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# A note on separability and intra-household resource allocation in a collective household model

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THE SCHOOL OF ECONOMICS, SMU

# A note on separability and intra-household resource allocation in a collective household model*

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**Abstract**

We consider a collective model of a household in which each member has a utility function satisfying the weak separability condition. We show that the separability at the individual level carries over to the household level and that the allocation of private goods in any Pareto-efficient allocation can be supported as a Pareto-efficient allocation of the private sub-problem. We also provide the necessary and sufficient condition for the Pareto weight for the private sub-problem to move in the same direction as the household Pareto weight.

**JEL classification codes:** C78, D01, D11

**Keywords:** Collective model, Intra-household resource allocation, Bargaining, Separability.

## 1 Introduction

Collective models of households with heterogeneous preferences among members have become a standard framework for analyzing household behavior. This is in part because a number of

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empirical studies have rejected the traditional unitary household model in favor of a collective model (e.g., Thomas (1990) and Lundberg, Pollak, and Wales (1997)). The theoretical development of the Pareto efficient approach to collective household models following the seminal papers by Chiappori (1988, 1992) has also enabled us to test the unitary model under a set of fairly weak assumptions (e.g., Browning and Chiappori (1998)). Blundell, Chiappori, and Meghir (2005) advanced the Pareto-efficient approach to households with children. They show that the sharing rule can be identified in the presence of either separability or one distribution factor.

In this note, we discuss the relationship between the Pareto-optimal allocation in the household and the allocation of private goods in the corresponding private sub-problem. Under the weak separability assumption, we first show the existence of a private sub-problem that supports the allocation of private goods in any Pareto-efficient outcome. We then provide a necessary and sufficient condition for the household Pareto weight and the Pareto weight in the private sub-problem to move in the same direction. When this condition is satisfied, we can focus on the space of private goods to track the changes in the resource allocation within a household.

Our results are related to Blundell, Chiappori, and Meghir (2005) in the sense that both studies deal with the intra-household resource allocation in the presence of public goods. However, the purpose of this note is not to identify the sharing rule but to provide an analytical framework that allows us to identify the factors that affect the Pareto weight.

In the next section, we formalize the intra-household allocation problem and derive the main results. In Section 3, we provide conclusions including a brief discussion on the empirical application of our results.

## 2 Main results

We consider a collective model for a household, which consists of a husband $h$, a wife $w$ and possibly children and other household members. There are $i$ private goods and $c$ public goods in the economy. The total number of goods in the economy is $n \equiv i + c$. Private goods are enjoyed either by the husband or by the wife, whereas the public goods are enjoyed in common. Public goods may include those goods consumed by the children and other household members, because the husband and the wife care about their consumption or welfare.

Let $\mathbb{R}^i_+$ and $\mathbb{R}^c_+$ be the consumption sets of private and public goods, respectively, for a household. Further, we let $Y$ be the household income and $p \in \Delta(\mathbb{R}^n_{++})$ the price vector of the $n$ goods, respectively, where $\Delta(\bullet)$ is a unit simplex for a set $\bullet$. With some slight abuse of notation, we denote by $p^i$ and $p^c$ the price vectors of private and public goods, respectively, with $p = (p^i, p^c) = (p_1, \ldots, p_i, p_{i+1}, \ldots, p_n)$. Similarly, we denote the index sets of private and public goods by $\mathbb{I}^i \equiv \{1, \ldots, i\}$ and $\mathbb{I}^c \equiv \{i+1, \ldots, n\}$, respectively.

Each member $m \in \{w, h\}$ has a cardinal utility function $U^m : \mathbb{R}^i_+ \times \mathbb{R}^c_+ \to \mathbb{R}_+$, which is strictly increasing, strictly concave, and twice continuously differentiable on $\mathbb{R}^n_+$. As with Blundell, Chiappori, and Meghir (2005), we assume that the allocation is Pareto-efficient for the husband and wife. Then, for any Pareto-efficient allocation $(\tilde{x}^w, \tilde{x}^h, \tilde{x}^c)$, there exists a unique *household Pareto weight* $\lambda \in [0, 1]$ such that $(\tilde{x}^w, \tilde{x}^h, \tilde{x}^c)$ is supported as follows:

$$
\begin{aligned}
(\tilde{x}^w, \tilde{x}^h, \tilde{x}^c) = \underset{(x^w, x^h, x^c)}{\operatorname{argmax}} \ & \lambda U^w(x^w, x^c) + (1-\lambda) U^h(x^h, x^c) \\
\text{s.t.} \quad & p^i(x^w + x^h) + p^c x^c \leq Y.
\end{aligned}
\tag{1}
$$

The household Pareto weight $\lambda$ describes how the resources are allocated within the household. In what follows, we shall exclude the degenerate cases where $\lambda \in \{0, 1\}$ because they are in effect cases of a single decision-maker.

We shall hereafter maintain the *weak separability* of $\mathbb{I}^m$ from $\mathbb{I}^c$ in $U^m$. Formally, for any $j, k \in \mathbb{I}^m$ and any $l \in \mathbb{I}^c$,

$$
\frac{\partial}{\partial x^c_l} \left( \frac{\partial U^m / \partial x^m_j}{\partial U^m / \partial x^m_k} \right) = 0,
$$

where $x^m_j$ and $x^m_k$ are the $j$-th and $k$-th components of the vector of $m$'s private goods consumption $x^m \in \mathbb{R}^i_+$, and $x^c_l$ is the $l$-th component of the vector of public goods consumption $x^c \in \mathbb{R}^c_+$. The weak separability condition states that the marginal rate of substitution between any two private goods does not depend on the level of public goods consumption.

Given the weak separability condition, there exist continuous, strictly increasing, and strictly quasi-concave functions $u^m$ and $\bar{U}^m$ such that $U^m(x^m, x^c) = \bar{U}^m(u^m(x^m), x^c)$.[1] We call $u^m$ the private sub-utility function because it represents the utility that the member $m$ derives from his private goods consumption. Under the weak separability condition, the

[1]See Theorem 3.3b in Blackorby, Primont, and Russell (1978).

maximization problem Eq. (1) can be written as follows:

$$\begin{aligned}\max_{(x^w,x^h,x^c)} \quad & \lambda \bar{U}^w(u^w(x^w),x^c) + (1-\lambda)\bar{U}^h(u^h(x^h),x^c) \\ \text{s.t.} \quad & p^i(x^w+x^h)+p^c x^c \leq Y.\end{aligned} \tag{2}$$

Let us denote the maximand of Eq. (2) by $W_\lambda(x^w,x^h,x^c)$. Then, it can be shown that $W_\lambda(x^w,x^h,x^c)$ is continuous, strictly increasing, and strictly quasi-concave. Further, for the $j$-th and $k$-th components in any $x^m \in \mathbb{R}^i_+$ and the $l$-th component in any $x^c \in \mathbb{R}^c_+$, the following holds:

$$\frac{\partial}{\partial x^c_l}\left(\frac{\partial W_\lambda/\partial x^m_j}{\partial W_\lambda/\partial x^m_k}\right) = \frac{\partial}{\partial x^c_l}\left(\frac{\partial U^m/\partial x^m_j}{\partial U^m/\partial x^m_k}\right) = \frac{\partial}{\partial x^c_l}\left(\frac{\partial u^m/\partial x^m_j}{\partial u^m/\partial x^m_k}\right) = 0.$$

The following proposition follows directly from this equation and the definition of weak separability:

**Proposition 1** *For $m \in \{h,w\}$, $\mathbb{I}^m$ is weakly separable from $\mathbb{I}^c$ in $W_\lambda$.*

Proposition 1 states that weak separability at the individual level is preserved in the household utility function $W_\lambda$.[2] This suggests that we can follow the changes in the resource allocation within the household by looking at the space of private goods. Let us now formally define the private sub-problem for the household problem Eq. (1) as follows:

**Definition 1** Consider a Pareto efficient allocation $(\tilde{x}^w,\tilde{x}^h,\tilde{x}^c)$. The problem $(u^w,u^h;y,\mu)$ is called *the private sub-problem* of the household problem $(U^w,U^h;Y,\lambda)$ if $(\tilde{x}^w,\tilde{x}^h)$ is supported in the following problem:

$$(\tilde{x}^w,\tilde{x}^h) = \underset{(x^w,x^h)}{\text{argmax}}\ \mu u^w(x^w)+(1-\mu)u^h(x^h) \quad \text{s.t.} \quad p^i(x^w+x^h) \leq y,$$

where $y(\leq Y)$ is the private-good expenditure and $\mu \in (0,1)$ is the private Pareto weight.

It can be shown that the private sub-problem exists under Pareto efficiency and the weak separability conditions.

**Theorem 1** *Suppose that $(\tilde{x}^w,\tilde{x}^h,\tilde{x}^c)$ maximizes the maximand $W_\lambda$ in Eq. (2). Then, a private sub-problem $(u^w,u^h;y,\mu)$ exists.*

[2]Notice, however, that $\mathbb{I}^w \cup \mathbb{I}^h$ is not necessarily weakly separable from $\mathbb{I}^c$. We do not need such a condition for our purpose.

**Proof:** Consider a Pareto efficient allocation $(\tilde{x}^w, \tilde{x}^h, \tilde{x}^c)$. This allocation must satisfy the first-order conditions of the maximization problem in Eq. (2). Hence, with some slight abuse of notation, we have the following for $m \in \{w, h\}$ and any $l_1, l_2, l_3, l_4 \in \mathbb{I}^i$:

$$\frac{\partial \bar{U}^m}{\partial u^m}\frac{\partial u^m}{\partial x^m_{l_1}} \Bigg/ \frac{\partial \bar{U}^m}{\partial u^m}\frac{\partial u^m}{\partial x^m_{l_2}} = \frac{\dfrac{\partial u^m}{\partial x^m_{l_1}}}{\dfrac{\partial u^m}{\partial x^m_{l_2}}} = \frac{p^i_{l_1}}{p^i_{l_2}}; \tag{3}$$

$$\frac{\lambda \dfrac{\partial \bar{U}^w}{\partial u^w}\dfrac{\partial u^w}{\partial x^w_{l_3}}}{(1-\lambda)\dfrac{\partial \bar{U}^h}{\partial u^h}\dfrac{\partial u^h}{\partial x^h_{l_4}}} = \frac{p^i_{l_3}}{p^i_{l_4}}. \tag{4}$$

This follows from the weak separability of $W_\lambda$.

Now, let us denote the marginal utility from the private sub-utility evaluated at $(\tilde{x}^m, \tilde{x}^c)$ by $\phi^m \equiv \partial \bar{U}^m / \partial u^m |_{(x^m, x^c) = (\tilde{x}^m, \tilde{x}^c)}$ for $m \in \{h, w\}$. Further, let $\mu^w \equiv \lambda \phi^w$ and $\mu^h \equiv (1-\lambda)\phi^h$, and set $\mu = \frac{\mu^w}{\mu^w + \mu^h}$ and $y = p^i(\tilde{x}^w + \tilde{x}^h)$. It is straightforward to verify that the first order conditions for the private sub-problem coincide with Eq. (3) and Eq. (4). Since $u^m$ is strictly concave, the solution to the first-order conditions corresponds to the unique allocation in $(u^w, u^h; y, \mu)$. This proves the existence of the private sub-problem. $\square$

Theorem 1 suggests that we can potentially track the changes of resource allocation within the household by focusing on the space of the private sub-problem. In general, however, the relationship between the household Pareto weight $\lambda$ and the private Pareto weight $\mu$ may not be monotonic. It turns out that the following condition on the elasticity $\rho$ of the ratio $\psi \equiv \phi^w / \phi^h$ of marginal utilities with respect to $\lambda$ is important to ensure that $\lambda$ and $\mu$ move in the same direction.

**EL:** The elasticity $\rho \equiv \psi / \lambda \cdot \partial \lambda / \partial \psi$ satisfies $\rho > -(1-\lambda)^{-1}$ for any $\lambda \in (0, 1)$, $Y \in \mathbb{R}_+$ and $p \in \Delta(\mathbb{R}^n_{++})$.

Note that $\phi^m$ is a function of $Y$, $p$ and $\lambda$, because $\tilde{x}^w$, $\tilde{x}^h$ and $\tilde{x}^c$ are functions of $Y$, $p$ and $\lambda$.

When $\lambda$ increases, the wife's sub-utility increases while the husband's decreases. Then, $\psi$ decreases because $\phi^m$ and $u^m$ tend to move in the opposite direction. Assumption **EL** requires that the proportional change in $\psi$ be sufficiently small in absolute terms relative to the proportional change in $\lambda$. Note that Assumption **EL** is violated only if $\rho < -1$. That is, it cannot be violated provided the proportional changes in $\lambda$ exceed those in $\psi$ in absolute terms. Under Assumption **EL**, the private Pareto weight $\mu$ moves in the same direction as the household Pareto weight $\lambda$ as shown in the following theorem.

**Theorem 2** $\partial\mu/\partial\lambda > 0$ *holds if and only if* **EL** *holds.*

**Proof:** We use the notations in the proof of Theorem 1. Then,

$$\frac{\partial\mu}{\partial\lambda} = \frac{\mu_\lambda^w \mu^h - \mu_\lambda^h \mu^w}{(\mu^w + \mu^h)^2} = \frac{\phi^w \phi^h}{(\mu^w + \mu^h)^2}\left[1 + \lambda(1-\lambda)\left(\frac{\phi_\lambda^w}{\phi^w} - \frac{\phi_\lambda^h}{\phi^h}\right)\right] = \frac{\phi^w \phi^h}{(\mu^w + \mu^h)^2}\left(1 + (1-\lambda)\rho\right).$$

Because $\phi^h$, $\phi^w$, and $1-\lambda$ are all positive, $\frac{\partial\mu}{\partial\lambda} > 0$ holds if and only if $\rho > -(1-\lambda)^{-1}$. □

As is clear from the construction of $\mu$ in the proof of Theorem 1, $\mu$ is a function of $p$, $Y$ and $\lambda$. Theorem 2 shows that after controlling for the changes in $p$ and $Y$ we only need to examine the changes in the private Pareto weight $\mu$ in order to follow those in the household Pareto weight $\lambda$.

# 3 Conclusion

We have shown that the allocation of private goods in any Pareto-efficient allocation in a household can be represented as a Pareto-efficient allocation in the private sub-problem. This result follows from the weak separability assumption. While a similar result is presented in Blundell, Chiappori, and Meghir (2005), they treat the private and public goods as a Hicksian composite good. As discussed in Hicks (1946, p.33), a sufficient condition for the existence of a Hicksian composite good is that the relative prices of all the component goods are constant, which is highly restrictive.[3] Without making such a restrictive assumption, we derive the relationship between the household and private Pareto weights and provide a meaningful interpretation of the private Pareto weight under Assumption **EL**.

These results are immediately applicable to the identification of changes in the resource allocation within a household. Using the analytical framework developed in this note, Fujii and Ishikawa (2012) estimate the impact of childbirth on intra-household allocation using Japanese panel data with measurements of consumption expenditure for the husband, wife, and other household members. They find that the private goods allocation tends to move towards the wife's disadvantage when a new baby is born. Their study also suggests that the wife may be substituting more say in child-rearing for private consumption. A number of other

[3]Weiss and Sharir (1978) show a slightly less restrictive sufficient condition, but their condition is still restrictive.

studies could be generated from similar datasets using the analytical framework developed in this note.

## References


Blackorby, C., D. Primont, and R. R. Russell (1978): *Duality, Separability, and Functional Structure: Theory and Economic Applications.* North-Holland, New York.

Blundell, R., P. Chiappori, and C. Meghir (2005): "Collective Labor Supply with Children," *Journal of Political Economy*, 113(6), 1277–1306.

Browning, M., and P. Chiappori (1998): "Efficient Intra-Household Allocations: A General Characterization and Empirical Tests," *Econometrica*, 66(6), 1241–1278.

Chiappori, P. (1988): "Rational Household Labor Supply," *Econometrica*, 56(1), 63–90.

——— (1992): "Collective Labor Supply and Welfare," *Journal of Political Economy*, 100(3), 437–467.

Fujii, T., and R. Ishikawa (2012): "How Does Childbirth Alter Intrahousehold Resource Allocation? Evidence From Japan," *Oxford Bulletin of Economics and Statistics*, Forthcoming.

Hicks, J. R. (1946): *Value and Capital: An inquiry into some fundamental principles of economic theory.* Oxford University Press, Oxford.

Lundberg, S., R. Pollak, and T. Wales (1997): "Do Husbands and Wives Pool Their Resources? Evidence from the United Kingdom Child Beneft," *Journal of Human Resources*, 32(3), 463–480.

Thomas, D. (1990): "Intra-Household Resource Allocation: An Inferential Approach," *Journal of Human Resources*, 25(4), 635–664.

Weiss, Y., and S. Sharir (1978): "A Composite Good Theorem for Simple Sum Aggregates," *Econometrica*, 46(6), 1499–1501.