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Statistical Arbitrage and Market Efficiency: Enhanced Theory, Robust Tests and Further Applications

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Abstract

Statistical arbitrage enables tests of market efficiency which circumvent the joint-hypotheses dilemma. This paper makes several contributions to the statistical arbitrage framework. First, we enlarge the set of statistical arbitrage opportunities in Hogan, Jarrow, Teo, and Warachka (2004) to avoid penalizing incremental trading profits with positive deviations from their expected value. Second, we provide a statistical methodology to remedy the lack of consistency and statistical power in their Bonferroni approach. In addition, this procedure allows for autocorrelation and non-normality in trading profits. Third, we apply our tests to a wide range of trading strategies based on stock momentum, stock value, stock liquidity, and industry momentum. Over 50% of these strategies are found to violate market efficiency. We also identify dominant trading strategies which converge to arbitrage most rapidly.

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1 Introduction

Tests of market efficiency have long been confounded by the joint-hypotheses dilemma, which states that conclusions regarding market efficiency are always conditioned on an equilibrium model for stock returns. According to Fama (1998), this caveat limits our profession's ability to confidently reject market efficiency despite numerous empirical challenges.

In view of this fundamental dilemma, Hogan, Jarrow, Teo, and Warachka (2004) (HJTW hereafter) develop an innovative technique for testing market efficiency which determines whether persistent anomalies constitute statistical arbitrage opportunities. Their statistical arbitrage framework replaces the standard t-statistic on the intercept of excess returns with a more stringent test of market efficiency that examines *multiple* t-statistics derived from dollar denominated trading profits. Moreover, an important advantage of the statistical arbitrage methodology is its ability to circumvent the joint-hypotheses dilemma. As with arbitrage opportunities, the definition of statistical arbitrage is independent of any equilibrium model or formulation for expected returns, and its existence contradicts market efficiency. Indeed, by appealing to arbitrage, assumptions on investor preferences are minimized.[1]

Statistical arbitrage is motivated by several generalizations of arbitrage (Benardo and Ledoit, 2000; Cochrane and Saá-Requejo, 2000; Carr, German, and Madan, 2001). Indeed, a standard finite horizon arbitrage opportunity is a special case of statistical arbitrage. Unlike the Arbitrage Pricing Theory (APT) of Ross (1976) derived from a cross-sectional limit of multiple assets at a specific timepoint, statistical arbitrage is a limiting condition across time.[2]

However, the statistical arbitrage framework of HJTW is deficient in several critical aspects. First, the HJTW definition of statistical arbitrage penalizes a trading strategy for producing profits with positive deviations from their expected value. Clearly, any investor benefits from positive deviations in their strategy's profitability. Bernardo and Ledoit (2000) raise a similar point with respect to the ubiquitous Sharpe ratio. Second, the statistical test employed by HJTW to detect statistical arbitrage opportunities is not consistent, resulting in a loss of power in empirical applications. Specifically, the Bonferroni approach in HJTW is only appropriate when the null hypothesis involves an intersection of sub-hypotheses, while the null hypothesis of market efficiency adopted

[1]For example, evaluating trading profits in "good" or "bad" states of the world is unnecessary.

[2]An alternative definition of statistical arbitrage found in Bondarenko (2003) is discussed in the next section.

in tests of statistical arbitrage is defined by a union.[3] Third, HJTW assume in their tests of statistical arbitrage that the trading profits tested are serially uncorrelated and normally distributed. However, serial correlation and departures from normality are well known regularities in empirical finance. Fourth, HJTW apply the statistical arbitrage framework to only two classes of anomalies: stock momentum and stock value based strategies.

This paper makes significant contributions to the literature on testing market efficiency by improving the theory underlying statistical arbitrage as well as its empirical implementation. On the theoretical front, we alter the fourth axiom in the HJTW definition of statistical arbitrage. Our modified axiom avoids penalizing a trading strategy for producing profits with positive deviations from their expected value. This improvement also eliminates a technical condition and the need to average the variance of trading profits by time when defining statistical arbitrage.

On the statistical front, we introduce a powerful testing methodology for detecting statistical arbitrage opportunities based on a Min-t statistic. Unlike the HJTW Bonferroni approach, our test procedure is statistically consistent when evaluating the null hypothesis of market efficiency. Specifically, the statistical arbitrage framework imposes several parametric constraints on an anomaly's trading profits, implying the null involves a union of sub-hypotheses. The elements of this union identify the t-statistics associated with the existence of statistical arbitrage. Intuitively, since rejecting even a single sub-hypothesis results in the acceptance of market efficiency, the Min-t statistic evaluates the "weakest" element in the union by focusing on the sub-hypothesis that is "closest" to being accepted. In applications of the statistical arbitrage framework, our improved statistical procedure allows for time-varying expectations, serial correlation, and non-normality in trading profit dynamics.

On the empirical front, we apply our robust statistical arbitrage tests to four broad classes of anomalies: individual stock momentum, individual stock value, individual stock liquidity, and industry momentum. Like HJTW, the stock momentum and stock value based strategies are based on those studied in Jegadeesh and Titman (1993) and Lakonishok, Shleifer, and Vishny (1994)

[3] This situation is aggravated when more sub-hypotheses are involved, causing a serious restriction on trading profit dynamics. For example, HJTW consider two processes for incremental trading profits; a constrained mean (CM) model with constant expected trading profits and a generalized unconstrained mean (UM) model. The UM model allows for time-varying expected profits but requires additional sub-hypotheses.

respectively. The stock liquidity strategies buy/short stocks with the lowest/highest past stock trading volume in the spirit of Brennan, Chordia, and Subrahmanyam (1998)[4] while the industry momentum strategies buy/short industries with the highest/lowest past returns as in Moskowitz and Grinblatt (1999). Furthermore, we also apply the statistical arbitrage tests to the Fama and French (1993) HML, SMB, and RMRF risk factors.[5] These empirical extensions demonstrate the applicability of the statistical arbitrage framework to strategies beyond those studied in HJTW.

Using our improved statistical arbitrage methodology, we find that over 50% of the strategies violate market efficiency. However, none of the risk factors are statistical arbitrages despite the fact that HML and RMRF both yield statistically positive trading profits. This result is consistent with their role as risk premiums which are justifiable in equilibrium.

By calculating the probability of loss for each statistical arbitrage opportunity, we also identify *dominant* strategies with rapidly declining loss probabilities within each class of anomalies. This is particularly relevant for short lived investors or those with limited capital who are concerned with incurring intermediate losses. Such investors include fund managers, who typically face the risk of retrenchment after a few years of poor performance (see Lakonishok and Vishny (1997)).

We find that the dominant stock momentum strategy is one with a formation period of six months and a holding period of nine months, while the dominant stock value strategy is one based on book-to-market ratios with a formation period of one year and a holding period of five years. Nonetheless, the value strategies which produce statistical arbitrage most consistently are those derived from sales growth. In addition, although the industry momentum strategies converge to arbitrage almost as quickly as those of stock momentum, their success is concentrated in portfolios with shorter three month formation periods while the success of the latter is concentrated in portfolios with longer six month formation periods.[6] In contrast, the stock liquidity strategies converge to arbitrage very slowly relative to the other types of anomalies.

[4]According to Brennan, Chordia, and Subrahmanyam (1998), stock trading volume provides incremental explanatory power on the cross-section of stock returns after adjusting for momentum, size, and book-to-market effects.

[5]RMRF is the market return in excess of the risk free rate. HML is the return of the top 30% of stocks minus the return of the bottom 30% of stocks sorted by book-to-market. SMB is the return of the bottom 30% of stocks minus the return of the top 30% of stocks sorted by market capitalization.

[6]This corroborates the observation by Moskowitz and Grinblatt (1999) that unlike individual stock momentum, industry momentum is strongest in the short term.

The remainder of this paper is structured as follows. Section 2 reviews the theoretical underpinnings of the statistical arbitrage framework for testing market efficiency. Section 3 provides our modified definition of statistical arbitrage while Section 4 describes our improved statistical methodology. The data is discussed in Section 5 with empirical results presented in Section 6. Section 7 concludes and offers directions for future research.

2 Review of Statistical Arbitrage

Previous empirical tests in the market efficiency literature focus exclusively on excess returns. However, positive excess returns may result from risk premiums associated with an equilibrium model. Thus, the joint-hypotheses dilemma confounds traditional market efficiency tests. In contrast, as with arbitrage opportunities, the existence of statistical arbitrage rejects all candidate models of market equilibrium.[7] In particular, their Sharpe ratio and contribution to expected utility are inconsistent with well functioning financial markets.

Bondarenko (2003) also develops a model-free test of market efficiency capable of circumventing the joint-hypotheses dilemma, and utilizes the statistical arbitrage terminology. However, several fundamental differences exist between the HJTW framework and Bondarenko (2003), despite both approaches appealing to arbitrage. First, the definition of statistical arbitrage in Bondarenko (2003) operates over a finite time horizon and allows for potentially negative payoffs. Specifically, Bondarenko (2003) constrains the average terminal payoff to be non-negative. In contrast, to study the implications of an anomaly's persistence, we require the probability of a loss to decline towards zero without specifying a terminal date. Second, Bondarenko requires an asset's risk neutral density, with the expected payoff conditioned on future information under the assumption that the pricing kernel is path independent. In contrast, we limit our attention to the empirical (or statistical) probability measure underlying observed trading profits. Third and most important, the two approaches differ significantly in their intended applications. To obtain an asset's risk neutral density and condition on its terminal payoff, the approach of Bondarenko (2003) is limited to

[7]Statistical arbitrage is a sufficient but not necessary condition for market inefficiency. Furthermore, although no statistical arbitrage does not imply market efficiency, we often refer to these concepts interchangeably given our empirical objective.

markets with liquid options while HJTW is suitable for testing a wide range of persistent anomalies.[8]

2.1 Trading Strategies and Statistical Arbitrage

A trading strategy determines the amount invested in the risky asset at each point in time. In our analysis, the risky asset consists of an equivalent buy/sell position in the long/short portfolios of previously documented anomalies. Therefore, a *trading* strategy is responsible for converting the returns of previously studied anomalies into dollar denominated profits, while we refer to the underlying returns themselves as being generated by *investment* strategies. Specifically, we consider investment strategies associated with stock momentum, stock value, stock liquidity, and industry momentum. Only the trading strategy subsequently applied to their returns is unique to the statistical arbitrage framework.

More importantly, one cannot challenge our empirical results by assuming a different trading strategy. For example, a trading strategy that rapidly decreases its exposure to the risky long minus short portfolio position is inappropriate since the *persistence* of the underlying anomaly is not addressed.[9] Furthermore, this strategy or any other alternative has no bearing on our empirical findings, since every potential strategy is not required to generate statistical arbitrage. Indeed, a single trading strategy is sufficient to confirm its existence (as with standard arbitrage opportunities). Appendix A discusses these issues in greater detail.

Instead, there are two alternatives when challenging our empirical results. First, our choice of trading strategy may be criticized. Second, the distributional assumptions we impose on trading profits may be scrutinized.

When implementing any trading strategy, the data snooping critique is relevant.[10] Therefore,

[8]Although options are available on a variety of US equities, they are not available on the trading profits of the anomalies investigated in the empirical finance literature.

[9]Such a strategy is the opposite of a doubling strategy which is excluded from standard (finite horizon) arbitrage theory.

[10]Haugen and Baker (1996) also investigate the survivorship and look-ahead bias as well as bid-ask bounce in their study of market efficiency. Section 5 details the caution undertaken when constructing our long and short portfolios to guard against the look-ahead bias. Naturally, if one rejects an underlying anomaly such as momentum due to data snooping considerations, then the statistical arbitrage test results are also not relevant. Indeed, the objective of the statistical arbitrage framework is to replace the intercept test on returns. However, the methodology does not overcome data snooping issues regarding the formation of anomalous returns.

the simplicity and long term viability of our approach are both important properties. More importantly, the statistical arbitrage framework is designed to replace the usual t-statistic on the intercept of abnormal returns. Consequently, our intention is not to "distort" the profitability of anomalies but examine whether their persistence enables them to produce arbitrage over a long horizon.

For example, the threat posed to market efficiency by investment strategies such as momentum stems from their simplicity and use of public information. Although extensive searching for ex-post zero-cost strategies which generate positive returns is problematic, momentum has survived this criticism. Analogously, when converting a sequence of momentum returns into dollar denominated trading profits, the subsequent trading strategy could manipulate returns in a manner that yields statistical arbitrage. However, the investment strategies which generate momentum returns are far more complex than our trading strategy which transforms these returns into trading profits.

In summary, the first constraint on a trading strategy emanates from the potential for data snooping, while the second requires the underlying anomaly's persistence to be captured. Third, when applied to a long term price process consistent with market efficiency, the chosen trading strategy cannot generate statistical arbitrage. For emphasis, a strategy which fails one of these conditions is not suitable for implementation. However, its existence does not invalidate the statistical arbitrage framework but merely eliminates this strategy from consideration.

2.2 Trading Strategy for Converting Returns into Trading Profits

When converting the returns of the stock momentum and value anomalies into dollar denominated trading profits, HJTW's approach has less exposure to the risky long minus short portfolio position than the trading strategy we implement. In particular, HJTW buy/sell a constant \$1 amount of the long/short portfolios throughout their entire sample period.

However, over our 35 year sample period (1965 to 2000), the value of \$1 declines. Therefore, we gradually increase our position in the risky portfolios by the accrued value of the money market account (initialized at \$1). Appendix A offers more details on the construction of trading profits. Specifically, our implementation utilizes equation (15) instead of equation (13) employed by HJTW.

Nonetheless, the simplicity of HJTW's trading strategy and their insistence on equivalent long/short positions in the risky portfolios are preserved to correctly examine an anomaly's persistence.

2.3 Definitions and Hypotheses

A statistical arbitrage opportunity requires the trading profits of a zero cost, self-financing trading strategy to satisfy four axioms. Specifically, cumulative discounted trading profits constitute a statistical arbitrage if they have a positive expectation, a declining time-averaged variance, and a probability of a loss converging to zero.

Let $\{v_i\}$ for $i = 1, \ldots, n$ be a sequence of discounted portfolio values generated by a self-financing trading strategy. We denote $v(n) = \sum_{i=1}^{n} \Delta v_i$ as the trading strategy's cumulative discounted trading profit, with its incremental components represented by Δv_i.

Definition 1 *A statistical arbitrage is a zero initial cost, self-financing trading strategy with cumulative discounted trading profits $v(n)$ such that:*

1. $v(0) = 0$
2. $\lim_{n\to\infty} E^P[v(n)] > 0$
3. $\lim_{n\to\infty} P(v(n) < 0) = 0$ *and*
4. $\lim_{n\to\infty} \frac{Var^P[v(n)]}{n} = 0 \quad$ *if* $P(v(n) < 0) > 0 \;\; \forall n < \infty$.

To test for statistical arbitrage, we begin by assuming the following process for incremental trading profits[11]

$$\Delta v_i = \mu i^{\theta} + \sigma i^{\lambda} z_i \, , \tag{1}$$

where z_i are i.i.d. $\mathcal{N}(0, 1)$ random variables, although the assumptions of normality and independence are subsequently relaxed. The initial quantities $z_0 = 0$ and Δv_0 are both zero by definition. The parameters σ and λ determine the volatility of incremental trading profits while the parameters μ and θ specify their corresponding expectation. In addition, observe that the process for incremental trading profits is nonstationary when θ or λ is nonzero.

[11]A geometric Brownian motion (lognormal distribution) which prevents negative values is inappropriate for modeling cumulative or incremental trading profits. Instead, an arithmetic Brownian motion is suitable for the difference between two portfolios (long minus short) over a Δ time interval. For a profitable strategy, the functions i^{θ} and i^{λ} alter this arithmetic process to account for the increasing investment in the riskfree asset over time. Further justification for this process is found in HJTW (using a Taylor series expansion) and Appendix A of this paper.

Although a trading profit process is required to facilitate empirical testing in the statistical arbitrage framework, every parametric statistical procedure has an underlying distributional assumption. Indeed, the linear regressions which dominate the traditional empirical anomalies literature generally assume normality.

It is critically important to emphasize that studying different trading profit processes in the statistical arbitrage framework is not comparable to assuming multiple models of market equilibrium. When testing for statistical arbitrage, this paper enables researchers to select the preferred trading profit dynamic depending on its time dependence, autocorrelation, and normality. Specifically, existing statistical procedures such as the Akaike Information Criteria (AIC) are capable of identifying the preferred model for describing the evolution of trading profits. In contrast, the traditional market efficiency approach specifies an equilibrium (or expected return) model apriori, the empirical validity of which is a maintained assumption that is not explicitly tested.

Furthermore, as with any statistical test, the parameters estimated from trading profits are derived from a specific set of data. Hence, the acceptance or rejection of market efficiency is with respect to a given sample period. Indeed, as discussed in HJTW, statistical arbitrage is intended to replace the usual t-statistic on the intercept of excess returns which has a similar limitation.

We implement two tests for statistical arbitrage under the assumption that trading profit innovations are uncorrelated with a normal distribution. The model described in equation (1) represents the unconstrained mean (UM) model which allows for time-varying expected trading profits. We also consider a more restrictive constrained mean (CM) model that assumes constant expected trading profits by setting θ equal to zero. Consequently, the CM version of statistical arbitrage has incremental trading profits evolving as

$$\Delta v_i = \mu + \sigma i^{\lambda} z_i \, . \tag{2}$$

According to HJTW, statistical arbitrage opportunities exist in the UM model when the following sub-hypotheses hold jointly:[12]

1. H1: $\mu > 0$

2. H2: $\lambda < 0$

[12]The third hypothesis $\theta > \max\{\lambda - \frac{1}{2}, -1\}$ actually contains two hypotheses but the second component $\theta > -1$ is a technicality (see Theorem 1 of HJTW) while the remaining three conditions have economic interpretations.

3. H3: $\theta > \max\left\{\lambda - \frac{1}{2}, -1\right\}$.

The first sub-hypothesis tests for positive expected profits while the second implies the trading strategy's time-averaged variance declines over time. The third sub-hypothesis ensures that a potential decline in expected trading profits does not prevent convergence to arbitrage. This restriction involves the trend in expected profits as well as volatility and allows for negative θ values. For the CM version of statistical arbitrage in equation (2), the third sub-hypothesis is eliminated.

2.4 Correlated Incremental Trading Profits

Given the manner in which financial anomaly portfolios are typically constructed, autocorrelation may be manifest in their incremental trading profits[13]. To address this issue, we allow the innovations of equation (1) to follow an MA(1) process given by

$$z_i = \epsilon_i + \phi\epsilon_{i-1}\,, \tag{3}$$

where ϵ_i are i.i.d. $\mathcal{N}(0,1)$ random variables. We abbreviate the incremental trading profit assumption in equation (1), modified to incorporate serially correlated innovations described by equation (3), as the UMC model. The corresponding model with constant expected incremental trading profits but serially correlated innovations is abbreviated CMC, and combines equation (2) with equation (3).

As proved in HJTW, the presence of an MA(1) process neither alters the conditions for statistical arbitrage nor increases the number of sub-hypotheses. However, including the additional parameter ϕ may improve the statistical efficiency of the remaining parameter estimates and avoid inappropriate standard errors.

2.5 Probability of Loss

The probability of a trading strategy generating a loss after n periods, as in the third axiom of Definition 1, depends on the μ, σ, λ, θ, and ϕ parameters as follows:

$$\Pr\{\text{Loss after } n \text{ periods}\} \quad = \quad \Phi\left(\frac{-\mu\sum_{i=1}^{n} i^{\theta}}{\sigma(1+\phi)\sqrt{\sum_{i=1}^{n} i^{2\lambda}}}\right)\,, \tag{4}$$

[13]This may be driven by negative serial correlation in stocks or cross-autocorrelation among stocks.

where $\Phi(\cdot)$ is the cumulative standard normal distribution function. This probability converges to zero at a rate which is faster than exponential as shown in HJTW. Observe that ϕ directly influences the convergence rate to arbitrage.

The UMC trading profit process includes all five parameters in equation (4) while the associated loss probability for the CMC model has θ set equal to zero. By ignoring serial correlation in trading profits, both the UM and CM models have $\phi = 0$ with the CM model further constraining θ to be zero. Thus, equation (4) is the most general expression for the convergence rate to arbitrage and nests the more restrictive trading profit assumptions.

3 Modified Definition of Statistical Arbitrage

This section illustrates the very conservative nature of the previous statistical arbitrage definition when rejecting market efficiency. In particular, the fourth axiom in Definition 1 is scrutinized. The example motivates a modified fourth axiom along with updated sub-hypotheses that are summarized in Proposition 1. We begin with the following example.

Example 1 *Consider a series of incremental trading profits that constitute statistical arbitrage, and the consequences of adding an independent Bernoulli process representing non-negative jumps. For example, over a trading interval* $(i-1, i]$, *assume incremental trading profits,* $\Delta v(i) = v(i) - v(i-1)$, *evolve as*

$$\Delta v(i) = \mu i^{\theta} + \sigma i^{\lambda} z_i + i^{\delta} B_i \,,$$

where μ, σ, *and* δ *are positive constants while* λ *is negative.* B_i *is a sequence of i.i.d. Bernoulli random variables with zero mean and unit variance. In the absence of the Bernoulli process* ($B_i = 0$ *for all* i), *the trading profits are consistent with statistical arbitrage provided* $\theta > \lambda - \frac{1}{2}$. *Denote the probability underlying the Bernoulli process as* $\pi \in (0,1)$. *Observe that the expected value of the trading profits increases from* μi^{θ} *to* $\mu i^{\theta} + i^{\delta}\pi$ *with the addition of positive jumps since* $E[B_i] = \pi$.

However, the variance of the jump component is $i^{2\delta}\pi(1-\pi)$ *for each increment. Therefore, for* $\delta \geq 0$, *the time-averaged variance of the jump process equals*

$$\pi(1-\pi)\,\frac{1}{n}\sum_{i=1}^{n} i^{2\delta} \,,$$

which does not converge to zero.

In economic terms, the above example presents a trading strategy which is rejected as being a statistical arbitrage opportunity, despite being more desirable than another which satisfies the required criteria. Hence, the HJTW statistical arbitrage definition is too conservative in rejecting market efficiency. The problem stems from the asymmetry between desirable positive deviations and detrimental negative deviations, a property which compromises the ability of variance to properly measure risk. This shortcoming motivates a modified fourth axiom that evaluates the semi-variance of incremental trading profits.

As a result of Example 1, consider the following definition for statistical arbitrage with a modified fourth axiom.

Definition 2 *A statistical arbitrage is a zero initial cost, self-financing trading strategy with cumulative discounted trading profits $v(n)$ and incremental discounted trading profits $\Delta v(n)$ such that:*

1. $v(0) = 0$
2. $\lim_{n\to\infty} E^P[v(n)] > 0$
3. $\lim_{n\to\infty} P(v(n) < 0) = 0$ *and*
4. $\lim_{n\to\infty} Var\left[\Delta v(n) | \Delta v(n) < 0\right] = 0.$

Observe that the first three axioms are identical to the previous statistical arbitrage definition. Only the fourth axiom is altered

$$\lim_{n\to\infty} Var\left[\Delta v(n) | \Delta v(n) < 0\right] = 0\,. \tag{5}$$

Under Definition 2, investors are only concerned about the variance of a potential "drawdown" in wealth. Provided the incremental trading profits are non-negative, their variability is not penalized. Therefore, when Definition 2 is applied to the trading profits of Example 1, large positive incremental trading profits caused by the Bernoulli process no longer prevent statistical arbitrage from being detected.

Three other observations are also worth emphasizing. First, Definition 2 continues to contain standard finite horizon arbitrage opportunities when the arbitrage profit is invested in the money market account. Second, in contrast to Definition 1, imposing the technical condition "if $P(v(n) <$

0) > 0 for all $n < \infty$" on the fourth axiom is no longer required. Third, since the fourth axiom pertains to incremental trading profits, normalizing the variance by time is unnecessary.

The economic content of the fourth axiom, in both Definition 1 as well as 2, stems from not having a finite horizon T at which point an arbitrage profit is realized. As compensation for this uncertainty, structure is imposed on the "risk" profile of the trading strategy across time. Specifically, both of the fourth axioms instill the limits of arbitrage concept into the statistical arbitrage framework by requiring intermediate trading profits to become less risky.

3.1 Statistical Implementation

The following proposition facilitates empirical tests of statistical arbitrage on incremental trading profits under Definition 2.

Proposition 1 *Under the modified fourth axiom in equation (5), a trading strategy generates statistical arbitrage if incremental trading profits satisfy the following conditions:*

H1: $\mu > 0\,,$

H2: $\lambda < 0$ ***or*** $\theta > \lambda\,,$

H3: $\theta > \max\left\{\lambda - \frac{1}{2}, -1\right\}\,.$

Appendix B provides the details which verify the $\theta > \lambda$ condition in $H2$. Observe that our proposed modification only applies to the UM and UMC models, enabling additional statistical arbitrage opportunities to be detected in circumstances where $\theta > 0$. Conversely, a negative point estimate for θ implies $H2$ reverts to the original hypothesis that $\lambda < 0$.

Intuitively, positive θ estimates are consistent with right-skewness in the incremental trading profits, a situation exploited by the earlier example. Figure 1 offers a visual illustration of the modified fourth axiom in terms of the boundary between no statistical arbitrage and statistical arbitrage. Observe that the upper half of the first quadrant (above the 45 degree line) is classified as a statistical arbitrage opportunity under the modified, but not the original, definition.

As a final observation, the probability of a loss in equation (4) is unaltered by the modified fourth axiom. Furthermore, the economic consequences of a statistical arbitrage opportunity are preserved in terms of its Sharpe ratio and contribution to expected utility. Thus, the original justification for statistical arbitrage contradicting market efficiency continues to apply under Definition 2.

To summarize, this section proposes a weaker set of axioms for testing market efficiency using statistical arbitrage that prevents positive fluctuations in incremental trading profits from being penalized. This modification preserves the important properties of the original definition for statistical arbitrage and yields a similar statistical test for its existence.

4 Robust Tests of Statistical Arbitrage

In this section, we provide a robust statistical methodology to test for statistical arbitrage. Although statistical tests may be conducted with either statistical arbitrage or no statistical arbitrage as the null, the accepted paradigm has the null hypothesis being market efficiency.

The hypothesis of market inefficiency, namely the existence of statistical arbitrage, consists of joint restrictions on the parameters underlying the evolution of trading profits. For the UM model, the following restrictions have to be satisfied simultaneously for a statistical arbitrage opportunity to exist:[14]

1. $R_1 : \mu > 0$ and
2. $R_2 : -\lambda > 0$ or $\theta - \lambda > 0,$ and
3. $R_3 : \theta - \lambda + \frac{1}{2} > 0$ and
4. $R_4 : \theta + 1 > 0\,.$

Thus, statistical arbitrage is defined by an *intersection* of sub-hypotheses. Conversely, the no statistical arbitrage null hypothesis involves a *union* of sub-hypotheses (a consequence of DeMorgan's Laws). In particular, the no statistical arbitrage null hypothesis is written as:

1. $R_1^c : \mu \leq 0$ or
2. $R_2^c : -\lambda \leq 0$ and $\theta - \lambda \leq 0,$ or
3. $R_3^c : \theta - \lambda + \frac{1}{2} \leq 0$ or
4. $R_4^c : \theta + 1 \leq 0\,.$

[14]A slight change of notation is adopted to separate H3 into two restrictions to facilitate the exposition of the proposed test.

Therefore, market efficiency is accepted provided a single sub-hypothesis R_i^c is satisfied. Statistically, the no statistical arbitrage null hypothesis presents a challenge as the Bonferroni procedure applies to an intersection, not union, of sub-hypotheses.[15] Appendix C examines the Bonferroni approach in HJTW and highlights its lack of power as the number of sub-hypotheses increase.

Given the limitations of the Bonferroni approach when testing for statistical arbitrage, this section proposes a new methodology centered on the Min-t statistic. We first consider trading profit innovations that are assumed to be normally distributed and serially uncorrelated. In these circumstances, critical values for the Min-t test procedure are estimated using Monte Carlo simulation. We then allow the innovations to be non-normal as well as serially dependent and estimate p-values for the Min-t statistics using a bootstrap procedure.

4.1 Monte Carlo Procedure for Uncorrelated Normal Errors

When each R_i is considered separately, the t-statistics $t(\hat{\mu})$, $\left\{t(-\hat{\lambda})\,,\,t(\hat{\theta}-\hat{\lambda})\right\}$, $t\left(\hat{\theta}-\hat{\lambda}+\frac{1}{2}\right)$, and $t(\hat{\theta}+1)$ test the restrictions R_1, R_2, R_3, and R_4 respectively, where hats denote the MLE parameter estimates.

Since all the restrictions in Proposition 1 must be simultaneously satisfied to reject the null hypothesis of no statistical arbitrage, the minimum of their associated t-statistics serves as the rejection criterion. Therefore, the accompanying test for statistical arbitrage based on Proposition 1 evaluates the following Min-t statistic[16]

$$\text{Min-}t = \text{Min}\left\{t(\hat{\mu}),\, t\left(\hat{\theta}-\hat{\lambda}+\tfrac{1}{2}\right),\, t(\hat{\theta}+1),\, \text{Max}\left[t(-\hat{\lambda}),\, t(\hat{\theta}-\hat{\lambda})\right]\right\}\,. \tag{6}$$

Intuitively, the Min-t statistic evaluates the "weakest" element in the union by focusing on the sub-hypothesis that is "closest" to being accepted. Thus, the null of no statistical arbitrage is rejected if Min-$t > t_c$, where the critical value t_c depends on the test's significance level denoted α.

For the CM models, equation (6) becomes

$$\text{Min-}t = \text{Min}\left\{t(\hat{\mu}),\, t(-\hat{\lambda})\right\}\,. \tag{7}$$

Therefore, as alluded to in the previous section, Definitions 1 and 2 both have identical implementations in the CM and CMC models.

[15] See Gourieroux and Monfort (1995), Chapter 19, for an exposition of testing joint hypotheses using the Bonferroni procedure.

[16] The original sub-hypotheses for Definition 1 may be tested using a special case of equation (6).

As the null of no statistical arbitrage involves a family of t-distributions, rather than a single distribution, the probability of rejecting the null varies across different parameter values. However, the probability of rejecting the null cannot exceed α. In other words, we require

$$\Pr\{\text{Min-}t > t_c | \mu, \lambda, \theta, \sigma\} \leq \alpha \tag{8}$$

for all $(\mu, \lambda, \theta, \sigma)$ combinations satisfying the null. Thus, two issues have to be addressed. First, while the individual t-statistics have asymptotic standard normal distributions, their joint distribution is unknown. Hence, the theoretical distribution of the Min-t statistic is intractable. We propose to overcome this difficulty using Monte Carlo simulation. Second, to achieve a size-α test as in equation (8), the critical value t_c is maximized over the null's parameter space.

We first consider the CM model whose two statistical arbitrage sub-hypotheses are R_1 and R_2. Obviously, t_c is maximized when $(\mu, \lambda) = (0, 0)$. Furthermore, as the t-statistics are scale free, we are able to select an arbitrary value of σ when estimating t_c. We assume $\sigma = 0.01$, which approximates its sample MLE estimate in our later empirical study. To estimate t_c, residuals z_i are obtained from a normal random number generator to form the incremental trading profits Δv_i in equation (2) based on assumed model parameters $(\mu, \lambda, \sigma) = (0, 0, 0.01)$. The estimated parameters, their individual t-statistics and the corresponding Min-t statistic are then computed. This procedure is repeated 5,000 times, from which t_c is estimated as the $100(1 - \alpha)$ percentile of the Min-t statistics.

Note that the distribution of Min-t is a function of the sample size n. As the series of trading profits used in our empirical study vary from 324 to 414 observations, sample sizes of 300 and 400 are examined. However, the results for both values of n are similar. Overall, critical values of 0.4754, 0.7484, and 1.2694 at the 10%, 5%, and 1% significance levels are utilized in subsequent tests of the CM model. These critical values correspond to the largest estimates in the Monte Carlo simulations.

For the UM model, there are five inequality restrictions involving three parameters and not all the restrictions are necessarily binding. Thus, a model within the null family and on the boundary of all inequality restrictions is not available. Nonetheless, as the t-statistics that comprise the Min-t statistic are monotonic in the underlying restrictions, it is appropriate to focus on their boundaries. Consequently, μ is set to -10^{-6} while the λ and θ parameters are varied along the boundary of the no statistical arbitrage / statistical arbitrage region as depicted in Figure 1.[17]

To control the probability of the Type I error at the stated nominal level, the maximum simulated critical values across different parameters are utilized in subsequent UM tests. These are 0.4034, 0.6004, and 0.9074 at the 10%, 5%, and 1% significance level respectively.[18]

4.2 Bootstrap Procedure for Correlated Non-Normal Errors

The previous methodology assumes the innovations in incremental trading profits are normally distributed and serially uncorrelated. However, both assumptions have been shown to be dubious in empirical finance (see, for instance, Affleck-Graves and McDonald (1989) and Lo and MacKinlay (1988)). Thus, we relax these assumptions by allowing trading profit innovations to be non-normal and serially correlated.

However, the MA(1) process for innovations described by equation (3) introduces an unspecified nuisance parameter ϕ. Consequently, searching for the maximum critical values using Monte Carlo methods becomes intractable. In particular, the influence of ϕ on the individual components of the Min-t statistic is unknown, offering little guidance for a search strategy. Therefore, we employ a bootstrap procedure to estimate the p-values.

Brock, Lakonishok, and LeBaron (1992) introduce the bootstrap technique into the empirical finance literature to study technical trading rules. Since then, this procedure has been adopted by many authors including Bessembinder and Chan (1998) as well as Sullivan, Timmermann, and White (1999). Ruiz and Pascual (2002) provide an excellent survey of the bootstrap method in empirical finance.

The steps we employ in our bootstrap procedure for the UMC model are:

1. Estimate the parameters of the UM model with MA(1) errors using quasi-MLE and calculate the residuals $\hat{\epsilon}_i$ using the following equations:

$$\hat{z}_i = \frac{\Delta v_i - \hat{\mu}\, i^{\hat{\theta}}}{\hat{\sigma}\, i^{\hat{\lambda}}}$$

[17]Note that the influence of θ disappears when $\mu = 0$. We also vary the values of μ from -0.01 to -0.0001 and obtain similar results.

[18]Since Monte Carlo simulation is employed to estimate the critical values of the Min-t statistic in finite samples, the nonstationarity of the UM model when $\theta \neq 0$ bears no consequence on our test procedure.

and

$$\hat{\epsilon}_i = \hat{z}_i - \hat{\phi}\,\hat{\epsilon}_{i-1}\,,$$

with the starting value of $\hat{\epsilon}_0$ being zero. In addition, the Min-t statistic in equation (6) is calculated.

2. Sample with replacement a set of n residuals denoted $\{\epsilon_1^*, \ldots, \epsilon_n^*\}$ from the original set of residuals $\{\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n\}$.

3. Generate a bootstrap sample of trading profits Δv_i^* with the parameter values $(\mu, \lambda, \theta, \sigma) = \left(-10^{-6}, -\frac{1}{2}, -1, 0.01\right)$ and the MLE estimate $\hat{\phi}$ using the equations:[19]

$$z_i^* = \epsilon_i^* + \hat{\phi}\,\epsilon_{i-1}^*$$

and

$$\Delta v_i^* = \mu\, i^{\theta} + \sigma\, i^{\lambda} z_i^*\,.$$

4. Calculate the MLE estimates for Δv_i^* and hence the Min-t statistic, denoted Min-$t^\sharp$.

5. Repeat Steps 2 to 4 a total of 1,000 times. The estimated p-value of the Min-t statistic is given by the empirical percentage of bootstrapped Min-$t^\sharp$ values that are larger than Min-t calculated in Step 1.

Implementing the bootstrap procedure for the CMC model follows in an identical fashion with $(\mu, \lambda) = (0, 0)$. Note that the guidelines provided by Hall and Wilson (1991) as well as Horowitz (2001) are adhered to in our procedure.[20] Section 6 confirms the convergence of the bootstrap procedure.

[19]Under the null of no statistical arbitrage with normally and serially uncorrelated errors, these parameter values provide the largest critical value t_c in the Monte Carlo simulations.

[20]Horowitz (2001) points out that bootstrapping should be used to estimate a test's critical value based on an asymptotically pivotal statistic whose asymptotic distribution under the null does not depend on any unknown parameters. This condition is satisfied by our test as the t-statistics are asymptotically standard normal, and thus pivotal. Furthermore, Hall and Wilson (1991) argue that the resampling in the bootstrap process should be conducted in a manner that reflects the null.

5 Data and Terminology

Our sample period starts in January 1965 and ends in December 2000. Monthly equity returns data are derived from the Center for Research in Security Prices at the University of Chicago (CRSP). Our analysis covers all stocks traded on the NYSE, AMEX, and NASDAQ that are ordinary common shares (CRSP sharecodes 10 and 11), excluding ADRs, SBIs, certificates, units, REITs, closed-end funds, companies incorporated outside the U.S., and Americus Trust Components.

The stock characteristics underlying the trading strategies include book-to-market equity, cash flow-to-price ratio, earnings-to-price ratio, annual sales growth, and monthly trading volume. To calculate book-to-market equity, book value per share is taken from the CRSP/COMPUSTAT price, dividend, and earnings database. We treat all negative book values as missing. We take the sum of COMPUSTAT data item 123 (Income before extraordinary items (SCF)) and data item 125 (Depreciation and amortization (SCF)) as cash flow. Only data 123 item is used to calculate the cash flow if data 125 item is missing. To compute earnings, we draw on COMPUSTAT data item 58 (Earnings per share (Basic) excluding extraordinary items) and to compute the sales we utilize COMPUSTAT data item 12 (sales (net)). Share volume is the number of shares traded divided by the number of shares outstanding. All price and number of outstanding common shares information employed in the calculation of the ratios are computed at the end of the year.

To ensure that the accounting variables are known before hand and to accommodate variation in fiscal year ends among firms, sorting on stock characteristics is performed in July of year t using the accounting information from year $t-1$. Hence, following Fama and French (1993), to construct the book-to-market deciles from July 1st of year t to June 30th of year $t+1$, stocks are sorted into deciles based on their book-to-market equity (BE/ME), where the book equity is in the fiscal year ending in year $t-1$ and the market equity is calculated in December of year $t-1$. Similarly, to construct the cash flow-to-price deciles from July 1st of year t to June 30th of year $t+1$, the stocks are sorted into deciles based on their cash flow-to-price, where the cash flow is in the fiscal year ending in year $t-1$ and the price is the closing price in December of year $t-1$. Earnings-to-price is calculated in a similar fashion. All portfolios are rebalanced every month as some firms disappear from the sample over the 12-month period.

The individual stock momentum strategies we implement follow Jegadeesh and Titman (1993).

These strategies buy the top return decile and short the bottom return decile based on formation and holding period combinations of 3, 6, 9, and 12 months. The individual stock value strategies follow Lakonishok, Shleifer, and Vishny (1994) and buy the top decile and short the bottom decile of stocks based on book-to-market, cash flow-to-price or earnings-to-price ratios of the past year along with past sales growth over the past three years. These portfolios are then held for 1, 3, and 5 years. The individual stock liquidity strategies are based on stock trading volume and buy the bottom trading volume decile and short the top trading volume decile of stocks in the spirit of Brennan, Chordia, and Subrahmanyam (1998). The industry momentum strategies follow Moskowitz and Grinblatt (1999). Stocks are first classified into 20 industries based on their SIC codes.[21] The industry momentum strategy buys the top three return industries and shorts the bottom three return industries as in Moskowitz and Grinblatt (1999). Like the stock momentum strategies, the stock liquidity, and industry momentum strategies are based on formation and holding period combinations of 3, 6, 9, and 12 months. For all strategies, once the long and short portfolio returns are generated, a self-financing condition is enforced by investing (borrowing) trading profits (losses) at the riskfree rate. Riskfree rate data is obtained from Kenneth French's website.

Given the possible permutations of formation and holding periods, we investigate 16 stock momentum strategies, 12 stock volume strategies, 16 stock liquidity strategies, and 16 industry momentum strategies. We adopt the notational convention of JTx_y for the stock momentum strategy with a formation period of x months and a holding period of y months. The book-to-market, cash flow-price, earnings-to-price, and sales growth based value portfolios with a holding period of y years are denoted BMy, CPy, EPy, and SALEy respectively. The formation period for all the sales growth strategies is three years while that for the other value strategies is one year. The stock liquidity and industry momentum portfolios with a formation period of x months and a holding period of y months are abbreviated VOLx_y and INDx_y respectively.

[21]The 20 industries are mining, food, apparel, paper, chemical, petroleum, construction, primary metals, fabricated metals, machinery, electrical equipment, transport equipment, manufacturing, railroads, other transportation, utilities, department stores, retail, financial, and others. We refer the interested reader to Moskowitz and Grinblatt (1999) for further details.

6 Empirical Results

We now discuss the results from applying our improved statistical arbitrage methodology to four anomalies described in the previous section: stock momentum, stock value, stock liquidity, and industry momentum strategies. Our analysis implements four trading profit models summarized in Section 2: CM (constrained mean), UM (unconstrained mean), CMC (constrained mean with correlation), and UMC (unconstrained mean with correlation). The UM model allows for time variation in expected trading profits, while its CM counterpart has these being constant. Their respective UMC and CMC extensions permit autocorrelation and non-normality in trading profits.

The effects of transaction costs, margin requirements, additional reserves for short-selling, higher borrowing rates than lending rates, and the exclusion of small stocks on statistical arbitrage opportunities are investigated in HJTW. Despite these market frictions, conclusions regarding their existence are not seriously compromised for the CM model. Therefore, we focus our attention in this paper on different trading profit assumptions, rather than on replicating previous robustness tests for the influence of market frictions.[22]

6.1 General Findings

The primary benefit of our Min-t test approach is the statistical power it provides when investigating trading profits dynamics with time-varying expectations as well as serially correlated non-normal innovations. Without this statistical power, more complex trading profit formulations cannot be reliably examined. For example, the statistical arbitrage test results for both the CM and the UM models are presented in Table 1.[23] Unlike the Bonferroni test procedure in HJTW which cannot detect any UM statistical arbitrage opportunities amongst the stock value anomalies, our results reveal a strong congruence between the CM and UM specifications for the sales strategies.

As discussed in Subsection 4.1, the critical Min-t values in Table 1 are estimated from a large scale Monte Carlo experiment, since the errors are normally distributed and uncorrelated in the UM

[22] Although the stock liquidity strategies involve buying stocks with low trading volume, they also short stocks with high trading volume. Thus, the short-selling costs associated with these strategies may be lower than those of the three anomalies.

[23] Allowing for serial correlation does not introduce a systematic bias into any of the CM and UM parameters versus those of the CMC and UMC models reported in Tables 2 and 3 respectively. Therefore, the estimated CM and UM parameters are omitted for brevity but available upon request.

and CM models. As a robustness check and to cross-validate the bootstrap methodology developed in Subsection 4.2 for the CMC and UMC models, we apply that methodology to the CM and UM models by constraining the autocorrelation coefficient ϕ to be zero. With minor exceptions, the resulting bootstrapped p-value estimates for the CM and UM models reported in Table 1 agree with those from the Monte Carlo procedure (which assumes $\phi = 0$ as well as normality). This reassuring result indicates convergence of the bootstrap procedure, and demonstrates the robustness of the Min-t statistic with respect to the assumption of normality.

It is also interesting to note that the CMC results for λ, σ, and ϕ in Table 2 and those of its UMC counterpart in Table 3 are similar. This consistency attests to our statistical procedure's accuracy. As expected, whether or not the θ parameter is calibrated influences the estimate of μ. However, negative θ estimates in Table 3 are not necessarily indicative of a diminishing anomaly. By construction, a profitable trading strategy increases the amount invested in the riskfree asset over time. Thus, declines in expected trading profits may reflect a smaller proportion of wealth being exposed to the risky long and short portfolios (see Appendix A for details).

In Table 4, we summarize the number of strategies that produce statistical arbitrage at the 5% and 10% significance levels. The numbers are presented by strategy class (stock momentum, stock value, stock liquidity, or industry momentum) and by trading profit specification (CM, UM, CMC, UMC). Table 5 provides additional information at the 5% significance level by detailing which of the four trading profit models is preferred according to the Akaike Information Criteria. With the exception of the stock momentum strategies which are sensitive to the estimation of θ, the statistical arbitrage results are generally consistent across the four trading profit dynamics.

Inferences regarding the presence of statistical arbitrage are usually unchanged after relaxing the twin assumptions of normality and serial independence. In particular, Table 4 indicates that most of the portfolios that test positive for statistical arbitrage under the CM (UM) formulation remain statistical arbitrages in the CMC (UMC) version of the test. With the exception of stock momentum, the significance of the autocorrelation coefficient estimates ϕ, and the lack of significance for the change in expected profits estimated by θ in Table 3, suggest that preference should usually be allocated to the CMC model. For stock momentum, the CM model is usually preferred as serial correlation in trading profit innovations is less prevalent.[24] These statements are reinforced

[24]The negative ϕ estimates for stock momentum may result from negative serial correlation in stock returns over

by the Akaike Information Criteria (AIC) results in Table 5.

Overall, Tables 4 and 5 report that out of 60 portfolios, almost half produce statistical arbitrage at the 5% significance level, and at least 37 portfolios yield statistical arbitrage at the 10% level. The number of statistical arbitrage opportunities cannot be attributed to the procedure's Type I error (even if the strategies are not independent). Hence, it is reasonable to conclude that our empirical results contradict the Efficient Markets Hypothesis.

6.2 Statistical Arbitrage Opportunities Across Trading Strategy Classes

In this subsection, we compare the statistical arbitrage opportunities across the four classes of trading strategies (stock momentum, stock value, stock liquidity, and industry momentum), and summarize their implications for market efficiency.

The stock liquidity strategies consistently exhibit statistical arbitrage opportunities as almost all of these 16 strategies test positive for statistical arbitrage at the 5% significance level across each of the four trading profit formulations. Thus, the capacity of the stock liquidity strategies to generate statistical arbitrage is largely independent of the formation and holding periods, as well as the specified trading profit process.

The stock momentum strategies of Jegadeesh and Titman (1993) exhibit less consistency across the four trading profit specifications. In particular, the CM and CMC test results differ from those of the unconstrained UM and UMC models. However, Table 3 indicates that the θ parameter is insignificant. Thus, the constrained models offer a more accurate description of trading profit dynamics as estimating the unnecessary θ parameter reduces the test's statistical power. Further evidence supporting the constrained models is provided by the Akaike Information Criteria in Table 5, while HJTW document the same effect using a Likelihood Ratio Test (LRT).

Of the stock value strategies, those based on past three-year sales growth exhibit statistical arbitrage with the greatest consistency. Indeed, all three of these strategies are statistical arbitrage opportunities at the 5% level, compared to one book-to-market strategy and one cash flow-to-price strategy. In contrast, none of the earnings-to-price strategies test positive for statistical arbitrage.

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short monthly horizons. In particular, the stocks in the long and short positions remain in these respective portfolios over several periods which could induce negative serial correlation in their returns. Karolyi and Kho (2004) find evidence of negative serial correlation in stock momentum returns.

Thus, our results suggest that value investors should consider past sales growth as an indication of value, as opposed to other popular metrics such as earnings-to-price.

It is intriguing to compare the results for the industry momentum strategies with the results for the stock momentum strategies. We find that the industry momentum strategies only test positive for statistical arbitrage with shorter formation periods such as three months. Moskowitz and Grinblatt (1999) observe a similar phenomenon as industry momentum appears strongest in the short term (at the one-month horizon). In the context of statistical arbitrage, while almost all the industry momentum portfolios have positive expected trading profits, only those with short formation periods yield statistical arbitrage. This pattern arises because with long formation periods, the volatility of the industry momentum profits fails to decline over time.

Within a given class of anomalies, the preferred description of trading profits according to the Akaike Information Criteria is identical for all but one strategy (SALE1). Although many statistical arbitrage opportunities under the UM and UMC models are revealed, allowing for time-varying trading profits is not warranted.

Furthermore, observe that when any of the four trading profit models detects statistical arbitrage for a given trading strategy, the preferred description usually yields statistical arbitrage. The exceptions to this generality are the book-to-market strategies with one and three year holding periods, along with four industry momentum portfolios.[25]

6.3 Probability of Loss

Another advantage of the statistical arbitrage methodology is its ability to yield the probability of a loss at specific time horizons. Shleifer and Vishny (1997) demonstrate the importance of capital constraints and intermediate losses to trading decisions. Given these considerations, not all statistical arbitrage opportunities are equally desirable and the convergence rates of the loss probabilities to zero (arbitrage) offer guidance regarding which strategies to pursue. From this perspective, our statistical procedures are of considerable practical importance as they identify dominant strategies in each of the four classes.

Table 5 records the number of months required for the loss probability to fall below five and one

[25] These are the IND3_12, IND6_3, IND6_6, and IND6_9 industry momentum strategies.

percent for each trading strategy that yields statistical arbitrage at the 5% level of significance.[26] All else being equal, equation (4) implies a positive autocorrelation coefficient ϕ reduces the convergence rate to zero. Conversely, even if time variation in expected incremental profits captured by θ is significantly negative, the rate of convergence to arbitrage may not be reduced as its calibration often results in a larger estimate of the profit parameter μ.

The dominant stock momentum and industry momentum strategies only require 71 and 66 months respectively for their loss probabilities to decline below five percent. The dominant stock momentum strategy has a formation period of six months and a holding period of nine months, while the dominant industry momentum strategy has a formation period of three months and a holding period of three months. Plots of these loss probabilities are found in Figures 2 and 3 respectively for each trading profit dynamic that yields statistical arbitrage at the 5% significance level.

The dominant value strategy derived from a book-to-market strategy with a formation period of one year and a subsequent five year holding period. Comparing across the four types of anomalies, this strategy experiences the most rapid convergence to arbitrage as only 41 months are required before its loss probability declines below 5%, with Figure 4 offering a visual illustration of this phenomena. Interestingly, the dominant industry momentum and value strategies consistently produce statistical arbitrage at the 5% significance level across all four trading profit specifications.

Observe that while fewer stock momentum, stock value, and industry momentum portfolios constitute statistical arbitrage opportunities in comparison to the stock liquidity portfolios, they converge to arbitrage more rapidly. The small expected trading profits associated with the stock liquidity strategies are responsible for their slow convergence rates. Thus, they require extremely patient investors.

[26] The entries in Table 5 are computed using equation (4) which assumes normality. Therefore, as a robustness check, distribution-free bootstrapped loss probabilities are also computed based on 10,000 trials. This bootstrap procedure searches every generated sample path to determine the relative frequency of incurring a loss at each monthly horizon. However, the results are nearly identical to those produced by equation (4) and are omitted for brevity but available upon request.

6.4 Test of Risk Premiums

Although a risk premium should enable investors to profit from bearing its risk, the resulting profits cannot generate statistical arbitrage and be compatible with an equilibrium model. Indeed, the excess compensation offered by a statistical arbitrage opportunity has no equilibrium justification. Thus, the role of a risk premium is not contradicted if its trading profits generate statistical arbitrage. Instead, such a result finds the premium's compensation excessive relative to that which is justifiable in equilibrium.[27]

Given the limited success of the BM1 strategy in producing statistical arbitrage, we investigate the profits from a Fama and French (1993) HML trading strategy as well as its SMB counterpart for comparative purposes.[28] In addition, we also analyze an equity premium proxied for by the market factor of Fama and French (1993) denoted RMRF. Appendix A demonstrates that an equity whose purchase is financed by riskfree borrowing cannot generate statistical arbitrage unless the volatility of equity is declining (while continuing to offer a positive premium). Consequently, testing the RMRF premium for statistical arbitrage serves as a robustness test of our empirical implementation. As with our earlier anomalies, the trading profits implied by the HML, SMB, and RMRF strategies are studied from January 1965 to December 2000.

Parameter estimates and statistical arbitrage test results for the three risk premiums are reported in Table 6. Empirically, none are found to produce statistical arbitrage, although the strategies are profitable since all their corresponding μ estimates are positive. Thus, the compensation these premiums provide is justifiable in equilibrium.

6.5 Final Observations

For comparison, we also implement HJTW's trading strategy (as described in equation (13) of Appendix A) when converting the returns generated by the four anomalies into dollar denominated trading profits. For every anomaly and all four trading profit formulations, the number of statistical

[27] As an extreme example, consider an economy in which the expected return of a stock is 30% per annum with a corresponding volatility of only 1%, while the riskfree rate is fixed at 2%. Intuitively, the magnitude of this equity premium is not compatible with equilibrium.

[28] Although, BM1 and HML have identical one year formation and holding periods, HML is derived from the 30^{th} and 70^{th} book-to-market percentiles of the NYSE, while BM1 is defined via the 10^{th} and 90^{th} book-to-market percentiles for the NYSE, NASDAQ, and AMEX markets.

arbitrage opportunities is found to increase. Thus, market efficiency is rejected even more strongly when HJTW's more conservative construction of trading profits is considered.[29]

Consequently, the single greatest improvement offered by this paper is the improved Min-t statistical procedure, and its associated advantage of allowing for serial correlation in trading profits. Indeed, when statistical arbitrage is detected for a trading strategy, the corresponding λ estimates are generally negative. Thus, trading profits are not sufficiently right-skewed in the strategies we implement to require the modified fourth axiom in Definition 2.

7 Conclusion

Given the importance of market efficiency to finance, every effort should be undertaken to accurately assess the validity of this fundamental tenet. Two important contributions for testing market efficiency using statistical arbitrage are proposed in this paper.

First, we modify one of the statistical arbitrage axioms. This theoretical improvement corrects the very conservative nature of the original statistical arbitrage definition when rejecting market efficiency. In addition, besides eliminating the need for imposing a technical condition on trading profits, our improved definition is more intuitive since averaging their variance by time is no longer required.

Second, a more powerful test procedure is provided that circumvents the limitations of Hogan, Jarrow, Teo, and Warachka (2004)'s Bonferroni approach. Empirically, we document the importance of our robust statistical tests on stock momentum, stock value, stock liquidity, and industry momentum strategies. Our improved methodology resolves the empirical disparity in Hogan, Jarrow, Teo, and Warachka (2004) by identifying statistical arbitrage opportunities when expected incremental trading profits are time-varying. The second contribution also allows for serial correlation and non-normality in trading profit innovations. This extension enables us to investigate the sensitivity of our decision to reject market efficiency with respect to these generalizations, without compromising our ability to detect statistical arbitrage opportunities.

By implementing our modified tests on four broad classes of well-known stock market anomalies, we uncover a large number of statistical arbitrage opportunities that are hard to reconcile with the

[29] Recall from Subsection 2.2 that HJTW maintain a constant \$1 position in the long and short portfolios over the entire sample period.

Efficient Markets Hypothesis. Furthermore, when ascertaining dominant strategies which converge most rapidly to standard arbitrage opportunities, incorporating autocorrelation into trading profits is crucial.

In summary, this paper improves the definition and implementation of statistical arbitrage to minimize the possibility of accepting market efficiency due to right-skewed trading profits or a lack of statistical power. Indeed, we confirm the appropriateness of statistical arbitrage as a test of market efficiency by describing the evolution of trading profits with time-varying processes that have autocorrelated and non-normal innovations.

Promising avenues for future research include testing other persistent anomalies, such as the abnormal returns from earnings announcements, analyst forecasts or changes in dividend policy, for statistical arbitrage.

References

[1] Affleck-Graves, J., and B. McDonald, 1989, "Nonnormalities and tests of asset pricing theories," *Journal of Finance*, 44, 889-908.

[2] Bernardo, A. E., and O. Ledoit, 2000, "Gain, loss, and asset pricing," *Journal of the Political Economy*, 108, 144–172.

[3] Bessembinder, H., and K. Chan, 1998, "Market efficiency and the returns to technical analysis," *Financial Management*, 21, 5–17.

[4] Bondarenko, O., 2003, "Statistical arbitrage and securities prices," *Review of Financial Studies*, 16, 875–919.

[5] Brennan, M. J., T. Chordia, and A. Subrahmanyam, 1998, "Alternative factor specifications, security characteristics and the cross-section of expected stock returns," *Journal of Financial Economics*, 49, 345–373.

[6] Brock, W. A., J. Lakonishok, and B. LeBaron, 1992, "Simple technical trading rules and the stochastic properties of stock returns," *Journal of Finance*, 47, 1731–1764.

[7] Carr, P., H. Geman, and D. Madan, 2001, "Pricing and hedging in incomplete markets," *Journal of Financial Economics*, 62, 131–167.

[8] Cochrane, J. H., and J. Saá-Requejo, 2000, "Beyond arbitrage: Good-deal asset price bounds in incomplete markets," *Journal of the Political Economy*, 108, 79–119.

[9] Fama, Eugene F., 1998, "Market efficiency, long-term returns, and behavioral finance," *Journal of Financial Economics*, 49, 283–306.

[10] Fama, Eugene F., and K. R. French, 1993, "Common risk factors in the returns on stocks and bonds," *Journal of Financial Economics*, 33, 3–56.

[11] Gourieroux, C., and A. Monfort, 1995, *Statistics and Econometric Models* (Vol 2), Cambridge Press, Cambridge, UK.

[12] Hall, P., and S. R. Wilson, 1991, "Two guidelines for bootstrap hypothesis testing," *Biometrics*, 47, 757–762.

[13] Haugen, R. A., and N. L. Baker, 1996, "Commonality in the determinants of expected stock returns," *Journal of Financial Economics*, 41, 401–439.

[14] Hogan, S., R. Jarrow, M. Teo, and M. Warachka, 2004, "Testing market efficiency using statistical arbitrage with applications to momentum and value trading strategies," *Journal of Financial Economics*, 73, 525–565.

[15] Horowitz, J. L., 2001, "The bootstrap and hypothesis tests in econometrics," *Journal of Econometrics*, 100, 37–40.

[16] Hull, J. C., 2000, *Options, Futures, and Other Derivative Securities* (4th ed.), Prentice Hall, New Jersey.

[17] Jegadeesh, N., and S. Titman, 1993, "Returns to buying winners and selling losers: Implications for stock market efficiency," *Journal of Finance*, 48, 65–91.

[18] Karolyi, G. A., and B. C. Kho, 2004, "Momentum strategies: Some bootstrap tests," *Journal of Empirical Finance*, 11, 509–536.

[19] Lakonishok, J., A. Shleifer, and R. W. Vishny, 1994, "Contrarian investment, extrapolation, and risk," *Journal of Finance*, 49, 1541–1578.

[20] Lo, A., and C. MacKinlay, 1988, "Stock market prices do not follow random walks: Evidence from a simple specification test," *Review of Financial Studies*, 1, 41-66.

[21] Margrabe, W., 1978, "The value of an option to exchange one asset for another," *Journal of Finance*, 33, 177–186.

[22] Moskowitz, T. J., and M. Grinblatt, 1999, "Do industries explain momentum?" *Journal of Finance*, 54, 1249–1290.

[23] Ross, S., 1976, "The arbitrage theory of capital asset pricing," *Journal of Economic Theory*, 13, 341–360.

[24] Ruiz, E., and L. Pascual, 2002, "Bootstrapping financial time series," *Journal of Economic Surveys*, 16, 271–300.

[25] Shleifer, A., and R. W. Vishny, 1997, "The limits of arbitrage," *Journal of Finance*, 52, 35–55.

[26] Sullivan, R., A. G. Timmermann, and H. L. White, 1999, "Data-snooping, technical trading rule performance and the bootstrap," *Journal of Finance*, 46, 1647–1691.

Appendices

A Trading Strategies and Statistical Arbitrage

The existence of statistical arbitrage opportunities is determined by the profits derived from a trading strategy. Although this strategy is not necessarily motivated by excess returns, consider the regression of raw returns y_{t_i} on factors f_{k,t_i}

$$y_{t_i} = \hat{\alpha}_0 + \sum_{k=1}^{K} \hat{\alpha}_k \, f_{k,t_i} + \hat{\epsilon}_{t_i} \,, \tag{9}$$

where the estimated $\hat{\alpha}_k$ coefficients for $K > 1$ are associated with premiums for market risk, size, book-to-market or other possibilities and $\hat{\epsilon}_{t_i}$ are i.i.d. errors. The multifactor regression representation in equation (9) contains CAPM and the Fama-French three factor model. If returns conform with the above specification, then $\hat{\alpha}_0$ is statistically insignificant.

There are two issues surrounding equation (9) in our context. First, there are no equilibrium models designed for long time horizons. Second, the parameter estimates are not traded assets since they cannot be bought or sold. Nonetheless, equation (9) may motivate an application of the statistical arbitrage methodology by returning to the portfolio(s) which generated y_{t_i}.

A.1 Buy and Hold Strategy

Let i represent the time index t_i for notational simplicity, and consider a \$1 investment in the long portfolio with \$1 of the short portfolio being sold at time zero. Define R^L and R^S as the return of the long and short portfolio respectively. Let $L(i) = \exp\left\{\sum_{j=1}^{i} R_j^L\right\}$ and $S(i) = \exp\left\{\sum_{j=1}^{i} R_j^S\right\}$ denote the long and short portfolios, which are linear combinations of limited liability assets whose prices cannot become negative.

A buy and hold strategy has cumulative trading profits equaling

$$V(i) \quad = \quad \$\left[L(i) - S(i)\right] \,, \tag{10}$$

with a discounted value $v(i) = V(i)e^{-ri}$ of

$$v(i) \quad = \quad \exp\left\{\sum_{j=1}^{i}\left(R_j^L - r\right)\right\} - \exp\left\{\sum_{j=1}^{i}\left(R_j^s - r\right)\right\} \quad \overset{def}{=} \quad l(i) - s(i)\,. \tag{11}$$

For ease of exposition, we return to continuous time and place a common lognormal structure on the two portfolios. This economy facilitates an illustrative analysis with the following definitions

$$\begin{aligned} l(t) \quad &= \quad l(0)\exp\left\{\left(R^L - r - \tfrac{1}{2}\sigma_L^2\right)t + \sigma_L W_t^L\right\} \\ s(t) \quad &= \quad s(0)\exp\left\{\left(R^S - r - \tfrac{1}{2}\sigma_S^2\right)t + \sigma_S W_t^S\right\} \,, \end{aligned}$$

where W_t^L and W_t^S are independent Brownian motions with corresponding volatilities denoted σ_L and σ_S. These dynamics are chosen to simplify computations although none of their individual parameters require calibration. Furthermore, having uncorrelated Brownian motions is also without loss of generality.

The increments of equation (11) are obtained via Ito's lemma (bi-variate version) on the function $v(l(t), s(t)) = l(t) - s(t)$. The difference between two lognormal processes has also been studied by

Margrabe (1978) in the context of an option to exchange two risky securities. Unlike the application of Ito's lemma in option pricing, there is no partial derivative with respect to time and the linear function has second derivatives equaling zero. Consequently, the increments of equation (11) over a Δ time interval are

$$\begin{aligned} dv(t) &= \left[(R^L - r)\,dt + \sigma_L\,dW_t^L\right] l(t) - \left[(R^S - r)\,dt + \sigma_S\,dW_t^S\right] s(t) \\ &= r\,v(t)\,dt + \left[R^L\,l(t) - R^S\,s(t)\right]\,dt + \left[\sigma_L\,l(t)\,dW_t^L - \sigma_S\,s(t)\,dW_t^S\right]\,, \end{aligned} \tag{12}$$

which are normally distributed but with a time-varying mean and variance that reflect the $l(t)$ and $s(t)$ portfolio values.

However, the buy and hold strategy yields trading profits that are very sensitive to the start date and the length of the time horizon being studied. For example, the long position increases exponentially as past gains increase the amount invested in this risky portfolio. Consequently, if profitable, this strategy has $l(t)$ becoming larger than $s(t)$, implying a disparity between the amount invested in the long portfolio and the amount sold of the short portfolio. Unfortunately, this property obscures the strategy's ability to capture an anomaly's persistence. Finally, equation (10) is not appropriate for frequent (monthly) realizations of intermediate gains and losses. These limitations are overcome in the next subsection.

To provide a connection with Appendix B of HJTW, note that the purchase of a risky (but limited liability) asset financed with riskfree borrowing has $L(t) = A(t)$ and $S(t) = \exp\{rt\}$, where r is the riskfree interest rate. Thus, $R^L = \mu$ and $\sigma_L = \sigma$ while $R^S = r$ and $\sigma_S = 0$, which reduces equation (12) to

$$dA(t) = (\mu - r)A(t)dt + \sigma A(t)dW_t\,.$$

As expected, the fourth axiom prevents the discounted geometric Brownian motion in the above equation from producing statistical arbitrage.

A.2 Trading Strategy Implementation

At time zero, there is no difference between our trading strategy and the previous buy and hold strategy. However, the trading strategy HJTW and we implement places the cumulative profit in the money market account each month. Thus, to properly capture an anomaly's persistence, we both maintain an equivalent buy/sell position in the risky long/short portfolios across time. As a result, only the R^L and R^S return sequences are necessary.

In contrast to equation (10), HJTW's trading strategy yields cumulative trading profits equaling

$$V(j) \quad = \quad \exp\{r\}\, V(j-1) + \$1 \left[\exp\left\{R_j^L\right\} - \exp\left\{R_j^S\right\}\right] \,. \tag{13}$$

Observe that this strategy ensures that \$1 of the risky long/short portfolio is bought/sold each month. Thus, profits (or losses) are harvested based on the returns of the long and short portfolios in the previous period. Consequently, trading profits are constructed recursively.

Furthermore, time-varying moments are induced by an allocation between the fraction of wealth invested in the riskfree asset versus the positions in the risky portfolios. In particular, \$1 is exposed to the risky long minus short position while the accumulated value $V_x(j-1)$ is deposited into (borrowed from) the money market account.

Overall, the return generated by equation (13) may be decomposed as

$$[1-\pi(j)]\exp\{r\} + \pi(j)\,\left[\exp\left\{R_j^L\right\} - \exp\left\{R_j^S\right\}\right] \tag{14}$$

for $\pi(j) \overset{def}{=} \frac{1}{1+V_x(j-1)}$ over a single time increment. Thus, the fraction $1-\pi(j)$ is invested in the riskfree asset while the remaining $\pi(j)$ percent is kept in the risky portfolios. A profitable strategy has $\pi(j) \to 0$, which justifies the trading profit specifications in Section 2 such as equation (1). Intuitively, this strategy creates a riskless "cash account" (with a zero investment) whose magnitude depends on the trading strategy's profitability.

Note that in applications of the statistical arbitrage methodology, the interest rate is not assumed to be constant nor is stationarity imposed on the asset returns. These features of the data provide additional sources of variability with respect to time.

Equation (13) may be altered by having an equivalent position, denoted $x(t)$, other than \$1 in the risky portfolios

$$V_x(j) = \exp\{r\}\, V_x(j-1) + \$x(j)\left[\exp\left\{R_j^L\right\} - \exp\left\{R_j^S\right\}\right] \,. \tag{15}$$

This strategy remains self-financing as $x(j)$ dollars of the long (short) portfolio are bought (sold) at time j. However, having $x(j) = B(j)$ gradually increases our exposure to the risky portfolios over time. As reported in Section 6, this trading strategy is less likely to induce statistical arbitrage.[30]

[30]Recall that our analysis of discounted incremental trading profits, denoted $\Delta v(t)$ in Section 2, accounts for the time-value-of-money.

The factor $\pi(j)$ in equation (14) is easily modified to become $\frac{B(j)}{B(j)+V_x(j-1)}$ with the intuition behind the risky and riskfree asset decomposition remaining. For clarification, HJTW have $x(j) \equiv 1$ although the value of \$1 declines over time.

For additional intuition, we may appeal to the most well-known application of arbitrage, the Black Scholes option pricing theory. Their trading strategy replicates a European call or put option, and cannot be separated from their final pricing formula. By way of contrast, Black Scholes derive the appropriate trading strategy for replicating an option assuming price dynamics to avoid arbitrage. Our framework requires the opposite perspective. We choose the simplest possible trading strategy and allow historical data to determine in-sample profit dynamics which may or may not produce arbitrage. Overall, the test for statistical arbitrage determines whether the simple self-financing $(x(j), -x(j))$ linear combination of portfolios yields a *traded asset* providing a positive expected profit, decreasing risk, and a declining loss probability whose incremental contributions are either positive or have declining variance.

A.3 Stock and Riskfree Asset

The purchase of an individual equity (or market index) financed by riskfree borrowing yields an equity premium. To simplify our analysis, we consider a linearized version of the cumulative trading profits generated by equation (15). Subsection 6.4 details the empirical implementation of the equity premium.

With $x(j) = B(j)$ defined as $(1+r)^j$ in equation (15), cumulative trading profits equal

$$\begin{aligned}
V_x(0) &= \$0 \\
V_x(1) &= \$B(1)\,\mathcal{N}\left(\mu - r, \sigma^2\right) + (1+r)V_x(0) \\
&= \$B(1)\,\mathcal{N}\left(\mu - r, \sigma^2\right) \\
V_x(2) &= \$B(2)\,\mathcal{N}\left(\mu - r, \sigma^2\right) + (1+r)V_x(1) \\
&= \$B(2)\left[\mathcal{N}\left(\mu - r, \sigma^2\right) + \mathcal{N}\left(\mu - r, \sigma^2\right)\right] \\
V_x(3) &= \$B(3)\,\mathcal{N}\left(\mu - r, \sigma^2\right) + (1+r)V_x(2) \\
&= \$B(3)\left[\mathcal{N}\left(\mu - r, \sigma^2\right) + \mathcal{N}\left(\mu - r, \sigma^2\right) + \mathcal{N}\left(\mu - r, \sigma^2\right)\right] ,
\end{aligned}$$

and so forth which leads to the following recursion

$$V_x(i) = \$B(i) \sum_{j=1}^{i} \mathcal{N}\left(\mu - r, \sigma^2\right) \ ,$$

with a discounted value of

$$v_x(i) = \$ \sum_{j=1}^{i} \mathcal{N}\left(\mu - r, \sigma^2\right) \ ,$$

whose increments $\Delta v_x(j)$ are distributed $\mathcal{N}\left(\mu - r, \sigma^2\right)$ and therefore fail to generate statistical arbitrage. Indeed, only if the variability of stock returns is decreasing over time (while the stock continues to offer a positive excess return) could statistical arbitrage ever be generated.

A.4 Constraint on Trading Strategy

For illustration, consider the influence of a general time-varying strategy on the distribution of incremental trading profits

$$\Delta v_x(j) \overset{d}{\sim} x(j)\mathcal{N}\left(\alpha_j, \nu_j^2\right) \ ,$$

where $\mathcal{N}\left(\alpha_j, \nu_j^2\right)$ describes the underlying returns of the anomaly being tested for statistical arbitrage. The cumulative (discounted) trading profit has the following distribution

$$v_x(i) = \mathcal{N}\left(\sum_{j=1}^{i} x(j)\alpha_j, \sum_{j=1}^{i} x^2(j)\nu_j^2\right) \ .$$

The above equation demonstrates the relationship between the trading strategy which converts returns into dollar denominated profits and its implied distribution.

However, a decreasing function $x(j)$ could satisfy the second axiom with

$$\sum_{j=1}^{i} x(j)\alpha_j$$

either not converging or converging to a positive number, with the fourth axiom satisfied through a reduction in the strategy's exposure to the risky position. For example, let α_j and ν_j^2 be positive constants and consider the trading strategy

$$x(j) = \frac{1}{j}$$

which has a sum that fails to converge,

$$\sum_{j=1}^{i} \frac{1}{j} \quad \rightarrow \quad \infty \, , \tag{16}$$

although its sum of squares

$$\sum_{j=1}^{i} \frac{1}{j^2} \quad \rightarrow \quad 0 \tag{17}$$

converges. This trading strategy satisfies the axioms of statistical arbitrage after drastically modifying the returns of the original anomaly.

However, recall that statistical arbitrage is intended to test whether the persistence of an anomaly is sufficient to yield arbitrage profits in the long run. Thus, we are testing whether the four classes of strategies in Section 5 violate market efficiency, not whether their returns are capable of being manipulated into a rejection of market efficiency. Indeed, the test for statistical arbitrage is designed to replace the single t-statistic on the intercept of excess returns, not distort the profitability of an existing anomaly. In addition, as with arbitrage, only a single trading strategy capable of generating statistical arbitrage is required for market efficiency to be violated.

In particular, a decreasing function $x^D(j) > x^D(j+1)$ is not viable in the long run as the role of the underlying anomaly in the analysis diminishes over time. Thus, an anomaly's persistence is not properly measured by a declining trading strategy. Indeed, such a strategy deliberately avoids the underlying anomaly as time progresses. Therefore, economic considerations dictate that trading strategies are constrained to have the property that $x(j+1) \geq x(j)$. Hence, the HJTW strategy is valid as well as equation (15) with $x(j) = B(j)$. However, only those rapidly declining strategies whose sum of squared terms converge, as in equation (17), are formally required to be excluded from consideration.

In the existing literature, doubling strategies are exogenously excluded from the set of arbitrage opportunities, a restriction that is justified by a wealth constraint.[31] In the context of statistical arbitrage, declining strategies are disallowed as they prevent the underlying anomaly's persistence from being measured. Moreover, such strategies would imply infinitesimally small transactions in the risky position which are not feasible, notwithstanding the transaction costs incurred to pursue no potential gain from the anomaly.[32]

[31] Even at the end of our 35 year sample period, $B(i)$, is less than \$9. Furthermore, the gradual increase in the risky position implied by equation (15) for $x(j) = B(j)$ does not pose a problem to a wealth constraint.

[32] For emphasis, our focus on purchasing/selling \$1 of the long/short portfolio also implies that fractions of *individual* securities are purchased or sold. However, scaling up this investment enables integer valued investments in the individual securities. Thus, the equivalent \$1 position is without loss of generality as we are ultimately concerned

B Verification of Semi-Variance Sub-Hypotheses

The quantity $Var\left[\Delta v(t)|\Delta v(t)<0\right]$ is computed from the distribution of $\Delta v(t)$, which equals $\mathcal{N}\left(\mu t^{\theta}, \sigma^2 t^{2\lambda}\right)$. The conditional variance is expressed as

$$
\begin{aligned}
Var\left[\Delta v(t)|\Delta v(t)<0\right] &= \frac{1}{\sqrt{2\pi\sigma^2 t^{2\lambda}}}\int_{-\infty}^{0}(x-\mu t^{\theta})^2 e^{\frac{-\left(x-\mu t^{\theta}\right)^2}{2\sigma^2 t^{2\lambda}}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-\mu t^{\theta}}{\sigma t^{\lambda}}}\left(\sigma t^{\lambda} y\right)^2 e^{\frac{-y^2}{2}}dy \\
&= \frac{\sigma^2 t^{2\lambda}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-\mu t^{\theta}}{\sigma t^{\lambda}}} y^2 e^{\frac{-y^2}{2}}dy \qquad (18)\\
&\leq \sigma^2 t^{2\lambda}\,, \qquad (19)
\end{aligned}
$$

after a change of variables $y=\frac{x-\mu t^{\theta}}{\sigma t^{\lambda}}$ which implies $\sigma t^{\lambda}dy = dx$. The inequality in equation (19) stems from

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-\mu t^{\theta}}{\sigma t^{\lambda}}} y^2 e^{\frac{-y^2}{2}}dy \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2 e^{\frac{-y^2}{2}}dy = 1\,,$$

since the second term equals the second moment (or variance) of a standard normal random variable. Thus, the constraint $\lambda < 0$ is a sufficient condition for the fourth axiom to hold. However, the integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-\mu}{\sigma}t^{\theta-\lambda}} y^2 e^{\frac{-y^2}{2}}dy$$

also converges to zero provided $\theta > \lambda$. Indeed, if $\theta > \lambda$, then $t^{\theta-\lambda}\to\infty$ as $t\to\infty$ which implies the range of integration declines to zero. Thus, a weaker version of the fourth axiom implies statistical arbitrage occurs if either $\lambda < 0$ or $\theta > \lambda$.

To provide an alternative perspective and confirm the above result, observe that the integral in equation (18) equals

$$\frac{\mu t^{\theta-\lambda}}{\sqrt{2\pi\sigma^2}}e^{\frac{-\mu^2 t^{2(\theta-\lambda)}}{2\sigma^2}} + N\left(\frac{-\mu t^{\theta-\lambda}}{\sigma}\right)\,. \qquad (20)$$

Although there is no closed form solution for the standard normal cdf, a polynomial approximation (for $x<0$) is available in Hull (2000) as

$$N(x) \quad = \quad N'(x)\left(a_1\frac{1}{1+\gamma x}+a_2\frac{1}{(1+\gamma x)^2}+a_3\frac{1}{(1+\gamma x)^3}+\text{ h.o.t. }\right)\,,$$

with transforming the returns generating by previously documented anomalies into associated trading profits.

where a_1, a_2, a_3, and γ are constants. Ignoring the constants a_1, σ, and μ as well as the contribution of $\frac{1}{1+\gamma x}$ implies the relevant terms of equation (20) are of the order

$$t^{\theta-\lambda}e^{-t^{2(\theta-\lambda)}} + e^{-t^{2(\theta-\lambda)}} .$$

The product $t^{2\lambda}$ from equation (18) or (19) results in the above expression becoming

$$t^{\theta+\lambda}e^{-t^{2(\theta-\lambda)}} + t^{2\lambda}e^{-t^{2(\theta-\lambda)}} .$$

Since the exponential function converges to zero for $\theta - \lambda$ faster than the power function increases towards ∞, the conditional semi-variance becomes zero in the limit as $t \to \infty$.

C Bonferroni Approach for Testing Multiple Hypotheses

This appendix discusses the Bonferroni approach for testing sub-hypotheses, with particular reference to testing for statistical arbitrage as in HJTW.

Let H_0 be the null hypothesis consisting of K sub-hypotheses $h_1, ..., h_K$, all of which are required to hold under H_0. Thus, the rejection of even one sub-hypothesis rejects the null H_0. As a consequence, H_0 is the *intersection* of sub-hypotheses given by

$$H_0 : \bigcap_{i=1}^{K} h_i .$$

In the Bonferroni procedure, each sub-hypothesis h_i is tested at a given level of significance α_i with a critical region denoted C_i so that $\Pr(C_i|H_0) = \alpha_i$. The critical region of the null hypothesis H_0 is the union $\bigcup_{i=1}^{K} C_i$. Let C_i^c be the complement of C_i. The null hypothesis H_0 is accepted if all the sub-hypotheses are accepted. Suppressing the conditioning notation, the probability of accepting H_0 is $\Pr\left(\bigcap_{i=1}^{K} C_i^c\right)$.

The Bonferroni inequality states that

$$\Pr\left(\bigcap_{i=1}^{K} C_i^c\right) \geq 1 - \sum_{i=1}^{K} \Pr(C_i) = 1 - \sum_{i=1}^{K} \alpha_i$$

from which we obtain

$$\sum_{i=1}^{K} \alpha_i \geq 1 - \Pr\left(\bigcap_{i=1}^{K} C_i^c\right) . \tag{21}$$

Therefore, $\sum_{i=1}^{K} \alpha_i$ is an upper bound on the size of the statistical test, that is, the probability of committing a Type I error.

If H_0 is not satisfied, then at least one sub-hypothesis, say h_j, is not satisfied. As

$$\Pr\left(\bigcup_{i=1}^{K} C_i\right) \geq \Pr(C_j)\,,$$

we observe that if all the sub-tests reject their sub-hypothesis with probability one as the sample size tends to infinity, $\Pr(C_j) \to 1$, then $\Pr\left(\bigcup_{i=1}^{K} C_i\right) \to 1$. As a result, the Bonferroni test is consistent.

However, in the statistical arbitrage test conducted by HJTW, the null hypothesis of no statistical arbitrage is a *union* of sub-hypotheses. This statement is a consequence of the fact that to reject no statistical arbitrage, all the sub-hypotheses must be rejected. Rejecting one sub-hypothesis is not sufficient to reject no statistical arbitrage. Thus, the null hypothesis is defined as

$$H_0^* : \bigcup_{i=1}^{K} h_i$$

and the probability of accepting H_0^* is $\Pr\left(\bigcup_{i=1}^{K} C_i^c\right)$. As the probability of a union is greater than its corresponding intersection, we have

$$\Pr\left(\bigcup_{i=1}^{K} C_i^c\right) \geq \Pr\left(\bigcap_{i=1}^{K} C_i^c\right) \tag{22}$$

which, when combined with equation (21), yields the relationship

$$\sum_{i=1}^{K} \alpha_i \geq 1 - \Pr\left(\bigcap_{i=1}^{K} C_i^c\right) \geq 1 - \Pr\left(\bigcup_{i=1}^{K} C_i^c\right)\,. \tag{23}$$

Thus, we conclude that $\sum_{i=1}^{K} \alpha_i$ is also an upper bound on the size of the test for the null hypothesis H_0^* defined in terms of a union. However, equation (23) implies the Bonferroni inequality is a weaker bound for H_0^* than for H_0. Furthermore, the Bonferroni test is generally not consistent for H_0^*, in contrast to H_0. Indeed, when K is large the actual size of the Bonferroni test for H_0^* may be far below $\sum_{i=1}^{K} \alpha_i$, resulting in a test with low power. Conversely, the Min-t test has the correct nominal size. To the extent that searching for the maximum probability of rejection over the parameter space H_0 results in the true maximum, the power of the test is also enhanced.

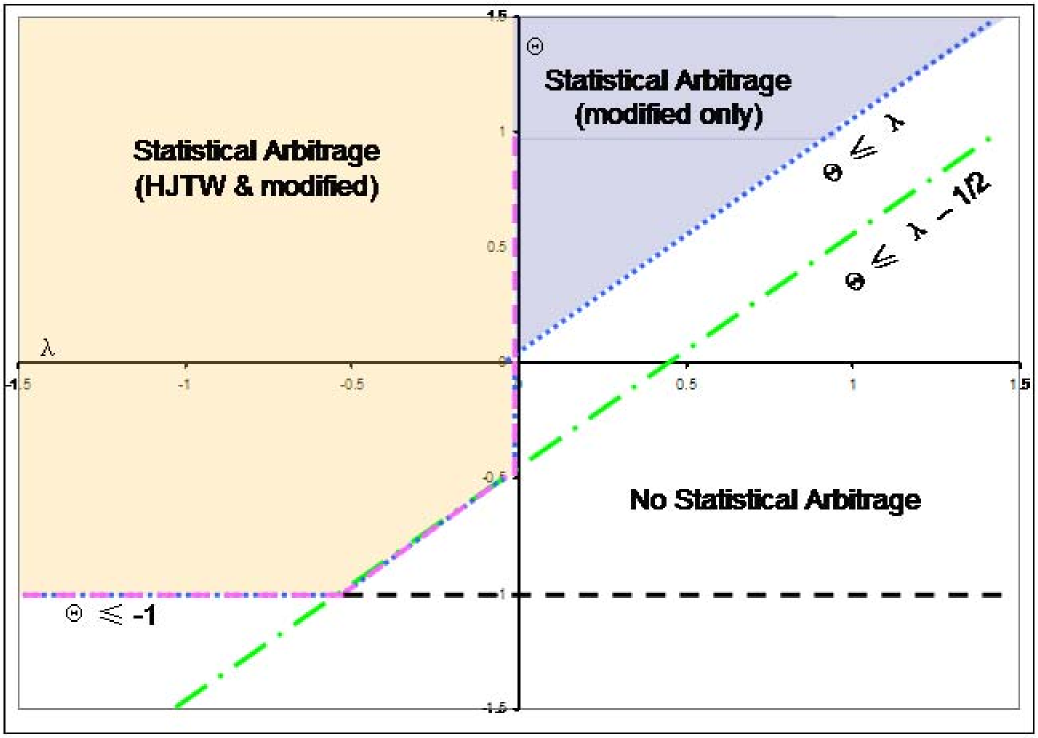

Figure 1: Regions corresponding to the null hypothesis of no statistical arbitrage as well as rejections of the null under both the Hogan, Jarrow, Teo, and Warachka (2004) definition of statistical arbitrage and our definition which modifies the fourth axiom.

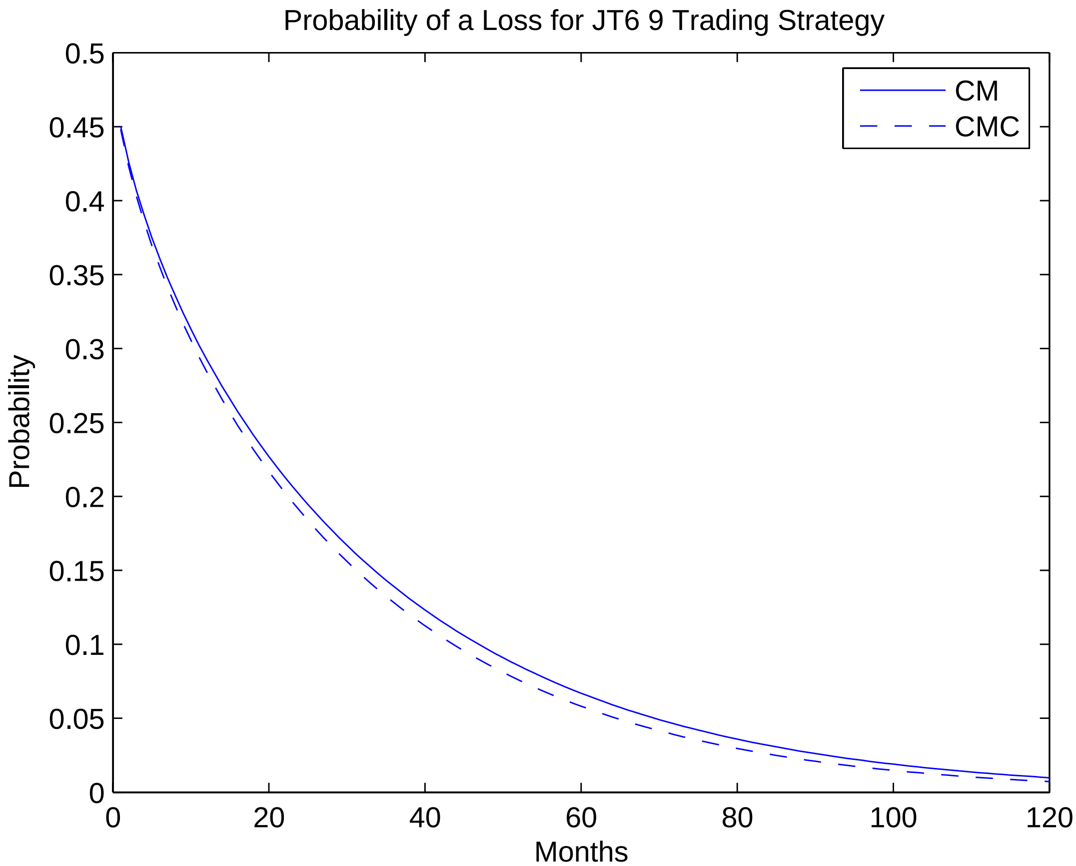

Figure 2: The trading strategy JT6_9 denotes a Jegadeesh and Titman (1993) stock momentum portfolio with a formation period of six months and a holding period of nine months. Plotted above, for the JT6_9 trading strategy, are loss probabilities derived from parameter estimates of the CM (constrained mean) and CMC (constrained mean with correlation) versions of statistical arbitrage. Both of these trading profit formulations result in positive tests for statistical arbitrage at the 5% significance level. The probability of a loss is computed according to equation (4) with the preferred model for trading profits being the CM version as reported in Table 5.

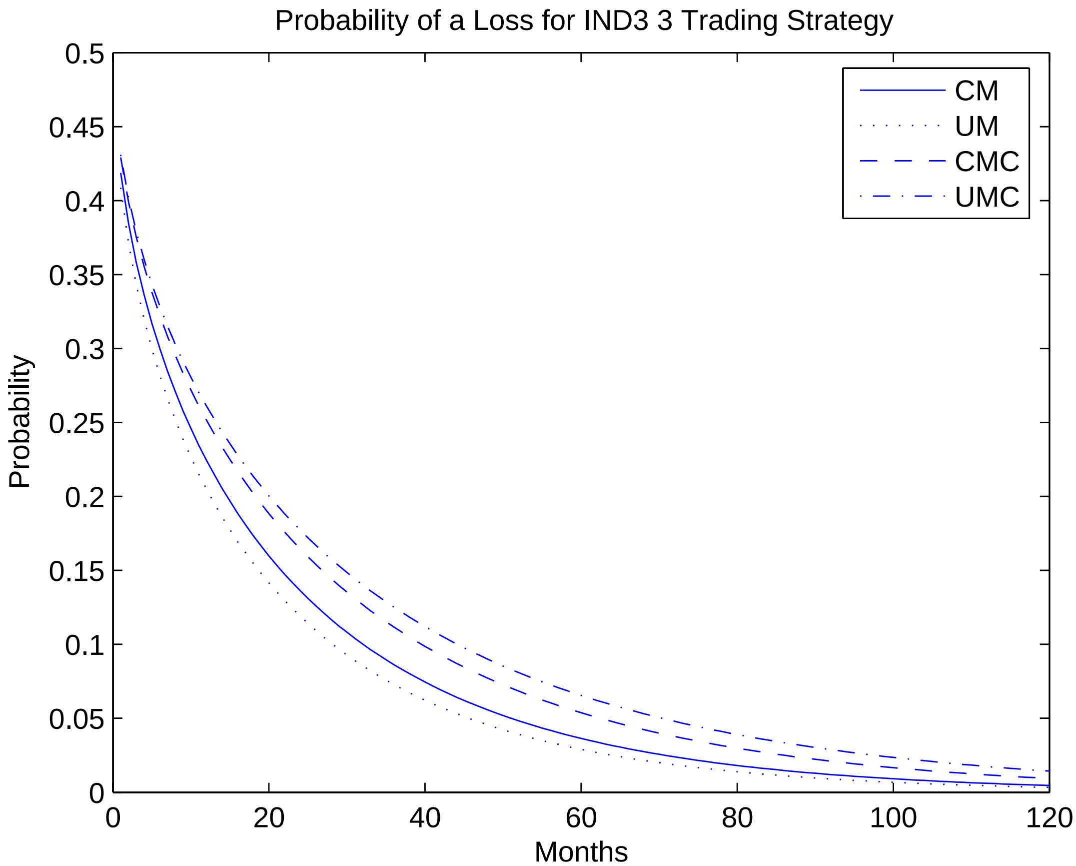

Figure 3: The industry momentum portfolio of Moskowitz and Grinblatt (1999) denoted IND3_3 has a three month formation and holding period. Plotted above, for the IND3_3 trading strategy, are loss probabilities derived from parameter estimates of the CM (constrained mean) and UM (unconstrained mean) models along with their counterparts CMC (constrained mean with correlation) and UMC (unconstrained mean with correlation). The probability of a loss is computed according to equation (4) with the preferred model for trading profits being the CMC version as reported in Table 5.

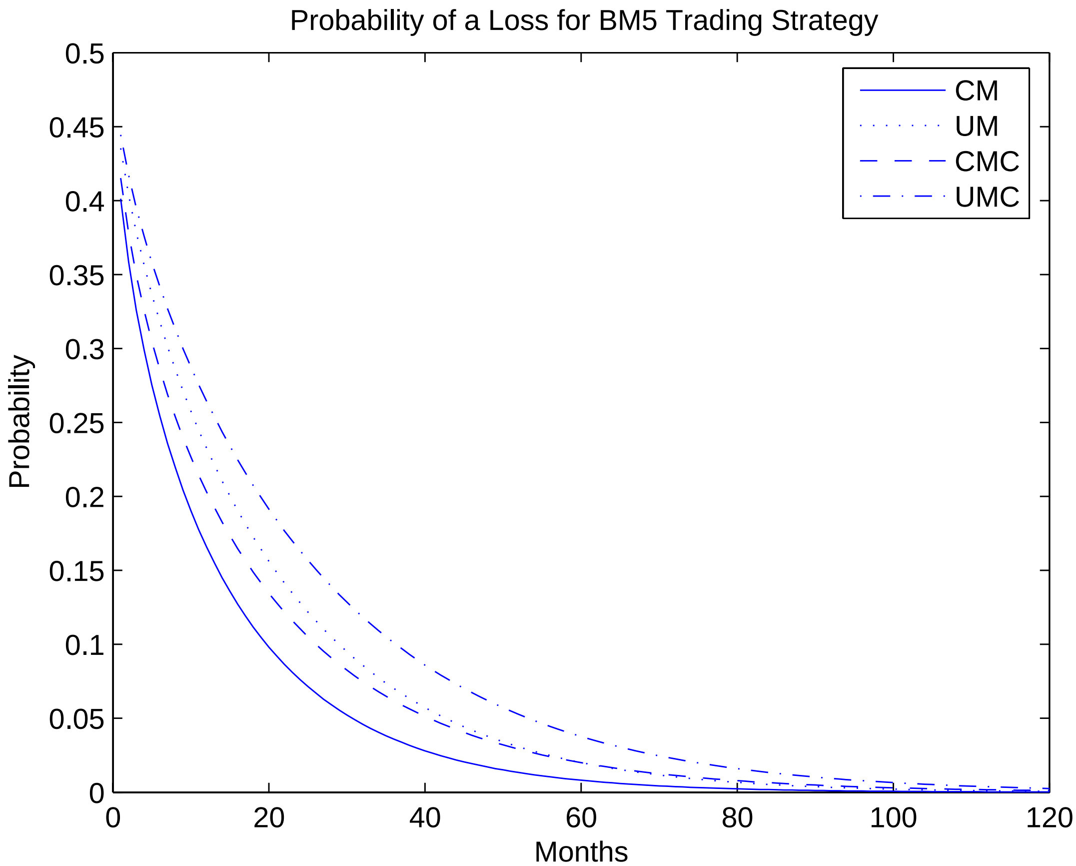

Figure 4: The book-to-market portfolio of Lakonishok, Shleifer, and Vishny (1994) denoted BM5 has a one year formation period and a five year holding period. Plotted above, for the BM5 trading strategy, are loss probabilities derived from parameter estimates of the CM (constrained mean) and UM (unconstrained mean) models along with their counterparts CMC (constrained mean with correlation) and UMC (unconstrained mean with correlation). The probability of a loss is computed according to equation (4) with the preferred model for trading profits being the CMC version as reported in Table 5.

Table 1: Tests of statistical arbitrage with the CM (constrained mean) and UM (unconstrained mean) models

For the sample period of January 1965 to December 2000, Min-t statistics and bootstrapped p-values for statistical arbitrage are presented. The UM (unconstrained mean) model allows for time-varying expected trading profits while its CM (constrained mean) counterpart has constant expected trading profits. Both models have uncorrelated trading profit innovations as described in equations (1) and (2). Four types of strategies are considered: stock momentum, value, stock liquidity, and industry momentum based strategies. The JTx_y portfolios are stock momentum portfolios with a formation period of x months and a holding period of y months as in Jegadeesh and Titman (1993). BMy, CPy, EPy, and SALEy are book-to-market, cash flow-to-price, earnings-to-price, and sales growth based value strategies respectively with a formation period of 1 year (3 years for sales growth) and a holding period of y years as in Lakonishok, Shleifer, and Vishny (1994). The VOLx_y and INDx_y are stock liquidity and industry momentum based strategies with a formation period of x months and a holding period of y months. The VOL portfolio buys the bottom decile of stocks and shorts the top decile of stocks sorted by share volume/shares outstanding. The IND portfolio buys the top 3 industries and shorts the bottom 3 industries sorted by industry return as in Moskowitz and Grinblatt (1999) who group US stocks into 20 industries by their SIC codes. The Min-t test statistics are defined in equations (6) and (7) for the respective UM and CM versions of statistical arbitrage. For emphasis, asterisks in parentheses denote the significance associated with bootstrapped p-values while those without parentheses denote significance generated by Monte Carlo simulation: * = significant at the 10% level; ** = significant at the 5% level; *** = significant at the 1% level.

Panel A: Momentum strategies

	Sample Size	CM Model		UM Model	
Portfolio	n	Min-t	p-value	Min-t	p-value
JT3_3	398	–1.128	0.690	–0.415	0.461
JT3_6	398	2.294 ***	0.000 (***)	0.099	0.247
JT3_9	398	3.803 ***	0.000 (***)	0.119	0.250
JT3_12	398	2.508 ***	0.000 (***)	0.348	0.140
JT6_3	398	1.760 ***	0.002 (***)	0.119	0.250
JT6_6	398	4.162 ***	0.000 (***)	0.118	0.255
JT6_9	398	3.049 ***	0.000 (***)	0.366	0.137
JT6_12	398	1.928 ***	0.001 (***)	0.375	0.100 (*)
JT9_3	398	3.153 ***	0.000 (***)	0.025	0.317
JT9_6	398	3.207 ***	0.000 (***)	0.371	0.116
JT9_9	398	2.239 ***	0.000 (***)	0.430 *	0.084 (*)
JT9_12	398	1.491 ***	0.006 (***)	0.445 *	0.100 (*)
JT12_3	398	2.924 ***	0.000 (***)	0.323	0.131
JT12_6	398	2.344 ***	0.001 (***)	0.405 *	0.095 (*)
JT12_9	398	1.700 ***	0.004 (***)	0.485 *	0.083 (*)
JT12_12	398	1.167 **	0.025 (**)	0.404 *	0.090 (*)

Panel B: Value strategies

BM1	414	0.111	0.168	1.005 ***	0.009 (***)
BM3	372	0.233	0.155	0.993 ***	0.006 (***)
BM5	324	1.293 ***	0.003 (***)	1.157 ***	0.003 (***)
CP1	414	–0.266	0.393	–10.801	0.984
CP3	372	0.827 **	0.056 (*)	0.451 *	0.106
CP5	324	1.017 **	0.029 (**)	0.371	0.095 (*)
EP1	414	–5.320	1.000	–0.149	0.955
EP3	372	–1.413	0.769	–0.140	0.851
EP5	324	–0.191	0.317	–1.371	0.800
SALE1	378	1.336 ***	0.003 (***)	1.198 ***	0.001 (***)
SALE3	336	1.530 ***	0.007 (***)	1.362 ***	0.000 (***)
SALE5	288	2.627 ***	0.000 (***)	1.073 ***	0.005 (***)

Panel C: Liquidity based strategies

	Sample Size	CM Model		UM Model	
Portfolio	n	Min-t	p-value	Min-t	p-value
VOL3_3	398	0.897 **	0.036 (**)	0.896 **	0.015 (**)
VOL3_6	398	0.881 **	0.025 (**)	0.882 **	0.018 (**)
VOL3_9	398	0.973 **	0.027 (**)	0.977 ***	0.008 (***)
VOL3_12	398	1.121 **	0.026 (**)	1.126 ***	0.006 (***)
VOL6_3	398	0.974 **	0.027 (**)	0.980 ***	0.007 (***)
VOL6_6	398	1.027 **	0.021 (**)	1.034 ***	0.006 (***)
VOL6_9	398	1.109 **	0.023 (**)	1.120 ***	0.005 (***)
VOL6_12	398	1.245 **	0.021 (**)	1.234 ***	0.002 (***)
VOL9_3	398	1.035 **	0.023 (**)	0.000	0.117
VOL9_6	398	1.098 **	0.019 (**)	1.094 ***	0.004 (***)
VOL9_9	398	1.219 **	0.018 (**)	0.870 **	0.016 (**)
VOL9_12	398	1.347 ***	0.014 (**)	1.244 ***	0.001 (***)
VOL12_3	398	1.166 **	0.011 (**)	1.174 ***	0.003 (***)
VOL12_6	398	1.214 **	0.014 (**)	0.960 **	0.010 (***)
VOL12_9	398	1.331 ***	0.005 (***)	1.141 ***	0.000 (***)
VOL12_12	398	1.455 ***	0.004 (***)	1.362 ***	0.000 (***)

Panel D: Industry momentum strategies

IND3_3	398	0.791 **	0.050 (**)	0.808 **	0.031 (**)
IND3_6	398	0.829 **	0.038 (**)	0.836 **	0.028 (**)
IND3_9	398	1.174 **	0.018 (**)	0.934 **	0.012 (**)
IND3_12	398	–0.065	0.278	0.852 **	0.013 (**)
IND6_3	398	0.350	0.119	0.353	0.112
IND6_6	398	0.336	0.121	0.715 **	0.044 (**)
IND6_9	398	–0.899	0.623	0.692 **	0.037 (**)
IND6_12	398	–0.823	0.595	0.426 *	0.086 (*)
IND9_3	398	1.870 ***	0.002 (***)	0.626 **	0.045 (**)
IND9_6	398	–0.321	0.364	0.556 **	0.055 (*)
IND9_9	398	–0.793	0.589	0.392	0.115
IND9_12	398	–0.561	0.457	0.269	0.179
IND12_3	398	–0.596	0.496	0.440 *	0.086 (*)
IND12_6	398	–0.728	0.564	0.055	0.231
IND12_9	398	–0.518	0.442	–0.174	0.307
IND12_12	398	0.152	0.190	–0.091	0.355

Table 2: Tests of statistical arbitrage with the CMC (constrained mean with correlation) model

For the sample period of January 1965 to December 2000, statistical arbitrage test results for the CMC model are presented. The CMC (constrained mean with correlation) model features correlated innovations in trading profits described by a MA(1) process and expected trading profits that are constant over time, as described in equations (2) and (3). The tests are applied to four types of strategies: stock momentum, value, stock liquidity, and industry momentum based strategies. The JTx_y portfolios are stock momentum portfolios with a formation period of x months and a holding period of y months as in Jegadeesh and Titman (1993). BMy, CPy, EPy, and SALEy are book-to-market, cash flow-to-price, earnings-to-price, and sales growth based value strategies respectively with a formation period of 1 year (3 years for sales growth) and a holding period of y years as in Lakonishok, Shleifer, and Vishny (1994). The VOLx_y and INDx_y are stock liquidity and industry momentum based strategies with a formation period of x months and a holding period of y months. The VOL portfolio buys the bottom decile of stocks and shorts the top decile of stocks sorted by share volume/shares outstanding. The IND portfolio buys the top 3 industries and shorts the bottom 3 industries sorted by industry return as in Moskowitz and Grinblatt (1999) who group US stocks into 20 industries by their SIC codes. For each trading strategy, the first row records the MLE parameter estimates of the CM model with MA(1) errors, while the second row records their t-statistics. The Min-t test statistic is defined in equation (7). Trading strategies that yield statistical arbitrage at the 10%, 5%, and 1% significance levels are denoted by *, **, and *** respectively.

Panel A: Momentum strategies

	Parameters (t-statistics)					
Portfolio	mean profit μ	growth rate of std dev λ	std dev σ	autocorrelation ϕ	Min-t	p-value
JT3_3	–0.002 (–1.11)	–0.183 (–3.59)	0.086 (3.51)	0.022 (0.41)	-1.107	0.700
JT3_6	0.003 (2.37)	–0.208 (–3.96)	0.085 (3.50)	–0.031 (–0.50)	2.374	0.000 (***)
JT3_9	0.005 (4.16)	–0.191 (–3.83)	0.070 (3.65)	–0.043 (–0.64)	3.831	0.000 (***)
JT3_12	0.006 (5.32)	–0.122 (–2.53)	0.045 (3.84)	–0.066 (–0.81)	2.534	0.000 (***)
JT6_3	0.004 (1.81)	–0.220 (–4.21)	0.125 (3.49)	–0.026 (–0.45)	1.811	0.003 (***)
JT6_6	0.008 (4.37)	–0.209 (–4.19)	0.108 (3.66)	–0.043 (–0.68)	4.190	0.000 (***)
JT6_9	0.009 (5.59)	–0.147 (–3.07)	0.073 (3.86)	–0.053 (–0.71)	3.073	0.000 (***)
JT6_12	0.008 (5.04)	–0.101 (–1.96)	0.055 (3.56)	–0.069 (–0.80)	1.961	0.001 (***)
JT9_3	0.007 (3.30)	–0.203 (–4.11)	0.125 (3.63)	–0.040 (–0.63)	3.298	0.000 (***)
JT9_6	0.010 (5.20)	–0.153 (–3.22)	0.090 (3.87)	–0.053 (–0.71)	3.221	0.000 (***)
JT9_9	0.009 (4.95)	–0.116 (–2.27)	0.071 (3.59)	–0.067 (–0.78)	2.270	0.000 (***)
JT9_12	0.007 (4.09)	–0.084 (–1.54)	0.057 (3.35)	–0.074 (–0.78)	1.540	0.002 (***)
JT12_3	0.009 (4.12)	–0.144 (–2.93)	0.097 (3.71)	–0.060 (–0.74)	2.928	0.000 (***)
JT12_6	0.009 (4.35)	–0.124 (–2.37)	0.084 (3.49)	–0.069 (–0.79)	2.373	0.001 (***)
JT12_9	0.008 (3.82)	–0.095 (–1.75)	0.069 (3.37)	–0.071 (–0.76)	1.750	0.001 (***)
JT12_12	0.006 (2.95)	–0.070 (–1.22)	0.058 (3.14)	–0.074 (–0.77)	1.221	0.012 (**)

Panel B: Value strategies

BM1	0.015 (5.51)	–0.004 (–0.06)	0.052 (2.78)	0.094 (1.41)	0.055	0.194
BM3	0.012 (5.38)	–0.017 (–0.23)	0.040 (2.70)	0.139 (2.38)	0.234	0.159
BM5	0.011 (5.36)	–0.069 (–1.23)	0.043 (3.55)	0.194 (3.06)	1.227	0.009 (***)
CP1	–0.001 (–0.25)	–0.200 (–3.40)	0.209 (3.74)	0.189 (2.33)	–0.251	0.406
CP3	0.002 (0.77)	–0.168 (–2.40)	0.111 (3.55)	0.077 (1.52)	0.768	0.071 (*)
CP5	0.002 (0.96)	–0.233 (–2.73)	0.125 (3.13)	0.080 (1.33)	0.956	0.043 (**)
EP1	–0.000 (–0.06)	0.291 (5.00)	0.013 (4.23)	0.205 (3.03)	–5.000	1.000
EP3	–0.001 (–0.19)	0.116 (1.36)	0.024 (2.76)	0.118 (2.09)	–1.359	0.725
EP5	–0.001 (–0.16)	–0.047 (–0.63)	0.044 (3.17)	0.173 (2.60)	–0.155	0.275
SALE1	0.009 (5.23)	–0.092 (–1.30)	0.050 (2.57)	0.066 (0.92)	1.301	0.007 (***)
SALE3	0.006 (3.94)	–0.127 (–1.60)	0.046 (2.44)	0.103 (1.47)	1.598	0.006 (***)
SALE5	0.005 (3.20)	–0.169 (–2.85)	0.049 (3.43)	0.148 (2.36)	2.846	0.000 (***)

Panel C: Liquidity based strategies

Portfolio	Parameters (*t*-statistics) mean profit μ	growth rate of std dev λ	std dev σ	autocorrelation ϕ	Min-*t*	*p*-value
VOL3_3	0.005 (1.68)	–0.040 (–1.03)	0.065 (5.07)	0.111 (2.00)	1.025	0.021 (**)
VOL3_6	0.006 (2.00)	–0.039 (–1.01)	0.063 (5.12)	0.115 (2.09)	1.010	0.025 (**)
VOL3_9	0.006 (2.14)	–0.043 (–1.09)	0.063 (4.98)	0.123 (2.27)	1.094	0.026 (**)
VOL3_12	0.006 (2.23)	-0.049 (–1.23)	0.064 (4.87)	0.130 (2.41)	1.230	0.008 (***)
VOL6_3	0.006 (2.07)	–0.043 (–1.11)	0.067 (5.14)	0.112 (2.04)	1.112	0.020 (**)
VOL6_6	0.007 (2.26)	–0.045 (–1.15)	0.066 (4.98)	0.118 (2.16)	1.149	0.024 (**)
VOL6_9	0.007 (2.30)	–0.050 (–1.22)	0.067 (4.81)	0.127 (2.36)	1.223	0.007 (***)
VOL6_12	0.007 (2.30)	–0.056 (–1.35)	0.067 (4.73)	0.133 (2.47)	1.349	0.012 (**)
VOL9_3	0.007 (2.19)	–0.046 (–1.14)	0.068 (4.89)	0.110 (2.03)	1.136	0.015 (**)
VOL9_6	0.007 (2.28)	–0.049 (–1.19)	0.068 (4.76)	0.121 (2.24)	1.193	0.012 (**)
VOL9_9	0.007 (2.28)	–0.054 (–1.30)	0.069 (4.66)	0.129 (2.41)	1.303	0.012 (**)
VOL9_12	0.007 (2.31)	–0.060 (–1.43)	0.069 (4.64)	0.134 (2.47)	1.426	0.002 (***)
VOL12_3	0.007 (2.24)	–0.051 (–1.26)	0.070 (4.79)	0.121 (2.24)	1.126	0.016 (**)
VOL12_6	0.007 (2.25)	–0.053 (–1.28)	0.069 (4.70)	0.125 (2.31)	1.283	0.009 (***)
VOL12_9	0.007 (2.27)	–0.058 (–1.40)	0.070 (4.65)	0.130 (2.40)	1.398	0.004 (***)
VOL12_12	0.007 (2.29)	–0.064 (–1.52)	0.070 (4.63)	0.135 (2.46)	1.522	0.007 (***)

Panel D: Industry momentum strategies

IND3_3	0.008 (4.41)	–0.049 (–0.97)	0.040 (3.74)	0.121 (2.19)	0.975	0.036 (**)
IND3_6	0.006 (3.66)	–0.058 (–1.11)	0.036 (3.62)	0.187 (3.50)	1.114	0.019 (**)
IND3_9	0.006 (4.35)	–0.063 (–1.31)	0.032 (4.10)	0.211 (3.89)	1.313	0.014 (**)
IND3_12	0.006 (4.43)	–0.012 (–0.23)	0.023 (3.90)	0.223 (3.79)	0.233	0.166
IND6_3	0.008 (3.66)	–0.031 (–0.67)	0.042 (4.17)	0.175 (3.34)	0.672	0.062 (*)
IND6_6	0.008 (3.94)	–0.023 (–0.53)	0.036 (4.51)	0.217 (4.44)	0.531	0.087 (*)
IND6_9	0.008 (4.14)	0.025 (0.51)	0.026 (4.09)	0.267 (5.14)	–0.514	0.454
IND6_12	0.006 (3.37)	0.026 (0.46)	0.025 (3.45)	0.256 (4.53)	–0.457	0.402
IND9_3	0.009 (3.86)	–0.080 (–1.87)	0.054 (4.46)	0.233 (4.76)	1.874	0.000 (***)
IND9_6	0.009 (3.84)	0.010 (0.23)	0.033 (4.54)	0.259 (5.23)	–0.225	0.338
IND9_9	0.007 (3.39)	0.033 (0.63)	0.028 (3.72)	0.253 (4.84)	–0.629	0.534
IND9_12	0.005 (2.53)	0.018 (0.36)	0.029 (3.84)	0.239 (4.38)	–0.356	0.399
IND12_3	0.010 (4.15)	0.010 (0.19)	0.036 (4.03)	0.261 (5.11)	–0.193	0.343
IND12_6	0.008 (3.49)	0.021 (0.41)	0.032 (3.81)	0.251 (4.91)	–0.409	0.404
IND12_9	0.006 (2.74)	0.010 (0.22)	0.033 (4.18)	0.252 (4.85)	–0.220	0.337
IND12_12	0.004 (1.90)	–0.020 (–0.43)	0.036 (4.00)	0.234 (4.49)	0.429	0.084 (*)

Table 3: Tests of statistical arbitrage with the UMC (unconstrained mean with correlation) model

For the sample period of January 1965 to December 2000, statistical arbitrage test results for the UMC model are presented. The UMC (unconstrained mean with correlation) model has correlated trading profit innovations described by a MA(1) process and expected trading profits that are time-varying, as described in equations (1) and (3). The tests are applied to four types of strategies: stock momentum, value, stock liquidity, and industry momentum based strategies. The JTx_y portfolios are stock momentum portfolios with a formation period of x months and a holding period of y months as in Jegadeesh and Titman (1993). BMy, CPy, EPy, and SALEy are book-to-market, cash flow-to-price, earnings-to-price, and sales growth based value strategies respectively with a formation period of 1 year (3 years for sales growth) and a holding period of y years as in Lakonishok, Shleifer, and Vishny (1994). The VOLx_y and INDx_y are stock liquidity and industry momentum based strategies with a formation period of x months and a holding period of y months. The VOL portfolio buys the bottom decile of stocks and shorts the top decile of stocks sorted by share volume/shares outstanding. The IND portfolio buys the top 3 industries and shorts the bottom 3 industries sorted by industry return as in Moskowitz and Grinblatt (1999) who group US stocks into 20 industries by their SIC codes. For each trading strategy, the first row records the MLE parameter estimates of the UM model with MA(1) errors, while the second row records their t-statistics. The Min-t test statistic is defined in equation (6). Trading strategies that yield statistical arbitrage at the 10%, 5%, and 1% significance levels are denoted by *, **, and *** respectively.

Panel A: Momentum strategies

	Parameters (t-statistics)						
Portfolio	mean profit μ	growth rate of std dev λ	growth rate of mean profit θ	std dev σ	autocorrelation ϕ	Min-t	p-value
JT3_3	–0.003 (–0.41)	–0.183 (–3.54)	–0.056 (–0.13)	0.086 (3.46)	0.022 (0.41)	–0.413	0.425
JT3_6	0.002 (0.10)	–0.198 (–3.67)	0.151 (0.08)	0.076 (3.49)	–0.031 (–0.40)	0.101	0.197
JT3_9	0.000 (0.12)	–0.194 (–3.87)	0.675 (0.47)	0.071 (3.66)	–0.047 (–0.69)	0.123	0.221
JT3_12	0.001 (0.37)	–0.122 (–2.51)	0.308 (0.62)	0.045 (3.81)	–0.069 (–0.83)	0.368	0.121
JT6_3	0.001 (0.19)	–0.161 (–3.74)	0.265 (0.56)	0.085 (4.58)	–0.040 (–0.70)	0.192	0.127
JT6_6	0.000 (0.12)	–0.213 (–4.17)	0.587 (0.41)	0.110 (3.59)	–0.046 (–0.72)	0.123	0.278
JT6_9	0.002 (0.38)	–0.147 (–3.03)	0.272 (0.57)	0.073 (3.80)	–0.055 (–0.73)	0.383	0.109
JT6_12	0.002 (0.40)	–0.102 (–1.93)	0.244 (0.52)	0.055 (3.49)	–0.071 (–0.81)	0.400	0.106
JT9_3	0.000 (0.02)	–0.207 (–3.77)	0.815 (0.10)	0.127 (3.31)	–0.042 (–0.65)	0.021	0.290
JT9_6	0.003 (0.39)	–0.154 (–3.18)	0.209 (0.44)	0.090 (3.80)	–0.054 (–0.72)	0.386	0.095 (*)
JT9_9	0.003 (0.45)	–0.116 (–2.24)	0.203 (0.50)	0.071 (3.54)	–0.068 (–0.79)	0.454	0.089 (*)
JT9_12	0.004 (0.47)	–0.084 (–1.53)	0.126 (0.32)	0.057 (3.33)	–0.075 (–0.78)	0.470	0.087 (*)
JT12_3	0.005 (0.34)	–0.144 (–2.88)	0.119 (0.22)	0.097 (3.64)	–0.060 (–0.74)	0.338	0.124
JT12_6	0.005 (0.43)	–0.124 (–2.35)	0.126 (0.29)	0.084 (3.45)	–0.069 (–0.79)	0.429	0.097 (*)
JT12_9	0.005 (0.51)	–0.095 (–1.74)	0.082 (0.22)	0.069 (3.35)	–0.071 (–0.76)	0.510	0.079 (*)
JT12_12	0.004 (0.42)	–0.070 (–1.22)	0.079 (0.18)	0.058 (3.13)	–0.074 (–0.77)	0.424	0.097 (*)

Panel B: Value strategies

BM1	0.006 (0.69)	–0.001 (–0.01)	0.167 (0.61)	0.051 (7.81)	0.093 (2.63)	0.593	0.004 (***)
BM3	0.005 (0.71)	–0.015 (–0.46)	0.154 (0.56)	0.040 (6.40)	0.138 (3.09)	0.608	0.007 (***)
BM5	0.007 (0.72)	–0.065 (–1.60)	0.093 (0.34)	0.042 (5.18)	0.193 (3.93)	0.722	0.005 (***)
CP1	–0.106 (–0.65)	–0.198 (–13.04)	–0.812 (–1.13)	0.206 (11.68)	0.186 (7.24)	–0.158	0.452
CP3	0.007 (0.19)	–0.168 (–7.45)	–0.227 (–0.20)	0.111 (8.09)	0.077 (1.45)	0.188	0.081 (*)
CP5	0.001 (0.08)	–0.232 (–8.54)	0.247 (0.11)	0.125 (6.73)	0.080 (1.17)	0.080	0.160
EP1	0.019 (0.17)	0.301 (14.03)	–1.647 (–0.32)	0.012 (8.23)	0.204 (6.15)	–0.378	0.797
EP3	–0.000 (–0.03)	0.116 (5.23)	0.620 (0.12)	0.024 (8.25)	0.117 (1.98)	–0.034	0.417
EP5	–0.000 (–0.01)	–0.047 (–2.14)	0.667 (0.04)	0.044 (8.57)	0.172 (2.58)	0.209	0.228
SALE1	0.009 (0.65)	–0.092 (–2.81)	0.001 (0.00)	0.050 (6.45)	0.066 (1.83)	0.649	0.004 (***)
SALE3	0.014 (0.60)	–0.129 (–3.39)	–0.184 (–0.54)	0.046 (5.77)	0.102 (2.28)	0.595	0.002 (***)
SALE5	0.016 (0.64)	–0.166 (–3.84)	–0.259 (–0.77)	0.048 (5.00)	0.147 (3.01)	0.635	0.003 (***)

Panel C: Liquidity based strategies

Portfolio	mean profit μ	growth rate of std dev λ	growth rate of mean profit θ	std dev σ	autocorrelation ϕ	Min-t	p-value
	Parameters (t-statistics)						
VOL3_3	0.004 (0.93)	–0.039 (–1.03)	0.034 (0.19)	0.065 (5.15)	0.111 (2.00)	0.929	0.011 (**)
VOL3_6	0.005 (1.01)	–0.039 (–1.02)	0.026 (0.15)	0.063 (5.19)	0.115 (2.09)	1.005	0.010 (***)
VOL3_9	0.005 (1.01)	–0.043 (–1.11)	0.025 (0.14)	0.063 (5.07)	0.123 (2.27)	1.005	0.006 (***)
VOL3_12	0.006 (1.06)	–0.049 (–1.24)	0.024 (0.15)	0.064 (4.94)	0.130 (2.41)	1.064	0.006 (***)
VOL6_3	0.006 (1.06)	–0.042 (–1.12)	0.015 (0.09)	0.067 (5.22)	0.112 (2.04)	1.058	0.003 (***)
VOL6_6	0.006 (1.05)	–0.045 (–1.16)	0.015 (0.09)	0.066 (5.07)	0.118 (2.16)	1.053	0.004 (***)
VOL6_9	0.006 (1.00)	–0.050 (–1.24)	0.013 (0.07)	0.066 (4.90)	0.127 (2.36)	1.000	0.005 (***)
VOL6_12	0.006 (0.95)	–0.055 (–1.37)	0.012 (0.06)	0.067 (4.82)	0.133 (2.47)	0.949	0.012 (**)
VOL9_3	0.006 (1.09)	–0.045 (–1.15)	0.009 (0.06)	0.068 (4.98)	0.110 (2.03)	1.087	0.003 (***)
VOL9_6	0.007 (1.31)	–0.049 (–1.21)	0.009 (0.07)	0.068 (4.82)	0.121 (2.24)	1.206	0.004 (***)
VOL9_9	0.006 (0.74)	–0.054 (–1.33)	0.011 (0.04)	0.069 (4.77)	0.129 (2.40)	0.743	0.039 (**)
VOL9_12	0.006 (0.99)	–0.059 (–1.44)	0.013 (0.07)	0.069 (4.71)	0.134 (2.47)	0.989	0.011 (**)
VOL12_3	0.006 (1.13)	–0.051 (–1.27)	0.014 (0.09)	0.070 (4.86)	0.121 (2.24)	1.126	0.005 (***)
VOL12_6	0.006 (1.17)	–0.053 (–1.30)	0.010 (0.07)	0.069 (4.76)	0.125 (2.31)	1.171	0.004 (***)
VOL12_9	–0.186 (–0.03)	–0.007 (–0.21)	–2.130 (–0.11)	0.054 (6.01)	0.138 (3.08)	–0.085	0.919
VOL12_12	0.006 (1.14)	–0.064 (–1.54)	0.010 (0.06)	0.070 (4.69)	0.135 (2.46)	1.142	0.003 (***)

Panel D: Industry momentum strategies

IND3_3	0.008 (0.74)	–0.049 (–1.78)	–0.012 (–0.05)	0.041 (7.27)	0.121 (2.50)	0.742	0.004 (***)
IND3_6	0.006 (0.55)	–0.058 (–1.12)	0.010 (0.03)	0.036 (3.67)	0.187 (3.48)	0.548	0.066 (*)
IND3_9	0.003 (0.78)	–0.060 (–1.27)	0.165 (0.70)	0.031 (4.15)	0.211 (3.88)	0.781	0.021 (**)
IND3_12	0.003 (0.86)	–0.008 (–0.15)	0.168 (0.79)	0.023 (3.79)	0.222 (3.77)	0.819	0.019 (**)
IND6_3	0.008 (0.70)	–0.031 (–0.67)	0.005 (0.02)	0.042 (4.16)	0.175 (3.33)	0.667	0.037 (**)
IND6_6	0.003 (0.59)	–0.021 (–0.48)	0.191 (0.61)	0.036 (4.46)	0.216 (4.43)	0.594	0.047 (**)
IND6_9	0.002 (0.54)	0.029 (0.57)	0.301 (0.89)	0.026 (3.89)	0.265 (5.09)	0.540	0.070 (*)
IND6_12	0.002 (0.42)	0.025 (0.45)	0.201 (0.45)	0.025 (3.45)	0.255 (4.51)	0.390	0.115
IND9_3	0.003 (0.52)	–0.078 (–1.83)	0.218 (0.62)	0.054 (4.47)	0.233 (4.74)	0.521	0.067 (*)
IND9_6	0.002 (0.45)	0.013 (0.27)	0.324 (0.80)	0.033 (4.33)	0.258 (5.19)	0.453	0.088 (*)
IND9_9	0.001 (0.31)	0.034 (0.63)	0.432 (0.74)	0.028 (3.68)	0.251 (4.78)	0.313	0.136
IND9_12	0.001 (0.20)	0.016 (0.32)	0.280 (0.30)	0.029 (3.86)	0.238 (4.36)	0.201	0.177
IND12_3	0.004 (0.58)	0.010 (0.20)	0.160 (0.50)	0.036 (3.98)	0.261 (5.11)	0.467	0.106
IND12_6	0.006 (0.35)	0.020 (0.38)	0.066 (0.12)	0.033 (3.74)	0.251 (4.92)	0.079	0.205
IND12_9	0.012 (0.28)	0.014 (0.26)	–0.154 (–0.20)	0.032 (3.58)	0.252 (4.85)	–0.206	0.286
IND12_12	0.033 (0.70)	–0.005 (–0.08)	–0.501 (–0.89)	0.034 (3.28)	0.236 (4.50)	0.007	0.321

Table 4: Summary of statistical arbitrage opportunities

The sample period is from January 1965 to December 2000. Statistical arbitrage test results are summarized for four models of trading profit processes: CM (constrained mean), UM (unconstrained mean), CMC (constrained mean with correlation), and UMC (unconstrained mean with correlation). The UM model allows for time variation in expected trading profits while its CM counterpart has constant expected trading profits. The UMC and CMC models feature, in addition, correlated innovations in trading profits that are described by a MA(1) process. The tests are applied to four types of strategies: stock momentum, value, stock liquidity, and industry momentum based strategies. The stock momentum strategies buy the highest return decile and short the lowest return decile of stocks as in Jegadeesh and Titman (1993). The stock value strategies buy the highest decile and short the lowest decile of stocks sorted on book-to-market, cash flow-to-price, earnings-to-price, and sales growth, as in Lakonishok, Shleifer, and Vishny (1994). The stock liquidity strategies buy the lowest trading volume decile and short the highest trading volume decile of stocks in the spirit of Brennan, Chordia, and Subrahmanyam (1998). The industry momentum strategies buy the top three return industries and short the bottom three return industries as in Grinblatt and Moskowitz (1999).

Panel A: Statistical arbitrage opportunities at the 10% level of significance

	Number	Trading profit model			
Type of trading strategy	tested	CM	UM	CMC	UMC
Stock momentum: Jegadeesh and Titman (1993)	16	15	6	15	6
Stock value: Lakonishok, Shleifer, and Vishny (1994)	12	6	7	6	7
Stock liquidity: Brennan, Chordia, and Subrahmanyam (1998)	16	16	15	16	15
Industry momentum: Moskowitz and Grinblatt (1999)	16	4	10	7	9
Total	60	41	38	44	37

Panel B: Statistical arbitrage opportunities at the 5% level of significance

	Number	Trading profit model			
Type of trading strategy	tested	CM	UM	CMC	UMC
Stock momentum: Jegadeesh and Titman (1993)	16	15	0	15	0
Stock value: Lakonishok, Shleifer, and Vishny (1994)	12	5	6	5	6
Stock liquidity: Brennan, Chordia, and Subrahmanyam (1998)	16	16	15	16	15
Industry momentum: Moskowitz and Grinblatt (1999)	16	4	7	4	5
Total	60	40	28	40	26

Table 5: Comparing loss probabilities across trading profit models

The sample period is from January 1965 to December 2000. The number of months until the loss probability declines below 1% and 5 % are recorded, using equation (4), for various statistical arbitrage models: UM, CM, UMC, and CMC. The UM (unconstrained mean) model allows for time-varying expected trading profits while the CM (constrained mean) model imposes constant expected trading profits. Both the UM and CM models have uncorrelated trading profit innovations as described in equations (1) and (2) respectively. In contrast, their UMC (unconstrained mean with correlation) and CMC (constrained mean with correlation) counterparts allow for serial correlation in trading profits through the addition of an MA(1) process given in equation (3). The models are applied to four groups of strategies: stock momentum, value, stock liquidity, and industry momentum based strategies. The JTx_y portfolios are stock momentum portfolios with a formation period of x months and a holding period of y months as in Jegadeesh and Titman (1993). BMy, CPy, EPy, and SALEy are book-to-market, cash flow-to-price, earnings-to-price, and sales growth based value strategies respectively with a formation period of 1 year (3 years for sales growth) and a holding period of y years as in Lakonishok, Shleifer, and Vishny (1994). The VOLx_y and INDx_y are stock liquidity and industry momentum based strategies with a formation period of x months and a holding period of y months. The VOL portfolio buys the top decile of stocks and shorts the bottom decile of stocks sorted by share volume/shares outstanding. The IND portfolio buys the top 3 industries and shorts the bottom 3 industries sorted by industry return as in Moskowitz and Grinblatt (1999) who group US stocks into 20 industries by their SIC codes. The Akaike Information Criteria (AIC) identifies the preferred model for describing incremental trading profit dynamics.

Panel A: Momentum strategies

	Preferred	Loss Probability below 5%				Loss Probability below 1%			
Portfolio	Model	CM	UM	CMC	UMC	CM	UM	CMC	UMC
JT3_3	CM	-	-	-	-	-	-	-	-
JT3_6	CM	286	-	273	-	468	-	447	-
JT3_9	CM	123	-	116	-	205	-	192	-
JT3_12	CM	73	-	64	-	128	-	112	-
JT6_3	CM	423	-	408	-	688	-	663	-
JT6_6	CM	122	-	113	-	200	-	185	-
JT6_9	CM	71	-	65	-	121	-	112	-
JT6_12	CM	74	-	66	-	132	-	117	-
JT9_3	CM	177	-	168	-	292	-	168	-
JT9_6	CM	81	-	73	-	138	-	125	-
JT9_9	CM	79	-	71	-	-	-	124	-
JT9_12	CM	100	-	86	-	100	-	156	-
JT12_3	CM	112	-	102	-	193	-	175	-
JT12_6	CM	99	-	88	-	173	-	154	-
JT12_9	CM	114	-	101	-	204	-	180	-
JT12_12	CM	170	-	149	-	313	-	273	-

Panel B: Value strategies

BM1	CMC	-	60	-	69	-	101	-	116
BM3	CMC	-	53	-	63	-	88	-	106
BM5	CMC	31	43	41	55	58	73	76	93
CP1	CMC	-	-	-	-	-	-	-	-
CP3	CMC	-	-	-	-	-	-	-	-
CP5	CMC	703	-	732	-	1137	-	1179	-
EP1	CMC	-	-	-	-	-	-	-	-
EP3	CMC	-	-	-	-	-	-	-	-
EP5	CMC	-	-	-	-	-	-	-	-
SALE1	CM	50	49	55	55	90	89	99	99
SALE3	CMC	78	40	92	49	137	85	160	106
SALE5	CMC	97	42	119	55	165	97	201	128

Panel C: Liquidity based strategies

	Preferred	Loss Probability below 5%				Loss Probability below 1%			
Portfolio	Model	CM	UM	CMC	UMC	CM	UM	CMC	UMC
VOL3_3	CMC	320	317	385	388	611	584	731	711
VOL3_6	CMC	229	232	279	282	437	432	530	522
VOL3_9	CMC	196	202	249	252	373	375	470	465
VOL3_12	CMC	185	187	230	234	350	347	432	428
VOL6_3	CMC	213	217	265	264	405	407	502	493
VOL6_6	CMC	179	183	224	227	357	342	422	422
VOL6_9	CMC	174	178	221	222	322	334	415	410
VOL6_12	CMC	174	176	222	225	312	328	414	414
VOL9_3	CMC	193	-	239	236	366	-	451	440
VOL9_6	CMC	177	182	223	224	334	342	418	416
VOL9_9	CMC	178	177	225	223	333	330	421	412
VOL9_12	CMC	173	176	219	222	322	325	408	407
VOL12_3	CMC	184	187	231	229	346	348	433	424
VOL12_6	CMC	183	184	231	231	344	344	432	427
VOL12_9	CMC	179	184	226	-	334	341	420	-
VOL12_12	CMC	179	176	222	228	326	361	410	418

Panel D: Industry momentum strategies

IND3_3	CMC	52	48	66	63	99	93	123	120
IND3_6	CMC	71	68	95	-	134	130	177	-
IND3_9	CMC	52	81	73	108	97	134	134	175
IND3_12	CMC	-	68	-	93	-	116	-	155
IND6_3	CMC	-	-	-	90	-	-	-	171
IND6_6	CMC	-	85	-	118	-	142	-	193
IND6_9	CMC	-	86	-	-	-	138	-	-
IND6_12	CMC	-	-	-	-	-	-	-	-
IND9_3	CMC	68	109	93	-	125	171	168	-
IND9_6	CMC	-	-	-	-	-	-	-	-
IND9_9	CMC	-	-	-	-	-	-	-	-
IND9_12	CMC	-	-	-	-	-	-	-	-
IND12_3	CMC	-	-	-	-	-	-	-	-
IND12_6	CMC	-	-	-	-	-	-	-	-
IND12_9	CMC	-	-	-	-	-	-	-	-
IND12_12	CMC	-	-	-	-	-	-	-	-

Table 6: Tests of statistical arbitrage on Fama and French (1993) risk factors

The sample period is from January 1965 to December 2000. Statistical arbitrage test results are reported for the Fama and French (1993) size and book-to-market based risk factors (SMB and HML) as well as the equity premium (RMRF). Estimated parameter values, with their individual t-statistics reported below in parentheses, are provided for each of the four trading profit models: UM, CM, UMC, and CMC. The UM (unconstrained mean) model allows for time-varying expected trading profits while the CM (constrained mean) model has constant expected trading profits. Both the UM and CM models have uncorrelated trading profit innovations as described in equations (1) and (2) respectively. In contrast, their UMC (unconstrained mean with correlation) and CMC (constrained mean with correlation) counterparts allow for serial correlation in trading profits through the addition of an MA(1) process given in equation (3). Statistical significance at the 10%, 5%, and 1% levels is denoted with *, **, and *** respectively. The Akaike Information Criteria (AIC) identifies the preferred model for describing incremental trading profit dynamics, which we highlight in boldface.

Risk Factors	Trading Profit Model	Parameters (t-statistics): mean profit μ	growth rate of std dev λ	growth rate of mean profit θ	std dev σ	autocorrelation ϕ	Test Statistics: Min-t	p-value
HML	CM	0.004	0.197	-	0.010	-	–4.679	1.000
		(2.98)	(4.68)	-	(4.77)	-		
	UM	0.004	0.200	–0.003	0.010	-	–1.184	0.714
		(1.30)	(4.68)	(–0.02)	(4.78)	-		
	CMC	0.004	0.198	-	0.010	0.153	–4.579	1.000
		(2.64)	(4.58)	-	(4.61)	(2.85)		
	UMC	0.004	0.198	–0.000	0.010	0.153	–0.607	0.901
		(0.63)	(6.49)	(–0.00)	(6.18)	(4.01)		
SMB	CM	0.002	0.050	-	0.026	-	–0.887	0.564
		(1.36)	(0.89)	-	(3.99)	-		
	UM	0.031	0.064	–0.525	0.024	-	–1.058	0.690
		(3.91)	(1.06)	(–4.02)	(3.68)	-		
	CMC	0.002	0.061	-	0.024	0.104	–1.073	0.623
		(1.31)	(1.07)	-	(3.98)	(1.27)		
	UMC	0.031	0.074	–0.523	0.022	0.100	–1.584	0.991
		(0.77)	(3.08)	(–1.38)	(7.32)	(3.58)		
RMRF	CM	0.005	0.054	-	0.034	-	–1.126	0.720
		(2.06)	(1.13)	-	(4.24)	-		
	UM	0.001	–0.050	0.132	0.095	-	0.160	0.189
		(0.16)	(–0.83)	(0.12)	(2.16)	-		
	CMC	0.005	0.057	-	0.033	0.053	–1.176	0.721
		(1.96)	(1.18)	-	(4.25)	(0.93)		
	UMC	0.001	–0.080	0.134	0.093	0.050	0.128	0.141
		(0.13)	(–1.97)	(0.10)	(4.70)	(1.24)		