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Un-Balanced Economic Growth

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Un-balanced Economic Growth

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"Un-balanced" Economic Growth *

By Hing-Man Leung

Abstract: Since the elasticity of substitution between capital and labor is not always one, and since technical progress is not always Harrod-neutral, it is desirable to have an endogenous growth model that admits all sizes of the elasticity and all known technology modes. We derive an equation to do just that, fully describing the per capita income growth rate at all times. It shows a typical economy needing hundreds if not thousands of years to reach its long term growth rate, leading to the conclusion that even the short run may be very long indeed.

JEL Classification: O10, O11, O12, 040

Keywords: The elasticity of substitution, Non-Harrod-neutral technology, short-run growth

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1. Introduction

Growth theory has traditionally focused on the so-called "balanced path." This presupposes either a unitary elasticity of substitution between capital and labor, or a Harrod-neutral mode of technical change, or both. Despite efforts to justify these presuppositions, we are a long way from being sure that they completely describe the world in which we live.[1]

If we are not sure about the size of the elasticity of substitution, then we should have a model that admits all its plausible values, and work towards finding out how growth rates behave under them. The same is true for the mode of technology. Thus is the aim of this paper: to build an endogenous growth model that explicitly allows all conceivable elasticities, and all forms of technical progress.

To do this, we derive a dynamic equation comprising four exogenous parameters on its right-hand side, together with time t

$$\hat{y}(t) = \Psi[\sigma, a, b, \phi(0), t], \tag{1}$$

where $\hat{y}$ is the endogenous rate of per capita income growth, σ is the elasticity of substitution between capital and labor; a and b are defined in a general production function

$$Y(t) = F[e^{at}K(t), e^{bt}L(t)], \tag{2}$$

where Y is income or output, while (2) in turn implies the technology modes set out in (3) below

[1] There is a growing recognition that the elasticity is typically not one, and often quite far from this number. For example, Arrow, Chenery, Minhas and Solow (1961, p. 226) showed that the United States of America in the first half of the twentieth century had "an over-all elasticity of substitution between capital and labor significantly less than unity". Many studies since then, surveyed in Yuhn (1991, p. 343) reported U.S. elasticities not exceeding 0.76. Yuhn however found the Republic of Korea having a significantly higher σ, probably much closer to 1. Duffy and Papageorgiou (2000, p. 87) rejected the Cobb-Douglas, and found labor and capital "more substitutable in the richest group of countries and are less substitutable in the poorest group of countries". Klump, McAdam and Wilmann (2004) found the elasticity of substitution well below unity for the U.S. economy from 1953 to 1998. Antràs (2004) concluded that the U.S. economy was not well described by a Cobb-Douglas aggregate production function. Zarembka (1970) and Bernt (1976) found elasticity close to but less than 1.

In addition, papers such as Acemoglu's "Directed Technical Change" (2002) remind us that technical change is not always Harrod-neutral.

$$\begin{cases} a = 0, \quad b > 0 & \Rightarrow \quad \text{Harrod neutrality} \\ a > 0, \quad b = 0 & \Rightarrow \quad \text{Solow neutrality} \\ a = b > 0 & \Rightarrow \quad \text{Hicks neutrality}; \end{cases} \tag{3}$$

the fourth variable, $\phi(0)$, is the initial share of income earned by capital. We could have used initial conditions $\hat{y}(0)$ instead of $\phi(0)$, but that changes little. The important point is that all of the exogenous variables in the system are on the right-hand side of (1); all those remaining and not included in the equation, namely savings, capital formation, and the factor shares are endogenously determined.

Using (1) we can plot the growth path precisely, and conduct comparative analysis in terms of σ, a, b and $\phi(0)$. Because of the familiarity with the balanced path, we contrarily call this the *un-balanced* path. "Unbalanced" does not mean that growth behaves in some unwieldy way. By contrast we find the economy neither collapses in the sense of failing to sustain itself, nor dwindles to stagnation; instead it converges towards an asymptotic, constant, and positive growth rate at least when the elasticity of substitution does not exceed one.

This dynamic equation has other uses, too. For instance, it helps us to find how long it takes to converge to the long-term growth rate, and to answer: Just how long is the long run? It turns out that the long run, and even the short run, is much longer than we previously thought. It takes hundreds or sometimes thousands of years for the economy to gravitate to its long-term state.

Some economists asked "How long is the long run?" a few decades ago, for example Atkinson (1969), Drandakis and Phelps (1966), and others. We give a very different answer here, and our answer is more accurate because both $\hat{K}$ and $\phi(t)$ are endogenous.

Our plan of the paper is as follows. Section two traces the factor shares, section three the income growth path, both under Hicks-neutral technology with any elasticity. Section four

generalizes it to all technology types, comparing and contrasting between them, finding out which technology mode is more growth-prone. Section five concludes the paper.

2. Tracing the Factor Shares

We want finally to be able to trace income growth, but to that end we first need to find the path of the factor share. This share is interesting in its own right, depicting as it does the income distribution as a nation grows.

We begin with a Hick-neutral technology. Write the homogeneous of degree one, strictly concave-in-factors production function as

$$Y(t) = A(t) \cdot F[K(t), L(t)], \tag{4}$$

where Y, K, L and A are output, capital, labor and technology, respectively. Define capital's share as

$$\phi(t) \equiv \frac{Y_K(t)K(t)}{Y(t)}, \tag{5}$$

where and henceforth a subscript denotes a derivative. From (4) we have

$$\phi(t) \equiv \frac{F_K(t)K(t)}{F(t)}. \tag{6}$$

Let us denote any variable z's time-derivative by $\dot{z}(t) \equiv dz(t)/dt$, its growth rate by $\hat{z}(t) \equiv \dot{z}(t)/z(t)$, and keep the time reference implicit when possible. Differentiating (6) we get

$$\hat{\phi} = \hat{F}_K + \hat{K} - \hat{F};$$

differentiating $F[K(t), L(t)]$ we find

$$\hat{F} = \phi(\hat{F}_K + \hat{K}) + (1-\phi)(\hat{F}_L + \hat{L});$$

combining the last two equations we get

$$\hat{\phi} = (1-\phi)(\hat{K} - \hat{L}) + \hat{F}_K . \tag{7}$$

All the variables in (7) are endogenous and can change over time, except $\hat{L}$, but of that we have no interests.

In its intensive form we can write $F(K,L) = Lf(k)$, where $k \equiv K/L$, and $F_K = f_k$. Differentiating it, and introducing the elasticity of substitution

$$\sigma \equiv -\frac{f_k(f - k f_k)}{k f f_{kk}} , \tag{8}$$

we have

$$\hat{F}_K = -\frac{1}{\sigma}(1-\phi)(\hat{K} - \hat{L}) . \tag{9}$$

Substituting (9) into (7) we get

$$\hat{\phi} = (1-\frac{1}{\sigma})(1-\phi)(\hat{K} - \hat{L}) . \tag{10}$$

A variant of equation (10) first appeared in Drandakis and Phelps (1966), and later in Atkinson (1969). Notice in obtaining (10) we have *not* used any growth models, not even the traditional assumption that a constant fraction of income is saved. We have merely assumed that *K* and *L* are paid their competitive returns.

Using (10), we could be tempted to perform "back of an envelop" calculations to find $\phi(t)$.[2] For instance, if σ is 0.7, $\hat{K}$ 20 per cent per year, $\hat{L}$ 2 per cent per year and ϕ 40 per cent, $\hat{\phi}$ would have fallen by about 4.6 per cent per year, and it would have taken merely 6.2 years for the capital share to fall by ten percentage points. However, this is wrong, since $\hat{K}$ and ϕ on the right-hand side of (10) are erroneously fixed. In fact we should be curious about (10): how could

[2] See Atkinson (ibid.).

the rate of technical progress, $\hat{A}$, not help determine capital's share, even when the elasticity is not unity? The correct answer is that indeed it should, as we will soon find out.

2.1 Integrating the differential equation

A straightforward first step is to eliminate $\phi(t)$ from the right-hand side of (10), and then replace it with its initial value. This turns out not to change much, but is nevertheless a necessary step to take. Equation (10) can be written as

$$\dot{\phi}(t) = \phi(t)[1-\phi(t)](1-\frac{1}{\sigma})(\hat{K}-\hat{L}). \tag{11}$$

Treating $\hat{K}$ and $\hat{L}$ as constant for now, we integrate (11) as a first-order differential equation. Writing $\hat{k} \equiv \hat{K} - \hat{L}$, this gives us

$$\phi(t) = \frac{e^{\hat{k}t}\phi(0)}{e^{\hat{k}t/\sigma}[1-\phi(0)] + e^{\hat{k}t}\phi(0)}, \tag{12}$$

where $\phi(0)$ is the initial capital share.

$\phi(t)$, according to (12), falls even faster than (10). We plot both paths using $\sigma = 0.7$, $\phi(0) = 0.4$, $\hat{K} = 0.2$ and $\hat{L} = 0.02$, and compare them in table 1 below. The $\phi(t)$ row uses equation (10), and the $\phi(t)*$ row (12). Integrating for $\phi(t)$ has not changed the numbers much, suggesting erroneously a short run that is far shorter than it is.

Table 1

Year(t)	0	10	20	30	40	50	60	70	80	90	100
$\phi(t)$	0.400	0.252	0.159	0.101	0.063	0.040	0.025	0.016	0.010	0.006	0.004
$\phi(t)*$	0.400	0.291	0.142	0.066	0.030	0.014	0.007	0.003	0.001	0.001	0.000

Key: the $\phi(t)$ column uses equation (10); the $\phi(t)*$ column uses equation (12).

2.2 Endogenizing $\hat{K}$

Just as the share $\phi(t)$ change over time, so does the rate of capital accumulation. While a poor nation has meager means to accumulate capital, an emerging country saves and accumulates vigorously to fuel growth, and a developed nation by contrast prefers spending to saving. Sub-Saharan Africa, China, and the United States are obvious cases in point. I have examined the World Development Indicators published by the World Bank, and found positive and statistically significant time-trends for gross capital formation as a percentage of GDP for both the low- and middle-income countries, but a negative trend for the developed ones. The need to endogenize $\hat{K}$ is clear.

It is customary to model savings as a consequence of individual citizens maximizing their discounted future utility from consumption. We normally take for granted the Ramsey utility function

$$u(t) = \frac{c(t)^{\alpha} - 1}{\alpha},$$

where α, between 0 and 1, defines $u(t)'s$ curvature and the speed at which marginal utility diminishes as consumption increases. However, new evidence has emerged to question the suitability of the Ramsey utility in studying growth. If we suddenly have twice as much to eat and to wear, marginal utility would undoubtedly diminish. But if we have twice as much to eat and wear in ten, twenty or even fifty year's time, marginal utility would not fall quite as much, or it may not fall at all since life-style, custom, and technology would have changed drastically by then. It becomes even more problematic to assume diminishing marginal utility over hundreds or more years. La Grandville (2006, 2007) has recently made a striking discovery that this Ramsey utility yields an optimal savings rate approaching 100 percent of income within a few years for any reasonable initial savings rates, leading him to question the appropriateness of the Ramsey

formula. He further shows that replacing $u(t)$ with $c(t)$, or in other words setting α to one, produces a much more reasonable picture. We will follow Grandville, and work with $c(t)$ instead of $u(t)$.

L are workers as well as consumers, who consume $c(t)$ in period t, giving rise to the aggregate objective function

$$\int_0^\infty e^{-\rho t} c(t) L(t) dt, \tag{13}$$

where ρ is a constant discount rate, and (13) is maximized subject to a savings-investment constraint

$$\dot{K}(t) = A(t) \cdot F\left[K(t), L(t)\right] - c(t)L(t). \tag{14}$$

Dynamic optimization yields the Euler's equation

$$A(t) F_K(t) = \rho. \tag{15}$$

Since ρ is constant, so is $A(t) F_K(t)$. Differentiating this and suppressing the time reference we have $\dot{F}_K / F_K = -\dot{A}/A = -\mu$. This constancy of AF_K governs optimal savings. If $\mu = 0$, people save and invest in order to keep F_K constant, and that means keeping $\dot{K}/K$ abreast with the exogenous $\dot{L}/L$. However if $\mu > 0$, we have an additional incentive to save and to add to the capital stock. Each unit of new capital alters the marginal product F_K in a way dictated by the elasticity of substation between capital and labor, for that reason the elasticity plays a crucial role in the process of endogenous growth.

In intensive form we have $y \equiv \frac{Y}{L} = A \cdot f(k)$ where $k \equiv \frac{K}{L}$, and thus $F_K(K, L) = f_k(k)$. Differentiating it we get $\dot{F}_K = f_{kk}\dot{k}$, where f_{kk} is the second-derivative with respect to k, and we can write

$$\dot{F}_K / F_K = [\, f_{kk} / f_k \,]\dot{k}\,. \tag{16}$$

Using (16), the elasticity of substitution $\sigma \equiv -\dfrac{f_k(f - k\,f_k)}{k\,f\,f_{kk}}$, and $\dot{F}_K / F_K = -\mu$ we have

$$\frac{\dot{k}}{k} = \mu\sigma\left[\frac{f}{f - k\,f_k}\right] = \mu\sigma\left[\frac{F}{LF_L}\right]. \tag{17}$$

Since F/LF_L is the inverse of the share of labor in national income, we can write $\dfrac{\dot{k}}{k} = \mu\sigma/(1-\phi)$. Substituting this into (12) we have

$$\dot{\phi}(t) = \phi(t)\mu(\sigma - 1)\,,$$

and from which we obtain an equation defining the time-path of capital's share

$$\phi(t) = \phi(0)\exp[\mu(\sigma - 1)t]\,. \tag{18}$$

Assume $\mu = 0.02$, $\sigma = 0.7$, and $\phi(0) = 0.4$ as before, we plotted $\phi(t)^{**}$ using (18), in table 2. The numbers from table 1 are included for comparison.

Table 2: The endogenous share of capital $\phi(t)$

Year(t)	0	10	20	30	40	50	60	70	80	90	100
$\phi(t)$	0.400	0.252	0.159	0.101	0.063	0.040	0.025	0.016	0.010	0.006	0.004
$\phi(t)^*$	0.400	0.291	0.142	0.066	0.030	0.014	0.007	0.003	0.001	0.001	0.000
$\phi(t)^{**}$	0.400	0.377	0.355	0.334	0.315	0.296	0.279	0.263	0.248	0.233	0.220

Key: $\phi(t)$ and $\phi(t)^*$ are values taken from table 1.

The difference between the bottom row and the other rows speaks volumes about the importance of endogenizing $\dot{K}(t)$. The technology progress rate μ plays a pivotal role in (18); for instance a larger μ speeds up the decline of capital's share, when capital is not good enough a substitute for labor (σ is less than 1). This is precisely what we would expect, because then

capital's marginal reward falls disproportionately faster than capital is added to production. In addition, capital accumulates more rapidly the larger is μ.

The long run seems quite long, taking hundreds of years for the factor shares to change significantly. The factor share path gives a glimpse of what a country's growth path might look like – per capita income probably takes just as long to change as the factor share. This conjecture will be proved correct in what follows. However, it is imperative to know precisely how $\hat{y}$ behaves over time because, as it turns out, the shares path misses something critical, making it a poor proxy for gauging the incomes path. We will soon verify this.

3. The Incomes Path

We will find in the next section a dynamic income growth equation that admits all three technology modes, but for the moment focus only on the Hicks-neutral technology. Differentiating $\phi = F_K K / F$ we may write

$$\hat{F} = \hat{F}_K + \hat{K} - \hat{\phi}. \tag{19}$$

Since technology is Hicks-neutral, we can use $\hat{\phi} = \mu(\sigma - 1)$ from (18) in (19) to get

$$\hat{F} = \hat{K} - \mu\sigma. \tag{20}$$

From (17) we also have

$$\hat{K} = \frac{\mu\sigma}{(1-\phi)} + \hat{L}. \tag{21}$$

Using (21) in (20), adding μ to both sides and noting $\hat{y} = \mu + \hat{F} - \hat{L}$, we have

$$\hat{y}(t) = \mu\left[1 + \sigma\left(\frac{\phi(t)}{1-\phi(t)}\right)\right]. \tag{22}$$

Substituting to eliminate $\phi(t)$ using (18), we have an income growth path entirely in terms of the exogenous parameters and *t*

$$\hat{y}(t) = \mu\left\{1+\phi(0)\sigma\left(\frac{\exp[\mu\sigma t]}{\exp[\mu t]-\exp[\mu\sigma t]\phi(0)}\right)\right\}. \tag{23}$$

We may sometimes want to consider

$$\frac{\hat{y}(t)}{\mu} = 1+\phi(0)\sigma\left(\frac{\exp[\mu\sigma t]}{\exp[\mu t]-\exp[\mu\sigma t]\phi(0)}\right), \tag{24}$$

which is the per capita income growth for each percentage technical improvement μ. There are three cases to consider.

3.1 $\sigma = 1$

Substituting $\sigma = 1$ into (24) we get

$$\frac{\hat{y}(t)}{\mu} = \frac{1}{1-\phi(0)}. \tag{25}$$

This is well known but still worth spelling out: when $\sigma = 1$, $\hat{y}$ stays constant at a *multiple* of the speed of technical progress. If the initial capital share is $\phi(0) = 0.4$, per capita income grows at $5/3$ times the technology growth rate. We know also, using (18), that $\phi(t)$ stays constant at all *t*. Thus we have a striking result: a country grows faster if a larger share of income goes to capital and a smaller share to labor, given a Hicksian technology and a unitary elasticity of substitution. Workers may still gain if the national cake enlarges fast enough to more than offset their reduced share, but the Hicks-neutral technology is biased towards capital. This is the only case precisely known hitherto.

3.2 $\sigma < 1$

Differentiating the fractional term in (24) with respect to *t* we get

$$d\left(\frac{\exp[\mu\sigma t]}{\exp[\mu t]-\exp[\mu\sigma t]\phi(0)}\right)\Big/dt = -\frac{\exp[\mu t(1+\sigma)]\mu(1-\sigma)}{\left\{\exp[\mu t]-\exp[\mu\sigma t]\phi(0)\right\}^2}, \tag{26}$$

which is negative for $\sigma < 1$. From this we know $\hat{y}(t)$ must be falling at all *t*.

It is easy to verify that for all $\sigma < 1$,

$$Lim\left(\frac{\exp[\mu\sigma t]}{\exp[\mu t]-\exp[\mu\sigma t]\phi(0)}\right)\Bigg|_{(t\to\infty)} = 0\,. \tag{27}$$

It follows that $\hat{y}$ is initially greater than μ, falling at all *t*, and asymptotically reaches μ.

If $\mu = 0.02$, $\sigma = 0.7$ and $\phi(0) = 0.4$, it takes between 700 to 800 years for $\hat{y}$ to reach μ, as shown in figure 1 below.

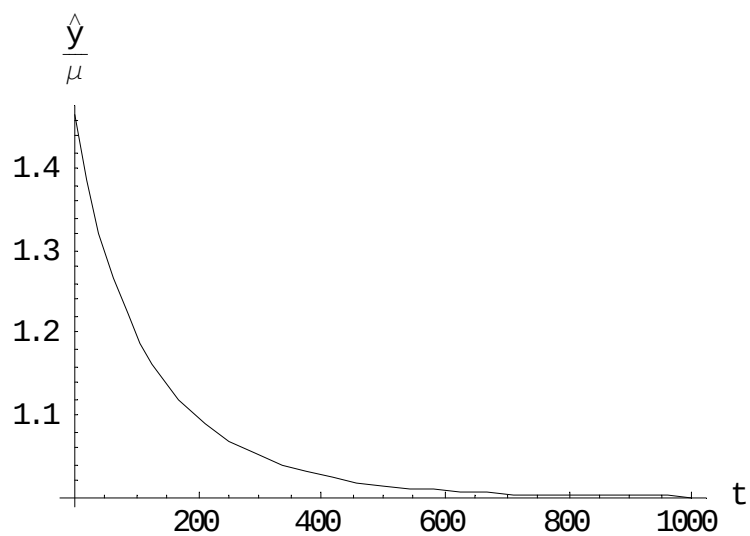

Figure 1: `Plot[1+ϕ0σ(e^(μσt)/(e^(μt) − e^(μσt)ϕ0)) /. {μ → .02, σ → .7, ϕ0 → .4}, {t, 0, 1000}]`

Increasing σ towards 1 lengthens the time needed to reach μ, which makes sense because of what we found in section 3.1 above. Figure 2 below shows that if σ goes from 0.7 to 0.8, $\hat{y}$ takes thousands instead of hundreds of years to reach μ.

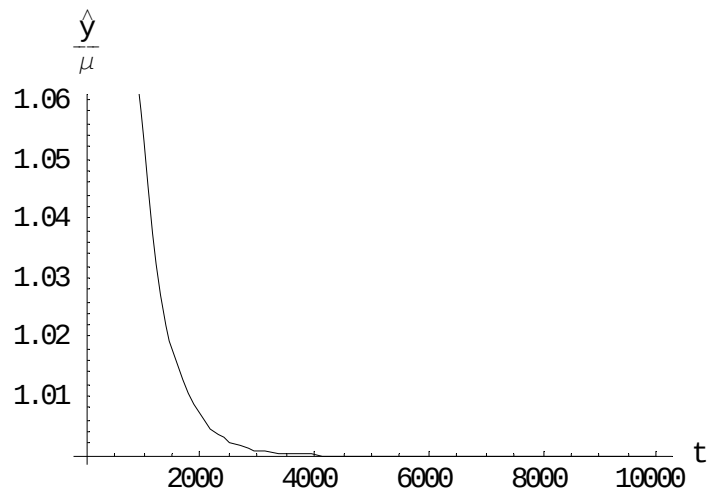

Figure 2: $\texttt{Plot}\left[1+\phi 0\,\sigma\left(\frac{e^{\mu\sigma t}}{e^{\mu t}-e^{\mu\sigma t}\phi 0}\right)\ /.\ \{\mu\to .02,\ \sigma\to .9,\ \phi 0\to .4\},\ \{t,\ 0,\ 10000\}\right]$

Using (24) we can plot any growth path using whatever parameter values we deem suitable. Generally speaking, a larger μ, a smaller σ, and a larger $\phi(0)$ make the curve steeper, shifting it to the left, shortening the short-run; and conversely. For as long as $\sigma<1$, $\hat{y}$ gravitates towards μ, and it takes longer to do so if σ is closer to 1. If σ reaches 1, however, $\hat{y}$ jumps abruptly and permanently to $1/[1-\phi(t)]$ times μ, whereupon $\phi(t)$ itself becomes a constant. In this way, σ controls a "gearing" mechanism, leveraging $\hat{y}$ upwards, using μ as a base.

3.3 $\sigma > 1$

It is common knowledge that $\sigma > 1$ leads to an upwardly explosive growth, and $d\hat{y}/dt > 0$ from (26) reinforces that belief. It is interesting, nonetheless, to plot a typical path using (24), as we have done in figure 3.

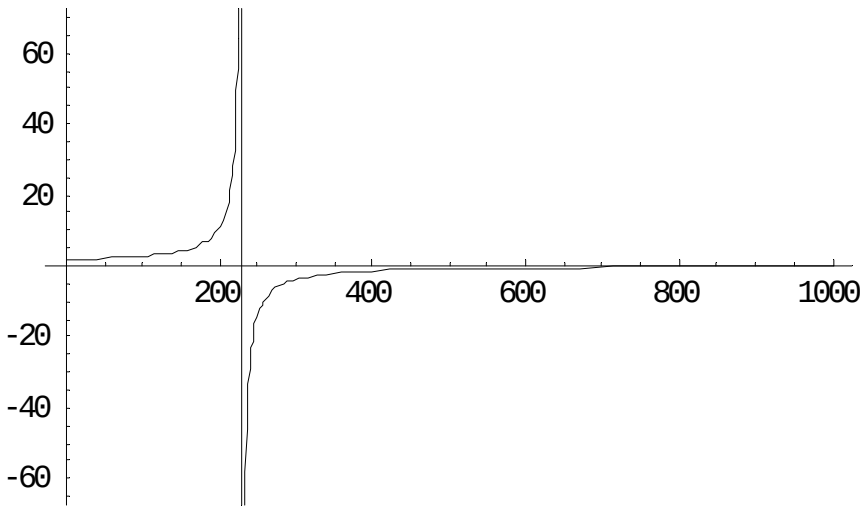

Figure 3: $\texttt{Plot}\left[1 + \phi 0\, \sigma \left(\frac{e^{\mu \sigma t}}{e^{\mu t} - e^{\mu \sigma t}\, \phi 0}\right) /. \{\mu \to .02, \sigma \to 1.2, \phi 0 \to .4\}, \{t, 0, 1000\}\right]$

It first appears that the path turns from $+\infty$ to $-\infty$ at some t above 200 years; but from (18), $\phi(t)$ becomes unity at exactly 229.073 years using the parameters in figure 3, and from that point on capital receives more than 100 per cent of national income. We must therefore rule out the region to the right of 229.073 years in figure 3.

Growth positively explodes at some finite time for all $\sigma > 1$. A larger μ, a larger σ, or a larger $\phi(0)$ bring that explosive date sooner. Although we know very little what explosive growth means in practice, a larger σ without doubt is growth-promoting.

4. A General Technology

Instead of the Hicks-neutral (4) suppose we have

$$Y(t) = F[G(t), H(t)], \quad G \equiv e^{at}K(t) \quad and \quad H(t) \equiv e^{bt}L(t), \quad a, b > 0 . \tag{28}$$

The variables G and H stand for efficiency units of capital and labor. If $a = b$ we are back to (4) and the conclusions reached there apply. The capital $\hat{K}(t)$ comes from endogenous savings as before, and $\hat{L}(t)$ is assumed exogenously constant. Our aim here is to discover which technology mode yields faster income growth. Recall from (3) that $a \gg b$ describes a more Solow-neutral mode, etc. We first need an equation for the factor shares.

4.1 The factor shares

We will show that the factor shares path is completely independent of b, but the incomes path is not. Consequently, the factor share is a poor proxy for the income growth path.

Differentiating (28), rearranging it, and analogous to (10) we get

$$\hat{\phi} = (1 - \frac{1}{\sigma})(1 - \phi)[(a - b) + (\hat{K} - \hat{L})] . \tag{29}$$

The variables ϕ, $\hat{K}$ and $\hat{L}$ on the right-hand side are endogenous, but we can find them using the method used before.

Notice that the aggregate consumption term in the investment equation is in terms of L and not H. Physical capital accumulates according to

$$\dot{K}(t) = F[G(t), H(t)] - c(t)L(t) , \tag{30}$$

and the Hamiltonian, denoted Γ since H is efficiency labor, is

$$\Gamma(t) = c(t)L(t) - \theta\{F[G(t), H(t)] - c(t)L(t)\} . \tag{31}$$

Keeping the time-reference implicit, the Euler equation is

$$e^{at}F_G = \rho\,. \tag{32}$$

Differentiating this gives

$$\hat{F}_G = a\,. \tag{33}$$

But the intensive form is in terms of H and not L. Hence we have

$$y \equiv \frac{F}{L} = \frac{e^{bt}}{H} F\left[\frac{G}{H}, 1\right] = e^{bt} f(g) \quad \text{where} \quad g \equiv \frac{G}{H} = e^{(a-b)t}\frac{K}{L} = e^{(a-b)t}k\,. \tag{34}$$

We may also write $F(G,H) = Hf$. Differentiating it we have $F_G G_g = Hf_g$. Using $G = gH$ and thus $G_g = H$ this becomes $F_G = f_g$. Differentiating we get $\dot{F}_G = f_{gg}\dot{g}$, or $\dot{F}_G / F_G = f_{gg}\dot{g} / f_g$. Introducing the elasticity of substitution $\sigma \equiv -\dfrac{f_g(f - g\,f_g)}{g\,f\,f_{gg}}$, and rearranging, we have

$$\frac{\dot{g}}{g} = a\sigma\frac{1}{(1-\phi)}\,. \tag{35}$$

From $g = e^{(a-b)t}k$ we know $\hat{g} = (a-b) + (\hat{K} - \hat{L})$. Substituting this into (35) and then into (29) we get $\hat{\phi}(t) = a(\sigma - 1)$. Integrating it, we finally have

$$\phi(t) = \phi(0)\exp[a(\sigma - 1)t]\,, \tag{36}$$

which is almost identical to (18). This is what we claimed earlier: the path of the factors' shares is completely independent of the human capital accumulation rate b.

Notice that (36) shows something we have known all along – that the factor shares will stay constant under either the Harrod-neutral technology $a = 0$, or the Cobb-Douglas technology $\sigma = 1$. What we did not know is the exactly manner in which $\phi(t)$ evolves, which is now displayed vividly in (36). $\phi(t)$ is a bad proxy for $y(t)$ because the latter is very much a function of *b*.

4.2 The income path

Differentiating $y = e^{bt} f(g)$ from (34) we have

$$\hat{y} = b + \hat{f} \ . \tag{37}$$

Notice $\dot{f} = f_g \dot{g}$, and thus

$$\frac{\dot{f}}{f} = \frac{g\, f_g}{f} \frac{\dot{g}}{g} = \phi \hat{g} \ . \tag{38}$$

Using this and (35) in (37) we have

$$\hat{y}(t) = b + a\sigma \frac{\phi(t)}{[1-\phi(t)]} \ . \tag{39}$$

Using (36), we finally have

$$\hat{y}(t) = b + a\sigma \frac{\phi(0)\exp[a(\sigma-1)t]}{1-\phi(0)\exp[a(\sigma-1)t]} \ . \tag{40}$$

It is straightforward to check that $\hat{y}(t)$ is constant if either $\sigma = 1$, or $a = 0$ and $b > 0$.

Two results emerge immediately. First, there is a one-to-one positive relationship between the human capital and the per capita incomes growth rate *b*, even though *b* has no impact on $\phi(t)$.

Second, the relations between $\hat{y}(t)$ and a is a complex one. If $\sigma < 1$, the exponential terms go to 0 as *t* goes to infinity, and asymptotically $\hat{y}(t)$ converges to *b*. Since that takes a very long time to happen, it is important to know more than just the asymptote. To get a clearer picture we can differentiate (40) with respect to *a* to get

$$\frac{d\hat{y}(t)}{da} = \frac{\phi(0)\exp[a(\sigma-1)t]\{1-at(1-\sigma)-\phi(0)\exp[a(\sigma-1)t]\}}{\{1-\phi(0)\exp[a(\sigma-1)t]\}^2} \ , \tag{41}$$

which when plotted yields figure 4.

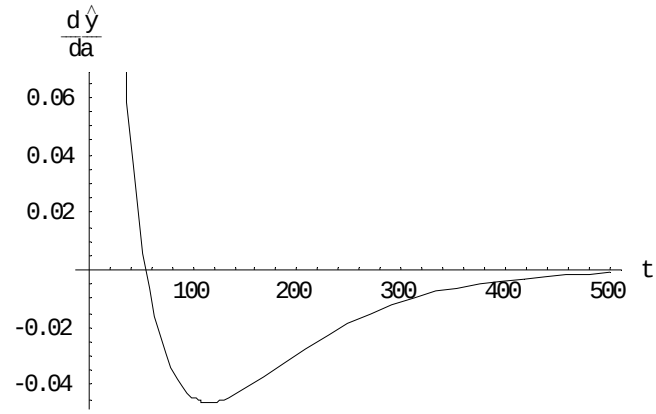

Figure 4:

$$\text{Plot}\left[-\frac{e^{at(-1+\sigma)}\,\sigma\,\phi 0\,(-1+a\,(t-t\,\sigma)+e^{at(-1+\sigma)}\,\phi 0)}{(-1+e^{at(-1+\sigma)}\,\phi 0)^2}\ /.\ \{\phi 0\to .4,\ \sigma\to .7,\ a\to .05\},\ \{t,\ 0,\ 500\}\right]$$

What this says is that raising the technology progress rate of physical capital, *a*, would raise the per capita income growth rate for some 50 years, but lowering it thereafter, and the net effect fizzles out eventually. Using other parameter values of $\phi(0)$ and a leaves the pattern largely unaltered – the time profile always has a positive section followed by a negative one.

For $\sigma > 1$ however, $d\hat{y}(t)/da$ is always positive, giving a picture such as figure 5.

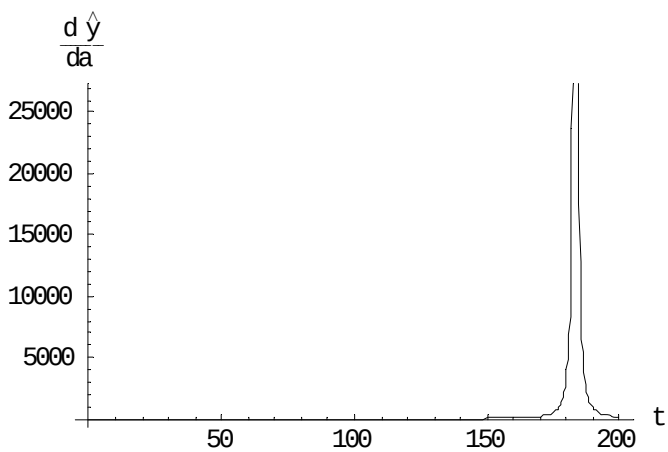

Figure 5:

$$\text{Plot}\left[-\frac{e^{at(-1+\sigma)}\,\sigma\,\phi 0\,(-1+a\,(t-t\,\sigma)+e^{at(-1+\sigma)}\,\phi 0)}{(-1+e^{at(-1+\sigma)}\,\phi 0)^2}\ /.\ \{\phi 0\to .4,\ \sigma\to 1.1,\ a\to .05\},\ \{t,\ 0,\ 200\}\right]$$

Using our previous reasoning, $d\hat{y}(t)/da$ after the spike at about 185 years is not meaningful since $\phi(t)$ would then have reached 1.

As an exercise suppose $a+b=0.06$. Figure 6 gives a slide-show of reducing *a* and increasing *b*, keeping their sum equal to 0.06. As the technology mode shifts towards Harrod-neutral from Solow-neutral (increasing *b* and decreasing *a*), the curve in figure 6 shifts towards the northeast direction. Improving human capital leads to a faster and longer lasting growth, than improving the efficiency of physical capital.

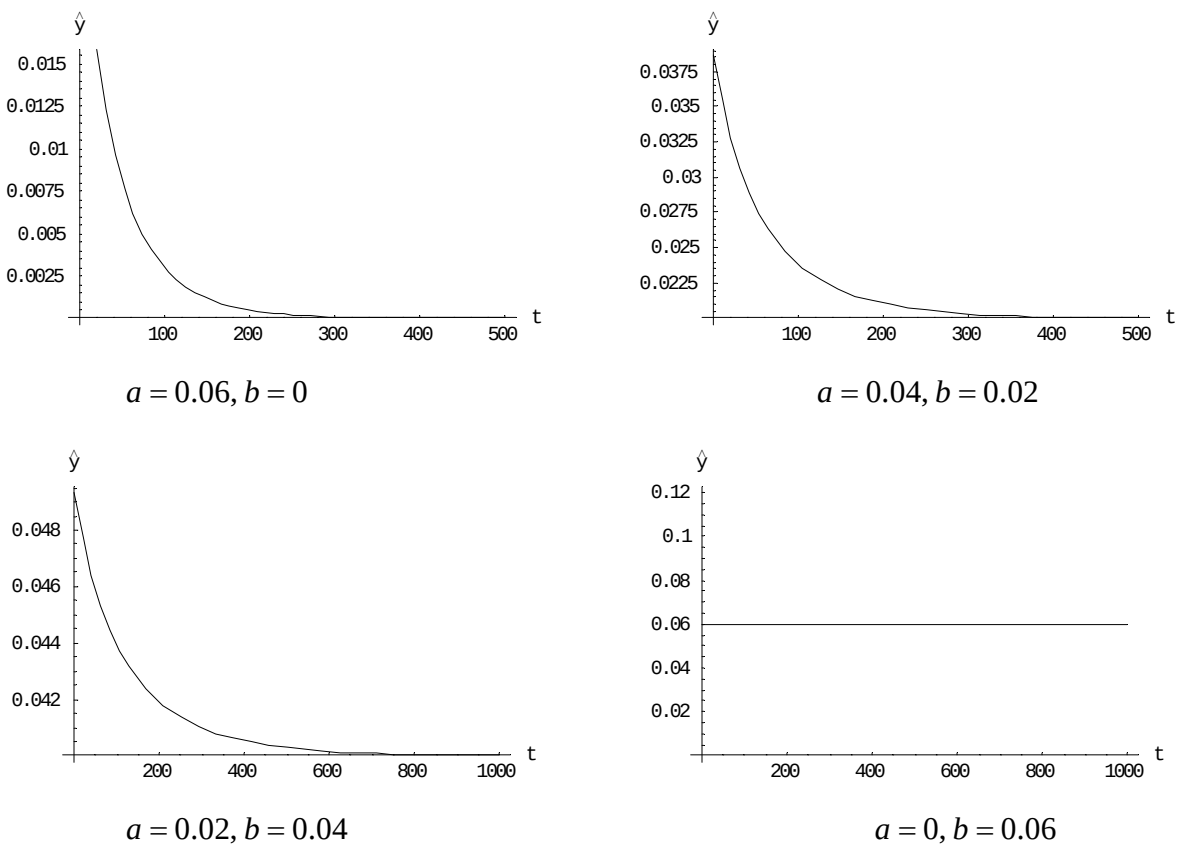

Figure 6: plotting $\hat{y}(t)$ using equation (40), and $\sigma=0.7$, $\phi(0)=0.4$

5. Conclusions

Some findings in the foregoing are familiar, the others are relatively new. The major ones may be summarized more systematically as follows.

1. A larger elasticity of substitution in general leads to faster and longer-lasting growth in per capita income.
2. A more Harrod-neutral type of technical progress has more growth-promoting properties, than a more Solow-neutral type.
3. It is quite sensible to talk about long-term growth when the elasticity of substitution between capital and labor is not one. Even the short-run is quite long, taking hundreds or more years for the growth rate to settle to its long-term constant rate. It is time we focused more on phenomena other than the "balanced path."
4. Similarly it is quite reasonable to talk about the non-Harrod-neutral growth. A rise in the technical progress rate has very long-lasting benefits on income growth.

The main theoretical contribution of this paper is the derivation of a dynamic growth equation that is much more general than what we have known from the literature. This equation allows all elasticities and all technology modes, as shown in (40). It should be straightforward to test this equation empirically, if we have good data, and good estimates of the elasticities and the other variables. The incomes and the factor shares paths have something in common – they both take hundreds of years to return to their long term level. Even when there are limited opportunities to substitute capital for labor—when the elasticity of substitution is low—and even when physical capital improves faster than human capital, each mode of technical change have prolonged boosting effects on income which, over time, accumulates to tremendous improvements on our standards of living.

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