\ell)$ with color difference $m/(k/\ell)$ and order ℓ. The only additional step we need to observe is that under our previous construction, when we replace the k-colored generalized Frobenius symbol λ of n with a k-colored generalized Frobenius symbol $\lambda^\dagger$ with only colors j with $1 \le j \le \ell$ (by identifying colors belong to the cycle containing j), the color difference of $\lambda^\dagger$ becomes $m/(k/\ell)$. This is because if a color j appears in λ, then the rest of the colors belonging to the cycle containing j are of the form $j + w\ell$, $1 \leq w < k/\ell$.

Using inversion formula similar to Lemma 4.7.3 with two-variable arithmetical functions replaced by three-variable arithmetical functions, we deduce from (4.7.9) that

$$\overline{c\phi}_k(m,n) = \sum_{\ell|k} \mu(\ell) c\phi_{k/\ell}\left(\frac{m}{\ell}, \frac{n}{\ell}\right). \tag{4.7.10}$$

Now, the function

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} c\phi_k(m,n) t^m q^n$$

is the constant term, i.e., coefficient of z^0 of the function

$$\prod_{j=1}^{k} (zt^j q; q)_\infty (z^{-1} t^{-j}; q)_\infty,$$

which we shall write as

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} c\phi_k(m,n) t^m q^n = \mathrm{CT}\left(\prod_{j=1}^{k} (zt^j q; q)_\infty (z^{-1} t^{-j}; q)_\infty\right). \tag{4.7.11}$$

See [4, pp. 4–6, Theorems 5.1 and 5.2] for examples of expressing generating functions of various partitions functions as constant term of infinite products involving z.

From (4.7.10) and (4.7.11), we deduce that

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \overline{c\phi}_k(m,n) t^m q^n &= \sum_{\ell|k} \mu(\ell) \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} c\phi_{k/\ell}\left(\frac{m}{\ell}, \frac{n}{\ell}\right) t^m q^n \\
&= \sum_{\ell|k} \mu(\ell) \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} c\phi_{k/\ell}\left(m, n\right) t^{\ell m} q^{\ell n} \\
&= \sum_{\ell|k} \mu(\ell) \mathrm{CT}\left(\prod_{j=1}^{k/\ell} (zt^{\ell j} q^\ell; q^\ell)_\infty (z^{-1} t^{-\ell j}; q^\ell)_\infty\right) \\
&= \sum_{\ell|k} \mu(\ell) \mathrm{CT}\left(\prod_{j=1}^{k/\ell} (z^\ell t^{\ell j} q^\ell; q^\ell)_\infty (z^{-\ell} t^{-\ell j}; q^\ell)_\infty\right),
\end{aligned} \tag{4.7.12}$$

where the last equality follows from the fact that (4.7.11) holds with z replaced by z^a for any positive integer a.

Next, we rewrite the left hand side of (4.7.11) as

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\overline{c\phi}_k(m,n)t^m q^n = \sum_{j=0}^{k-1}\sum_{s=-\infty}^{\infty}\sum_{n=0}^{\infty}\overline{c\phi}_k(sk+j,n)t^{sk+j}q^n$$
$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{k-1}\sum_{s=-\infty}^{\infty}\overline{c\phi}_k(sk+j,n)t^{sk+j}\right)q^n. \qquad (4.7.13)$$

Let

$$c_k(j,n) = \sum_{\substack{m=-\infty \\ m\equiv j \pmod{k}}}^{\infty} \overline{c\phi}_k(m,n).$$

Let $t=1$ in (4.7.13). Note that

$$\overline{c\phi}_k(n) = \sum_{m=-\infty}^{\infty}\overline{c\phi}_k(m,n).$$

We find that

$$\sum_{j=0}^{k-1} c_k(j,n) = \overline{c\phi}_k(n). \qquad (4.7.14)$$

Next, if $t=\zeta\neq 1$ is a primitive r-th of unity with $r|k$, then from (4.7.13), we deduce that

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\overline{c\phi}_k(m,n)\zeta^m q^n = \sum_{n=0}^{\infty}\sum_{j=0}^{k-1} c_k(j,n)\zeta^j q^n. \qquad (4.7.15)$$

To complete the proof of (4.7.4), we need the following lemma:

Lemma 4.7.5. *Let ζ_k be a primitive k-th root of unity. Then ζ_k^s is a root of*

$$P_n(t) := \sum_{j=0}^{k-1} c_k(j,n)t^j$$

for all $1\le s\le k-1$.

Assuming that Lemma 4.7.5 is true. It would imply that $P_n(t)$ is divisible by $Q(t)=1+t+\cdots+t^{k-1}$ and since the degrees of $P_n(t)$ and $Q(t)$ are the same, we must conclude that $c_k(j,n) = c_k(0,n)$ are all equal for $1 \le j \le k-1$. From (4.7.14), we conclude that

$$\overline{c\phi}_k(n) = kc_k(0,n).$$

Let S_0 be the set of k-colored generalized Frobenius symbols of n of order k with color difference divisible by k. Note that $|S_0| = c_k(0,n)$. If $\pi \in S_0$ then π under the action of $\sigma_k^j, 1 \le j \le k-1$ is also in S_0 since the residue of the color difference is invariant modulo k under the action of σ_k and the order of π is k. This implies that S_0 can be grouped into disjoint sets containing k elements in each set, which implies that k divides $c_k(0,n)$. Therefore,

$$\overline{c\phi}_k(n) \equiv 0 \pmod{k^2}$$

and this completes the proof of (4.7.4). □

It remains to prove Lemma 4.7.5.

Proof of Lemma 4.7.5. Given any integer j between 1 and $k-1$, there exists an integer $r|k$ such that ζ_k^j is a primitive r-th root of unity. Therefore, to prove Lemma 4.7.5, it suffices to prove that $P_n(\zeta) = 0$ for any primitive r-th root of unity with $r|k$. From (4.7.12) and (4.7.15), we deduce that

$$\sum_{n=0}^{\infty} P_n(\zeta)q^n = \sum_{\ell|k} \mu(\ell)\mathrm{CT}\left(\prod_{j=1}^{k/\ell}(z^\ell\zeta^{\ell j}q^\ell;q^\ell)_\infty(z^{-\ell}\zeta^{-\ell j};q^\ell)_\infty\right). \tag{4.7.16}$$

The presence of the factor $\mu(\ell)$ in (4.7.16) shows that we only need to consider divisors of the squarefree part of k. Fix a prime p which divides r and separate the sum in (4.7.16) into a sum over divisors of the form d where $(p,d) = 1$ and a sum over divisors of the form pd. We only need to show that the term corresponding to d cancels with the term corresponding to pd.

Observe that since d is squarefree and $(d,p) = 1$, we can write $d = ww'$ where $w|r$ and $(w',r) = 1$. Note that the term corresponding to $d = ww'$ is

$$\mathrm{CT}\left(\mu(ww')\prod_{j=1}^{k/(ww')}(z^{ww'}\zeta^{ww'j}q^{ww'};q^{ww'})_\infty(z^{-ww'}\zeta^{-ww'j};q^{ww'})_\infty\right)$$

$$= \mu(ww')\mathrm{CT}\left((z^{rw'}q^{rw'};q^{rw'})_\infty^{k/(rw')}(z^{-rw'};q^{rw'})_\infty^{k/(rw')}\right)$$

since ζ^w is a r/w-th primitive root of unity and

$$\prod_{j=0}^{\nu}(1-z\zeta_\nu^j) = (1-z^\nu).$$

Similarly, the term corresponding to $pd = pww'$ is

$$\mathrm{CT}\left(\mu(pww')\prod_{j=1}^{k/(pww')}(z^{pww'}\zeta^{pww'j}q^{pww'};q^{pww'})_\infty(z^{-pww'}\zeta^{-pww'j};q^{pww'})_\infty\right)$$
$$= \mu(pww')\mathrm{CT}\left((z^{rw'}q^{rw'};q^{rw'})_\infty^{k/(rw')}(z^{-rw'};q^{rw'})_\infty^{k/(rw')}\right).$$

Clearly these two terms cancel as $\mu(pww') = -\mu(ww')$. This completes the proof of the lemma. □

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Chapter 5

Hecke-Rogers Type Identities and False Theta Functions

5.1 Introduction

Hecke-Rogers type series are of the following type:

$$\sum_{(m,n)\in D} (-1)^{H(m,n)} q^{Q(m,n)+L(m,n)},$$

where H and L are linear forms, Q is a quadratic form, and D is some subset of $\mathbf{Z} \times \mathbf{Z}$. The following classical identity of Jacobi is of this type:

$$\sum_{n=-\infty}^{\infty} \sum_{m\geq|n|} (-1)^m q^{(m^2+m)/2} = \prod_{n=1}^{\infty} (1-q^n)^3.$$

Here and throughout this chapter, as usual we need to assume that $|q| < 1$. Motivated by the Jacobi identity, E. Hecke systematically investigated theta series related to indefinite quadratic forms [48]. For instance, Hecke [48, p. 425] found that

$$\sum_{n=-\infty}^{\infty} \sum_{|m|\leq n/2} (-1)^{n+m} q^{(n^2-3m^2)/2+(n+m)/2} = \prod_{n=1}^{\infty} (1-q^n)^2,$$

which is originally due to L.J. Rogers [76, p. 323]. The deepest work on this topic is that of V.C. Kac and D.H. Peterson who showed how to prove the Hecke identity using affine Lie algebra [53].

In q-series, Hecke-Rogers type series have played an important role and received a lot of attention. However, unlike such series associated with indefinite quadratic forms, there are few results on series associated with definite quadratic forms (c.f. [34, 80]). Recently, an interesting double series associated with definite quadratic forms has arisen from the work of G.E. Andrews, A. Dixit, D. Schultz, and A.J. Yee on partition functions associated with Ramanujan's third order mock theta function $\omega(q)$ [11], where

$$\omega(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}.$$

The first purpose of this chapter is to prove a double series identity, which is given in the following theorem.

Theorem 5.1.1. *We have*

$$\sum_{n=1}^{\infty}\frac{q^n(q;q^2)_n}{(-q;q^2)_n(1+q^{2n})}=\sum_{n=1}^{\infty}\sum_{|m|\leq n}(-1)^m q^{n^2+m^2}-\sum_{n=1}^{\infty}(-1)^n q^{2n^2}. \tag{5.1.1}$$

In [10], Andrews, Dixit and Yee discovered a new partition function associated with $\omega(q)$. The coefficient of q^n in $q\omega(q)$ counts the number $p_\omega(n)$ of partitions of n in which all odd parts are less than twice the smallest part [10]. In [11], Andrews, Dixit, Schultz and Yee considered an overpartition analogue $\overline{p}_\omega(n)$ of the partition function. Namely, $\overline{p}_\omega(n)$ enumerates the number of overpartitions of n such that all odd parts are less than twice the smallest part, and in which the smallest part is always overlined. They revealed various congruence properties of $\overline{p}_\omega(n)$.

The summation on the left hand side of Theorem 5.1.1 was studied to prove some mod 4 congruences of $\overline{p}_\omega(n)$. To be more specific, in [11], it was shown that

$$A(q)\equiv -\frac{1}{4}+\frac{1}{4}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}\right)^2+q\left(\sum_{n=0}^{\infty}q^{2n(n+1)}\right)^2 \pmod 4, \tag{5.1.2}$$

where $A(q)$ is the summation on the left-hand side of Theorem 5.1.1. Indeed, $A(q)$ also has the representation as follows:

$$A(q)=-\frac{1}{4}+\frac{1}{4}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}\right)^2+q\left(\sum_{n=0}^{\infty}(-1)^n q^{2n(n+1)}\right)^2, \tag{5.1.3}$$

which can be deduced from Theorem 5.1.1. (See Lemma 5.3.1.)

Note that (5.1.2) and (5.1.3) differ only in the the signs of their odd power terms. We also note that $A(q)$ has such a beautiful mod 2 dissection, i.e., the even power terms are the square of a theta series while the odd power terms are the square of a false theta series.

False theta functions, first introduced generally by Rogers [77], are series that are instances of classical theta series except for an alteration of the signs of some of the series' terms. While theta series enjoy modularity and are well studied, false theta series have strange behavior and receive less attention. However, several special cases were considered by Ramanujan in his notebooks [71] and lost notebook [73]. For instance, on page 13 of Ramanujan's lost notebook [73] (c.f. [8, Section 9.3, pp. 227–232], [13]), the following identities are given:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n\, q^{n(n+1)}}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}, \tag{5.1.4}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n^2\, q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)}, \tag{5.1.5}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n\, q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)/2}, \tag{5.1.6}$$

$$\sum_{n=0}^{\infty} \frac{(q;-q)_{2n}\, q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)}, \tag{5.1.7}$$

and

$$\sum_{n=0}^{\infty} \frac{(q;-q)_n (-q^2;q^2)_n\, q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)}. \tag{5.1.8}$$

By rewriting the summands on the left-hand side in (5.1.1) as

$$\frac{(-q^2;q^2)_{n-1}(q;q^2)_n q^n}{(-q;q)_{2n}},$$

we see that its denominator is analogous to that of the summands in (5.1.4)–(5.1.8). In addition, the following identity can be found in Ramanujan's lost notebook, Part 1 [8, p. 236, Entry 9.4.9]:

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n\, q^n}{(-q;q)_{2n}} = \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2}(1+q^{2n+1}), \tag{5.1.9}$$

which may be considered as a companion of the series on the left-hand side in (5.1.6), in which the denominator is replaced by $(-q;q)_{2n}$. Thus, it would be interesting to investigate further identities. In this chapter, along with (5.1.1), we will prove the following two new identities.

Theorem 5.1.2. *We have*

$$\sum_{n=1}^{\infty}\frac{q^{n^2}}{(-q;q^2)_n(1+q^{2n})}=\sum_{n=1}^{\infty}\sum_{|m|\le n/2}(-1)^m q^{n^2-2m^2}-\sum_{n=1}^{\infty}(-1)^n q^{2n^2}.$$

Theorem 5.1.3. *We have*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^n(q^2;q^2)_{n-1}}{(-q^2;q^2)_n}=\sum_{n=1}^{\infty}\sum_{|m|\le n/2}(-1)^m q^{n^2-2m^2}-\sum_{n=1}^{\infty}(-1)^n q^{2n^2}.$$

This chapter is organized as follows. In Section 5.2, we recall some q-series identities and formulas from the theory of basic hypergeometric series. Section 5.3 is devoted to proving Theorem 5.1.1 and (5.1.3). Theorems 5.1.2 and 5.1.3 will be proved in Sections 5.4 and 5.5, respectively. As an application of Theorem 5.1.1, in Section 5.6, we give new proofs of the following congruences in [11]:

$$\overline{p}_\omega(4n+3)\equiv 0 \pmod 4, \tag{5.1.10}$$

$$\overline{p}_\omega(8n+6)\equiv 0 \pmod 4. \tag{5.1.11}$$

Lastly, in Section 5.7, we provide new proofs of Ramanujan's identities (5.1.4)–(5.1.8) for false theta functions. It should be noted that the essential tools for our proofs are amazing formulas discovered by Z.G. Liu [64], which are recalled in the next section.

5.2 Preliminaries

In this section, we collect some identities on basic hypergeometric series from the literature for later use.

From [42, p. 15] (see also [2, p. 527]), we have

$$ {}_2\phi_1\left(\begin{matrix} a, & b \\ & c \end{matrix}; q, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty} \sum_{n=0}^{\infty} \frac{(a, c/b; q)_n}{(q, c, az; q)_n} (-bz)^n q^{\binom{n}{2}}. \tag{5.2.1} $$

From [42, p.40, Eq. (2.2.1)], [42, p.71, Eq. (3.2.5)] and [42, p.71, Eq. (3.2.6)], we find that

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & aq^n, & aq/bc \\ & aq/b, & aq/c \end{matrix}; q, q\right) = \frac{(b, c; q)_n}{(aq/b, aq/c; q)_n} \left(\frac{aq}{bc}\right)^n, \tag{5.2.2} $$

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & a, & b \\ & d, & e \end{matrix}; q, \frac{deq^n}{ab}\right) = \frac{(e/a;q)_n}{(e;q)_n} {}_3\phi_2\left(\begin{matrix} q^{-n}, & a, & d/b \\ & d, & aq^{1-n}/e \end{matrix}; q, q\right), \tag{5.2.3} $$

and

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & aq^n, & b \\ & d, & e \end{matrix}; q, \frac{de}{ab}\right) = \frac{(aq/d, aq/e; q)_n}{(d, e; q)_n} \left(\frac{de}{aq}\right)^n {}_3\phi_2\left(\begin{matrix} q^{-n}, & aq^n, & abq/de \\ & aq/d, & aq/e \end{matrix}; q, \frac{q}{b}\right). \tag{5.2.4} $$

We also need the following transformation formulas due to Liu: For $|\alpha ab/q| < 1$, [64, p.2089]

$$ \begin{aligned} &\frac{(\alpha q, \alpha ab/q; q)_\infty}{(\alpha a, \alpha b; q)_\infty} {}_3\phi_2\left(\begin{matrix} q/a, & q/b, & \beta \\ & c, & d \end{matrix}; q, \frac{\alpha ab}{q}\right) \\ &= \sum_{n=0}^{\infty} \frac{(1-\alpha q^{2n})(\alpha, q/a, q/b; q)_n (-\alpha ab/q)^n q^{\binom{n}{2}}}{(1-\alpha)(q, \alpha a, \alpha b; q)_n} {}_3\phi_2\left(\begin{matrix} q^{-n}, & \alpha q^n, & \beta \\ & c, & d \end{matrix}; q, q\right) \end{aligned} \tag{5.2.5} $$

and [64, Proposition 2.4]

$$ \begin{aligned} &(-1)^n \frac{(\alpha q;q)_n}{(q;q)_n} q^{\binom{n+1}{2}} {}_3\phi_2\left(\begin{matrix} q^{-n}, & \alpha q^{n+1}, & \alpha cd/q \\ & \alpha c, & \alpha d \end{matrix}; q, 1\right) \\ &= \sum_{j=0}^{n} (-1)^j \frac{(1-\alpha q^{2j})(\alpha, q/c, q/d; q)_j}{(1-\alpha)(q, \alpha c, \alpha d; q)_j} q^{j(j-3)/2} (\alpha cd)^j. \end{aligned} \tag{5.2.6} $$

We remark that $(-1)^n$ on the left-hand side is missing in the latter formula in [64].

5.3 Proof of Theorem 5.1.1

Proof of Theorem 5.1.1. Let

$$A(q)=\sum_{n=1}^{\infty}\frac{q^n(q;q^2)_n}{(-q;q^2)_n(1+q^{2n})}.$$

Then

$$1+2A(q)=\sum_{n=0}^{\infty}\frac{q^n(q;q^2)_n(-1;q^2)_n}{(-q;q^2)_n(-q^2;q^2)_n}={}_3\phi_2\left(\begin{matrix}q^2, & q, & -1\\ & -q, & -q^2\end{matrix};q^2,q\right). \tag{5.3.1}$$

Now, in (5.2.5), replace q by q^2 and take $(\alpha,\beta,a,b,c,d)=(q^2,-1,1,q,-q,-q^2)$. Then we obtain

$${}_3\phi_2\left(\begin{matrix}q^2, & q, & -1\\ & -q, & -q^2\end{matrix};q^2,q\right)=\sum_{n=0}^{\infty}(-1)^nq^{n^2}(1+q^{2n+1})\,{}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & -1\\ & -q, & -q^2\end{matrix};q^2,q^2\right). \tag{5.3.2}$$

Replacing q by q^2 and taking $(a,b,d,e)=(q^{2n+2},q,-q,-q^2)$ in (5.2.3), we deduce that

$${}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & q\\ & -q, & -q^2\end{matrix};q^2,1\right)=\frac{(-q^{-2n};q^2)_n}{(-q^2;q^2)_n}\,{}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & -1\\ & -q, & -q^2\end{matrix};q^2,q^2\right).$$

Note that

$$\frac{(-q^{-2n};q^2)_n}{(-q^2;q^2)_n}=\prod_{k=1}^{n}\frac{1+q^{-2k}}{1+q^{2k}}=q^{-n(n+1)}.$$

Thus,

$${}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & -1\\ & -q, & -q^2\end{matrix};q^2,q^2\right)=q^{n(n+1)}\,{}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & q\\ & -q, & -q^2\end{matrix};q^2,1\right). \tag{5.3.3}$$

Replacing q by q^2 and taking $(\alpha,c,d)=(1,-q,-q^2)$ in (5.2.6), we deduce that

$$(-1)^nq^{n(n+1)}\,{}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & q\\ & -q, & -q^2\end{matrix};q^2,1\right)=1+2\sum_{j=1}^{n}(-1)^jq^{j^2}. \tag{5.3.4}$$

Now we substitute (5.3.4) into (5.3.3), and then substitute the result into (5.3.2). We deduce that

$$\begin{aligned}
{}_3\phi_2\left(\begin{matrix} q^2, & q, & -1 \\ & -q, & -q^2 \end{matrix}; q^2, q\right) &= \sum_{n=0}^{\infty} q^{n^2}(1+q^{2n+1})\left(1+2\sum_{j=1}^{n}(-1)^j q^{j^2}\right) \\
&= \sum_{n=0}^{\infty} q^{n^2} \sum_{j=-n}^{n} (-1)^j q^{j^2} + \sum_{n=0}^{\infty} q^{(n+1)^2} \sum_{j=-n}^{n} (-1)^j q^{j^2} \\
&= 1 + \sum_{n=1}^{\infty} q^{n^2} \sum_{j=-n}^{n} (-1)^j q^{j^2} + \sum_{n=1}^{\infty} q^{n^2} \sum_{j=-n+1}^{n-1} (-1)^j q^{j^2} \\
&= 1 + 2\sum_{n=1}^{\infty} \sum_{j=-n+1}^{n} (-1)^j q^{n^2+j^2}. \qquad (5.3.5)
\end{aligned}$$

Therefore, by (5.3.1)

$$\sum_{n=1}^{\infty} \frac{q^n (q;q^2)_n}{(-q;q^2)_n (1+q^{2n})} = \sum_{n=1}^{\infty} \sum_{j=-n+1}^{n} (-1)^j q^{n^2+j^2},$$

which completes the proof. □

Remark 5.3.1. We note that (5.3.4) can be obtained by replacing q by q^{-1} in the identity (4.5) of Liu's paper [64], which is also equivalent to the identity (6.16) in Andrews' paper [7].

As mentioned in the introduction, (5.1.3) can be derived as a corollary of Theorem 5.1.1. We first need a lemma.

Lemma 5.3.1. *We have*

$$\sum_{n=1}^{\infty} \sum_{m=-n+1}^{n} (-1)^m q^{m^2+n^2} = -\frac{1}{4} + \frac{1}{4}\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}\right)^2 + q\left(\sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)}\right)^2.$$

Proof. First note that

$$\begin{aligned}
-\frac{1}{4} + \frac{1}{4}\Big(\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}\Big)^2 &= -\frac{1}{4} + \frac{1}{4}\Big(1 + 2\sum_{n=1}^{\infty} (-1)^n q^{2n^2}\Big)^2 \\
&= \sum_{n=1}^{\infty} (-1)^n q^{2n^2} + \Big(\sum_{n=1}^{\infty} (-1)^n q^{2n^2}\Big)^2
\end{aligned}$$

$$= \sum_{n=1}^{\infty}(-1)^n q^{2n^2} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m} q^{2n^2+2m^2}$$
$$= \sum_{n=1}^{\infty}(-1)^n q^{n^2+n^2} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n-m} q^{(n+m)^2+(n-m)^2}.$$

Let $k = n - m, l = n + m$. Then when (n, m) ranges over all positive integer pairs, (k, l) ranges on integer pairs of $\mathbf{Z}^2$ which satisfy

$$l \geq 2, \quad -l < k < l, \quad k \equiv l \pmod 2.$$

Therefore,

$$-\frac{1}{4} + \frac{1}{4}\Big(\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}\Big)^2 = \sum_{n=1}^{\infty} \sum_{\substack{-n+1 \leq m \leq n \\ m \equiv n \pmod 2}} (-1)^m q^{m^2+n^2}. \tag{5.3.6}$$

Similarly,

$$q\Big(\sum_{n=0}^{\infty}(-1)^n q^{2n(n+1)}\Big)^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m} q^{2n(n+1)+2m(m+1)+1}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m} q^{(n+m+1)^2+(n-m)^2}.$$

Now, let $k = n - m, l = n + m + 1$. Then, when (n, m) ranges over all nonnegative integer pairs, (k, l) ranges over all integer pairs in $\mathbf{Z}^2$ with

$$-l + 1 \leq k \leq l, \quad k \equiv l + 1 \pmod 2.$$

Therefore,

$$q\Big(\sum_{n=0}^{\infty}(-1)^n q^{2n(n+1)}\Big)^2 = \sum_{n=1}^{\infty} \sum_{\substack{-n+1 \leq m \leq n \\ m \equiv n+1 \pmod 2}} (-1)^m q^{m^2+n^2}. \tag{5.3.7}$$

Thus we complete the proof by (5.3.6) and (5.3.7). □

By Theorem 5.1.1 and Lemma 5.3.1, we obtain (5.1.3), which can be restated as follow:

Corollary 5.3.2. *We have*

$$\sum_{n=1}^{\infty} \frac{q^n(q;q^2)_n}{(-q;q^2)_n(1+q^{2n})} = -\frac{1}{4} + \frac{1}{4}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}\right)^2 + q\left(\sum_{n=0}^{\infty}(-1)^n q^{2n(n+1)}\right)^2.$$

5.4 Proof of Theorem 5.1.2

We first need two lemmas.

Lemma 5.4.1. *We have*

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)_n\,q^{n^2}}{(-q;q)_{2n}}=\sum_{n=0}^{\infty}q^{2n^2-n}(1+q^{4n+1})\frac{(q;q)_{2n}}{(-q;q)_{2n}}. \tag{5.4.1}$$

Proof. Note that

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)_n\,q^{n^2}}{(-q,q^2)_n(-q^2,q^2)_n}=\lim_{a\to 0}{}_3\phi_2\left(\begin{matrix}-1, & q^2, & q^2/a\\ & -q, & -q^2\end{matrix};q^2,-\frac{a}{q}\right).$$

Now, replacing q by q^2 and setting $(b,c,d,\alpha,\beta)=(1,-q,-q^2,-q,-1)$ in (5.2.5), we deduce that

$$\begin{aligned}&\frac{(-q^3,-a/q;q^2)_\infty}{(-aq,-q;q^2)_\infty}{}_3\phi_2\left(\begin{matrix}-1, & q^2, & q^2/a\\ & -q, & -q^2\end{matrix};q^2,-\frac{a}{q}\right)\\ =&\sum_{n=0}^{\infty}\frac{(1+q^{4n+1})(-q,q^2/a,q^2;q^2)_n(a/q)^n q^{2\binom{n}{2}}}{(1+q)(q^2,-aq,-q;q^2)_n}{}_3\phi_2\left(\begin{matrix}q^{-2n}, & -q^{1+2n}, & -1\\ & -q, & -q^2\end{matrix};q^2,q^2\right).\end{aligned}$$

By letting $a\to 0$, we obtain

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)_n q^{n^2}}{(-q,q^2)_n(-q^2,q^2)_n}=\sum_{n=0}^{\infty}(-1)^n(1+q^{4n+1})q^{2n^2-n}{}_3\phi_2\left(\begin{matrix}q^{-2n}, & -q^{1+2n}, & -1\\ & -q, & -q^2\end{matrix};q^2,q^2\right). \tag{5.4.2}$$

Replacing q by q^2 and set $(a,b,d,e)=(-q^{1+2n},q,-q,q)$ in (5.2.3), we obtain

$${}_3\phi_2\left(\begin{matrix}q^{-2n}, & -q^{1+2n}, & q\\ & -q, & q\end{matrix};q^2,1\right)=\frac{(-q^{-2n};q^2)_n}{(q;q^2)_n}{}_3\phi_2\left(\begin{matrix}q^{-2n}, & -q^{1+2n}, & -1\\ & -q, & -q^2\end{matrix};q^2,q^2\right). \tag{5.4.3}$$

Replacing q by q^2 and set $(\alpha,c,d)=(-q^{-1},q^2,-q^2)$ in (5.2.6), we obtain

$${}_3\phi_2\left(\begin{matrix}q^{-2n}, & -q^{1+2n}, & q\\ & -q, & q\end{matrix};q^2,1\right)=(-1)^n\frac{(q^2;q^2)_n}{(-q;q^2)_n}q^{-n(n+1)}. \tag{5.4.4}$$

Substituting (5.4.4) into (5.4.3), we get

$$ {}_3\phi_2\left(\begin{matrix} q^{-2n}, & -q^{1+2n}, & -1 \\ & -q, & -q^2 \end{matrix}; q^2, q^2\right) = (-1)^n \frac{(q;q^2)_n (q^2;q^2)_n}{(-q;q^2)_n(-q^2;q^2)_n}. \tag{5.4.5} $$

Substituting (5.4.5) into (5.4.2), we obtain (5.4.1). □

Lemma 5.4.2. *We have*

$$ \sum_{n=1}^{\infty} \sum_{|m|\le n/2} (-1)^m q^{n^2-2m^2} - \sum_{n=1}^{\infty}(-1)^n q^{2n^2} = \sum_{n=1}^{\infty}(-1)^n q^{2n^2} \sum_{r=-n+1}^{n} (-1)^r q^{-r^2}. \tag{5.4.6} $$

Proof.

$$ \begin{aligned} &\sum_{n=1}^{\infty} \sum_{|m|\le n/2} (-1)^m q^{n^2-2m^2} - \sum_{n=1}^{\infty}(-1)^n q^{2n^2} \\ &= \sum_{n=1}^{\infty} q^{n^2} + 2\sum_{n=1}^{\infty} \sum_{1\le m\le n/2} (-1)^m q^{n^2-2m^2} - \sum_{n=1}^{\infty}(-1)^n q^{2n^2} \quad (\text{replace } n \text{ by } 2m+r) \\ &= \sum_{n=1}^{\infty}(-1)^{n-1} q^{2n^2} + \sum_{n=1}^{\infty} q^{n^2} + 2\sum_{m=1}^{\infty}\sum_{r=0}^{\infty}(-1)^m q^{2m^2+4mr+r^2} \quad (\text{let } n = m+r) \\ &= \sum_{n=1}^{\infty}(-1)^{n-1} q^{2n^2} + \sum_{n=1}^{\infty} q^{n^2} + 2\sum_{n=1}^{\infty}(-1)^n q^{2n^2}\sum_{r=0}^{n-1}(-1)^r q^{-r^2} \\ &= \sum_{n=1}^{\infty}(-1)^n q^{2n^2} \sum_{r=-n+1}^{n} (-1)^r q^{-r^2}. \end{aligned} $$

□

We are now ready to prove Theorem 5.1.2. Let us recall the theorem:

$$ \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q^2)_n(1+q^{2n})} = \sum_{n=1}^{\infty} \sum_{|m|\le n/2} (-1)^m q^{n^2-2m^2} - \sum_{n=1}^{\infty}(-1)^n q^{2n^2}. $$

Proof of Theorem 5.1.2. Let

$$ X(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q^2)_n(1+q^{2n})}. \tag{5.4.7} $$

Then we have

$$ 1 + 2X(q) = \sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n^2}}{(-q^2;q^2)_n(-q;q^2)_n} \tag{5.4.8} $$

$$= \sum_{n=0}^{\infty} q^{2n^2-n}(1+q^{4n+1})\frac{(q;q)_{2n}}{(-q;q)_{2n}},$$

where the second equality follows from Lemma 5.4.1. We now show that

$$\sum_{n=0}^{\infty} q^{2n^2-n}(1+q^{4n+1})\frac{(q;q)_{2n}}{(-q;q)_{2n}} = 1-2\sum_{n=1}^{\infty}\frac{(q;q)_{n-1}(-1)^n q^{n(n+1)/2}}{(-q;q)_n}. \tag{5.4.9}$$

The right hand side of (5.4.9) is

$$\begin{aligned}
&1-2\sum_{n=1}^{\infty}\frac{(q;q)_{n-1}(-1)^n q^{n(n+1)/2}}{(-q;q)_n}\\
&=1-2\sum_{n=1}^{\infty}\frac{q^{2n^2+n}(q;q)_{2n-1}}{(-q;q)_{2n}}+2\sum_{n=1}^{\infty}\frac{q^{2n^2-n}(q;q)_{2n-2}}{(-q;q)_{2n-1}}\\
&=1+2\sum_{n=1}^{\infty}\frac{q^{2n^2-n}(q;q)_{2n}}{(-q;q)_{2n}}-2\sum_{n=1}^{\infty}\frac{q^{2n^2-n}(q;q)_{2n-1}}{(-q;q)_{2n}}+2\sum_{n=1}^{\infty}\frac{q^{2n^2-n}(q;q)_{2n-2}}{(-q;q)_{2n-1}}\\
&=1+\sum_{n=1}^{\infty}\frac{q^{2n^2-n}(q;q)_{2n}}{(-q;q)_{2n}}+\sum_{n=1}^{\infty}\frac{q^{2n^2-n}(q;q)_{2n-2}(1+q^{2n-1}+q^{2n}+q^{4n-1})}{(-q;q)_{2n}}\\
&=1+\sum_{n=1}^{\infty}\frac{q^{2n^2-n}(q;q)_{2n}}{(-q;q)_{2n}}+\sum_{n=1}^{\infty}\frac{q^{2n^2-n}(q;q)_{2n-2}(1+q^{2n-1})(1+q^{2n})}{(-q;q)_{2n}}\\
&=1+\sum_{n=1}^{\infty}\frac{q^{2n^2-n}(q;q)_{2n}}{(-q;q)_{2n}}+\sum_{n=1}^{\infty}\frac{q^{2n^2-n}(q;q)_{2n-2}}{(-q;q)_{2n-2}}.
\end{aligned}$$

Also, the left hand side of (5.4.9) is

$$\sum_{n=0}^{\infty} q^{2n^2-n}(1+q^{4n+1})\frac{(q;q)_{2n}}{(-q;q)_{2n}} = 1+\sum_{n=1}^{\infty} q^{2n^2-n}\frac{(q;q)_{2n}}{(-q;q)_{2n}}+\sum_{n=1}^{\infty} q^{2n^2-n}\frac{(q;q)_{2n-2}}{(-q;q)_{2n-2}}.$$

Thus (5.4.9) holds true. We now recall the following identity from [67, Eq. (2.11)]:

$$\sum_{n=1}^{\infty}\frac{(q;q)_{n-1}(-1)^n q^{n(n+1)/2}}{(-q;q)_n} = \sum_{n=1}^{\infty}\sum_{j=-n+1}^{n}(-1)^{n+j+1}q^{2n^2-j^2}, \tag{5.4.10}$$

which with (5.4.6) and (5.4.9) completes the proof. □

5.5 Proof of Theorem 5.1.3

Let us recall the identity in Theorem 5.1.3:

$$\sum_{n=1}^{\infty}\frac{(-q)^n(q^2;q^2)_{n-1}}{(-q^2;q^2)_n} = -\sum_{n=1}^{\infty}\sum_{|m|\le n/2}(-1)^m q^{n^2-2m^2}+\sum_{n=1}^{\infty}(-1)^n q^{2n^2}. \tag{5.5.1}$$

Proof of Theorem 5.1.3. Let $Y(q)$ denote the left hand side of (5.5.1). We have

$$\begin{aligned} Y(q) &= \sum_{n=1}^{\infty} \frac{(-q)^n (q^2;q^2)_{n-1}}{(-q^2;q^2)_n} \\ &= \frac{-q}{1+q^2} \sum_{n=0}^{\infty} \frac{(-q)^n (q^2;q^2)_n}{(-q^4;q^2)_n}. \end{aligned} \tag{5.5.2}$$

Replacing q by q^2 and setting $(a,b,c,z) = (q^2, q^2, -q^4, -q)$ in (5.2.1), we obtain

$$\begin{aligned} Y(q) &= -\frac{q}{(1+q)(1+q^2)} \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2+2n}}{(-q^3,-q^4;q^2)_n} \\ &= -\frac{q}{(1+q)(1+q^2)} \left(1 + q^{-1} \sum_{n=2}^{\infty} \frac{(-q^2;q^2)_{n-1} q^{n^2}}{(-q^3,-q^4;q^2)_{n-1}}\right) \\ &= -\frac{q}{(1+q)(1+q^2)} \left(1 + \frac{(1+q)(1+q^2)}{2q} \sum_{n=2}^{\infty} \frac{(-1;q^2)_n q^{n^2}}{(-q;q)_{2n}}\right) \\ &= -\frac{q}{(1+q)(1+q^2)} - \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1;q^2)_n q^{n^2}}{(-q;q)_{2n}} \\ &= -\frac{q}{(1+q)(1+q^2)} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n^2}}{(-q;q)_{2n}} + \frac{1}{2}\left(1 + \frac{2q}{(1+q)(1+q^2)}\right) \\ &= \frac{1}{2}\left(1 - \sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n^2}}{(-q;q)_{2n}}\right). \end{aligned} \tag{5.5.3}$$

Therefore, we have

$$1 - 2Y(q) = \sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n^2}}{(-q;q)_{2n}}. \tag{5.5.4}$$

Comparing (5.5.4) with (5.4.8), we conclude that $Y(q) = -X(q)$ where $X(q)$ is defined in (5.4.7). Therefore, Theorem 5.1.3 follows from Theorem 5.1.2. □

From (5.5.3) we deduce that

$$\begin{aligned} Y(q) &= \frac{1}{2}\left(1 - \sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n^2}}{(-q;q)_{2n}}\right) \\ &= -\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1;q^2)_n q^{n^2}}{(-q;q^2)_n (-q^2;q^2)_n} \\ &= -\sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q^2)_n (1+q^{2n})}. \end{aligned} \tag{5.5.5}$$

Replacing q by $-q$ and recall the definition of $Y(q)$, we obtain the following interesting identity.

Corollary 5.5.1. *We have*

$$\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_n(1+q^{2n})} = -\sum_{n=1}^{\infty} \frac{q^n(q^2;q^2)_{n-1}}{(-q^2;q^2)_n}.$$

We also find another fascinating result.

Corollary 5.5.2. *We have*

$$\sum_{n=-\infty}^{-1} \frac{q^{n(n+1)/2}}{1+q^{2n}} - \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{1+q^{2n}} = \sum_{n=1}^{\infty} \sum_{|m|\le n/2} (-1)^m q^{n^2-2m^2} - \sum_{n=1}^{\infty} (-1)^n q^{2n^2}. \quad (5.5.6)$$

Proof. By Theorem 5.1.3, it suffices to show that

$$Y(q) = -\sum_{n=1}^{\infty} \frac{(1-q^n)q^{n(n+1)/2}}{1+q^{2n}}. \quad (5.5.7)$$

We have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-q)^n(q^2;q^2)_{n-1}}{(-q^2;q^2)_n} &= -\frac{q}{1+q^2} \sum_{n=0}^{\infty} \frac{(q^2;q^2)_n(-q)^n}{(-q^4;q^2)_n} \\
&= -\frac{q}{1+q^2} \sum_{n=0}^{\infty} \frac{(q,-q;q)_n}{(q^2\,i,-q^2\,i;q)_n}(-q)^n \\
&= -\frac{q}{1+q^2}\, {}_3\phi_2 \left(\begin{matrix} q, & -q, & q \\ & q^2 i, & -q^2 i \end{matrix} ; q, -q \right). \quad (5.5.8)
\end{aligned}$$

In (5.2.5), we set $(\alpha,\beta,a,b,c,d) = (q^2,q,1,-1,q^2i,-q^2i)$. Then

$$\begin{aligned}
&\frac{(q^3,-q;q)_\infty}{(q^2,-q^2;q)_\infty} {}_3\phi_2 \left(\begin{matrix} q, & -q, & q \\ & q^2 i, & -q^2 i \end{matrix} ; q, -q \right) \\
&= \sum_{n=0}^{\infty} \frac{(1-q^{2n+2})(q^2,q,-q;q)_n q^{n(n+1)/2}}{(1-q^2)(q,q^2,-q^2;q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, & q^{n+2}, & q \\ & q^2 i, & -q^2 i \end{matrix} ; q, q \right).
\end{aligned}$$

So

$${}_3\phi_2 \left(\begin{matrix} q, & -q, & q \\ & q^2 i, & -q^2 i \end{matrix} ; q, -q \right) = \sum_{n=0}^{\infty} (1-q^{n+1}) q^{n(n+1)/2} {}_3\phi_2 \left(\begin{matrix} q^{-n}, & q^{n+2}, & q \\ & q^2 i, & -q^2 i \end{matrix} ; q, q \right). \quad (5.5.9)$$

By (5.5.8) and (5.5.9),

$$\sum_{n=1}^{\infty} \frac{(-q)^n (q^2;q^2)_{n-1}}{(-q^2;q^2)_n} = -\frac{q}{1+q^2} \sum_{n=0}^{\infty} (1-q^{n+1}) q^{n(n+1)/2} {}_3\phi_2 \left(\begin{matrix} q^{-n}, & q^{n+2}, & q \\ & q^2 i, & -q^2 i \end{matrix} ; q, q \right). \tag{5.5.10}$$

Setting $(a,b,c) = (q^2, -iq, iq)$ in (5.2.2), we deduce that

$$ {}_3\phi_2 \left(\begin{matrix} q^{-n}, & q^{n+2}, & q \\ & q^2 i, & -q^2 i \end{matrix} ; q, q \right) = \frac{(-iq, iq; q)_n q^n}{(iq^2, -iq^2; q)_n} = \frac{(-q^2;q^2)_n q^n}{(-q^4;q^2)_n} = \frac{(1+q^2) q^n}{1+q^{2n+2}}. \tag{5.5.11}$$

Substituting (5.5.11) into (5.5.10), we conclude that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-q)^n (q^2;q^2)_{n-1}}{(-q^2;q^2)_n} &= -\sum_{n=0}^{\infty} \frac{(1-q^{n+1}) q^{(n+1)(n+2)/2}}{1+q^{2n+2}} \\ &= -\sum_{n=1}^{\infty} \frac{(1-q^n) q^{n(n+1)/2}}{1+q^{2n}} \\ &= -\sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{1+q^{2n}} + \sum_{n=-\infty}^{-1} \frac{q^{n(n+1)/2}}{1+q^{2n}}. \end{aligned} \tag{5.5.12}$$

□

Remark 5.5.1. It would be interesting if one can find a direct proof of Corollary 5.5.2. This will lead to a new proof of Theorem 5.1.3.

5.6 An application

Let

$$P(q) := \sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_n (q^{2n+2};q^2)_\infty}{(1+q^n)(-q^{n+1};q)_n (-q^{2n+2};q^2)_\infty}.$$

From [11, Lemma 4.1], we have that

$$P(q) = \sum_{n=1}^{\infty} \frac{q^n (q;q^2)_n}{(-q;q^2)_n (1+q^{2n})} - \sum_{n=1}^{\infty} (-1)^n q^{2n^2},$$

from which with Corollary 5.3.2, it follows that

$$P(q) = -\frac{1}{4} + \frac{1}{4}\Big(1 + 2\sum_{n=1}^{\infty}(-1)^n q^{2n^2}\Big)^2 + q\Big(\sum_{n=0}^{\infty}(-1)^n q^{2n(n+1)}\Big)^2 - \sum_{n=1}^{\infty}(-1)^n q^{2n^2}$$
$$= \Big(\sum_{n=1}^{\infty}(-1)^n q^{2n^2}\Big)^2 + q\Big(\sum_{n=0}^{\infty}(-1)^n q^{2n(n+1)}\Big)^2. \tag{5.6.1}$$

Recall from [11] that

$$\sum_{n=1}^{\infty}\overline{p}_\omega(n)q^n = \sum_{n=1}^{\infty}\frac{q^n(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}.$$

From [11, Eq. (4.14)] we know that

$$\sum_{n=1}^{\infty}\overline{p}_\omega(n)q^n \equiv P(q) \pmod 4.$$

Thus by (5.6.1) we have

$$\sum_{n=1}^{\infty}\overline{p}_\omega(n)q^n \equiv \Big(\sum_{n=1}^{\infty}(-1)^n q^{2n^2}\Big)^2 + q\Big(\sum_{n=0}^{\infty}(-1)^n q^{2n(n+1)}\Big)^2 \pmod 4, \tag{5.6.2}$$

which yields the congruences (5.1.10) and (5.1.11) for $\overline{p}_\omega(n)$ given in [11].

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5.7 New proofs of Ramanujan's identities on false theta functions

We will give new proofs to (5.1.4)–(5.1.8) in this section.

Proof of (5.1.4). We observe that

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{n(n+1)}}{(-q^2;q^2)_n(-q^3;q^2)_n}$$

$$= \lim_{a\to 0}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(q^2/a;q^2)_n a^n}{(-q^2;q^2)_n(-q^3;q^2)_n}$$
$$= \lim_{a\to 0} {}_3\phi_2\left(\begin{matrix} q, & q^2, & q^2/a \\ & -q^2, & -q^3 \end{matrix}; q^2, a\right). \tag{5.7.1}$$

Replacing q by q^2 and setting $(b,c,d,\alpha,\beta) = (1,-q^2,-q^3,q^2,q)$ in (5.2.5), we get

$$\frac{(q^4,a;q^2)_\infty}{(q^2a,q^2;q^2)_\infty} {}_3\phi_2\left(\begin{matrix} q, & q^2, & q^2/a \\ & -q^2, & -q^3 \end{matrix}; q^2, a\right)$$
$$= \sum_{n=0}^{\infty}\frac{(1-q^{4n+2})(q^2,q^2/a,q^2;q^2)_n(-a)^n q^{n(n-1)}}{(1-q^2)(q^2,q^2a,q^2;q^2)_n} \times {}_3\phi_2\left(\begin{matrix} q^{-2n}, & q^{2n+2}, & q \\ & -q^2, & -q^3 \end{matrix}; q^2, q^2\right). \tag{5.7.2}$$

Taking $a\to 0$ in (5.7.2) and substituting this into (5.7.1), we obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{n(n+1)}}{(-q^2;q^2)_n(-q^3;q^2)_n}$$
$$= \sum_{n=0}^{\infty}(1-q^{4n+2})q^{2n^2} {}_3\phi_2\left(\begin{matrix} q^{-2n}, & q^{2n+2}, & q \\ & -q^2, & -q^3 \end{matrix}; q^2, q^2\right). \tag{5.7.3}$$

Replacing q by q^2 and setting $(a,b,c) = (q^2,-q^2,-q)$ in (5.2.2), we deduce that

$${}_3\phi_2\left(\begin{matrix} q^{-2n}, & q^{2n+2}, & q \\ & -q^2, & -q^3 \end{matrix}; q^2, q^2\right) = \frac{(-q,-q^2;q^2)_n q^n}{(-q^2,-q^3;q^2)_n} = \frac{(1+q)q^n}{1+q^{2n+1}}. \tag{5.7.4}$$

Substituting (5.7.4) into (5.7.3) and dividing both sides by $1+q$, we deduce that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{n(n+1)}}{(-q;q)_{2n+1}} &= \sum_{n=0}^{\infty}(1-q^{2n+1})q^{2n^2+n} \\
&= \sum_{n=0}^{\infty} q^{2n^2+n} - \sum_{n=0}^{\infty} q^{(2n+1)(n+1)} \\
&= \sum_{n=0}^{\infty} q^{2n(2n+1)/2} - \sum_{n=0}^{\infty} q^{(2n+1)(2n+2)/2} \\
&= \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}.
\end{aligned}$$

□

Proof of (5.1.5). We observe that

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n^2q^n}{(-q;q)_{2n+1}}=\sum_{n=0}^{\infty}\frac{(q;q^2)_n^2q^n}{(1+q)(-q^2,-q^3;q^2)_n}$$
$$=\frac{1}{1+q}{}_3\phi_2\left(\begin{matrix}q & q^2 & q\\ & -q^2 & -q^3\end{matrix};q^2,q\right). \tag{5.7.5}$$

Replacing q by q^2 and setting $(a,b,c,d,\alpha,\beta)=(aq,1,-q^2,-q^3,q^2/a,aq)$ in (5.2.5), we obtain

$$\frac{(a^{-1}q^4,q;q^2)_\infty}{(q^3,a^{-1}q^2;q^2)_\infty}{}_3\phi_2\left(\begin{matrix}a^{-1}q & q^2 & aq\\ & -q^2 & -q^3\end{matrix};q^2,q\right) \tag{5.7.6}$$
$$=\sum_{n=0}^{\infty}\frac{(1-a^{-1}q^{4n+2})(a^{-1}q^2,a^{-1}q,q^2;q^2)_n(-1)^nq^{n^2}}{(1-a^{-1}q^2)(q^2,q^3,a^{-1}q^2;q^2)_n}$$
$$\times\,{}_3\phi_2\left(\begin{matrix}q^{-2n} & a^{-1}q^{2+2n} & aq\\ & -q^2 & -q^3\end{matrix};q^2,q^2\right).$$

Replacing q by q^2 and taking $(a,b,c)\to(a^{-1}q^2,-a^{-1}q^2,-a^{-1}q)$ in (5.2.2), we deduce that

$${}_3\phi_2\left(\begin{matrix}q^{-2n} & a^{-1}q^{2+2n} & aq\\ & -q^2 & -q^3\end{matrix};q^2,q^2\right)=\frac{(-a^{-1}q,-a^{-1}q^2;q^2)_n}{(-q^2,-q^3;q^2)_n}(aq)^n. \tag{5.7.7}$$

Substituting this identity into (5.7.6) and setting $a=1$, we obtain

$$\frac{(q^4,q;q^2)_\infty}{(q^3,q^2;q^2)_\infty}{}_3\phi_2\left(\begin{matrix}q & q^2 & q\\ & -q^2 & -q^3\end{matrix};q^2,q\right)$$
$$=\sum_{n=0}^{\infty}\frac{(1-q^{4n+2})(q^2,q,q^2;q^2)_n(-1)^nq^{n^2}}{(1-q^2)(q^2,q^3,q^2;q^2)_n}\cdot\frac{(-q,-q^2;q^2)_nq^n}{(-q^2,-q^3;q^2)_n}$$
$$=\sum_{n=0}^{\infty}\frac{1-q^{4n+2}}{1-q^2}\cdot\frac{1-q}{1-q^{2n+1}}\cdot\frac{1+q}{1+q^{2n+1}}\cdot(-1)^nq^{n^2+n}$$
$$=\sum_{n=0}^{\infty}(-1)^nq^{n^2+n}. \tag{5.7.8}$$

It is not difficult to show that the first line of (5.7.8) agrees with the right hand side of (5.7.5). Hence we have completed the proof of (5.1.5). □

Proof of (5.1.6). We observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^n}{(-q;q)_{2n+1}}&=\frac{1}{1+q}\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^n}{(-q^2;q^2)_n(-q^3;q^2)_n}\\&=\frac{1}{1+q}{}_3\phi_2\left(\begin{matrix}q^2, & q, & 0\\ & -q^2, & -q^3\end{matrix};q^2,q\right).\end{aligned}\tag{5.7.9}$$

Replacing q by q^2 and setting $(a,b,c,d,\alpha,\beta)=(1,q,-q^2,-q^3,q^2,0)$ in (5.2.5), we deduce that

$$\begin{aligned}&\frac{(q^4,q;q^2)_\infty}{(q^2,q^3;q^2)_\infty}{}_3\phi_2\left(\begin{matrix}q^2, & q, & 0\\ & -q^2, & -q^3\end{matrix};q^2,q\right)\\=&\sum_{n=0}^{\infty}\frac{(1-q^{4n+2})(q^2,q^2,q;q^2)_n(-q)^nq^{n(n-1)}}{(1-q^2)(q^2,q^2,q^3;q^2)_n}{}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & 0\\ & -q^2, & -q^3\end{matrix};q^2,q^2\right)\\=&\sum_{n=0}^{\infty}(-1)^nq^{n^2}\cdot\frac{1+q^{2n+1}}{1+q}\cdot{}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & 0\\ & -q^2, & -q^3\end{matrix};q^2,q^2\right).\end{aligned}\tag{5.7.10}$$

Next, we recall the following formula in [42, p.71]:

$${}_2\phi_1\left(\begin{matrix}q^{-n}, & d/b\\ & d\end{matrix};q,bq/e\right)=(-1)^nq^{-\binom{n}{2}}(e;q)_ne^{-n}{}_3\phi_2\left(\begin{matrix}q^{-n}, & b, & 0\\ & d, & e\end{matrix};q,q\right).\tag{5.7.11}$$

Replacing q by q^2 and setting $(b,d,e)=(q^{2n+2},-q^2,-q^3)$ in (5.7.11), we get

$$\begin{aligned}&{}_2\phi_1\left(\begin{matrix}q^{-2n}, & -q^{-2n}\\ & -q^2\end{matrix};q^2,-q^{2n+1}\right)\\=&q^{-n(n+2)}(-q^3;q^2)_n{}_3\phi_2\left(\begin{matrix}q^{-2n}, & q^{2n+2}, & 0\\ & -q^2, & -q^3\end{matrix};q^2,q^2\right).\end{aligned}\tag{5.7.12}$$

Now we need the following famous transformation formula of Heine for ${}_2\phi_1$ series (see [42, p.13]):

$${}_2\phi_1\left(\begin{matrix}a, & b\\ & c\end{matrix};q,z\right)=\frac{(b,az;q)_\infty}{(c,z;q)_\infty}{}_2\phi_1\left(\begin{matrix}c/b, & z\\ & az\end{matrix};q,b\right).\tag{5.7.13}$$

Replacing q by q^2 and setting $(a,b,c,z)=(q^{-2n},-q^{-2n},-q^2,-q^{2n+1})$ in (5.7.13), we obtain

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} q^{-2n}, & -q^{-2n} \\ & -q^2 \end{matrix}; q^2, -q^{2n+1}\right) \\
=&\frac{(-q^{-2n},-q;q^2)_\infty}{(-q^2,-q^{2n+1};q^2)_\infty}{}_2\phi_1\left(\begin{matrix} q^{2n+2}, & -q^{2n+1} \\ & -q \end{matrix}; q^2, -q^{-2n}\right) \\
=&q^{-n(n+1)}(-q;q^2)_n(-q^2;q^2)_n{}_2\phi_1\left(\begin{matrix} -q^{2n+1}, & q^{2n+2} \\ & -q \end{matrix}; q^2, -q^{-2n}\right).
\end{aligned}
\tag{5.7.14}
$$

Next, we will apply the following identity of W.N. Bailey and J.A. Daum [42, Eq. (1.8.1)]:

$$
{}_2\phi_1\left(\begin{matrix} a, & b \\ & aq/b \end{matrix}; q, -\frac{q}{b}\right)=\frac{(-q;q)_\infty(qa,aq^2/b^2;q^2)_\infty}{(aq/b,-q/b;q)_\infty}.
\tag{5.7.15}
$$

Replacing q by q^2 and setting $(a,b)=(-q^{2n+1},q^{2n+2})$ in (5.7.15) and assuming that $1-2n\equiv r \pmod 4$ with $r=1$ or 3, we deduce that

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} -q^{2n+1}, & q^{2n+2} \\ & -q \end{matrix}; q^2, -q^{-2n}\right) \\
=&\frac{(-q^2;q^2)_\infty(-q^{2n+3},-q^{1-2n};q^4)_\infty}{(-q,-q^{-2n};q^2)_\infty} \\
=&\frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty}\cdot\frac{(-q^{2n+3};q^4)_\infty(-q^r;q^4)_\infty(1+q^{1-2n})(1+q^{5-2n})\cdots(1+q^{r-4})}{(-q^2;q^2)_\infty(1+q^{-2n})(1+q^{-2n+2})\cdots(1+q^{-2})} \\
=&\frac{q^{n(n+1)}(-q^r;q^4)_\infty(-q^{4-r};q^4)_\infty}{(-q;q^2)_\infty(-q^2;q^2)_n}q^{-\left((4-r)+\cdots+(2n-5)+(2n-1)\right)} \\
=&\frac{q^{n(n+1)/2}}{(-q^2;q^2)_n}.
\end{aligned}
\tag{5.7.16}
$$

Substituting (5.7.16) into (5.7.14), we deduce that

$$
{}_2\phi_1\left(\begin{matrix} q^{-2n}, & -q^{-2n} \\ & -q^2 \end{matrix}; q^2, -q^{2n+1}\right)=q^{-n(n+1)/2}(-q;q^2)_n.
\tag{5.7.17}
$$

Now substituting (5.7.17) into (5.7.12), we obtain

$$
{}_3\phi_2\left(\begin{matrix} q^{-2n}, & q^{2n+2}, & 0 \\ & -q^2, & -q^3 \end{matrix}; q^2, q^2\right)=\frac{(1+q)q^{(n^2+3n)/2}}{1+q^{2n+1}}.
\tag{5.7.18}
$$

Finally, substituting (5.7.18) into (5.7.10), we deduce that

$$\frac{1}{1+q}{}_3\phi_2\left(\begin{matrix} q^2, & q, & 0 \\ & -q^2, & -q^3 \end{matrix};q^2,q\right)=\sum_{n=0}^{\infty}(-1)^n q^{3n(n+1)/2}. \tag{5.7.19}$$

By (5.7.9), we complete our proof. □

Proof of (5.1.7). We observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q,-q)_{2n}q^n}{(-q;q)_{2n+1}} &= \frac{1}{1+q}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q^2;q^2)_n q^n}{(-q^2;q^2)_n(-q^3;q^2)_n} \\ &= \frac{1}{1+q}\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^n}{(-q^3;q^2)_n} \\ &= \frac{1}{1+q}{}_2\phi_1\left(\begin{matrix} q^2, & q \\ & -q^3 \end{matrix};q^2,q\right).\end{aligned} \tag{5.7.20}$$

Replacing q by q^2 and setting $(a,b,c,z)=(q^2,q,-q^3,q)$ in (5.2.1), we deduce that

$$\begin{aligned}{}_2\phi_1\left(\begin{matrix} q^2, & q \\ & -q^3 \end{matrix};q^2,q\right) &= \frac{(q^3;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(q^2,-q^2;q^2)_n}{(q^2,-q^3,q^3;q^2)_n}\cdot(-q^2)^n q^{n(n-1)} \\ &= \frac{1}{1-q}\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n(-1)^n q^{n(n+1)}}{(-q^3;q^2)_n(q^3;q^2)_n} \\ &= \frac{1}{1-q}\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n}{(-q^3;q^2)_n(q^3;q^2)_n}\lim_{a\to 0}(\frac{q^2}{a};q^2)_n a^n \\ &= \frac{1}{1-q}\lim_{a\to 0}{}_3\phi_2\left(\begin{matrix} q^2/a, & q^2, & -q^2 \\ & -q^3, & q^3 \end{matrix};q^2,a\right).\end{aligned} \tag{5.7.21}$$

Replacing q by q^2 and setting $(b,c,d,\alpha,\beta)=(1,-q^3,q^3,q^2,-q^2)$ in (5.2.5), we get

$$\begin{aligned}&\frac{(q^4,a;q^2)_\infty}{(q^2a,q^2;q^2)_\infty}{}_3\phi_2\left(\begin{matrix} q^2/a, & q^2, & -q^2 \\ & -q^3, & q^3 \end{matrix};q^2,a\right) \\ &=\sum_{n=0}^{\infty}\frac{(1-q^{4n+2})(q^2,q^2/a,q^2;q^2)_n(-a)^n q^{n(n-1)}}{(1-q^2)(q^2,q^2a,q^2;q^2)_n}{}_3\phi_2\left(\begin{matrix} q^{-2n}, & q^{2n+2}, & -q^2 \\ & -q^3, & q^3 \end{matrix};q^2,q^2\right).\end{aligned} \tag{5.7.22}$$

We can now take the limit $a \to 0$ so that

$$\lim_{a\to 0} {}_3\phi_2 \left(\begin{matrix} q^2/a, & q^2, & -q^2 \\ & -q^3, & q^3 \end{matrix} ; q^2, a \right) = \sum_{n=0}^{\infty} (1-q^{4n+2}) q^{2n^2} {}_3\phi_2 \left(\begin{matrix} q^{-2n}, & q^{2n+2}, & -q^2 \\ & -q^3, & q^3 \end{matrix} ; q^2, q^2 \right). \tag{5.7.23}$$

Replacing q by q^2 and setting $(a, b, c) = (q^2, -q, q)$ in (5.2.2), we get

$${}_3\phi_2 \left(\begin{matrix} q^{-2n}, & q^{2n+2}, & -q^2 \\ & -q^3, & q^3 \end{matrix} ; q^2, q^2 \right) = \frac{(-q, q; q^2)_n}{(-q^3, q^3; q^2)_n} (-q^2)^n = \frac{1-q^2}{1-q^{4n+2}} (-1)^n q^{2n}. \tag{5.7.24}$$

Substituting (5.7.24) into (5.7.23), we deduce that

$$\lim_{a\to 0} {}_3\phi_2 \left(\begin{matrix} -q^2, & q^2/a, & q^2 \\ & -q^3, & q^3 \end{matrix} ; q^2, a \right) = (1-q^2) \sum_{n=0}^{\infty} (-1)^n q^{2n^2+2n}. \tag{5.7.25}$$

Substituting (5.7.25) into (5.7.21) and then combining with (5.7.20), we complete the proof of (5.1.7). □

Proof of (5.1.8). We observe that

$$\sum_{n=0}^{\infty} \frac{(q;-q)_n (-q^2;q^2)_n q^n}{(-q;q)_{2n+1}} = \frac{1}{1+q} \sum_{n=0}^{\infty} \frac{(q;-q)_n q^n}{(-q^3;q^2)_n}. \tag{5.7.26}$$

We will next consider the sum on the right with q replaced by $-q$. For this sum we note that

$$\sum_{n=0}^{\infty} \frac{(-q;q)_n (-q)^n}{(q^3;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-q;q)_n (-q)^n}{(q^{3/2};q)_n (-q^{3/2};q)_n} = {}_3\phi_2 \left(\begin{matrix} q, & -q, & 0 \\ & q^{3/2}, & -q^{3/2} \end{matrix} ; q, -q \right). \tag{5.7.27}$$

Setting $(a, b, c, d, \alpha, \beta) = (1, -1, q^{3/2}, -q^{3/2}, q^2, 0)$ in (5.2.5), we deduce that

$$\begin{aligned} &\frac{(q^3, -q; q)_\infty}{(q^2, -q^2; q)_\infty} {}_3\phi_2 \left(\begin{matrix} q, & -q, & 0 \\ & q^{3/2}, & -q^{3/2} \end{matrix} ; q, -q \right) \\ &= \sum_{n=0}^{\infty} \frac{1-q^{2n+2}}{1-q^2} \cdot \frac{(q^2, q, -q; q)_n q^n q^{n(n-1)/2}}{(q, q^2, -q^2; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, & q^{n+2}, & 0 \\ & q^{3/2}, & -q^{3/2} \end{matrix} ; q, q \right) \end{aligned}$$

$$=\sum_{n=0}^{\infty}\frac{(1-q^{n+1})q^{n(n+1)/2}}{1-q}{}_3\phi_2\left(\begin{matrix}q^{-n}, & q^{n+2}, & 0\\ & q^{3/2}, & -q^{3/2}\end{matrix};q,q\right). \tag{5.7.28}$$

Setting $(b,d,e)=(q^{n+2},q^{3/2},-q^{3/2})$ in (5.7.11), we deduce that

$$\begin{aligned}&{}_2\phi_1\left(\begin{matrix}q^{-n}, & q^{-n-\frac{1}{2}}\\ & q^{3/2}\end{matrix};q,-q^{n+\frac{3}{2}}\right)\\ =&(-1)^nq^{-\binom{n}{2}}(-q^{3/2};q)_n(-q^{3/2})^{-n}{}_3\phi_2\left(\begin{matrix}q^{-n}, & q^{n+2}, & 0\\ & q^{3/2}, & -q^{3/2}\end{matrix};q,q\right)\\ =&q^{-n(n+2)/2}(-q^{3/2};q)_n{}_3\phi_2\left(\begin{matrix}q^{-n}, & q^{n+2}, & 0\\ & q^{3/2}, & -q^{3/2}\end{matrix};q,q\right).\end{aligned} \tag{5.7.29}$$

Setting $(a,b)=(q^{-n},q^{-n-1/2})$ in (5.7.15), we deduce that

$${}_2\phi_1\left(\begin{matrix}q^{-n}, & q^{-n-1/2}\\ & q^{3/2}\end{matrix};q,-q^{n+\frac{3}{2}}\right)=\frac{(-q;q)_\infty(q^{1-n};q^2)_\infty(q^{n+3};q^2)_\infty}{(q^{3/2};q)_\infty(-q^{n+3/2};q)_\infty}. \tag{5.7.30}$$

It is clear that if $n\geq 1$ is odd, then the left hand side of (5.7.30) equals 0. Now we assume that n is even. We have

$$\begin{aligned}&\frac{(-q;q)_\infty(q^{1-n};q^2)_\infty(q^{n+3};q^2)_\infty}{(q^{3/2};q)_\infty(-q^{n+3/2};q)_\infty}\\ =&\frac{(-q;q)_\infty(1-q^{1-n})\cdots(1-q^{-3})(1-q^{-1})(q;q^2)_\infty(q;q^2)_\infty}{(1-q)(1-q^3)\cdots(1-q^{n+1})}\\ &\cdot\frac{1}{(q^{3/2};q)_\infty}\cdot\frac{(1+q^{3/2})(1+q^{5/2})\cdots(1+q^{n+1/2})}{(-q^{3/2};q)_\infty}\\ =&\frac{(q;q^2)_\infty(-1)^{n/2}q^{-n^2/4}(1-q)(1-q^3)\cdots(1-q^{n-1})}{(1-q)(1-q^3)\cdots(1-q^{n+1})}\cdot\frac{1}{(q^{3/2};q)_\infty}\cdot\frac{(-q^{3/2};q)_n}{(-q^{3/2};q)_\infty}\\ =&\frac{(1-q)(-1)^{n/2}q^{-n^2/4}(-q^{3/2};q)_n}{1-q^{n+1}}.\end{aligned} \tag{5.7.31}$$

Substituting (5.7.30) and (5.7.31) into (5.7.29), we obtain

$${}_3\phi_2\left(\begin{matrix}q^{-n}, & q^{n+2}, & 0\\ & q^{3/2}, & -q^{3/2}\end{matrix};q,q\right)=\begin{cases}0 & n \text{ is odd};\\ \frac{1-q}{1-q^{n+1}}(-1)^{n/2}q^{n^2/4+n} & n \text{ is even}.\end{cases} \tag{5.7.32}$$

Substituting (5.7.32) into (5.7.28) and combining with (5.7.27), we deduce that

$$\frac{1}{1-q}\sum_{n=0}^{\infty}\frac{(-q;q)_n(-q)^n}{(q^3;q^2)_n}=\sum_{n=0,\,n\text{ even}}^{\infty}(-1)^{n/2}q^{\frac{3}{4}n^2+\frac{3}{2}n}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{3n^2+3n}. \tag{5.7.33}$$

Replacing q by $-q$ in (5.7.33) and noting that $3n^2 + 3n = 3n(n+1)$ is always even, by (5.7.26) we complete our proof. □

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