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A Flexible Payment Scheme in Hotel Business

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by

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Abstract

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This paper introduces a flexible payment scheme in the hotel business. When a customer makes a reservation for a hotel room, the hotel offers an optional payment scheme (Scheme O). If the customer chooses the Scheme O, he/she makes a non-refundable down payment immediately. Meanwhile, the hotel offers a discount if the customer actually checks in to the hotel. Thus, the payment at check-out time is much lower than the original rental rate. Alternatively, if the customer rejects the Scheme O, the reservation is made under a traditional Scheme (Scheme T), where no down payment is required. However, the customer choosing Scheme T must make a full payment without any discount when he checks out from the hotel. The value of Scheme O depends on customers' cancelation or no-show due to the competition from nearby hotels. We consider two scenarios: 1). the hotel knows the expected value of competitor's rental rate (deterministic case); 2). the hotel knows the stochastic distribution of competitor's rental rate (stochastic case). We have obtained optimal solutions for Scheme O for both deterministic and stochastic cases. Moreover, we also study the interaction between designing a flexible payment scheme and pricing on the rental rate of hotel room.

Key words: pricing; payment scheme; revenue management; hotels

Contents

List of Tables

List of Figures

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Chapter 1

Introduction

Nowadays, reservations are widely used in hospitality industries. Customers can make reservations for hotel rooms through Internet or phone calls before his/her check-in date. For example, if a customer wants to make a reservation through Internet, he can go to his favorite hotel's web-site and make a reservation. Customers can also make reservations through some agencies' web-site (such as www.zuji.com, www.booking.com, etc.).

Reservations are often accepted freely by hotels who normally make some cancelation policies. According to some hotel's cancelation policies, customers may be charged with a certain amount of money known as penalty if they cancel their reservations or they fail to show up on their check-in date. However, Some hotels don't charge customers (zero penalty) for their cancelation or no shows. Thus, reservations provide a form of insurance that customers can 'lock in' hotel rooms for their future check-in. Some price-sensitive customers would cancel their reservations or just never show up if they find out other hotel rooms with a lower price (Quan 2002). Meanwhile, some hotels would ask customers to make a non-refundable full down payment at the time of making a reservation for a promotion service, e.g. discount rate.

Our motivation for this paper comes from some questions that are based on the situation we mention above: 1) how should hotels handle the risk of cancelations or no shows? 2) should hotels always charge customers a full down payment for their promotion service, i.e. offering a discount rate? 3) how many discounts should hotels offer to customers for their promotion service? So far, we notice that few literatures discuss such kind of problems.

We explore a scenario where a hotel introduces a flexible payment scheme (we call it Scheme O). According to the price the hotel announces, customers make reservations through Internet or phone calls for their single-night stay. If the customer chooses the Scheme O, he/she should make a down payment when he makes the reservation. This down payment will not be returned to the customer if he cancels his reservation or he does not show up eventually. However, if the customer checks in to the hotel, he/she can enjoy some discounts and pays for his room at a lower rental rate when he/she checks out. If the customer does not choose Scheme O, the reservation is made under a traditional scheme (we call it Scheme T) where no down payment is required. However, the customer needs to make full payment without any discounts when he checks out from the hotel.

Meanwhile, the hotel faces competitions from other nearby hotels with a similar grade. Customers may cancel their reservations or not show up eventually if they find a lower rental rate from other hotels. This is possible even if customers have already made down payments under scheme O, as long as the rate offered by the hotel's competitors is lower enough (for example lower than the remaining amount due). Our goal is to find an optimal payment scheme for the hotel to maximize its revenue.

We first study the case where a hotel knows the expected value of its competitor's rental rate. We call this case the deterministic case. Our results show that two-payment scheme policy is optimal for the hotel. In other words, the hotel should offer both optional payment scheme (Scheme O) and traditional payment scheme (Scheme T) to customers. In addition, full down payment is optimal for Scheme O when there is no competition for the hotel from other nearby hotels. However, if the hotel faces competitions from other hotels, full down payment scheme may not be optimal for the hotel.

In the other case, we assume that the hotel knows the distribution of his competitor's rental rate (we call this case stochastic case). Our results show that two-payment scheme is still optimal for the hotel even if his competitor's rental rate is also stochastic. However, full down payment may not be optimal for the hotel under this stochastic case.

The remainder of this paper is organized as follows. Chapter 2 provides a brief review of literatures. Payment scheme based on known expected value of the competitor's rental rate and stochastic distribution of competitor's rental rate are studied in Chapter 3 and Chapter 4, respectively. Chapter 5 presents the interaction between designing a flexible payment scheme and pricing on the rental rate of hotel room. And Chapter 6 concludes this paper and discusses some possible directions for further research.

Chapter 2

Literature Review

There are three main streams of literatures related to our paper: literatures related the hotel industry, customer choice behavior and reservations.

For literatures related to the hotel industry, Liberman and Yechiali(1978) consider a hotel problem of finding an optimal over-booking strategy to maximize the hotel's expected total net profit as well as its expected discounted net profit. They obtained an optimal strategy of allocating rooms with over-booking problem based on the inventory level and new requests. Bitran and Mondshein (1995) study optimal policies for renting hotel rooms that are given a fixed capacity to various classes of customers arriving in a stochastic and dynamic way from different market segments within a finite horizon. They consider how to maximize the hotel's revenue by intelligently matching capacity with demand in a general order way from different market segments, multiple types of rooms with the possibility of downgrading, and multiple-night stays. Bitran and Gilbert (1996) present a realistic model of the hotel reservation problem with the assumption of all customers arriving simultaneously on the targeted booking date. And they formalize the relationship between the reservation control problem at the tactical level and the capacity allocation problem at the operational level with customer's single-night stay in single-room. Optimal solution policies have been derived which are consistent with the intuitive approaches that are used in practices. Ding et al.(2009) study the optimal pricing policy for hotels with a single type of rooms when customers requiring multiple-day stays. Their results show that hotels should substantially raise the rental rate for the high demand days while lowering the rental rate for their neighboring days with lower demand.

Schwartz and Cohen (2003) extend the basic model proposed by Bodily and Weatherford (1995) to the hotel revenue management problem with group discount room rates, and address questions like how many discounts a hotel can offer to a group of people while still maintaining the hotel's contribution margin. In recent group discount research, Choi (2006) develops a model to evaluate the group profitability for hotels with its objective to decide when to accept group customers and how much the minimum group rate should be. Hanks et al. (1992) and Boger et al.(1999) use empirical research methods to study the problem of discounting business rates among lodging companies. Koide and Ishii (2005) consider a problem for hotel rooms allocation with early discount, cancelations and overbooking, where customers can book rooms at a discounted price if they make reservations before a certain deadline. They model the expected total sales function and prove that their objective function is unimodal with respect to the number of rooms allocated for early discount as well as to the number of overbooking, under a condition. They also obtain a range where optimal solutions exist. Their work is an capacity allocation problem while our work focuses on the pricing strategy and our objective is to determine optimal payment scheme. Besides, Rothstein (1974), Ladany(1976), Varda Liberman and UriYechiali(1978) also address the the hotel reservation problem.

In recent years, more and more researchers study the problem related customer behavior. Shen and Su (2007) conducted a recent survey on customer behavior modeling in revenue management. There is another set of papers that focus on how a company sets the optimal pricing strategy in the presence of strategic customers. Elmaghraby (2008) designs a structure of the optimal markdown mechanism in the presence of strategic customers with multi-unit demands. These people compare the difference of seller's profit under the optimal markdown prices and the optimal single price. Aviv and Pazgal (2008) study the optimal pricing policy with a finite inventory facing strategic customers and myopic customers. They consider the problem where seller offers two classes of pricing strategies: *contingent* and *announced fixed-discount*. They find contingent pricing policies perform essentially the same as announced fixed-discount pricing policies for myopic consumers. However, under strategic consumer behavior, announced pricing policies can be more profitable to the seller than contingent pricing policies. Liu and van Ryzin (2008) study the problem whether it is optimal for a company to create rationing risk by deliberately under-stocking products. They consider a two-period model where customers have heterogeneous valuations for the firm's products and face declining prices over periods, and customers behave strategically to decide immediate purchase or delay their purchases. Su and Zhang (2008) address the impact of strategic customer behavior on supply chain performance. The seller initially charges a regular price, but after demand is realized, they may salvage the leftover at a lower salvage price. Customers anticipate future sales and decide to purchase at a regular price or purchase at a salvage price to maximize their expected surplus. Lai et al. (2009) examine the impact of posterior price matching on profit with strategic consumers. The seller promises to reimburse the price difference to consumers who buy a item before the seller marks it down. Cachon and Swinney (2009) also study the problem where retailer sells a product with uncertain demand over a finite selling season in the presence of strategic consumers. They discuss the value of quick response and demonstrate that which is more profitable to the retailer in the presence of strategic consumers. Su and Zhang (2009) analyze the role of product availability effect on the customer purchase behavior. The seller sets an observable price and an unobservable stocking quantity, consumers determine whether to visit the seller and incur sunk costs if they do. The authors analyze two strategies: *commitment* and *availability* guarantees. Their results show that the seller can improve profits by using a combination of commitment and availability guarantees. Yin et al. (2009) also focuses on the aspect of inventory-related information in the presence of strategic consumers. Seller uses one of two inventory display formats: display all and display one at a time. Two classes of customers (one with a higher valuation and the other with a lower valuation) decide the time of purchase strategically. Su (2010) proposes the optimal pricing policies with speculators and strategic consumers, customers may strategically determine when to purchase, and they may also decide whether to purchase from the firm or from speculators.

One of the early papers considering granting the buyers reservation right is discussed by Png (1989). The price strategy is set as a form of reservation that induces high valuation customer to exercise their purchase option while those with lower valuation do not exercise. Biyalogorsky and Gerstner (2004) address the idea of using 'contingent pricing' to reduce price risks, where a buyer has an option to reserve the item at a lower price that will obligate him to buy it if the seller has not sold the item in a specific period. They argue that 'contingent pricing' is beneficial to both the seller and the buyer. Gallego and Kou (2008) address the concept of 'callable product', which is a unit of item sold to the self-selected low-fare customers who willingly grant the seller the option to 'call' the item at a pre specified recall price. The idea is similar to the reservation option, where the seller has the right to 'call' back a item from the 'callable' agreement customer and sell it to the high price customers. Alexandrov and Lariviere (2008) study the problem whether a restaurant should offer a reservation right to customers. They show that a restaurant will never offer reservations when the market size is known. For mark-size uncertainty case, they develop conditions under which reservations are recommended. Recently, Elmaghraby et al. (2009) analyze two operating regimes: "no reservation regime" and "reservation regime". The reservation regime offers customers an extra option than no reservation regime. Under the reservation regime, customers have an option to reserve an item at a clean price, while who should obligate to purchase the item if it remains unsold at the end of the selling season. They show that more purchasing options do not necessarily benefit customers.

The most related paper to ours is Quan (2002). He show that for price-sensitive customers, reservations can provide a price insurance that customers can use to 'lock in' a lower price for the future delivery of the room. While those customers may cancel their reservation if they find other hotels offering a lower price. He suggests that in order to redeem the lost of cancelation, hotels may offer two reservation choices to customers. One is standard reservation, whereby the price is quoted for a specific check-in date, and that price includes the price of the reservation option; another choice requires the guest to make the non-refundable price for the room at the time the reservation is made. And this non-refundable price does not include the price of the reservation option (which is similar to that the hotel offers a discount rate to customers). Our paper differs from the model developed by this paper in the following ways. First this paper suggests to use Black-Scholes option-pricing model to calculate the reservation price as a form of discount, while our paper models the discount as a decision variable base on the customer choice behavior model. Secondly, Quan's paper suggests that for the second reservation choice, hotel should let customers pay full down payment when making reservations, while our paper models the down payment as a decision variable. And we obtain optimal solutions for these two decision variables.

Chapter 3

Deterministic Case

In this chapter, we study the payment scheme problem based on the situation where the hotel knows the expected value of its competitor's rental rate for the same period.

3.1 Problem Formulation

At first, we summarize the notations used in this chapter in Table 3.1.

Where in Table 3.1, $i \in \{T, O\}$, represents the traditional scheme and optional scheme, respectively.

We consider a single-period problem. Let $p \geq 0$ denotes the full rental rate for a room for the entire period. Let $x \geq 0$ and $y \geq 0$ denote the down payment and discount, respectively, for scheme O. We assume that $x + y \leq p$. A customer who chooses scheme O pays $p - x - y$ upon checking out from the hotel. A customer who chooses scheme T pays p upon checking out from the hotel. Table 3.2 illustrates the payment schemes.

Suppose that the expected value of a rental rate for a similar room offered by the hotel's competitors in the same period is $p_c \geq 0$. Let d_T and d_O denote the customer's

Table 3.1: **Notations for deterministic case (in Order of Appearance)**

p	full rental rate for a room
x	down payment for scheme O
y	discount for scheme O
p_c	lowest rental rate for a similar room offered by competitors
d_i	customer's payoff if he selects the hotel under scheme i instead of the competitor
λ	demand of the hotel
a	base demand of the hotel
b	coefficient of price sensitivity
U	additional utility from choosing scheme O instead of scheme T
θ	customer's preference for down payment
α	coefficient of customer's preference for down payment
z	the probability that a customer choose scheme O
λ_i^o	original demand of scheme i
e	coefficient of cancellation probabilities due to competition
γ	probability of no shows due to some other reasons i
λ_i	actual demand of scheme i
f	revenue of the hotel
V	maximum revenue of the hotel

Table 3.2: **Payment schemes of a hotel for deterministic case**

	Down payment	**Final payment**
Scheme T	0	p
Scheme O	x	$p - x - y$

payoff for selecting the hotel, under schemes T and O respectively, instead of the other competitors. These payoffs can be determined as follows.

$$\begin{aligned} d_O &= [(p - x - y) - p_c]^+; \\ d_T &= [p - p_c]^+. \end{aligned}$$

We assume deterministic demand for the hotel. Let λ denote the demand of the hotel and it is modeled as follows.

$$\lambda = a - bp$$

Where $a > 0$ represents the base demand and $b > 0$ represents the sensitivity of the demand to the hotel's price. In order to ensure that $\lambda > 0$, we assume $0 < p < \frac{a}{b}$.

We can derive the customer's additional utility $U = y - \theta\alpha x$ from choosing scheme O instead of scheme T, where $\theta \sim Unif[0, 1]$ and $\alpha \in [0, 1]$ are down-payment-sensitivities coefficient. If $U > 0$, then the customer chooses scheme O, otherwise, he chooses scheme T. Let z denotes the probability that a customer chooses scheme O and it is can be determined as follows.

$$\begin{aligned} z &= \Pr\{U > 0\} = \Pr\{y - \theta\alpha x > 0\} \\ &= \Pr\{\theta < \frac{y}{\alpha x}\} = \min\{\frac{y}{\alpha x}, 1\} \end{aligned}$$

If the discount is sufficient large or down payment is sufficient small (ie. $y > \alpha x$), all of the customers will choose scheme O. However, for the two payment scheme problem, we assume $z = \frac{y}{\alpha x} \leq 1$. This customer utility model is similar to Cattani et al. (2006).

Let λ_O^o and λ_T^o denote the number of customers who choose schemes T and O respectively and they can be determined as follows.

$$\begin{aligned} \lambda_O^o &= \lambda \frac{y}{\alpha x} \\ \lambda_T^o &= \lambda(1 - \frac{y}{\alpha x}). \end{aligned}$$

We assume that the probability for a customer to cancel his reservation due to the hotel's competition is proportional to his payoff. Under this assumption, the probability of a cancelation due to the hotel's competition under schemes T and O are ed_T and ed_O respectively, where e is coefficient. Let γ denotes the probability of no-shows due to some other reasons. We assume $0 \leq ed_T \leq 1$, $0 \leq ed_O \leq 1$, and $0 \leq \gamma \leq 1$.

By taking into account of cancelations and no-shows, the actual demand $\lambda_T \geq 0$ and $\lambda_O \geq 0$ of schemes T and O, respectively, can be expressed as follows.

$$\begin{aligned}\lambda_T &= \lambda_T^o[1 - ed_T]^+(1-\gamma);\\ \lambda_O &= \lambda_O^o[1 - ed_O]^+(1-\gamma).\end{aligned}$$

Notice that $z = \frac{y}{\alpha x}$, where x is in the denominator. Thus, for the technical convenience, we use z to replace $\frac{y}{\alpha x}$ as well as use $z\alpha x$ to replace y in the following discussions.

The revenue of the hotel can then be expressed as

$$f(x,z) = \lambda_T p + \lambda_O(p - z\alpha x) + (\lambda_O^o - \lambda_O)x.$$

Our objective is to determine the maximum revenue V for the hotel by optimally setting x, and z. This can be achieved through solving the following optimization problem:

$$V = \max_{x,z}\{f(x,z)\} \tag{3.1}$$

subject to

$$\begin{aligned} x + z\alpha x &\leq p; \\ z &\leq 1; \\ x, z &\geq 0.; \end{aligned}$$

3.2 Solution and Analysis

Given p_c, the customer's payoffs d_T and d_O have different values as the hotel's price changes. So we divide the analysis of the Problem (3.1) into three different cases. In section 3.2.1, we analyze the case of $p \leq p_c$, where $d_T = d_O = 0$, which means that there is no competition between the hotel and other nearby hotels with a similar grade. Whereas, when $p > p_c$, the hotel faces competition from other hotels and $d_T = p - p_c$, which means there is competition between scheme T and other hotels with a similar grade. However, d_O still has different values: it equals to 0 or $p - x - z\alpha x - p_c$, where $d_O = 0$ and $d_O = p - x - z\alpha x - p_c$ mean that there is no competition, and there is competition, respectively, between scheme O and other hotels with a similar grade). We analyze such problems in section 3.2.2 and 3.2.2 separately. Table 3.3 illustrates these three different cases.

Table 3.3: **Customer's payoff for selecting the hotel under different cases**

	3.2.1 $p \leq p_c$	**3.2.2** $p > p_c$ **and** $p - x - z\alpha x \leq p_c$	**3.2.2** $p - x - z\alpha x > p_c$
d_T	0	$p - p_c$	$p - p_c$
d_O	0	0	$p - x - z\alpha x - p_c$

In every case, we use a sequential decision procedure to solve Problem (3.1): We first assume z is fixed and find an optimal response for x. We denote this optimal value as $x^*(z)$. We then plug $x^*(z)$ into the objective function and reduces the number of variables to one, e.g. z. After that, we obtain an optimal value z^* for z.

3.2.1 The Hotel Faces No Competitions

In this case, we assume $p \le p_c$, $d_T = d_O = 0$, and there is no competition between the hotel and other nearby hotels with a similar grade. Then,

$$\begin{aligned} f(x,z) &= \lambda_T p + \lambda_O(p - z\alpha x) + (\lambda_O^o - \lambda_O)x; \\ &= \lambda\Big\{(1-z)(1-\gamma)p + z(1-\gamma)(p - z\alpha x) + z\gamma x\Big\} \\ &= \lambda\Big\{(1-\gamma)p + z[\gamma - (1-\gamma)z\alpha]x\Big\} \end{aligned} \tag{3.2}$$

Step 1: Obtain optimal value $x^*(z)$ for down payment x

Lemma 3.1. *Given z, the optimal down payment value $x^*(z)$ can be obtained through maximizing $f(x,z)$ over x in Equation (3.2).*

$$x^*(z) = \begin{cases} \frac{p}{1+z\alpha} & \text{if } \gamma > (1-\gamma)z\alpha. \\ 0 & \text{Otherwise}; \end{cases}$$

Proof. See Appendix. □

Lemma 3.1 indicates that for any given z, if the hazard rate is large enough (e.g. $\gamma > (1-\gamma)z\alpha$), then the hotel should set a positive down payment. Otherwise, it should just set the down payment equal to zero. The reason is that when the hazard rate is large enough to make the sequestration of the down payment from customers (γx) exceeds the lost of the discount ($(1-\gamma)z\alpha x$)given to customers ($\gamma x > (1-\gamma)z\alpha z$), a large down payment is optimal. Additionally, given z means that the percentage of customers who choose Scheme O is fixed. Then the hotel just needs to set the down payment as large as possible for there is no competition from other hotels. Otherwise, the lowest down payment is optimal for the hotel.

Since $x = 0$ means that hotel only offer Scheme T for the customer, then we just consider the case when $x^*(z) = \frac{p}{1+z\alpha}$ in the following Step.

Step 2: Obtain optimal value z^* for ratio z

Plugging $x^*(z)$ in lemma 3.1 into $f(x, z)$ in Equation (3.2), the optimization problem (3.1) becomes a maximization problem over a single variable z.

$$\max_z \{f(x^*(z), z)\} \tag{3.3}$$

subject to

$$\begin{aligned} (1-\gamma)z\alpha &< \gamma; \\ 0 \le z &\le 1. \end{aligned}$$

Solving the problem (3.3), we can obtain optimal value z^* as follows.

Lemma 3.2. *Maximizing the problem (3.3) over z, the optimal ratio z^* can be obtained as follows*

$$z^* = \begin{cases} \frac{1}{\alpha\sqrt{1-\gamma}} - \frac{1}{\alpha} & \textit{if } \gamma < 1 - \frac{1}{(1+\alpha)^2}. \\ 1 & \textit{Otherwise}; \end{cases}$$

Proof. See Appendix. □

Lemma 3.2 suggests an optimal ratio for the hotel regarding the Scheme O. We can observe that the ratio z^* increases in γ. If $\gamma \geq 1 - \frac{1}{(1+\alpha)^2}$, then $z = 1$, which means that the hotel should set a high discount in order to attract all customers to choose Scheme O under a high hazard rate situation (eg.$\gamma \geq 1 - \frac{1}{(1+\alpha)^2}$). Additionally, Lemma 3.2 indicates that $z^* > 0$ as long as $\gamma < 1$. And $z = 0$ means that the hotel only offers Scheme T or sets an extremely high down payment to keep all customers staying at Scheme T. However, the later case is meaningless for the hotel. Thus, we can have the following strategy for the hotel.

Corollary 3.1. *It is optimal for the hotel to offer flexible payment scheme to customers when there is no competition from other hotels.*

We formally summarize the results obtained from the above analysis in the following theorem.

Theorem 3.1. *If there is no competition for the hotel,*
(1) Lemma 3.1 and 3.2 solve the problem (3.1);
(2) It is optimal for the hotel to offer a flexible scheme with full down payment to customers using z^* and $x^*(z^*)$.

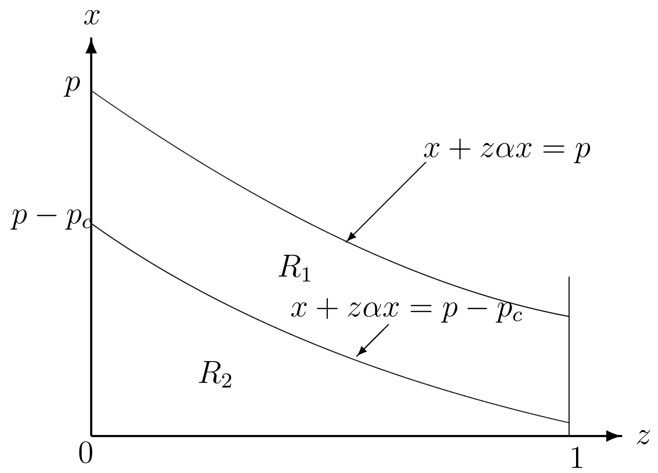

Figure 3.1: **The feasible region of (x, z) for the competition case.** Region R_1: $p-p_c \leq x+z\alpha x \leq p; 0 \leq z \leq 1; x \geq 0$, Region R_2: $x+z\alpha x \leq p-p_c; 0 \leq z \leq 1; x \geq 0$.

3.2.2 When the Hotel Faces Competition from Other Hotels

All analysis in section 3.2.1 is based on the condition of $p \leq p_c$. However, if p is larger than the expected value of the hotel's competitor's rental rate p_c, then the results in the section 3.2.1 are no longer optimal. In section 3.2.2 we solve the problem when $p > p_c$.

If $p > p_c$, then $d_T = p - p_c$, but d_O may be equal to 0 or $p - x - z\alpha x - p_c$. So we separate the analysis into Case 1 and Case 2. In Case 1, we consider $p-x-z\alpha x \leq p_c$, when there is competition between Scheme T and other nearby hotels and there is no competition between Scheme O and other nearby hotels. In Case 2, we consider the case $p - x - z\alpha x > p_c$, when there is competition from other hotels in both Scheme T and O.

Figure 3.1 shows the feasible region of (x, z) for the competition case. Region R_1 $(p - p_c \leq x + z\alpha x \leq p; 0 \leq z \leq 1; x \geq 0)$ corresponds to the case where competition is only in Scheme T, and the region where competition exists both in Scheme T and O is represented by R_2 $(x + z\alpha x \leq p - p_c; 0 \leq z \leq 1; x \geq 0)$.

Case 1: Competition exists only in Scheme T

In this section, we consider the case where $p - x - z\alpha x \leq p_c$. Then we can have $d_T = p - p_c, d_O = 0$, where competition only exists in Scheme T and the hotel faces competition from other hotels with a similar grade. But there is no competition from other hotels in Scheme O. Then we can have

$$\begin{aligned} f(x,z) &= \lambda_T p + \lambda_O(p - z\alpha x) + (\lambda_O^o - \lambda_O)x; \\ &= \lambda\Big\{(1-z)[1-e(p-p_c)](1-\gamma)p + z(1-\gamma)(p - z\alpha x) + z\gamma x\Big\} \\ &= \lambda\Big\{(1-\gamma)p - (1-z)e(p-p_c)(1-\gamma)p + z[\gamma - (1-\gamma)z\alpha]x\Big\} \end{aligned} \tag{3.4}$$

Then optimization problem (3.1) can be modeled as below:

$$\max_{x,z|(x,z)\in R_1} \{f(x,z)\} \tag{3.5}$$

Step 1: Obtain optimal value $x^*(z)$ for down payment x

Lemma 3.3. *Given z, the optimal value $x^*(z)$ of down payment x is as follows, which maximizes $f(x,z)$ in Equation (3.4).*

$$x^*(z) = \begin{cases} \frac{p}{1+z\alpha} & \textit{if } \gamma > (1-\gamma)z\alpha. \\ \frac{p-p_c}{1+z\alpha} & \textit{Otherwise}; \end{cases}$$

Proof. See Appendix. □

Lemma 3.3 indicates that for any given z, if the hazard rate is large enough (eg. $\gamma > (1-\gamma)z\alpha$), then the hotel should set a full down payment; otherwise it should just set the down payment equal to its lower bound. The reason is that when the hazard rate is large enough to make the sequestration of the down payment from customer (γx) exceeds the lost of the discount ($(1-\gamma)z\alpha x$)given to customer ($\gamma x > (1-\gamma)z\alpha z$), upper bound of down payment ($x = p - z\alpha x$) is optimal; otherwise, the lower bound of down payment ($x = p - z\alpha x - p_c$) is optimal for the hotel.

Notice that x is always positive in this case, so we can have the following corollary.

Corollary 3.2. *When $x + z\alpha x + p_c \geq p > p_c$, it is optimal for the hotel to offer the flexible payment scheme to its customers.*

Step 2: Obtain optimal value z^* for z

Let $x_u^*(z) = \frac{p}{1+z\alpha}$, $x_l^*(z) = \frac{p-p_c}{1+z\alpha}$, $Z_u \equiv \{z : (1-\gamma)z\alpha < \gamma, 0 \leq z \leq 1\}$, and $Z_l \equiv \{z : (1-\gamma)z\alpha \geq \gamma, 0 \leq z \leq 1\}$, then we consider these two cases of $x^*(z)$ separately as follows.

Subcase 1: $x^* = x_u^*(z)$

Plugging $x_u^*(z)$ back into $f(x, z)$ in Equation (3.4), the optimization problem (3.5) becomes a maximization problem over the single variable z:

$$\max_{z|z\in Z_u} \{f(x_u^*(z), z)\} \tag{3.6}$$

solving the problem (3.6), we have,

Lemma 3.4. *The optimal ratio z^* is as follows, which is the solution of problem (3.6).*

If $\frac{\gamma}{\alpha(1-\gamma)} > 1$, *then*

$$z^* = \begin{cases} \hat{z_u} & \textit{if } \hat{z_u} < 1; \\ 1 & \textit{Otherwise.} \end{cases}$$

Otherwise,

$$z^* = \begin{cases} \hat{z_u} & \textit{if } \hat{z_u} < \frac{\gamma}{\alpha(1-\gamma)}; \\ arg \max_{z|z \in Z_l} \{f(x_l^*(z), z)\} & \textit{Otherwise.} \end{cases}$$

where $\hat{z_u} = \frac{1}{\alpha}(\frac{1}{\sqrt{(1-\gamma)[1-e(p-p_c)]}} - 1)$.

Proof. See Appendix. □

Lemma 3.4 suggests an optimal ratio for problem (3.6). If $\frac{\gamma}{\alpha(1-\gamma)} > 1$, then the condition of $x^*(z) = x_u^*(z)$ is always satisfied, so z^* is either equal to $\hat{z_u}$ which satisfies $\frac{df(x_u^*(z),z)}{dz} = 0$ or equal to its upper bound. Otherwise, we need to consider whether $\hat{z_u} < \frac{\gamma}{\alpha(1-\gamma)}$ is satisfied. If it is satisfied, then $\hat{z_u}$ is the optimal value of z^*; otherwise we have to solve the case where $x^*(z) = x_l^*(z)$. Moreover, $\hat{z_u}$ has the following implications: If hazard rate (γ) is high or the competition ($e(p-p_c)$) in Scheme T and other nearby hotels is fierce, then the hotel should set a smaller down payment and a larger discount (since $x^*(z)$ and $z\alpha x^*(z)$ are decreasing and increasing in z, respectively.) to attract customer to choose Scheme O.

Subcase 2: $x^* = x_l^*(z)$

Plugging $x_l^*(z)$ back into $f(x,z)$ in Equation (3.4), the optimization problem (3.5) becomes a maximization problem over the single variable z:

$$\max_{z|z\in Z_l}\{f(x_l^*(z),z)\} \tag{3.7}$$

Solving problem (3.7), we have,

Lemma 3.5. *The optimal ratio z^* is as follows, which is the solution of problem (3.7).*

If $\frac{\gamma}{\alpha(1-\gamma)} > 1$, then $z^* = arg\max_z\{f(x_u^*(z),z)\}$;

Otherwise,

If $1-ep \le 0$, then $z^* = 1$;

Otherwise,

$$z^* = \begin{cases} arg\max_{z|z\in Z_u}\{f(x_u^*(z),z)\} & \text{if } \hat{z}_l < \frac{\gamma}{\alpha(1-\gamma)}; \\ \hat{z}_l & \text{if } \frac{\gamma}{\alpha(1-\gamma)} \le \hat{z}_l < 1; \\ 1 & \text{Otherwise.} \end{cases}$$

where $\hat{z}_l = \frac{1}{\alpha}(\frac{1}{\sqrt{(1-\gamma)(1-ep)}} - 1)$.

Proof. See Appendix. □

Lemma 3.5 suggests an optimal ratio for the problem (3.7). If $\frac{\gamma}{\alpha(1-\gamma)} > 1$, then the condition of $x^*(z) = x_l^*(z)$ always been violated, so we need to go to solve the case where $x^*(z) = x_u^*(z)$. Otherwise, we need to consider the value of $1-ep$: if $1-ep \le 0$, then $z^* = 1$, which implies that we should set a very small down payment and very large discount to attract all of the customer to choose Scheme O for a larger value of p ($p \ge \frac{1}{e}$), since the rental rate of the hotel is very high, then the competition is comparatively fierce. If $1 - ep > 0$, then we need to consider whether condition $\hat{z}_l \geq \frac{\gamma}{\alpha(1-\gamma)}$ is satisfied, if it is satisfied, then the optimal value of z^* is either equal to $\hat{z}_l$ or equal to upper bound of z ($z^* = 1$); otherwise we have to go to solve the case where $x^*(z) = x_u^*(z)$. Moreover, $\hat{z}_l$ has the following implications: if hazard rate (γ) is high or competition (ep) in Scheme T and other nearby hotels is comparatively fierce, then the hotel should set a smaller down payment and a larger discount to attract customers to choose Scheme T.

By integrating Case 1 ($x^*(z) = x_u^*(z)$) and Case 2 ($x^*(z) = x_l^*(z)$), we can have the following results.

Proposition 3.1. *The optimal payment scheme $(x^*(z), z)$ is as follows, which is the solution of problem (3.5).*

If $\gamma > \frac{\alpha}{1+\alpha}$, then $(x^*(z), z^*) = (x_u^*(z), z_u^*)$

Otherwise,

$$(x^*(z), z^*) = \begin{cases} (x_l^*(z), z_l^*) & \text{if } \gamma < e(p - p_c); \\ arg \max\limits_{x(z), z}\{f(x_u^*(z), z_u^*), f(x_l^*(z), z_l^*)\} & \text{if } e(p - p_c) \leq \gamma < ep; \\ (x_u^*(z), z_u^*) & \text{Otherwise.} \end{cases}$$

where

$$z_u^* = \begin{cases} \hat{z}_u & \text{if } \frac{1}{\sqrt{(1-\gamma)(1-e(p-p_c))}} < 1 + \alpha; \\ 1 & \text{Otherwise.} \end{cases}$$

$$z_l^* = \begin{cases} \hat{z}_l & \text{if } 1 - ep > 0 \text{ and } \frac{1}{\sqrt{(1-\gamma)(1-ep)}} < 1 + \alpha; \\ 1 & \text{Otherwise.} \end{cases}$$

Proof. See Appendix. □

Proposition 3.1 shows the optimal payment scheme for the case of competition only in Scheme T, which is determined by hazard rate (γ) and competition factor ($e(p-p_c)$). If $\gamma > \frac{\alpha}{1+\alpha}$, the hotel should offer a full down payment $x_u^*(z)$ to customer since the high hazard rate. And z^* is determined by comparing the value of $\hat{z_u}$ with the upper bound of z. Otherwise, we need to compare the hazard rate (γ) and competition factor ($e(p-p_c)$): if the hard rate is too low (ie., $\gamma < e(p-p_c)$), a small down payment (ie., $x_l^*(z)$) is optimal for the hotel, and optimal ratio is determined by the value of ep, $\hat{z}_l$ and upper bound of z. On the other hand, if the hard rate is too high (ie., $\gamma \geq ep$), a high down payment (ie., $x_u^*(z)$) is preferred by the hotel. If hazard rate is medial (ie., $e(p-p_c) \leq \gamma < ep$), the optimal payment scheme is determined by comparing $f(x_u^*(z), z_u^*)$ with $f(x_l^*(z), z_l^*)$.

Case 2: Competition exists in both scheme T and scheme O

In this section, we consider the case where $p - x - z\alpha x > p_c$. Then we have $d_O = p - x - z\alpha x - p_c$ and $d_T = p - p_c$, and there is competition existing in both on Scheme T and Scheme O. Then the problem becomes

$$\begin{aligned} f(x,z) &= \lambda_T p + \lambda_O(p - z\alpha x) + (\lambda_O^o - \lambda_O)x; \\ &= \lambda\Big\{(1-z)[1-e(p-p_c)](1-\gamma)p \\ &+ z[1-e(p-x-z\alpha x-p_c)](1-\gamma)(p-z\alpha x) \\ &+ z[1-(1-e(p-x-z\alpha x-p_c))(1-\gamma)]x\Big\} \end{aligned} \tag{3.8}$$

Then optimization problem (3.1) can be solved by solving the following optimization problem:

$$\max_{x,z|(x,z)\in R_2} \{f(x,z)\} \tag{3.9}$$

Step 1: obtain optimal value $x^*(z)$ for down payment x

Given z, we can obtain the optimal down payment value $x^*(z)$ as follows,

Lemma 3.6. *Given z, optimal down payment $x^*(z)$ can be obtained as follows, which maximizes $f(x,z)$ in Equation (3.8).*

If $(1-\gamma)z\alpha < \gamma$ or $p - \hat{x}(z) - z\alpha\hat{x}(z) \leq p$, then

$$x^*(z) = arg \max_{x|(x,z)\in R_1} \{f(x,z)\}$$

Otherwise,

$$x^*(z) = \begin{cases} \hat{x}(z) & \textit{if } \hat{x}(z) > 0. \\ 0 & \textit{Otherwise}; \end{cases}$$

where

$$\hat{x}(z) \quad = \quad \frac{p}{1+z\alpha} - \frac{1}{2(1+z\alpha)}[\frac{(1-\gamma)z\alpha-\gamma}{e(1-\gamma)(1+z\alpha)} + p_c]$$

Proof. See Appendix. □

Lemma 3.6 shows that for any given z, if the hazard rate is high enough (eg. $\gamma > (1-\gamma)z\alpha$), then we should go back to solve the case $p - x - z\alpha x \leq p$, which implies that higher down payment is preferred by the hotel. $p - \hat{x}(z) - z\alpha\hat{x}(z) \leq p$

means that there is no interior solutions exists for this case and higher down payment is preferred by the hotel as well. If $\hat{x}(z) < 0$, there is also no interior solutions exist and lower bounder of x is optimal due to the concavity of Equation (3.8) in x. Otherwise, $\hat{x}(z)$ is optimal for the hotel that satisfies $\frac{\partial f(x,z)}{\partial x} = 0$.

Proposition 3.2. *If* $x^*(z) = \hat{x}(z)$*, then* $\frac{dx^*(z)}{dz} \leq 0$*, and* $\frac{d(x^*(z)+z\alpha x^*(z))}{dz} \leq 0$.

Proof. See Appendix □

Proposition 3.2 indicates two intuitive properties for the local optimal response of down payment $x^*(z)$. First property is that the local optimal response of down payment decreases in z, which is intuitive because in order to attract customers to choose Scheme O, the hotel have to set a lower down payment rather than a higher down payment. Another intuitive property is that the sum of local optimal response of down payment and discount is also decreasing in z, which indicates that the amount due of customers who choose Scheme O $(p - x^*(z) - z\alpha x^*(z))$ is increasing in z, and consequently the probability of the customers who choose Scheme O and have no-shows due to the competition $(e(p - x^*(z) - z\alpha x^*(z) - p_c))$ is increasing in z. This result implies that the probability of customers who choose Scheme O and have no-shows due to the competition is increasing in the down payment.

Step 2: obtain optimal value z^* for z

Subcase 1: $x^*(z) = arg \max\limits_{x|(x,z)\in R_1} \{f(x,z)\}$

In this case, we need to go back to solve the problem $\max\limits_{x|(x,z)\in R_1} \{f(x,z)\}$ to get optimal value of x.

Subcase 2: $x^*(z) = 0$

If $x^*(z) = 0$, then $z\alpha x^*(z) = 0$, the optimal policy for the hotel is only to offer Scheme T.

Subcase 3: $x^*(z) = \hat{x}(z)$

Plugging $x^*(z)$ back into $f(x, z)$ in Equation (3.8), the optimization problem (3.9) becomes a maximization over the single variable z:

$$\max_z \{f(x^*(z), z)\} \tag{3.10}$$

subject to

$$\begin{aligned}
(1-\gamma)z\alpha &> \gamma; \\
p - x^*(z) - z\alpha x^*(z) &> p_c; \\
x^*(z) &> 0; \\
z &\leq 1; \\
z &\geq 0.
\end{aligned}$$

Solving $f(x^*(z), z)$ in problem (3.10), we have,

Lemma 3.7. *The local optimal ratio $\hat{z}$ is as follows, which maximizing $f(x^*(z), z)$ in problem (3.10).*

$$\hat{z} = \begin{cases} \frac{-1-2t+\sqrt{1+8t}}{2t\alpha} & \text{if } t > 0; \\ \frac{-1-2t-\sqrt{1+8t}}{2t\alpha} & \text{if } -\frac{1}{8} < t \leq 0; \\ 1 & \text{Otherwise.} \end{cases}$$

where

$$t \;=\; (1 - e(2p - p_c))(1 - \gamma).$$

Proof. See Appendix. □

Lemma 3.7 shows the local optimal ratio of problem (3.10), which is determined by the hazard rate (γ) and competition factor ($e(2p - p_c)$).

Notice that $x^*(z) > 0$ is always satisfied when $z = \hat{z}$ (refer to proof of Proposition 3.3), then the following result holds.

Corollary 3.3. *When $p - x - z\alpha x > p_c$, it is optimal for the hotel to offer the flexible payment scheme to customers.*

By combining Lemma 3.7 with boundary conditions in problem (3.10), we have,

Proposition 3.3. *The optimal payment scheme $(x^*(z), z)$ is as follows, which is the solution of problem (3.9).*

If $(1-\gamma)\hat{z}\alpha < \gamma$ or $1 - ep_c \le 0$ or $\{1 - ep_c > 0$ and $\hat{z}\alpha \le \frac{\gamma + ep_c(1-\gamma)}{(1-ep_c)(1-\gamma)}\}$, then

$$(x^*(z), z^*) \;=\; arg \max_{x,z|(x,z)\in R_1} \{f(x,z)\}$$

Otherwise

$$(x^*(z), z^*) = \begin{cases} (\hat{x}(z), \hat{z}) & \textit{if } \hat{z} < 1; \\ (\hat{x}(z), 1) & \textit{Otherwise.} \end{cases}$$

Proof. See Appendix. □

Proposition 3.3 suggests the optimal payment scheme for the problem (3.9). If local optimal ratio $\hat{z}$ is the value makes $(1-\gamma)\hat{z}\alpha < \gamma$ or $p - x^*(z) - z\alpha x^*(z) \leq p_c$, then the optimal payment scheme can be obtained by Proposition 3.1. Otherwise, the down payment $\hat{x}(z)$ is optimal for the hotel, and optimal ratio is determined by comparing the value of $\hat{z}$ and upper bound of z.

We formally summarize the results obtained in above analysis in the following Theorem.

Theorem 3.2. *In competition case,*
(1) Proposition 3.1 and Proposition 3.3 solve the problem (3.1);
(2) It is optimal for the hotel to offer a flexible payment scheme to customers using z^* and $x^*(z^*)$.

Reminding that full down payment is always optimal for the no competition case. However, in the competition case, full down payment may not always be optimal, all three values of the down payment: $\hat{x_u}(z)$, $\hat{x_l}(z)$ and $\hat{x}(z)$ are indicated in Proposition 3.1 and Proposition 3.3 could be a optimal down payment.

Chapter 4

Stochastic Case

In this chapter, we study the payment scheme problem based on the situation where the hotel knows the stochastic distribution of competitor's rental rate for the same period.

4.1 Problem Formulation

At first, we summarize the notations used in this paper in Table 4.1.

Where in Table 4.1, $i \in \{T, O\}$, represent the traditional scheme (Scheme T) and optional scheme (Scheme O), respectively.

We consider a single-period problem. Let $p \geq 0$ denote the full rental rate for a room for the entire period. Let $x \geq 0$ and $y \geq 0$ denote the down payment and discount, respectively, for Scheme O. We assume $x + y \leq p$. A customer who chooses scheme O pays $p - x - y$ upon checking out from the hotel. A customer who chooses scheme T pays p upon checking out. Table 4.2 illustrates the payment schemes.

We assume that the rental rate for a similar room offered by competitors of the hotel for the same period is p_c and consider p_c is stochastic with its range from A to

Table 4.1: **Notation for stochastic case (in Order of Appearance)**

p	full rental rate for a room
x	down payment for scheme O
y	discount for scheme O
p_c	lowest rental rate for a similar room offered by competitors
A	lower bound of p_c
B	upper bound of p_c
d_i	customer's payoff if he selects the hotel under schemes i instead of the competitor
λ	demand of the hotel
a	base demand of the hotel
b	coefficient of price sensitivity
U	additional utility from choosing scheme O instead of scheme T
θ	customer's preference for down payment
α	coefficient of customer's preference for down payment
z	the probability that a customer choose scheme O
λ_i^0	original demand of scheme i
e	coefficient of cancellation probabilities due to competition
γ	probability of no shows due to some other reasons i
λ_i	actual demand of scheme i
E	expected revenue of the hotel
V	maximum expected revenue of the hotel

Table 4.2: **Payment schemes of a hotel for stochastic case**

	Down payment	**Final payment**
Scheme T	0	p
Scheme O	x	$p - x - y$

B, i.e. $p_c \in [A, B]$, where A and B are lower bound and upper bound of p_c. $\phi(.)$ and $\Phi(.)$ are *pdf* and *cdf* of p_c respectively.

Let d_T and d_O denote the customer's payoff if he chooses the hotel, under schemes T and O respectively. These payoffs are modeled as follows.

$$\begin{aligned} d_O &= [(p - x - y) - p_c]^+; \\ d_T &= [p - p_c]^+. \end{aligned}$$

We assume that the demand for the hotel is deterministic. Let λ denote the demand of the hotel and it is modeled as follows.

$$\lambda = a - bp$$

Where $a > 0$ represents the base demand and $b > 0$ represents the sensitivity of the demand to the hotel's price. To ensure that $\lambda > 0$, we assume $0 < p < \frac{a}{b}$.

The customer derives additional utility $U = y - \theta\alpha x$ from choosing scheme O instead of scheme T, where $\theta \sim Unif[0, 1]$ and $\alpha > 0$ are down-payment-sensitivities coefficient. If $U > 0$, then the customer chooses scheme O; otherwise, he chooses scheme T. Let z denote the probability that a customer chooses scheme O and it is determined as follows.

$$\begin{aligned} z &= \Pr\{U > 0\} = \Pr\{y - \theta\alpha x > 0\} \\ &= \Pr\{\theta < \frac{y}{\alpha x}\} = \min\{\frac{y}{\alpha x}, 1\} \end{aligned}$$

If the discount is sufficient large or down payment is sufficient small (ie. $y > \alpha x$), all of the customers will choose scheme O. However, for the two payment scheme problem, we assume $z = \frac{y}{\alpha x} \leq 1$. This customer utility model is similar to Cattani et al. (2006).

Let λ_O^o and λ_T^o denote the number of customers who choose schemes T and O respectively and they are determined as follows.

$$\begin{aligned} \lambda_O^o &= \lambda z; \\ \lambda_T^o &= \lambda(1 - z). \end{aligned}$$

Assume the probability for a customer to cancel his reservation due to competition is proportional to his payoff. Under this assumption, the probability of a cancelation due to competition under schemes T and O are ed_T and ed_O respectively, where e is coefficient. Let γ denote the probability of no-shows due to some other reasons. We assume $0 \leq ed_T \leq 1$, $0 \leq ed_O \leq 1$, and $0 \leq \gamma \leq 1$.

By taking into account of cancelations and no-shows, the actual demand $\lambda_T \geq 0$ and $\lambda_O \geq 0$ of schemes T and O, respectively, can be expressed as follows.

$$\begin{aligned} \lambda_T &= \lambda_T^o[1 - ed_T]^+(1 - \gamma); \\ \lambda_O &= \lambda_O^o[1 - ed_O]^+(1 - \gamma). \end{aligned}$$

Notice that $z = \frac{y}{\alpha x}$, where x is in the denominator. Thus, for the technical convenience, we use $z\alpha x$ replace y as follows.

The expected revenue of the hotel can be expressed as follows due to the stochastic property of p_c,

$$\begin{aligned} E(x,z) &= E[\lambda_T p + \lambda_O(p-y) + (\lambda_O^o - \lambda_O)x]. \\ &= \lambda\Big\{ \int_A^{p-x-z\alpha x} f_3(x,z)\phi(p_c)dp_c + \int_{p-x-z\alpha x}^{p} f_2(x,z)\phi(p_c)dp_c \\ &+ \int_p^B f_1(x,z)\phi(p_c)dp_c \Big\} \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} f_1(x,z) &= (1-z)(1-\gamma)p + z(1-\gamma)(p-z\alpha x) + z\gamma x \\ f_2(x,z) &= (1-z)[1-e(p-p_c)](1-\gamma)p + z(1-\gamma)(p-z\alpha x) + z\gamma x \\ f_3(x,z) &= (1-z)[1-e(p-p_c)](1-\gamma)p + z[1-e(p-x-z\alpha x-p_c)](1-\gamma)(p-z\alpha x) \\ &\quad + z[1-(1-e(p-x-z\alpha x-p_c))(1-\gamma)]x \end{aligned}$$

Then our objective is to determine the maximum expected revenue V of the hotel by optimally setting x and z. This can be achieved by solving the following optimization problem:

$$V = \max_{x,z}\{E(x,z)\} \tag{4.2}$$

subject to

$$\begin{aligned} x + z\alpha x &\leq p; \\ 0 &\leq z \leq 1; \\ x &\geq 0. \end{aligned}$$

4.2 Solution and Analysis

In this section, we use the following approach to analyze the Problem (4.1): we first assume z is fixed and find an optimal value for x. Denote this optimal value as $x^*(z)$. We then substitute $x^*(z)$ into the objective function and reduces the number of variables to one. After that, we can find an optimal value of z.

By rearranging equation (4.1), we have

$$E(x,z) = E_T(x,z) + E_O(x,z) \tag{4.3}$$

where

$$\begin{aligned} E_T(x,z) &= \lambda\Big\{(1-z)[1-\int_A^p e(p-p_c)\phi(p_c)dp_c](1-\gamma)p\Big\} \\ E_O(x,z) &= \lambda\Big\{z[1-\int_A^{p-x-z\alpha x} e(p-x-z\alpha x-p_c)\phi(p_c)dp_c](1-\gamma)(p-z\alpha x) \\ &\quad +z[\gamma+(1-\gamma)\int_A^{p-x-z\alpha x} e(p-x-z\alpha x-p_c)\phi(p_c)dp_c]x\Big\} \end{aligned}$$

$E_T(x,z)$ is the expected revenue the hotel can have from Scheme T, where $\int_A^p e(p-p_c)\phi(p_c)dp_c$ is the percentage of customers who cancel their reservations due to the competition. Alternatively, $E_O(x,z)$ is the expected revenue the hotel can get from the Scheme O, where $\int_A^{p-x-z\alpha x} e(p - x - z\alpha x - p_c)\phi(p_c)dp_c$ is the percentage of customers who cancel their reservations due to the competition as well. So $z[1 - \int_A^{p-x-z\alpha x} e(p - x - z\alpha x - p_c)\phi(p_c)dp_c](1 - \gamma)$ corresponds to the percentage of customers who chose Scheme O and check in finally, contrarily, the percentage of customers who chose Scheme O but have no shows eventually is represented by $z[\gamma + (1 - \gamma)\int_A^{p-x-z\alpha x} e(p - x - z\alpha x - p_c)\phi(p_c)dp_c]$.

4.2.1 Obtain optimal value $x^*(z)$ for down payment x

Consider the first and second partial derivatives of $E(x, z)$ with respect to x:

$$\begin{aligned}\frac{\partial E(x,z)}{\partial x} &= \lambda\Big\{ - z(1-\gamma)z\alpha + z\gamma \\ &\quad + z(1+z\alpha)e(1-\gamma)\int_A^{p-x-z\alpha x}[2(p-x-z\alpha x) - p_c]\phi(p_c)dp_c\Big\}\end{aligned} \tag{4.4}$$

$$\frac{\partial^2 E(x,z)}{\partial x^2} = -\lambda z(1+z\alpha)^2 e(1-\gamma)M(x,z) \leq 0$$

where $M(x, z) = 2\Phi(p - x - z\alpha x) + (p - x - z\alpha x)\phi(p - x - z\alpha x)$. (In the following part, the item $2\Phi(p - x - z\alpha x) + (p - x - z\alpha x)\phi(p - x - z\alpha x)$ is always represented by $M(x, z)$).

Thus, we can obtain the following result:

Proposition 4.1. *Given z, $E(x, z)$ is concave in x. The optimal value of down payment x is determined uniquely by first-order-condition of $E(x, z)$ over x: $\frac{\partial E(x,z)}{\partial x} = 0$.*

Remark 4.1. *Notice from Equation (4.4) that:*

(1) if $\gamma \geq (1 - \gamma)z\alpha$, then $\frac{\partial E(x,z)}{\partial x}$ is non-negative, which implies that upper bound of

x is optimal for $E(x,z)$, i.e. $x^*(z) = \frac{p-A}{1+z\alpha}$. The reason is that when the hazard rate is large enough to make the sequestration of the down payment from customers (γx) exceeds the lost of the discount given to customers ($(1-\gamma)z\alpha x$), a large down payment is optimal. Additionally, at this point, no customer of Scheme O will cancel his reservation due to the competition, since the remaining amount due of these customer is non-larger than the lowest rental rate offered by the competitor.

(2)given $x^*(z) = \frac{p-A}{1+z\alpha}$, $\frac{dE(x^*(z),z)}{dz} = -[1-\int_A^p e(p-p_c)\phi(p_c)dp_c](1-\gamma)p + (1-\gamma)A + \frac{p-A}{(1+z\alpha)^2}$, which is decreasing in z, implying that z^* can be determined uniquely by $\frac{dE(x^*(z),z)}{dz} = 0$.

(3) given $x^*(z) = \frac{p-A}{1+z\alpha}$, $\frac{dx^*(z)}{dz} \leq 0$, $\frac{dz\alpha x^*(z)}{dz} \geq 0$, and $\frac{d(x^*(z)+z\alpha x^*(z))}{dz} \leq 0$.

In the following discussion, we consider the case when $x^*(z)$ is determined by $\frac{\partial E(x,z)}{\partial x} = 0$. And $x = 0$ means the hotel only offers Scheme T, so we have the following result:

Corollary 4.1. *If $x^*(z) > 0$, then the two payment schemes policy is optimal for the hotel; otherwise, the hotel should only offer a traditional scheme.*

Note from Equation (4.3) that if $p \leq A$, then $E(x,z) = \lambda\{(1-z)(1-\gamma)p + z(1-\gamma)(p - z\alpha x) + z\gamma x\} = \lambda\{(1-\gamma)p + z(\gamma - (1-\gamma)z\alpha)x\}$. It implies that there is no competition between the hotel and the other nearby hotels with a similar grade, We can get the following result.

Corollary 4.2. *It is optimal for the hotel to offer a full down payment optional scheme to customers if there is no competition between the hotel and the other nearby hotels with a similar grade.*

Proof. See Appendix. $\square$

Quan (2002) suggests the hotel charges a full down payment at the time the reservation is made, while Corollary 4.2 and Proposition 4.1 indicate that full down payment can be optimal for the hotel when there is no competition among the nearby hotels but may not optimal when competition exists.

By taking the first and second partial derivative of $E(x,z)$ in Equation (4.3) with respect to z, we can obtain the following result.

Proposition 4.2. *$E(x,z)$ is concave in z at $x = x^*(z)$.*

Proof. See Appendix. □

Proposition 4.2 indicates that for given x, the expected revenue function is concave in z only when x is at its optimal value $x^*(z)$, i.e. $x = x^*(z)$

Proposition 4.3. $\frac{dx^*(z)}{dz} \leq 0$.

Proof. See Appendix. □

Proposition 4.3 shows that optimal down payment is decreasing in z. This result is intuitive. Because in order to attract customers to choose Scheme O, the hotel has to set a lower down payment rather than a higher down payment.

Proposition 4.4. $\frac{d(x^*(z)+z\alpha x^*(z))}{dz} \leq 0$.

Proof. See Appendix. □

Proposition 4.4 shows that the sum of down payment and discount is deceasing in z, it indicates the amount due $(p - x - z\alpha x)$ to customers who choose Scheme O is increasing in z, which implies that the percentage of customers who cancel their reservations due to the competition $(\int_A^{p-x-z\alpha x} e(p-x-z\alpha x-p_c)\phi(p_c)dp_c)$ is increasing in z as well. This result implies that the probability of customers who choose Scheme O and have no-shows due to the competition is increasing in the down payment.

Remark 4.2. *For $z\alpha x^*(z)$, we have,*

$$\frac{d^2(z\alpha x^*(z))}{dz^2}\Big|_{\frac{d(z\alpha x^*(z))}{dz}=0} = \frac{\alpha^2}{(1+z\alpha)^4 e(1-\gamma)M(x^*(z),z)^2}f(n)$$

where $n = p - x^*(z) - z\alpha x^*(z)$, which is increasing in z; $f(n) = -4\Phi(n) + (3p - 5n)\phi(n) + n(p-n)\phi'(n)$, which is a polynomial function of n. Thus, $z\alpha x^*(z)$ is quasi-concave for some values of z, and quasi-convex for some other values of z, which implies that monotone property is broken for $z\alpha x^*(z)$.

Proof. See Appendix for technical detail of Remark 4.2. □

4.2.2 Obtain optimal value z^* for ratio z

Plugging $x = x^*(z)$ into problem (4.2), the optimization problem becomes a maximization problem over the single variable z:

$$V = \max_z\{E(x^*(z), z)\} \tag{4.5}$$

subject to

$$\begin{aligned} x^*(z) + z\alpha x^*(z) &\leq p; \\ 0 &\leq z \leq 1. \end{aligned}$$

Consider the first derivative of $E(x^*(z), z)$ over z (For simplicity, we use x^* represents $x^*(z)$ in the following discussions.):

$$\begin{aligned}
\frac{dE(x^*, z)}{dz} &= \frac{\partial E(x^*, z)}{\partial z} + \frac{\partial E(x^*, z)}{\partial x}\frac{dx^*}{dz} = \frac{\partial E(x^*, z)}{\partial z} \\
&= \lambda \Big\{ -[1 - \int_A^p e(p - p_c)\phi(p)dp_c](1-\gamma)p \\
&\quad +(1-\gamma)[1 - \int_A^{p-x^*-z\alpha x^*} e(p - x^* - z\alpha x^* - p_c)\phi(p_c)dp_c](p - 2z\alpha x^*) \\
&\quad +[\gamma + (1-\gamma)\int_A^{p-x^*-z\alpha x^*} e(p - x^* - z\alpha x^* - p_c)\phi(p_c)dp_c]x^* \\
&\quad +e(1-\gamma)z\alpha x^*(p - x^* - z\alpha x^*)\Phi(p - x^* - z\alpha x^*) \Big\} \\
&= \lambda \Big\{ -[1 - \int_A^p e(p - p_c)\phi(p)dp_c](1-\gamma)p \\
&\quad +(1-\gamma)(p - 2z\alpha x^*) + \gamma x^* \\
&\quad -(1-\gamma)[\int_A^{p-x^*-z\alpha x^*} e(p - x^* - z\alpha x^* - p_c)\phi(p_c)dp_c](p - x^* - z\alpha x^*) \\
&\quad +(1-\gamma)z\alpha x^* \int_A^{p-x^*-z\alpha x^*} e[2(p - x^* - z\alpha x^*) - p_c]\phi(p_c)dp_c \Big\}
\end{aligned}$$

where the second equality is by $x^*(z)$ satisfies the first-order-condition: $\frac{\partial E(x^*(z),z)}{\partial x} = 0$.

Reminding that from $\frac{\partial E(x,z)}{\partial x} = 0$, we have $\int_A^{p-x^*-z\alpha x^*} e[2(p-x^*-z\alpha x^*)-p_c]\phi(p_c)dp_c =$

$\frac{(1-\gamma)z\alpha-\gamma}{(1+z\alpha)(1-\gamma)}$, then,

$$\begin{aligned}\frac{dE(x^*,z)}{dz} &= \lambda\Big\{-[1-\int_A^p e(p-p_c)\phi(p)dp_c](1-\gamma)p+\frac{x^*}{1+z\alpha}\\&+[1-\int_A^{p-x^*-z\alpha x^*} e(p-x^*-z\alpha x^*-p_c)\phi(p_c)dp_c](1-\gamma)(p-x^*-z\alpha x^*)\Big\}\\&= \lambda\Big\{-[1-\int_A^p e(p-p_c)\phi(p)dp_c](1-\gamma)p\\&+[1-\int_A^{p-x^*-z\alpha x^*} e(p-x^*-z\alpha x^*-p_c)\phi(p_c)dp_c](1-\gamma)(p-z\alpha x^*)\\&+[\gamma+(1-\gamma)\int_A^{p-x^*-z\alpha x^*} e(p-x^*-z\alpha x^*-p_c)\phi(p_c)dp_c]x^*\\&-\frac{z\alpha}{1+z\alpha}x^*\Big\}\end{aligned}$$

The above equation represents the marginal revenue of z. First item of right hand side of the second equality corresponds to the revenue lost from the Scheme T as z increases one unit; second and third item correspond to the revenue earned from Scheme O as z increases one unit; while, x^* is decreasing in z, so changing z will affects the marginal revenue of Scheme O, which is represented by the last item.

As Theorem 4.1 demonstrates, $E(x^*,z)$ might have multiple points that satisfy the first-order optimality condition, depending on the parameters of the problem.

Theorem 4.1. *The optimal down payment $x^*(z)$ is specified by Proposition 4.1, and the optimal ratio z^* is determined by a polynomial function $f(n)$:*
(1)If $f(n)>0$ or $\frac{df(n)}{dn}|_{f(n)=0}<0$, then z^* is the unique z that makes a change of sign for $\frac{dE(x^*,z)}{dz}$ from positive to negative value and that satisfies $\frac{dE(x^*,z)}{dz}=0$.
(2) Otherwise, there are at most $\lfloor\frac{i}{2}\rfloor+1$ points of z that achieve local maximum of $E(x^*,z)$ that satisfies $\frac{dE(x^*,z)}{dz}=0$, where i is the rank of $f(n)$.
And $f(n)=(6p-9n)\phi(n)+2n(p-n)\phi'(n)-6\Phi(n)$, $n=p-x^*-z\alpha x^*$.

Proof. See Appendix. □

Part (1) of Theorem 4.1 indicates that there is an unique local maximum under two conditions. Condition $f(n) > 0$ guarantees that $\frac{dE(x^*,z)}{dz}$ is quasi-concave in z, implying that $\frac{dE(x^*,z)}{dz} = 0$ at most has two roots. The larger of the two makes a change of sign for $\frac{dE(x^*,z)}{dz}$ from positive to negative value that corresponds to a local maximum of $E(x^*, z)$. The second condition in (1) indicates that $f(n)$ changes its sign from positive to negative at most one time, implying that $\frac{dE(x^*,z)}{dz} = 0$ at most has three roots. The second of the three makes a change of sign for $\frac{dE(x^*,z)}{dz}$ from positive to negative value that corresponds to a local maximum of $E(x^*, z)$. Part (2) shows that $E(x^*, z)$ might have multiple points that satisfy $\frac{dE(x^*,z)}{dz} = 0$, depending on a polynomial function $f(n)$.

We consider three general distributions of competitor's rental rate: Uniform, Exponential, and Normal in the Corollary 4.3, 4.4, and 4.5 respectively.

Corollary 4.3. *For Uniform distribution of competitor's rental rate, z^* is the unique z makes a change of sign for $\frac{dE(x^*,z)}{dz}$ from positive to negative value that satisfies $\frac{dE(x^*,z)}{dz} = 0$.*

Proof. See Appendix. □

Corollary 4.4. *For Exponential distribution of competitor's rental rate, z^* is the unique z makes a change of sign for $\frac{dE(x^*,z)}{dz}$ from positive to negative value that satisfies $\frac{dE(x^*,z)}{dz} = 0$.*

Proof. See Appendix. □

Corollary 4.5. *For Normal distribution of competitor's rental rate, there are at most 3 points of z correspond to the local maximum of $E(x^*, z)$ that satisfies $\frac{dE(x^*,z)}{dz} = 0$.*

Proof. See Appendix. □

Chapter 5

Interaction of Payment Scheme and Rental Rate

After solving the problem of payment scheme in Chapter 3 and 4, we next study the pricing of the hotel's rental rate based on the optimal payment schemes in this chapter.

We consider the deterministic demand, where $\lambda = a - bp$. Then, the expected revenue $E(x, z, p)$ is as follows.

$$
\begin{aligned}
E(x, z, p) \;=\; & (a - bp)\Big\{(1 - z)[1 - \int_A^p e(p - p_c)\phi(p_c)dp_c](1 - \gamma)p \\
& + z[1 - \int_A^{p-x-z\alpha x} e(p - x - z\alpha x - p_c)\phi(p_c)dp_c](1 - \gamma)(p - z\alpha x) \\
& + z[\gamma + (1 - \gamma)\int_A^{p-x-z\alpha x} e(p - x - z\alpha x - p_c)\phi(p_c)dp_c]x\Big\}
\end{aligned}
$$

Let $E_{OT}(p)$ denote the hotel's expected revenue if the hotel only offers a traditional scheme to customers. Then, it can be determined as follows.

$$E_{OT}(p) \quad = \quad (a-bp)\left\{[1-\int_A^p e(p-p_c)\phi(p_c)dp_c](1-\gamma)p\right\}$$

We use the following algorithm to find the optimal rental rate for the hotel:

Algorithm 5.1. *(1) Step 1: Find the initial optimal price p from $E_{OT}(p)$;*

(2) Step 2: Plug p into $E(x,z,p)$, and obtain x^* and z^*;

(3) Step 3: Plug x^* and z^* back into $E(x,z,p)$, and obtain optimal p;

(4) Step 4: If p is convergent or hits its upper bound, then end; otherwise, go to Step 2.

We use Uniform distribution of p_c to do the iteration. The parameters are as follows. Case 1: $A = 50, B = 400, a = 250, b = 0.5, e = 0.004, \gamma = 0.15, \alpha = 0.5$. From Case 2 to 15, We change one parameter at a time. Where Case 2, 3 : $A = \{0, 100\}$; Case 4, 5: $B = \{500, 300\}$; Case 6, 7: $a = \{280, 220\}$; Case 8, 9: $b = \{0.45, 0.55\}$; Case 10, 11: $e = \{0.005, 0.003\}$; Case 12, 13: $\gamma = \{0.3, 0\}$; Case 14, 15: $\alpha = \{0.6, 0.4\}$. The results are summarized in Table 5.1, 5.2 and 5.3.

Where the column 'OS' indicates the optimal solution after iterations. The column '0' shows the initial solution at the beginning of iterations, which is the solution when the hotel only offers Scheme T. The column '1', '3', and '5' represent the gap at its iteration time and the optimal solution. From Table 5.1, we can see that price p converges in an optimal point very fast. 5 cases have reached its optimal points after 5 iterations. The expected revenue $E(x,z,p)$, and down payment x, ratio z are determined as p reaches its optimal points. The results are presented in Table 5.2 and 5.3, respectively. Comparing with the case where the hotel only offers a traditional scheme, the increasing revenue of offering two payment schemes are presented in column 'RI' in Table 5.2, which can be as high as 15.88%.

Table 5.1: **The results of iterations for p**

	p				
Case	OS	0	1	3	5
1	220.95	194.37	4.41%	0.86%	0.19%
2	239.01	184.98	12.77%	5.31%	0.00%
3	219.35	204.56	1.56%	0.11%	0.01%
4	220.81	203.86	1.88%	0.13%	0.01%
5	237.25	180.39	14.82%	5.21%	0.00%
6	264.83	205.16	13.56%	5.69%	0.00%
7	194.48	181.06	1.46%	0.09%	0.01%
8	262.6	204.44	13.30%	5.92%	0.00%
9	200.41	184.54	2.02%	0.15%	0.01%
10	235.88	185.22	12.94%	6.28%	2.01%
11	221.08	205.15	1.62%	0.10%	0.00%
12	228.9	194.37	4.48%	0.53%	0.07%
13	209.13	194.37	2.05%	0.22%	0.02%
14	213.64	194.37	2.52%	0.26%	0.03%
15	240.48	194.37	10.30%	4.13%	1.30%

Table 5.2: **The results of iterations for** $E(x, z, p)$

$E(x, z, p)$						
Case	OS	0	1	3	5	RI
1	23543.65	22240.28	1.00%	0.02%	0.00%	5.86%
2	22705.61	20528.64	2.74%	0.41%	0.00%	10.60%
3	24569.18	23812.91	0.39%	0.00%	0.00%	3.18%
4	24020.98	22958.2	0.45%	0.00%	0.00%	4.63%
5	23012.8	21170.41	2.67%	0.03%	0.00%	8.70%
6	29042.58	26683.23	2.76%	0.57%	0.00%	8.84%
7	18732.6	17969.81	0.38%	0.00%	0.00%	4.24%
8	25746.24	23714.42	2.59%	0.52%	0.00%	8.57%
9	21841.31	20884.63	0.48%	0.00%	0.00%	4.58%
10	23253.31	21542.79	2.30%	0.44%	0.13%	7.94%
11	24090.67	23055.51	0.40%	0.00%	0.00%	4.49%
12	21223.8	18315.52	2.38%	0.07%	0.01%	15.88%
13	26740.2	26165.03	0.39%	0.00%	0.00%	2.20%
14	23276.67	22240.28	0.63%	0.00%	0.00%	4.66%
15	24054.32	22240.28	1.99%	0.22%	0.05%	8.16%

Table 5.3: **The results of iterations for z and x**

	z				x			
	OS	1	3	5	OS	1	3	5
1	0.6276	34.32%	6.18%	1.34%	82.03	20.94%	2.32%	0.49%
2	1	49.47%	16.77%	0.00%	79.4	21.95%	0.58%	0.00%
3	0.433	20.83%	1.52%	0.12%	82.36	8.38%	0.63%	0.04%
4	0.4446	23.89%	1.60%	0.11%	107.9	21.96%	0.93%	0.06%
5	1	49.14%	4.63%	0.00%	78.33	4.77%	24.49%	0.00%
6	1	50.71%	14.20%	0.00%	78.73	14.35%	6.99%	0.00%
7	0.4131	19.32%	1.33%	0.07%	99.07	13.39%	0.96%	0.06%
8	1	51.26%	16.76%	0.00%	77.25	17.13%	4.09%	0.00%
9	0.4571	25.22%	1.77%	0.13%	93.31	23.15%	0.94%	0.06%
10	1	52.30%	20.62%	1.84%	67.23	21.02%	0.12%	7.90%
11	0.4241	21.15%	1.39%	0.07%	113.13	17.50%	0.95%	0.05%
12	0.7313	16.76%	1.94%	0.25%	131	15.51%	1.43%	0.18%
13	0.4689	22.01%	2.30%	0.26%	56.08	2.57%	0.16%	0.02%
14	0.468	26.60%	2.59%	0.30%	85.41	16.16%	1.00%	0.12%
15	1	48.47%	16.50%	3.39%	78.01	27.18%	2.86%	2.37%

Chapter 6

Conclusion

In this paper, we introduce a flexible payment scheme in the hotel business. The hotel offers an optional payment scheme to customers when they make reservations. If a customer chooses this optional scheme, then he/she makes a down payment immediately, and enjoys a discount from the hotel when the customer actually checks in to the hotel. If the customer does not choose the optional scheme, then the payment is under the traditional scheme where the customer does not need to pay any down payment at the time the reservation is made, and he makes full payment without any discounts upon his checking out from the hotel. The hotel also faces competitions from other nearby hotels with a similar grade. As customers may cancel their reservations or may not show up eventually if they find a lower rental rate from other hotels (Quan (2002)) or due to other reasons, the hotel can redeem the loss from the customer's cancelation or no-shows by introducing Scheme O.

We first study the case where the hotel knows the expected value of his competitor's price (deterministic case), and obtain optimal solutions for the two payment schemes. We find that, when there is no competition between the hotel and other nearby hotels, the optimal solution is the upper bound of down payment, which implies that full down payment policy is optimal for the hotel. And the optimal discount can be obtained uniquely. However, if there is competition from other hotels, full down payment policy may not be optimal. The optimal down payment is either an uniquely interior solution or an unique solution that hits its boundary. It is determined by the parameters of the problem. There is an unique optimal solution for discount, the value of which is also determined by the parameters of the problem.

For the stochastic case where the hotel knows the stochastic distribution of his competitor's price, we can find optimal solutions for down payment and discount as well. In this case, the full down payment policy is still optimal for the hotel if there is no competition between the hotel and other nearby hotels. When there is competition from other hotels, full down payment policy is not optimal anymore for the hotel. We also obtain some intuitive properties for the optimal response of down payment ($x^*(z)$). For example, we find that optimal down payment is decreasing in ratio (z), which is intuitive because in order to attract customers to choose Scheme O, the hotel should lower down its down payment. Another interesting property is that the sum of down payment and discount ($x^*(z) + z\alpha x^*(z)$) is also decreasing in ratio (z), which indicates that the amount due of customers who choose Scheme O ($p - x^*(z) - z\alpha x^*(z)$) is increasing in ratio (z), and consequently the probability of the customers who choose Scheme O and have no-shows due to the competition ($\int_A^{p-x^*(z)-z\alpha x^*(z)} e(p - x^*(z) - z\alpha x^*(z) - p_c)\phi(p_c)dp_c$) is increasing in the ratio (z). This result implies that the probability of customers who choose Scheme O and have no-shows due to the competition is increasing in the down payment, which is intuitive. After obtaining the optimal response of down payment, we plug it back into the hotel's revenue function, and find that the concavity of the expected revenue with respect to the discount is broken. There could be multiple points for discount that satisfy the first-order optimality condition, depending on the parameters of the problem. The optimal discount is determined by a polynomial function, and in some conditions, this polynomial function guarantees the first derivative of expected revenue is quasi-concave in the down payment. It implies that there is an unique optimal solution for discount that corresponds to the local maximum of the expected revenue that satisfies the first-order optimality condition. In addition, we also prove that the optimal discount which corresponds to the local maximum of the expected revenue and satisfies the first-order optimality condition is unique for the Uniform and Exponential distribution of competitor's price. However, there can be up to three optimal points for discount that correspond to the local maximum of the expected revenue, which satisfy the first-order optimality condition for the Normal distribution of competitor's price.

This paper also studies the interaction between payment scheme and rental rate. We design an algorithm to get the optimal rental rate numerically. We find that the rental rate converges to its optimal solution very fast. In our numerical example, $\frac{1}{3}$ cases reaches its optimal point after 5 times of iterations.

The tradeoff between the down payment and discount can not only be applied in the hotel business, but also be applied in other businesses. For example, supper markets (or food courts, barber shops, etc.) may ask customers to apply for membership cards and save money in the card advanced. Those shops then offer some discounts to members when they come for their services. Such kind of the problem is very similar to our problem.

Future research could be extended to problems that consider customers requesting for multi-day stay or consider the hotel having multi-type of rooms. Under such cases, the hotel may charge and offer different down payments and/or discounts, for different number of days stay or different types of rooms.

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Appendix A

Proof of Deterministic Case

Proof of Lemma 3.1

Proof. Since $f(x, z)$ in Equation (3.2) is a linear function of x, then $x^*(z)$ is either equal to its upper bound ($x^*(z) = p - z\alpha x$) or equal to its lower bound ($x^*(z) = 0$), which depends on the sign of $\gamma - (1 - \gamma)z\alpha$. □

Proof of Lemma 3.2

Proof. Considering the first derivative of $f(x^*(z), z)$ with respect to z, we have,

$$\frac{df(x^*(z), z)}{dz} = \lambda p\{\frac{1}{(1 + z\alpha)^2} - (1 - \gamma)\}$$

Notice that $\frac{df(x^*(z),z)}{dz}$ is decreasing in z, so $f(x^*, z)$ is concave in z. We can then obtain an unique optimal z which maximizes $f(x^*(z), z)$ by solving the $\frac{df(x^*(z),z)}{dz} =$

0.

$$\hat{z} = \frac{1}{\alpha\sqrt{1-\gamma}} - \frac{1}{\alpha}$$

We find that $\hat{z}$ always satisfies the first condition in problem (3.3); for the second condition , we find that if $\gamma < 1 - \frac{1}{(1+z\alpha)^2}$, then $\hat{z} < 1$, otherwise $\hat{z} > 1$.

Then this Lemma is proved. $\square$

Proof of Lemma 3.3

Proof. Since $f(x, z)$ in Equation (3.4) is a linear function of x, then $x^*(z)$ is either equal to its upper bound ($x^*(z) = p - z\alpha x$) or equal to its lower bound ($x^*(z) = p - z\alpha x - p_c$), which depends on the sign of $\gamma - (1-\gamma)z\alpha$. $\square$

Proof of Lemma 3.4

Proof. By considering the first derivative of $f(x_u^*(z), z)$ with respect to z, we have,

$$\frac{df(x_u^*(z), z)}{dz} = \lambda p \Big\{ \frac{1}{(1+z\alpha)^2} - (1-\gamma)[1 - e(p - p_c)] \Big\}$$

Notice that $\frac{df(x_u^*(z),z)}{dz}$ is decreasing in z, so $f(x_u^*(z), z)$ is concave in z, we can obtain an unique local optimal z by solving the $\frac{df(x_u^*(z),z)}{dz} = 0$.

$$\hat{z_u} = \frac{1}{\alpha}\Big(\frac{1}{\sqrt{(1-\gamma)[1-e(p-p_c)]}} - 1\Big)$$

By combining $\hat{z_u}$ with the boundary condition in problem (3.6), this lemma is proved. $\square$

Proof of Lemma 3.5

Proof. Notice that if $\frac{\gamma}{\alpha(1-\gamma)} > 1$, then $\frac{\gamma}{\alpha(1-\gamma)} > z$ would always hold, so $x^*(z) = x_u^*(z)$. We should go back to solve problem (3.6); otherwise, considering the first derivative of $f(x_l^*(z), z)$ with respect to z, we have,

$$\frac{df(x_l^*(z), z)}{dz} = \lambda(p - p_c)\Big\{\frac{1}{(1 + z\alpha)^2} - (1 - \gamma)(1 - ep)\Big\}$$

Notice that when $1 - ep \leq 0$, $\frac{df(x_l^*(z),z)}{dz} > 0$, then $f(x_l^*(z), z)$ is increasing in z, upper bound of z is the solution of problem (3.7)($z^* = 1$); when $1 - ep > 0$, $\frac{df(x_l^*(z),z)}{dz}$ is decreasing in z, so $f(x_l^*(z), z)$ is concave in z, we can get a unique local optimal z by solving the $\frac{df(x_l^*(z),z)}{dz} = 0$.

$$\hat{z_l} = \frac{1}{\alpha}(\frac{1}{\sqrt{(1 - \gamma)(1 - ep)}} - 1)$$

By combining $\hat{z_l}$ with the boundary condition in problem (3.7), this lemma is proved. $\square$

Proof of Proposition 3.1

Proof. From $\hat{z_u} < 1$, we can get the inequality $\frac{1}{\sqrt{(1-\gamma)[1-e(p-p_c)]}} < 1 + \alpha$, then the results for $\frac{\gamma}{\alpha(1-\gamma)} > 1$ can be obtained easily by combining Lemma 3.4 and Lemma

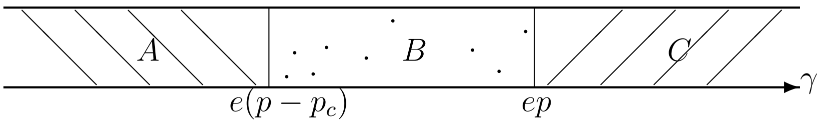

Figure A.1: **The value of $x^*(z)$ for different scenarios.** Region A: $z_l^* > \frac{\gamma}{\alpha(1-\gamma)}$ and $\hat{z_u} > \frac{\gamma}{\alpha(1-\gamma)}$; Region B: $z_l^* \geq \frac{\gamma}{\alpha(1-\gamma)}$ and $\hat{z_u} \leq \frac{\gamma}{\alpha(1-\gamma)}$; Region C: $z_l^* < \frac{\gamma}{\alpha(1-\gamma)}$ and $\hat{z_u} < \frac{\gamma}{\alpha(1-\gamma)}$.

3.5. For the case $\frac{\gamma}{\alpha(1-\gamma)} \leq 1$, we need to consider it in two cases $1-ep \leq 0$ and $1-ep > 0$ respectively.

(1)If $1-ep \leq 0$, then by solving inequality $\hat{z_u} < \frac{\gamma}{\alpha(1-\gamma)}$, we get $\gamma > e(p-p_c)$, and by combining Lemma 3.4 and Lemma 3.5, we can obtain that $(x_l^*(z), 1)$ is optimal for $\gamma \leq e(p-p_c)$ easily; for $\gamma > e(p-p_c)$, both condition $\hat{z_u} < \frac{\gamma}{\alpha(1-\gamma)}$ and $z^* = 1 > \frac{\gamma}{\alpha(1-\gamma)}$ are satisfied, then we can get optimal payment scheme by comparing $f(x_u^*(z), \hat{z_u})$ and $f(x_l^*(z), 1)$;

(2) If $1-ep > 0$, then by solving inequality $\hat{z_u} < \frac{\gamma}{\alpha(1-\gamma)}$ and $\hat{z_l} > \frac{\gamma}{\alpha(1-\gamma)}$, we can get $\gamma > e(p-p_c)$ and $\gamma < ep$, respectively. Then as Figure A.1 shows: in region A, $x^*(z) = x_u^*(z)$; in region C, $x^*(z) = x_l^*(z)$; and in region B, $x^*(z) = x_u^*(z)$ or $x^*(z) = x_l^*(z)$ which is determined by $\max_{x(z),z}\{f(x_u^*(z), \hat{z_u}), f(x_l^*(z), z_l^*)\}$. Besides, by solving $\hat{z_l} < 1$, we can get $\frac{1}{\sqrt{(1-\gamma)(1-ep)}} < 1+\alpha$.

Summarizing the above analysis, we have,

If $\frac{\gamma}{\alpha(1-\gamma)} > 1$, then

$$(x^*(z), z^*) = \begin{cases} (x_u^*(z), \hat{z_u}) & \text{if } \frac{1}{\sqrt{(1-\gamma)[1-e(p-p_c)]}} < 1+\alpha; \\ (x_u^*(z), 1) & \text{Otherwise.} \end{cases}$$

Otherwise,

If $1 - ep \leq 0$, then

$$(x^*(z), z^*) = \begin{cases} arg \max\limits_{x(z),z} \{f(x_u^*(z), \hat{z_u}), f(x_l^*(z), 1)\} & \text{if } \gamma > e(p - p_c); \\ (x_l^*(z), 1) & \text{Otherwise.} \end{cases}$$

Otherwise,

$$(x^*(z), z^*) = \begin{cases} (x_l^*(z), z_l^*) & \text{if } \gamma < e(p - p_c); \\ arg \max\limits_{x(z),z} \{f(x_u^*(z), \hat{z_u}), f(x_l^*(z), z_l^*)\} & \text{if } e(p - p_c) \leq \gamma < ep; \\ (x_u^*(z), \hat{z_u}) & \text{Otherwise.} \end{cases}$$

where

$$z_l^* = \begin{cases} \hat{z}_l & \text{if } \frac{1}{\sqrt{(1-\gamma)(1-ep)}} < 1 + \alpha; \\ 1 & \text{Otherwise.} \end{cases}$$

Thus, by rearranging the above primary results, this proposition is proved. □

Proof of Lemma 3.6

Proof. By consider the first and seconde partial derivative of $f(x, z)$ in Equation (3.8) with respect to x, we have,

$$\frac{\partial f(x,z)}{\partial x} = \lambda z \Big\{ - (1-\gamma)z\alpha + \gamma + (1 + z\alpha)e(1-\gamma)[2(p - x - z\alpha x) - p_c] \Big\} \tag{A.1}$$

$$\frac{\partial^2 f(x,z)}{\partial x^2} = -\lambda z(1 + z\alpha)^2 e(1-\gamma)2 \leq 0$$

So, for given z, $f(x,z)$ is concave in x, by solving $\frac{\partial f(x,z)}{\partial x} = 0$, we have,

$$\begin{aligned} p - \hat{x} - z\alpha\hat{x} &= \frac{1}{2}[\frac{(1-\gamma)z\alpha - \gamma}{e(1-\gamma)(1+z\alpha)} + p_c] \\ \hat{x} &= \frac{p}{1+z\alpha} - \frac{1}{2(1+z\alpha)}[\frac{(1-\gamma)z\alpha - \gamma}{e(1-\gamma)(1+z\alpha)} + p_c] \end{aligned}$$

Notice from Equation (A.1) that if $(1-\gamma)z\alpha < \gamma$, then $\frac{\partial f(x,z)}{\partial x} > 0$, upper bound of x is optimal for maximizing $f(x,z)$, since $(x = p - z\alpha x - p_c, z) \in R_1$, then we can get the optimal response of x by solving the problem $\max_{x|(x,z)\in R_1}\{f(x,z)\}$; And if $p - \hat{x} - z\alpha\hat{x} \leq p_c$, we also need to solve problem $\max_{x|(x,z)\in R_1}\{f(x,z)\}$ to get the optimal response of x; Otherwise we can get the optimal response of x by checking the other boundary condition $\hat{x}(z) > 0$. □

Proof of Proposition 3.2

Proof. By considering the derivative of x^* respect to z, we have,

$$\begin{aligned} \frac{dx^*}{dz} &= -\frac{\alpha}{1+z\alpha}\left\{(p - \frac{1}{2}p_c)\frac{1}{1+z\alpha} + \frac{1}{2e(1-\gamma)}\frac{\gamma-(1-\gamma)z\alpha}{(1+z\alpha)^2} + \frac{1}{2e(1-\gamma)(1+z\alpha)^2}\right\} \\ &= -\frac{\alpha}{1+z\alpha}\left\{x^* + \frac{1}{2e(1-\gamma)(1+z\alpha)^2}\right\} \\ &\leq 0 \end{aligned}$$

$$\begin{aligned} \frac{d(x^* + z\alpha x^*)}{dz} &= \frac{d(p - \frac{1}{2}p_c + \frac{\gamma-(1-\gamma)z\alpha}{2e(1-\gamma)(1+z\alpha)})}{dz} \\ &= -\frac{\alpha}{2e(1-\gamma)(1+z\alpha)^2} \\ &\leq 0 \end{aligned}$$

$$\begin{aligned}
\frac{dz\alpha x^*}{dz} &= \alpha\Big\{x^* + z\frac{dx^*}{dz}\Big\} \\
&= \frac{\alpha}{1+z\alpha}\Big\{x^* - \frac{z\alpha}{2e(1-\gamma)(1+z\alpha)^2}\Big\} \\
&= \frac{\alpha}{1+z\alpha}\Big\{\frac{1}{1+z\alpha}(p-\frac{1}{2}p_c) - \frac{(1-\gamma)z\alpha-\gamma}{2e(1-\gamma)(1+z\alpha)^2} - \frac{z\alpha}{2e(1-\gamma)(1+z\alpha)^2}\Big\}
\end{aligned}$$

Thus, x^* and $x^* + z\alpha x^*$ are decreasing in x, but the sign of $\frac{dz\alpha x^*}{dz}$ is depend on z and other parameters.

Remark A.1. *We can also get the monotone property of x^* by tacking the cross derivative of $f(x,z)$ respect to x and z,*

$$\begin{aligned}
\frac{\partial^2 f(x,z)}{\partial x \partial z} = \lambda\Big\{ &-2z\alpha(1-\gamma) + \gamma - z\alpha x(1+z\alpha)e(1-\gamma)2 \\
&+ (1+2z\alpha)e(1-\gamma)[2(p-x-z\alpha x) - p_c]\Big\}
\end{aligned} \tag{A.2}$$

From $\frac{\partial f(x,z)}{\partial x} = 0$, we can get

$$e[2(p-x-z\alpha x) - p_c] = \frac{(1-\gamma)z\alpha - \gamma}{(1+z\alpha)(1-\gamma)} \tag{A.3}$$

Plug it back into Equation (A.2), we have

$$\begin{aligned}
\frac{\partial^2 f(x,z)}{\partial x \partial z}\Big|_{x=x^*(z)} &= \lambda\Big\{ -2z\alpha(1-\gamma) + \gamma - z\alpha x(1+z\alpha)e(1-\gamma)2 \\
&\quad + \frac{(1-\gamma)z\alpha - \gamma}{1+z\alpha}(1+2z\alpha)\Big\} \\
&= -\lambda\Big\{\frac{z\alpha}{1+z\alpha} + z\alpha x(1+z\alpha)e(1-\gamma)2\Big\} \\
&\le 0
\end{aligned}$$

□

Proof of Lemma 3.7

Proof. By considering the first derivative of $f(x^*(z), z)$ with respect to z, we have (For simplicity, we use x^* represents $x^*(z)$ in this proof.),

$$\begin{aligned}
\frac{df(x^*, z)}{dz} &= \frac{\partial f(x^*, z)}{\partial z} + \frac{\partial f(x^*, z)}{\partial x}\frac{dx^*}{dz} \\
&= \frac{\partial f(x^*, z)}{\partial z} \\
&= \lambda\Big\{ -[1 - e(p - p_c)](1 - \gamma)p \\
&\quad +(1 - \gamma)[1 - e(p - x^* - z\alpha x^* - p_c)](p - 2z\alpha x^*) \\
&\quad +[\gamma + (1 - \gamma)e(p - x^* - z\alpha x^* - p_c)]x^* + e(1 - \gamma)z\alpha x^*(p - x^* - z\alpha x^*)\Big\} \\
&= \lambda\Big\{ -[1 - e(p - p_c)](1 - \gamma)p + (1 - \gamma)(p - 2z\alpha x^*) + \gamma x^* \\
&\quad -(1 - \gamma)[e(p - x^* - z\alpha x^* - p_c)](p - x^* - z\alpha x^*) \\
&\quad +(1 - \gamma)z\alpha x^* e[2(p - x^* - z\alpha x^*) - p_c]\Big\}
\end{aligned}$$

where the second equality is by $x^*(z)$ satisfies the first-order-condition: $\frac{\partial f(x^*(z),z)}{\partial x} = 0$.

From $\frac{\partial f(x,z)}{\partial x} = 0$, we have $e[2(p - x^* - z\alpha x^*) - p_c] = \frac{(1-\gamma)z\alpha-\gamma}{(1+z\alpha)(1-\gamma)}$. Thus,

$$
\begin{aligned}
\frac{df(x^*,z)}{dz} &= \lambda\Bigg\{ - [1 - e(p - p_c)](1-\gamma)p \\
&\quad +(1-\gamma)(p - 2z\alpha x^*) + \gamma x^* + z\alpha x^* \frac{(1-\gamma)z\alpha - \gamma}{1+z\alpha} \\
&\quad -(1-\gamma)[e(p - x^* - z\alpha x^* - p_c)](p - x^* - z\alpha x^*)\Bigg\} \\
&= \lambda\Bigg\{ - [1 - e(p - p_c)](1-\gamma)p \\
&\quad +[1 - e(p - x^* - z\alpha x^* - p_c)](1-\gamma)(p - x^* - z\alpha x^*) + \frac{x^*}{1+z\alpha}\Bigg\} \\
&= \lambda\Bigg\{ - [1 - e(p - p_c)](1-\gamma)(x^* + z\alpha x^*) \\
&\quad +e(1-\gamma)(p - x^* - z\alpha x^*)(x^* + z\alpha x^*) + \frac{x^*}{1+z\alpha}\Bigg\} \\
&= \lambda(x^* + z\alpha x^*)\Bigg\{ - [1 - e(p - p_c)](1-\gamma) + e(1-\gamma)(p - x^* - z\alpha x^*) \\
&\quad +\frac{1}{(1+z\alpha)^2}\Bigg\} \\
&= \lambda(x^* + z\alpha x^*)\Bigg\{ - [1 - e(p - p_c)](1-\gamma) \\
&\quad +e(1-\gamma)\frac{1}{2}[\frac{(1-\gamma)z\alpha - \gamma}{e(1-\gamma)(1+z\alpha)} + p_c] + \frac{1}{(1+z\alpha)^2}\Bigg\} \\
&= \lambda\frac{x^*}{2(1+z\alpha)} f_d(z)
\end{aligned}
$$

where

$$
\begin{aligned}
f_d(z) &= -[1 - e(2p - p_c)](1-\gamma)(z\alpha)^2 \\
&\quad -(2[1 - e(2p - p_c)](1-\gamma) + 1)z\alpha + 1 - [1 - e(2p - p_c)](1-\gamma)
\end{aligned}
$$

Since $\lambda\frac{x^*}{2(1+z\alpha)}$ is nonnegative for all values of z, then analyzing the quadratic function $f_d(z)$ is sufficient for determining the shape of $f(x^*, z)$.

Let $t = (1 - e(2p - p_c))(1 - \gamma)$, then $f_d(z)$ can be represented as follows.

$$f_d(z) \quad = \quad -t(z\alpha)^2 - (2t + 1)z\alpha + 1 - t$$

which is a quadratic function, then we let $\Delta = (1 + 2t)^2 + 4t(1 - t) = 1 + 8t$.

(1) If $t > 0$, then $\Delta > 0$, $-\frac{1+2t}{2t} < 0$.

And since $1 - t = \gamma + e(2p - p_c)(1 - \gamma) > 0$, then $f_d(z) = 0$ has two roots: one is negative $\frac{-1-2t-\sqrt{1+8t}}{2t\alpha}$, and the other one is positive $\frac{-1-2t+\sqrt{1+8t}}{2t\alpha}$ which is a local optimal value of z.

Then we get that $\hat{z} = \frac{-1-2t+\sqrt{1+8t}}{2t\alpha}$ when $t > 0$.

(2) If $t \leq 0$.

(2.1) If $t \leq -\frac{1}{8}$, then $\Delta \leq 0$. So we have $f_d(z) \geq 0$, which indicate that $\frac{df(x^*,z)}{dz} \geq 0$ for all z. Thus $z = 1$ is a local optimal value of z.

(2.2) If $t > -\frac{1}{8}$, then $\Delta > 0$ and $-\frac{1+2t}{2t} > 3 > 0$, so $f_d(z) = 0$ has two positive roots, one is $\frac{-1-2t-\sqrt{1+8t}}{2t\alpha}$, and the other one is $\frac{-1-2t+\sqrt{1+8t}}{2t\alpha}$. Since $t \leq 0$, then $\frac{-1-2t-\sqrt{1+8t}}{2t\alpha}$ is a local optimal value of z.

Then we get

$$\hat{z} \quad = \quad \begin{cases} \frac{-1-2t-\sqrt{1+8t}}{2t\alpha} & \text{if } -\frac{1}{8} < t \leq 0; \\ 1 & t \leq -\frac{1}{8}. \end{cases}$$

$\square$

Proof of Proposition 3.3

Proof. We consider the boundary conditions in Problem (3.10) as follows.

(1) $p - x^*(z) - z\alpha x^*(z) > p_c$

Inequality $p - x^*(z) - z\alpha x^*(z) > p_c$ can be simplified to $(1 - ep_c)(1 - \gamma)z\alpha > \gamma + ep_c(1 - \gamma)$. So,

(1.1) If $1 - ep_c \leq 0$, then $(1 - ep_c)(1 - \gamma)z\alpha \leq \gamma + ep_c(1 - \gamma)$, which correspond to $p - x^*(z) - z\alpha x^*(z) \leq p_c$.

(1.2) If $1 - ep_c > 0$, then by solving inequality $(1 - ep_c)(1 - \gamma)z\alpha > \gamma + ep_c(1 - \gamma)$ we can get that if $z\alpha > \frac{\gamma+ep_c(1-\gamma)}{(1-\gamma)(1-ep_c)}$, then $p - x^*(z) - z\alpha x^*(z) > p_c$; otherwise, $p - x^*(z) - z\alpha x^*(z) \leq p_c$.

(2) $x^*(z) > 0$

Inequality $x^*(z) > 0$ can be simplified to $tz\alpha < 1-t$, where $t = (1-e(2p-p_c))(1-\gamma)$. So,

(2.1) If $t \leq 0$, then $tz\alpha < 1 - t$, which correspond to $x^*(z) > 0$.

(2.2) If $t > 0$, then by solving inequality $tz\alpha < 1 - t$ we can get that if $z\alpha < \frac{1-t}{t}$, then $x^*(z) > 0$; otherwise, $x^*(z) \leq 0$.

Reminder that $\hat{z} = \frac{-1-2t+\sqrt{1+8t}}{2t\alpha}$ when $t > 0$, and notice that inequality $z\alpha < \frac{1-t}{t}$ is always satisfied when $t > 0$ and $z = \hat{z} = \frac{-1-2t+\sqrt{1+8t}}{2t\alpha}$.

Thus, we get that $x^*(z) > 0$ is always satisfied by $z = \hat{z}$, which implies that subcase 2 ($x^*(z) = 0$) will never happened.

At last, by combining $\hat{z}$ with other two boundary conditions $(1 - \gamma)z\alpha > \gamma$ and $z \leq 1$, this proposition is proved completely. $\square$

Appendix B

Proof of Stochastic Case

Proof of Corollary 4.2

Proof. Since $E(x, z)$ is a linear function of x when $p \leq A$, then $x^*(z)$ is either equal to its upper bound ($x^*(z) = p - z\alpha x$) or equal to its lower bound ($x^*(z) = 0$), which depends on the sign of $\gamma - (1-\gamma)z\alpha$.

$$x^*(z) = \begin{cases} \frac{p}{1+z\alpha} & \text{if } \gamma > \frac{z\alpha}{1+z\alpha}. \\ 0 & \text{Otherwise;} \end{cases}$$

First we consider the case $x^*(z) = \frac{p}{1+z\alpha}$.

Substituting $x^*(z)$ back into the equation $E(x, z) = \lambda\{(1-\gamma)p + z(\gamma - (1-\gamma)z\alpha)x\}$. And considering the first derivative of $E(x^*(z), z)$ with respect to z, we have,

$$\frac{dE(x^*(z), z)}{dz} = \lambda p\{\frac{1}{(1+z\alpha)^2} - (1-\gamma)\}$$

Notice that $\frac{dE(x^*(z),z)}{dz}$ is decreasing in z, so $E(x^*, z)$ is concave in z, we can obtain an unique local optimal z which maximizes $E(x^*(z), z)$ by solving the $\frac{dE(x^*(z),z)}{dz} = 0$.

$$\hat{z} = \frac{1}{\alpha\sqrt{1-\gamma}} - \frac{1}{\alpha}$$

And if $\gamma < 1 - \frac{1}{(1+z\alpha)^2}$, then $\hat{z} < 1$, otherwise $\hat{z} > 1$. Thus, we can get that

$$z^* = \begin{cases} \hat{z} & \text{if } \gamma < 1 - \frac{1}{(1+\alpha)^2}. \\ 1 & \text{Otherwise;} \end{cases}$$

We find that the condition of $\gamma > (1-\gamma)z\alpha$ is always satisfied when $z = \hat{z}$. And if $z = 1$, then $1 - \frac{1}{(1+\alpha)^2} > \frac{z\alpha}{1+z\alpha} = \frac{\alpha}{1+\alpha}$, so the condition of $\gamma > \frac{z\alpha}{1+z\alpha}$ will be satisfied as long as $\gamma \geq 1 - \frac{1}{(1+\alpha)^2}$.

Additionally, notice that $z^* > 0$.

Thus, $x = 0$ will never be optimal and $x^*(z) = \frac{p}{1+z\alpha}$ is always optimal for the hotel, it is equivalent to that it is optimal for the hotel to offer a full down payment optional scheme to customer under z^* and $x^*(z)$ when $p \leq A$. □

Proof of Proposition 4.2

Proof. Considering the first and second partial derivatives of $E(x,z)$ with respect to z:

$$\begin{aligned}\frac{\partial E(x,z)}{\partial z} &= \lambda\Big\{ -[1-\int_A^p e(p-p_c)\phi(p)dp_c](1-\gamma)p \\ &+ (1-\gamma)[1-\int_A^{p-x-z\alpha x} e(p-x-z\alpha x-p_c)\phi(p_c)dp_c](p-2z\alpha x) \\ &+ [\gamma+(1-\gamma)\int_A^{p-x-z\alpha x} e(p-x-z\alpha x-p_c)\phi(p_c)dp_c]x \\ &+ e(1-\gamma)z\alpha x(p-x-z\alpha x)\Phi(p-x-z\alpha x)\Big\}\end{aligned} \tag{B.1}$$

$$\begin{aligned}\frac{\partial^2 E(x,z)}{\partial z^2} &= -\lambda(1-\gamma)\alpha x\Big\{2\Big\{1-\int_A^{p-x-z\alpha x} e[2(p-x-z\alpha x)-p_c]\phi(p_c)dp_c\Big\} \\ &+ ez\alpha x M(x,z)\Big\}\end{aligned} \tag{B.2}$$

From $\frac{\partial E(x,z)}{\partial x}=0$, we can obtain

$$\int_A^{p-x-z\alpha x} e[2(p-x-z\alpha x)-p_c]\phi(p_c)dp_c = \frac{(1-\gamma)z\alpha-\gamma}{(1+z\alpha)(1-\gamma)} \tag{B.3}$$

Combining Equation (B.3) and (B.2), we have,

$$\begin{aligned}\frac{\partial^2 E(x,z)}{\partial z^2}|_{x=x^*(z)} &= -\lambda\Big\{\frac{2\alpha x^*(z)}{1+z\alpha} + e(1-\gamma)\alpha x^*(z)z\alpha x^*(z)M(x^*(z),z)\Big\} \\ &\leq 0\end{aligned}$$

□

Proof of Proposition 4.3

Proof. Taking cross partial derivative of $E(x,z)$ with respect to x and z, we have,

$$\begin{aligned}\frac{\partial^2 E(x,z)}{\partial x \partial z} = \lambda \Big\{ &-2z\alpha(1-\gamma)+\gamma - z\alpha x(1+z\alpha)e(1-\gamma)M(x,z) \\ &+(1+2z\alpha)e(1-\gamma)\int_A^{p-x-z\alpha x}[2(p-x-z\alpha x)-p_c]\phi(p_c)dp_c\Big\}\end{aligned} \tag{B.4}$$

By combining Equation (B.3) and (B.4), we have,

$$\begin{aligned}\frac{\partial^2 E(x,z)}{\partial x \partial z}|_{x=x^*(z)} &= \lambda\Big\{ -2z\alpha(1-\gamma)+\gamma - z\alpha x^*(z)(1+z\alpha)e(1-\gamma)M(x^*(z),z) \\ &\quad + \frac{(1-\gamma)z\alpha-\gamma}{1+z\alpha}(1+2z\alpha)\Big\} \\ &= -\lambda\Big\{\frac{z\alpha}{1+z\alpha}+z\alpha x^*(z)(1+z\alpha)e(1-\gamma)M(x^*(z),z)\Big\} \\ &\leq 0\end{aligned}$$

Then by Topkis's (1998) theorem , we can get that $x^*(z)$ is non-increasing in z. □

Proof of Proposition 4.4

Proof. Taking the first order condition, we know that at the optimum, $\frac{\partial E(x^*(z),z)}{\partial x} = 0$, differentiating the first order condition, with respect to z and using the implicit function theorem, we find that

$$\frac{\partial^2 E(x^*(z),z)}{\partial x^2}\frac{dx^*(z)}{dz}+\frac{\partial^2 E(x^*(z),z)}{\partial x \partial z} = 0 \tag{B.5}$$

From Equation (B.5), we can get

$$\begin{aligned}\frac{dx^*(z)}{dz} &= -\frac{\frac{\partial^2 E(x^*(z),z)}{\partial x \partial z}}{\frac{\partial^2 E(x^*(z),z)}{\partial x^2}} \\ &= -\frac{\frac{z\alpha}{1+z\alpha} + z\alpha x^*(z)(1+z\alpha)(1-\gamma)eM(x^*(z),z)}{z(1+z\alpha)^2(1-\gamma)eM(x^*(z),z)}\end{aligned} \tag{B.6}$$

Then, we have,

$$\begin{aligned}\frac{d}{dz}(x^*(z)+z\alpha x^*(z)) &= \frac{d}{dz}((1+z\alpha)x^*(z)) \\ &= \frac{d(1+z\alpha)}{dz}x^*(z) + (1+z\alpha)\frac{dx^*(z)}{dz}\end{aligned}$$

By plugging $\frac{dx^*(z)}{dz}$ in the above equation, we have

$$\begin{aligned}\frac{d}{dz}(x^*(z)+z\alpha x^*(z)) &= -\frac{z\alpha}{z(1+z\alpha)^2(1-\gamma)eM(x^*(z),z)} \\ &\leq 0\end{aligned} \tag{B.7}$$

which is non-positive. □

Technical detail of Remark 4.2

Proof. Considering the derivative of $z\alpha x^*(z)$ with respect to z, we have (For simplicity, we use x^* represent $x^*(z)$ in the following parts),

$$\begin{aligned}\frac{d}{dz}(z\alpha x^*) &= \alpha x^* + z\alpha\frac{dx^*}{dz} \\ &= \alpha x^* + z\alpha\frac{-\frac{z\alpha}{1+z\alpha} - z\alpha x^*(1+z\alpha)(1-\gamma)eM(x^*,z)}{z(1+z\alpha)^2(1-\gamma)eM(x^*,z)} \\ &= \frac{\alpha}{(1+z\alpha)^2(1-\gamma)eM(x^*,z)}[e(1-\gamma)(1+z\alpha)x^*M(x^*,z) - \frac{z\alpha}{1+z\alpha}]\end{aligned}$$

Let $A(z) = (1 + z\alpha)x^*M(x^*, z)$, then,

$$\frac{dA(z)}{dz} = \frac{d(x^* + z\alpha x^*)}{dz}M(x^*, z) + (x^* + z\alpha x^*)\frac{dM(x^*, z)}{dz} \tag{B.8}$$

where

$$\begin{aligned}
\frac{dM(x^*, z)}{dz} &= \frac{\partial M(x^*, z)}{\partial z}\frac{dx^*}{dz} + \frac{\partial M(x^*, z)}{\partial z} \\
&= -(1 + z\alpha)T(z)\frac{dx^*}{dz} - \alpha x^*T(z) \\
&= -(1 + z\alpha)T(z)\left(-\frac{\frac{z\alpha}{1+z\alpha} + z\alpha x^*(1 + z\alpha)(1 - \gamma)eM(x^*, z)}{z(1 + z\alpha)^2(1 - \gamma)eM(x^*, z)}\right) - \alpha x^*T(z) \\
&= \frac{z\alpha T(z)}{z(1 + z\alpha)^2e(1 - \gamma)M(x^*, z)}
\end{aligned}$$

where

$$T(z) = 3\phi(p - x^* - z\alpha x^*) + (p - x^* - z\alpha x^*)\phi'(p - x^* - z\alpha x^*)$$

Plugging $\frac{d}{dz}(x^* + z\alpha x^*)$ which is presented by Equation (B.7) and $\frac{dM(x^*,z)}{dz}$ back into the Equation (B.8), we have

$$\frac{dA(z)}{dz} = \frac{z\alpha x^*T(z)}{z(1 + z\alpha)e(1 - \gamma)M(x^*, z)} - \frac{z\alpha}{z(1 + z\alpha)^2e(1 - \gamma)}$$

Thus,

$$\begin{aligned}
\frac{d^2(z\alpha x^*)}{dz^2}\Big|_{\frac{d(z\alpha x^*)}{dz}=0} &= \frac{\alpha}{(1 + z\alpha)^2(1 - \gamma)eM(x^*, z)}\left[e(1 - \gamma)\frac{dA(z)}{dz} - \frac{\alpha}{(1 + z\alpha)^2}\right] \\
&= \frac{\alpha}{(1 + z\alpha)^2(1 - \gamma)eM(x^*, z)}\left[\frac{\alpha x^*T(z)}{(1 + z\alpha)M(x^*, z)} - \frac{2\alpha}{(1 + z\alpha)^2}\right] \\
&= \frac{\alpha^2}{(1 + z\alpha)^4e(1 - \gamma)M(x^*, z)^2}[(1 + z\alpha)x^*T(z) - 2M(x^*, z)]
\end{aligned}$$

Let $n = p - x^* - z\alpha x^*$, then

$$\frac{d^2(z\alpha x^*)}{dz^2}\Big|_{\frac{d(z\alpha x^*)}{dz}=0} = \frac{\alpha^2}{(1+z\alpha)^4 e(1-\gamma)M(x^*,z)^2} f(n)$$

where

$$\begin{aligned} f(n) &= (1+z\alpha)x^*T(z) - 2M(x^*,z) \\ &= (p-n)[3\phi(n) + n\phi'(n)] - 2[2\Phi(n) + n\phi(n)] \\ &= (3p-5n)\phi(n) + n(p-n)\phi'(n) - 4\Phi(n) \end{aligned}$$

□

Proof of Theorem 4.1

Proof. Remaind that we have,

$$\begin{aligned} \frac{dE(x^*,z)}{dz} = \lambda \Big\{ & -[1 - \int_A^p e(p-p_c)\phi(p)dp_c](1-\gamma)p + \frac{x^*}{1+z\alpha} \\ & + [1 - \int_A^{p-x^*-z\alpha x^*} e(p - x^* - z\alpha x^* - p_c)\phi(p_c)dp_c](1-\gamma)(p - x^* - z\alpha x^*) \Big\} \end{aligned}$$

In order to find the value of z that satisfies this first-order optimality condition, we let $R(x^*, z) = \frac{dE(x^*,z)}{dz}$, then we have,

$$\begin{aligned}
\frac{\partial R(x^*, z)}{\partial x} &= \lambda\Big\{ -(1-\gamma)(1+z\alpha) + \frac{1}{1+z\alpha} \\
&\quad +(1-\gamma)(1+z\alpha)\int_A^{p-x^*-z\alpha x^*} e[2(p-x^*-z\alpha x^*)-p_c]\phi(p_c)dp_c\Big\} \\
&= \lambda\Big\{ -(1-\gamma)(1+z\alpha) + \frac{1}{1+z\alpha} + (1-\gamma)(1+z\alpha)\frac{(1-\gamma)z\alpha-\gamma}{(1+z\alpha)(1-\gamma)}\Big\} \\
&= -\lambda\frac{z\alpha}{1+z\alpha}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial R(x^*, z)}{\partial z} &= \lambda\Big\{ -(1-\gamma)\alpha x^* + \frac{-\alpha x^*}{(1+z\alpha)^2} \\
&\quad +(1-\gamma)\alpha x^*\int_A^{p-x^*-z\alpha x^*} e[2(p-x^*-z\alpha x^*)-p_c]\phi(p_c)dp_c\Big\} \\
&= \lambda\Big\{ -(1-\gamma)\alpha x^* + \frac{-\alpha x^*}{(1+z\alpha)^2} + (1-\gamma)\alpha x^*\frac{(1-\gamma)z\alpha-\gamma}{(1+z\alpha)(1-\gamma)}\Big\} \\
&= -\lambda\frac{\alpha x}{1+z\alpha}\Big\{\frac{1}{1+z\alpha}+1\Big\}
\end{aligned}$$

$$\begin{aligned}
\frac{dR(x^*, z)}{dz} &= \frac{\partial R(x^*, z)}{\partial x}\frac{dx^*}{dz} + \frac{\partial R(x^*, z)}{\partial z} \\
&= -\lambda\frac{z\alpha}{1+z\alpha}\Big(-\frac{\frac{z\alpha}{1+z\alpha} + z\alpha x^*(1+z\alpha)(1-\gamma)eM(x^*, z)}{z(1+z\alpha)^2(1-\gamma)eM(x^*, z)}\Big) \\
&\quad -\lambda\frac{\alpha x}{1+z\alpha}\Big\{\frac{1}{1+z\alpha}+1\Big\} \\
&= -\lambda\frac{z\alpha}{z(1+z\alpha)^3 e(1-\gamma)M(x^*, z)}\Big\{2e(1-\gamma)(1+z\alpha)x^*M(x^*, z) \\
&\quad -\frac{z\alpha}{1+z\alpha}\Big\}
\end{aligned}$$

Let $A(z) = (1 + z\alpha)x^* M(x^*, z)$, then,

$$\frac{dA(z)}{dz} = \frac{d(x^* + z\alpha x^*)}{dz} M(x^*, z) + (x^* + z\alpha x^*) \frac{dM(x^*, z)}{dz}$$

where

$$\begin{aligned}
\frac{dM(x^*, z)}{dz} &= \frac{\partial M(x^*, z)}{\partial z} \frac{dx^*}{dz} + \frac{\partial M(x^*, z)}{\partial z} \\
&= -(1 + z\alpha)T(z)\frac{dx^*}{dz} - \alpha x^* T(z) \\
&= -(1 + z\alpha)T(z)\left(-\frac{\frac{z\alpha}{1+z\alpha} + z\alpha x^*(1 + z\alpha)(1 - \gamma)eM(x^*, z)}{z(1 + z\alpha)^2(1 - \gamma)eM(x^*, z)}\right) - \alpha x^* T(z) \\
&= \frac{z\alpha T(z)}{z(1 + z\alpha)^2 e(1 - \gamma)M(x^*, z)}
\end{aligned}$$

where

$$T(z) = 3\phi(p - x^* - z\alpha x^*) + (p - x^* - z\alpha x^*)\phi'(p - x^* - z\alpha x^*)$$

Thus, we have,

$$\frac{dA(z)}{dz} = \frac{z\alpha x^* T(z)}{z(1 + z\alpha)e(1 - \gamma)M(x^*, z)} - \frac{z\alpha}{z(1 + z\alpha)^2 e(1 - \gamma)}$$

$$\begin{aligned}
\Rightarrow \frac{d^2R(x^*,z)}{dz^2}\Big|_{\frac{dR(x^*,z)}{dz}=0} &= -\lambda\frac{z\alpha}{z(1+z\alpha)^3e(1-\gamma)M(x^*,z)}\Bigg\{2e(1-\gamma)\frac{dA(z)}{dz} \\
&\quad -\frac{\alpha}{(1+z\alpha)^2}\Bigg\} \\
&= -\lambda\frac{\alpha^2}{(1+z\alpha)^5e(1-\gamma)M(x^*,z)^2}\Bigg\{2(1+z\alpha)x^*T(z) \\
&\quad -3M(x^*,z)\Bigg\}
\end{aligned}$$

Let $n = p - x^* - z\alpha x^*$, then $\frac{d^2R(x^*,z)}{dz^2}|_{\frac{dR(x^*,z)}{dz}=0}$ can be represented as follows.

$$\frac{d^2R(x^*,z)}{dz^2}\Big|_{\frac{dR(x^*,z)}{dz}=0} = -\lambda\frac{\alpha^2}{(1+z\alpha)^5e(1-\gamma)M(x^*,z)^2}f(n)$$

where

$$\begin{aligned}
f(n) &= 2(1+z\alpha)x^*T(z) - 3M(x^*,z) \\
&= (2p-n)[3\phi(n)+n\phi'(n)] - 3[2\Phi(n)+n\phi(n)] \\
&= (6p-9n)\phi(n) + 2n(p-n)\phi'(n) - 6\Phi(n)
\end{aligned}$$

Notice that $x^* + z\alpha x^*$ is decreasing in z, so n is increasing in z. Thus, analyzing the polynomial function $f(n)$ is sufficient for determining the sign of $\frac{d^2R(x^*,z)}{dz^2}|_{\frac{dR(x^*,z)}{dz}=0}$.

(1) If $f(n) > 0$, then $\frac{d^2R(x^*,z)}{dz^2}|_{\frac{dR(x^*,z)}{dz}=0} < 0$, $R(x^*,z)$ is quasi-concave in z, which implies that $R(x^*,z) = \frac{dE(x^*,z)}{dz}$ has at most two roots, the larger of the two corresponds to a local maximum and the smaller of the two corresponds to a local minimum of $E(x^*,z)$, and the larger one makes a change of sign for $\frac{dE(x^*,z)}{dz}$ from

positive to negative. Since $f(0) = 6p\phi(0) \geq 0$, then we needn't to consider the case $f(n) \leq 0$ here.

(2) If $\frac{df(n)}{dn}|_{f(n)=0}$, then $f(n)$ will be always a negative value as long as $f(n)$ changes its sign from positive to negative as n increasing. Since $f(0) = 6p\phi(0) \geq 0$, then $\frac{df(n)}{dn}|_{f(n)=0}$ means that $R(x^*, z)$ will changes it's shape from quasi-concave to quasi-convex at one time, which implies that $R(x^*, z) = \frac{dE(x^*,z)}{dz}$ has at most three roots, the root which makes a change of sign for $\frac{dE(x^*,z)}{dz}$ from positive to negative corresponds to a local maximum of $E(x^*, z)$.

(3) Otherwise, the shape of $R(x^*, z)$ is determined by the rank of $f(n)$. Suppose i is the rank of $f(n)$, then there are at most i roots for $f(n) = 0$, and the first root indicates a change of sign for $f(n)$ from positive to negative since $f(0) = 6p\phi(0) \geq 0$, which implies that $R(x^*, z)$ has most $i+2$ roots and most $\lfloor \frac{i+2}{2} \rfloor$ of them indicate a change of sign for $R(x^*, z)$ from positive to negative. □

Proof of Corollary 4.3

Proof. For Uniform distribution of the hotel's competitor's rental rate, we have $p_c \backsim U[A, B]$, then $\phi(n) = \frac{1}{B-A}$, $\Phi(n) = \frac{n-A}{B-A}$, $\phi'(n) = 0$, and

$$\begin{aligned} f(n) &= (6p - 9n)\phi(n) + 2n(p-n)\phi'(n) - 6\Phi(n) \\ &= \frac{3}{B-A}[2(p+A) - 5n] \end{aligned}$$

Thus, $f(n)$ changes its sign from positive to negative as n increasing from 0 to p at most one time. So, there are two cases. In the first case, $f(n)$ is always positive as z increasing, it is equivalent to that $R(x^*, z)$ is quasi-concave function in z, implying that $R(x^*, z) = dE(x^*, z)/dz$ has at most two roots, the second one corresponds to a local maximum and another correspond to a local minimum of $E(x^*, z)$; In the second case, $f(n)$ changes its sign from positive to negative as n increasing from 0 to p at one time, it is equivalent to that $R(x^*, z)$ changes from quasi-concave to quasi-convex as z increasing, implying that $R(x^*, z) = dE(x^*, z)/dz$ has at most three roots, the root which makes a change of sign for $R(x^*, z)$ from positive to negative corresponds to a local maximum and others correspond to a local minimum of $E(x^*, z)$. And, in each case, there is only one value of z makes a change of sign for $R(x^*, z)$ from positive to negative, which corresponds to the unique local maximum point of $E(x^*, z)$. □

Proof of Corollary 4.4

Proof. For Exponential distribution of the hotel's competitor's rental rate, we have $p_c \backsim E(1/\theta)$, then $\phi(n) = \frac{1}{\theta}e^{-\frac{n}{\theta}}$, $\Phi(n) = 1 - e^{-\frac{n}{\theta}}$, $\phi'(n) = -\frac{1}{\theta}\phi(n) = -\frac{1}{\theta^2}e^{-\frac{n}{\theta}}$, and

$$\begin{aligned} f(n) &= (6p - 9n)\phi(n) + 2n(p-n)\phi'(n) - 6\Phi(n) \\ &= \left\{2n^2 - (2p + 9\theta)n + 6\theta(\theta + p)\right\}\frac{1}{\theta^2}e^{-\frac{n}{\theta}} - 6 \end{aligned}$$

Let $f_t(n) = 2n^2 - (2p + 9\theta)n + 6\theta(\theta + p)$, then $f(n) = f_t(n)\frac{1}{\theta^2}e^{-\frac{n}{\theta}} - 6$.

As Figure B.1 shows, $f_t(n) = 0$ has two roots: n_1 and n_2. And $n_1 = \frac{2p+9\theta-\sqrt{\Delta}}{4}$, $n_2 = \frac{2p+9\theta+\sqrt{\Delta}}{4}$, where $\Delta = 33\theta^2 - 12p\theta + 4p^2 > 0$. If $n < n_0$ ($n_0 = \frac{2p+9\theta}{4}$), $f_t(n)$ is decreasing in n. Notice that $\frac{1}{\theta^2}e^{-\frac{n}{\theta}}$ is a decreasing function of n and $f(0) = 6\frac{p}{\theta} > 0$, $f(n_0) = -\frac{\Delta}{8}\frac{e^{-\frac{n_0}{\theta}}}{\theta^2} - 6 < 0$. Thus,$f(n)$ is decreasing from a positive value to a negative value as n increasing from 0 to n_0. Moreover, $f_t(n)$ is negative when $n_0 < n < n_2$, implying that $f(n)$ is negative when $n_0 < n < n_2$. Integrating with the above

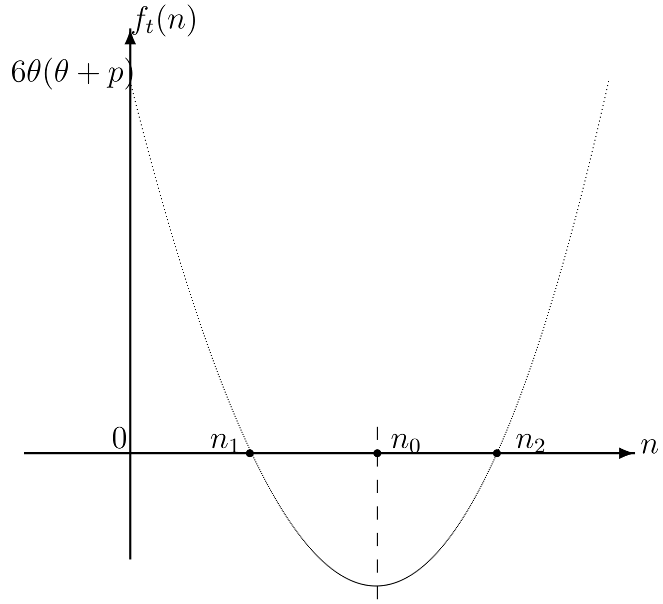

Figure B.1: **The figure of $f_t(n)$.** $n_1 = \frac{2p+9\theta-\sqrt{\Delta}}{4}$, $n_0 = \frac{2p+9\theta}{4}$, $n_2 = \frac{2p+9\theta+\sqrt{\Delta}}{4}$, where $\Delta = 33\theta^2 - 12p\theta + 4p^2$.

analysis, we can obtain that $f(n)$ changes its sign from positive to negative at one time as n increasing from 0 to n_2. And it is very easy to prove that $n_2 > p$.

Then, we propose that $f(n)$ changes its sign from positive to negative at one time as n increasing from 0 to p, which is equivalent to that $R(x^*, z)$ changes from quasi-concave to quasi-convex at one time as z increasing, implying that $R(x^*, z) = dE(x^*, z)/dz$ has at most three roots. The root which makes a change of sign for $R(x^*, z)$ from positive to negative corresponds to the unique local maximum point and the other two roots correspond to the local minimum points of $E(x^*, z)$. □

Proof of Corollary 4.5

Proof. For Normal distribution of the hotel's competitor's rental rate, we have, $\phi(n) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(n-\mu)^2}{2\sigma^2}}$, $\Phi(n) = \int_A^n \phi(p_c)dp_c$, $\phi'(n) = -\frac{n-\mu}{\sigma^2}\phi(n)$ and $\phi''(n) = [(\frac{n-\mu}{\sigma^2})^2-$

$\frac{1}{\sigma^2}]\phi(n)$. Then, we can obtain

$$f(n) = (6p - 9n)\phi(n) + 2n(p - n)\phi'(n) - 6\Phi(n)$$

$$\frac{df(n)}{dn} = \Big\{ - 15 - (8p - 13n)\frac{n - \mu}{\sigma^2} + 2n(p - n)[(\frac{n - \mu}{\sigma^2})^2 - \frac{1}{\sigma^2}]\Big\}\phi(n)$$

Using $f_1(n)$ to represent the item in the brace of above equation, since $\phi(n) > 0$, then analyzing $f_1(n)$ is sufficient for determining the shape of $\frac{df(n)}{dn}$. Notice that $f_1(n)$ is a polynomial function of n, and its rank is 4, so $f_1(n) = 0$ has at most 4 roots, and $f(n) = 0$ has at most 5 roots. Thus, $R(x^*, z) = 0$ at most has 7 roots. Additionally, $f(0) = 6p\phi(0) \geq 0$, thus, there are at most 3 roots which make $R(x^*, z) = dE(x^*, z)/dz$ change its sign from positive to negative that correspond to the local maximum of $E(x^*, z)$. $\square$